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q-ANALOGUE OF MITTAG-LEFFLER OPERATORS

GÜRHAN İÇÖZ AND BAYRAM ÇEKİM

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Abstract. In this paper, we study $q-$extension of Mittag-Leffler operators. We establish moments of these operators and estimate convergence results with the help of classical modulus of continuity. Also, we give weighted approximation property and $A-$statistically convergence of the operators $L_{n,q}^{(\beta)}$.

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1. INTRODUCTION

We first mention some notations of q-calculus. Throughout the present paper, assume that q is a real number satisfying the inequality $0 < q \leq 1$. For $n \in \mathbb{N}$, we have

$$[n]_q = [n] := \begin{cases} (1-q^n)/(1-q) & , \quad q \neq 1 \\ n & , \quad q = 1 \end{cases},$$

$$[n]_q! = [n]! := \begin{cases} [n]\,[n-1]\ldots[1] & , \quad n \geq 1 \\ 1 & , \quad n = 0 \end{cases},$$

and

$$(1+x)_q^n := \begin{cases} \prod_{j=0}^{n-1} \left(1+q^j x\right) & , \quad n = 1,2,\ldots \\ 1 & , \quad n = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}.$$

The q-analogue of integration, introduced by Jackson [7] in the interval $[0,a]$, is defined by

$$\int_0^a f(x)\, d_q x := a(1-q) \sum_{n=0}^{\infty} f\left(aq^n\right) q^n, \; 0 < q < 1 \text{ and } a > 0.$$

The q-improper integral used in the present paper is defined as

$$\int_0^{\infty/A} f(x)\,d_qx := (1-q)\sum_{n=0}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\ A>0,$$

provided the sum converges absolutely. The two q-Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1}E_q(-qt)\,d_qt \text{ and } \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1}e_q(-t)\,d_qt. \tag{1.1}$$

There are two q-analogues of the exponential function e^x, see [8],

$$e_q(x) = \sum_{k=0}^{\infty}\frac{x^k}{[k]!} = \frac{1}{(1-(1-q)x)_q^{\infty}},\ |x| < \frac{1}{1-q},\ |q|<1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]!} = (1+(1-q)x)_q^{\infty},\ |q|<1.$$

By Jackson [7], it was shown that the q-Beta function defined in the usual formula

$$B_q(t,s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}.$$

The q-integral representation of q-Beta function, which is a q-analogue of Euler's formula, is

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}\,d_qx,\ t,s>0. \tag{1.2}$$

In 1903, G.M. Mittag-Leffler [11] defined the Mittag-Leffler function by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)};\ (z\in\mathbb{C},\ R(\alpha)>0).$$

In 1905, A. Wiman [15] gave the two-index Mittag-Leffler function definition by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)};\ (z,\beta\in\mathbb{C},\ R(\alpha)>0).$$

Note that, for $\beta=1$ then we get $E_{\alpha,1}(z) = E_\alpha(z)$.

M.A. Özarslan [12] investigated properties of the following Mittag-Leffler operators

$$L_n^{(\beta)}(f;x) = \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)}\sum_{k=0}^{\infty} f\left(\frac{kb_n}{n}\right)\frac{(nx)^k}{b_n^k\,\Gamma(k+\beta)}, \tag{1.3}$$

M.H. Annaby and Z.S. Mansour [1] defined $q-$analogues of the Mittag-Leffler functions by

$$e_{\alpha,\beta}(z;q)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma_q(\alpha k+\beta)},\ \left|z(1-q)^{\alpha}\right|<1,$$

and

$$E_{\alpha,\beta}(z;q)=\sum_{k=0}^{\infty}\frac{q^{\alpha\frac{k(k-1)}{2}}z^k}{\Gamma_q(\alpha k+\beta)},\ z\in\mathbb{C}$$

where $\beta\in\mathbb{C}$ and $R(\alpha)>0$. Here we will use q-Mittag-Leffler function for $\alpha=1$ as

$$E_{1,\beta}\left(\frac{nz}{b_n};q\right)=\sum_{k=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nz}{b_n}\right)^k.$$

Now we introduce the following operators

$$L_{n,q}^{(\beta)}(f;x)=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}f\left(\frac{[k]\,b_n}{n}\right)\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k \tag{1.4}$$

where (b_n) is a sequence of positive real numbers, $\beta>1$ is fixed, $n\in\mathbb{N}$, and

$$f\in E:=\left\{f\in C[0,\infty):\ \lim_{x\to\infty}\frac{f(x)}{1+x^2}\text{ is finite}\right\},$$

where $C[0,\infty)$ is the space of continuous functions defined on $[0,\infty)$. The norm of Banach lattice E is given by

$$\|f\|_*:=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}.$$

It is obvious that the operators $L_{n,q}^{(\beta)}(f;x)$ are linear and positive.

