

**CERTAIN FAMILIES OF MULTIVARIABLE
CHAN-CHYAN-SRIVASTAVA POLYNOMIALS**

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Abstract. In this paper we introduce r -parameter Srivastava polynomials in r -variable by inserting new indices. These polynomials include the Lagrange polynomials in several variables, which are also known as Chan-Chyan-Srivastava polynomials (W.-C.C. Chan, C.-J. Chyan and H.M. Srivastava, The Lagrange polynomials in several variables, *Integral Transforms and Special Functions*, 12 (2001) 139–148). We prove several two sided linear generating relations between r -variable and $(r-1)$ -variable Chan-Chyan-Srivastava polynomials. Some special cases of the main results are also presented.

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1. INTRODUCTION

In 1972, Srivastava [21] introduced the following family of polynomials,

$$S_n^N(z) := \sum_{k=0}^{\lfloor \frac{n}{N} \rfloor} \frac{(-n)_{Nk}}{k!} A_{n,k} z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \quad (1.1)$$

where \mathbb{N} is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$, and $(\lambda)_v$, $(\lambda)_0 \equiv 1$, denotes the Pochhammer symbol defined by

$$(\lambda)_v := \frac{\Gamma(\lambda + v)}{\Gamma(\lambda)}$$

by means of familiar Gamma functions. After, González et al. [13] extended the Srivastava polynomials $S_n^N(z)$ as follows:

$$S_{n,m}^N(z) := \sum_{k=0}^{\lfloor \frac{n}{N} \rfloor} \frac{(-n)_{Nk}}{k!} A_{n+m,k} z^k \quad (m, n \in \mathbb{N}_0; N \in \mathbb{N}), \quad (1.2)$$

and investigated their properties extensively. In [3], the following family of bivariate polynomials has been introduced

$$S_n^{m,N}(x,y) = \sum_{k=0}^{\lfloor \frac{n}{N} \rfloor} A_{m+n,k} \frac{x^{n-Nk}}{(n-Nk)!} \frac{y^k}{k!} \quad (m, n \in \mathbb{N}_0, N \in \mathbb{N}),$$

and it has shown that the polynomials $S_n^{m,N}(x,y)$ includes many well known polynomials such as Lagrange-Hermite polynomials, Lagrange polynomials and Hermite-Kampé de Fériét polynomials (see also [20]). In [23], Srivastava et al. have introduced the three-variable polynomials

$$S_n^{m,M,N}(x,y,z) = \sum_{k=0}^{\lfloor \frac{n}{N} \rfloor} \sum_{l=0}^{\lfloor \frac{k}{M} \rfloor} A_{m+n,k,l} \frac{x^l}{l!} \frac{y^{k-Ml}}{(k-Ml)!} \frac{z^{n-Nk}}{(n-Nk)!} \quad (m, n \in \mathbb{N}_0; M, N \in \mathbb{N}), \quad (1.3)$$

where $\{A_{m,n,k}\}$ be a triple sequence of complex numbers. Suitable choices of $\{A_{m,n,k}\}$ in equation (1) gives three variable version of well-known polynomials (see also [14]). In [15], the multivariable extension of the Srivastava polynomials in r -variable was introduced:

$$S_n^{m,N_1,N_2,\dots,N_{r-1}}(x_1,x_2,\dots,x_r) \quad (1.4)$$

$$= \sum_{k_{r-1}=0}^{\lfloor \frac{n}{N_{r-1}} \rfloor} \sum_{k_{r-2}=0}^{\lfloor \frac{k_{r-1}}{N_{r-2}} \rfloor} \dots \sum_{k_1=0}^{\lfloor \frac{k_2}{N_1} \rfloor} A_{m+n,k_{r-1},k_1,\dots,k_{r-2}} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-N_1k_1}}{(k_2-N_1k_1)!} \dots \frac{x_r^{n-N_{r-1}k_{r-1}}}{(n-N_{r-1}k_{r-1})!}$$

$$(m, n \in \mathbb{N}_0; N_1, N_2, \dots, N_{r-1} \in \mathbb{N}),$$

where $\{A_{m,k_{r-1},k_1,k_2,\dots,k_{r-2}}\}$ be a sequence of complex numbers. Recently, in [16], two-parameter one-variable Srivastava polynomials

$$S_n^{m_1,m_2}(x) := \sum_{k=0}^n \frac{(-n)_k}{k!} A_{m_1+m_2+n, m_2+k} x^k \quad (1.5)$$

$$(m_1, m_2, n, k \in \mathbb{N}_0),$$

two-parameter two-variable Srivastava polynomials

$$S_n^{m_1,m_2}(x,y) = \sum_{k=0}^n A_{m_1+m_2+n, m_2+k} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} \quad (1.6)$$

$$(m_1, m_2, n, k \in \mathbb{N}_0),$$

and two-parameter three-variable Srivastava polynomials

$$S_n^{m_1, m_2, M}(x, y, z) = \sum_{k=0}^n \sum_{l=0}^{\lfloor \frac{k}{M} \rfloor} A_{m_1+m_2+n, m_2+k, l} \frac{x^l}{l!} \frac{y^{k-Ml}}{(k-Ml)!} \frac{z^{n-k}}{(n-k)!} \tag{1.7}$$

$(m_1, m_2, n, k, l \in \mathbb{N}_0, M \in \mathbb{N})$

were introduced. These polynomials include the family of polynomials which were introduced or investigated in [3, 13, 15, 17, 19, 21, 23]. In this paper we introduce r -parameter Srivastava polynomials in r -variables by inserting new indices. These polynomials include the Lagrange polynomials in several variables, which are also known as Chan-Chyan-Srivastava polynomials [4]. We prove several two sided linear generating relations and obtain various generating relations for the polynomials.

2. R-PARAMETER R-VARIABLE SRIVASTAVA POLYNOMIALS

In this section we define r -parameter r -variable Srivastava polynomials as follows:

$$S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r) \tag{2.1}$$

$$= \sum_{k_{r-1}=0}^n \sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{\lfloor \frac{k_2}{M} \rfloor} A_{\Omega} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-Mk_1}}{(k_2-Mk_1)!} \dots \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!},$$

$(m_1, m_2, \dots, m_{r-1}, n \in \mathbb{N}_0; M \in \mathbb{N}),$

where $\Omega = m_1 + m_2 + \dots + m_{r-1} + n, m_2 + \dots + m_{r-1} + k_{r-1}, k_1, m_{r-1} + k_2, m_{r-2} + m_{r-1} + k_3, \dots, m_3 + \dots + m_{r-1} + k_{r-2}$. Note that appropriate choices of the sequence $\{A_{m_1+m_2, m_2, k_1, m_{r-1}, \dots, m_3}\}$ in (2.1) give the r -variable versions of the well known polynomials.

Remark 1. The Lagrange polynomials in several variables, which are known as Chan-Chyan-Srivastava polynomials [4] are defined by the generating function relation

$$\prod_{j=1}^r (1 - x_j t)^{-\alpha_j} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_r)}(x_1, x_2, \dots, x_r) t^n$$

$$(\alpha_j \in \mathbb{C} (j = 1, \dots, r); |t| < \min \{|x_1|^{-1}, \dots, |x_r|^{-1}\}).$$

The polynomials are given explicitly by

$$g_n^{(\alpha_1, \alpha_2, \dots, \alpha_r)}(x_1, x_2, \dots, x_r)$$

$$= \sum_{k_{r-1}=0}^n \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} (\alpha_1)_{k_1} (\alpha_2)_{k_2-k_1} \dots (\alpha_{r-1})_{k_{r-1}-k_{r-2}} (\alpha_r)_{n-k_{r-1}}$$

$$\times \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-k_1}}{(k_2-k_1)!} \cdots \frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!} \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!}.$$

Many authors have studied the properties of these polynomials. For example, the bilateral generating functions for these polynomials and miscellaneous properties are given in Liu et al. [12, 18]. In [8], the orthogonality properties and various integral representations for these polynomials are given (see also [1, 2, 5–7]). Furthermore, these polynomials are used in approximation theory. In [11], Duman et al. investigated some approximation properties of positive linear operators constructed by these polynomials (see also [9, 10]). In [22] Srivastava et al. investigated umbral calculus presentations of these polynomials.

Remark 2. If we set $M = 1$ and

$A_{m,n,k_2,\dots,k_{r-2},k_{r-1}} = (\alpha_1)_{k_2}(\alpha_2)_{k_3-k_2}\cdots(\alpha_{r-2})_{k_{r-1}-k_{r-2}}(\alpha_{r-1})_{n-k_{r-1}}(\alpha_r)_{m-n}$
in equation (2.1), we find that

$$\begin{aligned} & A_{m_1+\dots+m_{r-1}+n, m_2+\dots+m_{r-1}+k_{r-1}, k_1, m_{r-1}+k_2, m_{r-2}+m_{r-1}+k_3, \dots, m_3+\dots+m_{r-1}+k_{r-2}} \\ &= (\alpha_1)_{k_1}(\alpha_2)_{m_{r-1}+k_2-k_1}(\alpha_3)_{m_{r-2}+k_3-k_2}\cdots(\alpha_{r-1})_{m_2+k_{r-1}-k_{r-2}}(\alpha_r)_{m_1+n-k_{r-1}} \\ &= (\alpha_2)_{m_{r-1}}(\alpha_3)_{m_{r-2}}\cdots(\alpha_{r-1})_{m_2}(\alpha_r)_{m_1}(\alpha_1)_{k_1}(\alpha_2+m_{r-1})_{k_2-k_1}(\alpha_3+m_{r-2})_{k_3-k_2} \\ &\quad \times \cdots (\alpha_{r-1}+m_2)_{k_{r-1}-k_{r-2}}(\alpha_r+m_1)_{n-k_{r-1}} \end{aligned}$$

and therefore

$$\begin{aligned} & S_n^{m_1, m_2, \dots, m_{r-1}, 1}(x_1, x_2, \dots, x_r) = \\ & (\alpha_2)_{m_{r-1}}(\alpha_3)_{m_{r-2}}\cdots(\alpha_{r-1})_{m_2}(\alpha_r)_{m_1} g_n^{(\alpha_1, \alpha_2+m_{r-1}, \alpha_3+m_{r-2}, \dots, \alpha_r+m_1)}(x_1, \dots, x_r). \end{aligned}$$

Now, let recall an infinite series identities which were obtained by Srivastava et al. [23].