When $q=1$, we have the Mittag-Leffler operators given by Özarslan [12]. When $q=1$ and $\beta=1$, we get the modified Szász-Mirakjan operators.

Lemma 1. *For each* $0<q<1$, $x\geq 0$ *and* $n\in\mathbb{N}$, *we have*

$$L_{n,q}^{(\beta)}(1;x)=1, \tag{1.5}$$

$$\left|L_{n,q}^{(\beta)}(t;x)-x\right|\leq\frac{[\beta-1]b_n}{n}, \tag{1.6}$$

$$\left|L_{n,q}^{(\beta)}(t^2;x)-x^2\right|\leq\frac{b_n}{n}\left(x+\frac{[\beta-1]b_n}{n}\right)+2x\frac{b_n[\beta-1]}{n}, \tag{1.7}$$

and

$$L_{n,q}^{(\beta)}((t-x)^2;x)\leq\frac{b_n}{n}(1+4[\beta-1])\,x+\left(\frac{b_n}{n}\right)^2[\beta-1]. \tag{1.8}$$

Proof. From definition of the operators $L_{n,q}^{(\beta)}$ given by (1.4), we get

$$L_{n,q}^{(\beta)}(1;x)=1.$$

Using $[k]=[k+\beta-1]-q^k[\beta-1]$ and from (1.4), we have

$$\begin{aligned}
L_{n,q}^{(\beta)}(t;x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\frac{[k]b_n}{n}\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k \\
&= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=1}^{\infty}\frac{b_n}{n}\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta-1)}\left(\frac{nx}{b_n}\right)^k \\
&\quad -\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\frac{b_n}{n}\frac{q^{\frac{k(k-1)}{2}}q^k}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k \\
&\leq \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\frac{b_n}{n}\frac{q^{\frac{k(k+1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^{k+1} \\
&\quad +\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\frac{b_n}{n}\frac{q^{\frac{k(k-1)}{2}}q^k}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k.
\end{aligned}$$

For $q^k\leq 1$, we get

$$\begin{aligned}
L_{n,q}^{(\beta)}(t;x) &\leq \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\frac{b_n}{n}\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^{k+1} \\
&\quad +\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\frac{b_n}{n}\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k \\
&\leq x+\frac{[\beta-1]b_n}{n}.
\end{aligned}$$

So we can write

$$\left|L_{n,q}^{(\beta)}(t;x)-x\right|\leq\frac{[\beta-1]b_n}{n}.$$

Using definition of the operators $L_{n,q}^{(\beta)}$, we have

$$L_{n,q}^{(\beta)}(t^2;x)=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\left(\frac{[k]b_n}{n}\right)^2\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k.$$

Using the fact that $[k]=q[k-1]+1$ and $[k]=[k+\beta-1]-q^k[\beta-1]$, we obtain

$$L_{n,q}^{(\beta)}(t^2;x)=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\left(\frac{b_n}{n}\right)^2(q[k-1][k]+[k])\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k$$

$$=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2[k-1][k]\frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k$$

$$+\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=0}^{\infty}\left(\frac{b_n}{n}\right)^2[k]\frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k$$

$$=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2[k-1]\Big([k+\beta-1]-q^k[\beta-1]\Big)\frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k$$

$$+\frac{b_n}{n}L_{n,q}^{(\beta)}(t;x)$$

$$=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2[k-1]\frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta-1)}\left(\frac{nx}{b_n}\right)^k+\frac{b_n}{n}L_{n,q}^{(\beta)}(t;x)$$

$$-\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=1}^{\infty}\left(\frac{b_n}{n}\right)^2[k-1]\frac{q^{\frac{k(k-1)}{2}+1+k}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k$$

$$=\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2\Big([k+\beta-2]-q^{k-1}[\beta-1]\Big)\frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta-1)}\left(\frac{nx}{b_n}\right)^k$$

$$-\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2[k-1]\frac{q^{\frac{k(k-1)}{2}+1+k}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k+\frac{b_n}{n}L_{n,q}^{(\beta)}(t;x)$$

$$=\frac{b_n}{n}L_{n,q}^{(\beta)}(t;x)+\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2\frac{q^{\frac{k(k-1)}{2}+1}}{\Gamma_q(k+\beta-2)}\left(\frac{nx}{b_n}\right)^k$$

$$-\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=2}^{\infty}\left(\frac{b_n}{n}\right)^2\frac{q^{\frac{k(k-1)}{2}+k}}{\Gamma_q(k+\beta-1)}\left(\frac{nx}{b_n}\right)^k$$

$$-\frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)}\sum_{k=1}^{\infty}\left(\frac{b_n}{n}\right)^2[k-1]\frac{q^{\frac{k(k-1)}{2}+1+k}}{\Gamma_q(k+\beta)}\left(\frac{nx}{b_n}\right)^k$$

$$\leq \frac{b_n}{n} L_{n,q}^{(\beta)}(t;x) + \frac{x^2}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} q^{3k+3}}{\Gamma_q(k+\beta)} \left(\frac{nx}{b_n}\right)^k$$

$$+ \frac{[\beta-1]}{E_{1,\beta}\left(\frac{nx}{b_n};q\right)} \sum_{k=2}^{\infty} \left(\frac{b_n}{n}\right)^2 \frac{q^{\frac{k(k-1)}{2}+k}}{\Gamma_q(k+\beta-1)} \left(1 + \frac{q[k-1]}{[k+\beta-1]}\right) \left(\frac{nx}{b_n}\right)^k.$$

For $q^k \leq 1$ and $\frac{[k-1]}{[k+\beta-1]} \leq 1$, we have

$$L_{n,q}^{(\beta)}(t^2;x) \leq x^2 + \frac{b_n}{n}\left(x + \frac{[\beta-1]b_n}{n}\right) + 2x\frac{b_n[\beta-1]}{n}. \tag{1.9}$$

So we can write

$$\left|L_{n,q}^{(\beta)}(t^2;x) - x^2\right| \leq \frac{b_n}{n}\left(x + \frac{[\beta-1]b_n}{n}\right) + 2x\frac{b_n[\beta-1]}{n}.$$

Finally, using the (1.6) and (1.7), we acquire that

$$\begin{aligned}
L_{n,q}^{(\beta)}((t-x)^2;x) &= L_{n,q}^{(\beta)}(t^2;x) - 2xL_{n,q}^{(\beta)}(t;x) + x^2 L_{n,q}^{(\beta)}(1;x) \\
&\leq \left|L_{n,q}^{(\beta)}(t^2;x) - x^2\right| + 2x\left|L_{n,q}^{(\beta)}(t;x) - x\right| \\
&\leq \frac{b_n}{n}\left(x + \frac{[\beta-1]b_n}{n}\right) + 2x\frac{b_n[\beta-1]}{n} + \frac{2[\beta-1]b_n}{n}x \\
&= \frac{b_n}{n}(1+4[\beta-1])\,x + \left(\frac{b_n}{n}\right)^2[\beta-1].
\end{aligned}$$

□

2. Weighted approximation

Here, we give weighted approximation theorem for the operator $L_{n,q}^{(\beta)}$.

Theorem 1. *Let $q = q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$, then for each $f \in E$ and $\lim_{n\to\infty} \frac{b_n}{n} = 0$, we get*

$$\lim_{n\to\infty} \left\| L_{n,q_n}^{(\beta)}(f;x) - f(x) \right\|_* = 0$$

where $\beta > 1$.

Proof. From (1.5), we know that $L_{n,q}^{(\beta)}(1;x) = 1$, so

$$\left\| L_{n,q_n}^{(\beta)}(1;x) - 1 \right\|_* = 0.$$

From (1.6), for $n > 1$, we get

$$\left\| L_{n,q_n}^{(\beta)}(t;x) - x \right\|_* = \sup_{x\in[0,\infty)} \frac{\left|L_{n,q_n}^{(\beta)}(t;x) - x\right|}{1+x^2}$$

$$\leq \frac{[\beta-1]b_n}{n} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}$$
$$\leq \frac{[\beta-1]b_n}{n}.$$

So we can write

$$\lim_{n\to\infty} \left\| L^{(\beta)}_{n,q_n}(t;x) - x \right\|_* = 0.$$

Finally, for (1.7), we get

$$\begin{aligned}\left\| L^{(\beta)}_{n,q_n}(t^2;x) - x^2 \right\|_* &= \sup_{x\in[0,\infty)} \frac{\left| L^{(\beta)}_{n,q_n}(t^2;x) - x^2 \right|}{1+x^2} \\ &\leq (1+2[\beta-1]) \frac{b_n}{n} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &\quad + \left(\frac{b_n}{n}\right)^2 [\beta-1] \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq (1+2[\beta-1]) \frac{b_n}{2n} + \left(\frac{b_n}{n}\right)^2 [\beta-1].\end{aligned}$$

Then, we acquire that

$$\left\| L^{(\beta)}_{n,q_n}(t^2;x) - x^2 \right\|_* \to 0 \text{ as } n\to\infty.$$

□

3. Rate of convergence

Lemma 2. *For all $x\in[0,\infty)$ and $n\in\mathbb{N}$, we get*

$$\omega(x) L^{(\beta)}_{n,q}\left(\frac{1}{\omega};x\right) \leq \mu(\beta)$$

where $\left(\frac{b_n}{n}\right)$ is a bounded sequence of positive numbers and $\beta>1$ is fixed. Also $\mu(\beta)$ is a constant and $\omega(x) = \frac{1}{1+x^2}$. Additionally for all $f\in E$ and $q\in(0,1)$, we can obtain

$$\left\| L^{(\beta)}_{n,q}(f) \right\|_* \leq \mu(\beta) \left\| f \right\|_*.$$

Proof. Using (1.5) and (1.9), we get

$$\begin{aligned}\omega(x) L^{(\beta)}_{n,q}\left(\frac{1}{\omega};x\right) &= \frac{1}{1+x^2}\left[L^{(\beta)}_{n,q}(1;x) + L^{(\beta)}_{n,q}(t^2;x) \right] \\ &\leq \frac{1}{1+x^2}\left(1 + x^2 + \frac{b_n}{n}(1+2[\beta-1])x + \left(\frac{b_n}{n}\right)^2[\beta-1] \right)\end{aligned}$$

$$\leq 1+\frac{b_n}{n}\left(1+2[\beta-1]\right)+\left(\frac{b_n}{n}\right)^2[\beta-1]$$
$$=\mu(\beta).$$

Also we have

$$\begin{aligned}\omega(x)\left|L_{n,q}^{(\beta)}(f;x)\right| &= \omega(x)\left|L_{n,q}^{(\beta)}\left(\omega\frac{f}{\omega};x\right)\right| \\ &\leq \|f\|_*\,\omega(x)L_{n,q}^{(\beta)}\left(\frac{1}{\omega};x\right) \\ &\leq \mu(\beta)\,\|f\|_*\,.\end{aligned}$$

If we take supremum both sides of above inequality over $x\in[0,\infty)$, then the proof is done. □

Now, we acquire that the rate of convergence of the operators $L_{n,q}^{(\beta)}f$ to f with the help of usual modulus of convergence.

Firstly, we remember that the modulus of continuity of f on the closed interval $[0,D]$ is given by

$$\omega_D(f,\delta)=\sup_{\substack{x,t\in[0,D]\\|t-x|\leq\delta}}\left|f(t)-f(x)\right|.$$

We always know that, for a function f belongs to E, there is a limit equation as $\lim\limits_{\delta\to\infty}\omega_D(f,\delta)=0$.

Theorem 2. *Assume that* $\left(\frac{b_n}{n}\right)$ *is a bounded sequence of positive numbers with* $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$ *and* $\beta>1$. *For each* $f\in E$, $q\in(0,1)$ *and* $x\in[0,D]\subset[0,\infty)$; $(D>0)$,

$$\left\|L_{n,q}^{(\beta)}(f;x)-f(x)\right\|_{C[0,D]}\leq M_f(\beta,D)\epsilon_n^2(\beta,D)+2\omega_{D+1}(f,\epsilon_n(\beta,D))$$

where

$$\epsilon_n(\beta,D)=\frac{b_n}{n}\left(1+4[\beta-1]\right)D+\left(\frac{b_n}{n}\right)^2[\beta-1]$$

and $M_f(\beta,D)$ *is an absolute constant.*

Proof. From the hypothesis, $x\in[0,D]$ and $t\leq D+1$, we get

$$|f(t)-f(x)|\leq\omega_{D+1}(f,|t-x|)\leq\left(1+\frac{|t-x|}{\delta}\right)\omega_{D+1}(f,\delta)$$

where $\delta>0$. Also we can write

$$\begin{aligned}|f(t)-f(x)| &\leq K_f(1+x^2+t^2)\leq K_f(2+3x^2+2(t-x)^2)\\ &\leq 6K_f(1+D^2)(t-x)^2\end{aligned}$$

for $x \in [0,D]$ and $t > D+1$, getting $t-x > 1$ and K_f is a constant depending on f. From the above, we have

$$|f(t)-f(x)| \leq 6K_f(1+D^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{D+1}(f,\delta)$$

for $x \in [0,D]$ and $t \geq 0$. By the well-known Caucy-Schwarz inequality and from (1.8), we have

$$\begin{aligned}
\left|L_{n,q}^{(\beta)}(f;x)-f(x)\right| &\leq 6K_f(1+D^2)L_{n,q}^{(\beta)}\left((t-x)^2;x\right) \\
&\quad + \omega_{D+1}(f,\delta)\left(1+\frac{1}{\delta}\left(L_{n,q}^{(\beta)}\left((t-x)^2;x\right)\right)^{1/2}\right) \\
&\leq 6K_f(1+D^2)\left[\frac{b_n}{n}(1+4[\beta-1])\,x+\left(\frac{b_n}{n}\right)^2[\beta-1]\right] \\
&\quad + \omega_{D+1}(f,\delta)\left[1+\frac{1}{\delta}\left(\frac{b_n}{n}(1+4[\beta-1])\,x+\left(\frac{b_n}{n}\right)^2[\beta-1]\right)^{\frac{1}{2}}\right] \\
&\leq M_f(\beta,D)\epsilon_n^2(\beta,D)+2\omega_{D+1}(f,\epsilon_n(\beta,D)),
\end{aligned}$$

where

$$\epsilon_n(\beta,D) = \left(\frac{b_n}{n}(1+4[\beta-1])\,D+\left(\frac{b_n}{n}\right)^2[\beta-1]\right)^{1/2}$$

and $M_f(\beta,D) = 6(1+D^2)K_f$. So, the proof is done. □

4. A-STATISTICAL CONVERGENCE

Recently, some authors use $A-$statistical convergence by [2, 3, 12, 13]. Here for a non-negative regular summability matrix $A = (a_{jk})$, the definition of A-density of a subset K of $\mathbb{N}$ is as

$$\delta_A(K) = \lim_j \sum_{k\in K} a_{j,k},$$

where limit exists (see [5]). If a sequence $x = (x_n)$ convergent to l meaning $A-$statistically converges symbolized by $st_A - \lim x = l$ where every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x-l| \geq \varepsilon\} = 0$ (see [4, 14]).

Substituting $A = C_1$ which C_1 is the Cesaro matrix of order one, then we obtain the statistical convergence instead of $A-$statistically convergence [6, 10]. Choosing $A = I$, which is the identity matrix, then we get the ordinary convergence instead of $A-$statistically convergence. In the case of $\lim_j \max_n \left|a_{j,n}\right| = 0$, Kolk [9] showed that $A-$statistical convergence is stronger than ordinary convergence.

Assuming that

$$st_A - \lim_n \frac{b_n}{n} = 0,$$

then one can easily see that

$$st_A - \lim_n \frac{b_n^2}{n^2} = 0.$$

Taking $A = C_1$ and defining

$$b_n := \begin{cases} n; & n = m^2 \ (m \in \mathbb{N}) \\ 1; & \text{otherwise} \end{cases},$$

so we can easily obtain that $st_A - \lim_n \frac{b_n}{n} = st_A - \lim_n \frac{b_n^2}{n^2} = 0$.

Theorem 3. *Assume that $A = (a_{jk})$ is a non-negative regular summability matrix, $q \in (0,1)$ and $\beta > 1$. We have*

$$st_A - \left\| L_{n,q}^{(\beta)}(f;x) - f(x) \right\|_{C[0,D]} = 0,$$

if $st_A - \lim_n \frac{b_n}{n} = 0$ and each $f \in E$.

Proof. We choose $\varepsilon > 0$ satisfying $\varepsilon < r$ where $r > 0$. Defining the following sets as

$$T := \{n : \epsilon_n(\beta, D) \geq r\},$$
$$T_1 := \left\{ n : \frac{b_n}{n}(1 + 4[\beta - 1])\, D \geq \frac{r - \varepsilon}{2} \right\},$$
$$T_2 := \left\{ n : \left(\frac{b_n}{n}\right)^2 [\beta - 1] \geq \frac{r - \varepsilon}{2} \right\}.$$

Then one can see $T \subset T_1 \cup T_2$. Thus we can write

$$\sum_{k \in T} a_{jk} \leq \sum_{k \in T_1} a_{jk} + \sum_{k \in T_2} a_{jk}.$$

We get $\lim_j \sum_{k \in T} a_{jk} = 0$ for $j \to \infty$ in the above and $st_A - \lim_n \frac{b_n}{n} = 0$. Hence we have that $st_A - \lim_n \epsilon_n(\beta, D) = 0$ and this showed that

$$st_A - \lim_n \omega_{D+1}(f, \epsilon_n(\beta, D)) = 0.$$

From the Theorem 2, the proof is done. □

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Authors' addresses

Gürhan İçöz
Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey
E-mail address: gurhanicoz@gazi.edu.tr

Bayram Çekim
Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey
E-mail address: bayramcekim@gazi.edu.tr