Lemma 1 (see [23], Lemma 1). *Let $N_1, N_2, \dots, N_{r-1} \in \mathbb{N}$, $r = \{2, 3, \dots\}$. Then*

$$\begin{aligned} & \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} A(n_1, n_2, \dots, n_r) \\ &= \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\lfloor \frac{n_r}{N_{r-1}} \rfloor} \cdots \sum_{n_1=0}^{\lfloor \frac{n_2}{N_1} \rfloor} A(n_1, n_2 - N_1 n_1, \dots, n_r - N_{r-1} n_{r-1}) \end{aligned}$$

and

$$\sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\lfloor \frac{n_r}{N_{r-1}} \rfloor} \cdots \sum_{n_1=0}^{\lfloor \frac{n_2}{N_1} \rfloor} A(n_1, n_2, \dots, n_r)$$

$$= \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \dots \sum_{n_1=0}^{\infty} A(n_1, n_2 + N_1 n_1, n_3 + N_2 N_1 n_1 + N_2 n_2, \dots, n_r + \prod_{j=1}^{r-1} N_j n_1 + \prod_{j=2}^{r-1} N_j n_2 + \dots + N_{r-1} n_{r-1})$$

where $\{A(n_1, n_2, \dots, n_r)\}$ is a bounded r -tuple sequence of real or complex numbers.

The main result of this paper is given by the following theorem.

Theorem 1. Let $\{f(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then

$$\begin{aligned} & \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) \tag{2.2} \\ & \times S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \\ & = \sum_{k_1, m_1, \dots, m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + M k_1) \\ & \times A_{m_1+m_2+\dots+m_{r-1}+M k_1, m_2+\dots+m_{r-1}+M k_1, k_1, m_{r-1}+M k_1, \dots, m_3+\dots+m_{r-1}+M k_1} \\ & \times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{(x_r t + w_1)^{m_1}}{m_1!} \frac{(x_{r-1} t + w_2)^{m_2}}{m_2!} \dots \frac{(x_2 t + w_{r-1})^{m_{r-1}}}{m_{r-1}!} \end{aligned}$$

provided that each member of the series identity (2.2) exists.

Proof. Let the left hand side of (2.2) be denoted by $\Psi(x_1, \dots, x_r)$. Then using the definition of $S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r)$ on the left hand side of (2.2), we have

$$\begin{aligned} & \Psi(x_1, \dots, x_r) \\ & = \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) \\ & \times \sum_{k_{r-1}=0}^n \sum_{k_{r-2}=0}^{k_{r-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{\lfloor \frac{k_2}{M} \rfloor} A_{\Omega} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2 - M k_1}}{(k_2 - M k_1)!} \dots \frac{x_r^{n - k_{r-1}}}{(n - k_{r-1})!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n. \end{aligned}$$

Let define

$$F(r-1) := M k_1 + k_2 + k_3 + \dots + k_{r-1}.$$

By applying Lemma 1, we find

$$\Psi(x_1, \dots, x_r)$$

$$= \sum_{k_1, \dots, k_{r-1}, m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n + F(r-1)) \\ \times A_{\Omega_1} \frac{x_1^{k_1}}{k_1!} \dots \frac{x_{r-1}^{k_{r-1}}}{k_{r-1}!} \frac{x_r^n}{n!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^{n+Mk_1+k_2+\dots+k_{r-1}}$$

where $\Omega_1 = m_1 + m_2 + \dots + m_{r-1} + n + F(r-1), m_2 + \dots + m_{r-1} + k_{r-1} + F(r-2),$

$k_1, m_{r-1} + k_2 + Mk_1, \dots, m_3 + \dots + m_{r-1} + k_{r-2} + F(r-3).$

Let $m_1 \rightarrow m_1 - n, m_2 \rightarrow m_2 - k_{r-1}, m_3 \rightarrow m_3 - k_{r-2},$

$\dots, m_{r-2} \rightarrow m_{r-2} - k_3, m_{r-1} \rightarrow m_{r-1} - k_2$ then

$$\Psi(x_1, \dots, x_r) \\ = \sum_{k_1, m_1, \dots, m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + Mk_1) \\ \times A_{m_1+\dots+m_{r-1}+Mk_1, m_2+\dots+m_{r-1}+Mk_1, k_1, m_{r-1}+Mk_1, \dots, m_3+\dots+m_{r-1}+Mk_1} \\ \times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{\sum_{n=0}^{m_1} \binom{m_1}{n} (x_r t)^n w_1^{m_1-n}}{m_1!} \frac{\sum_{k_{r-1}=0}^{m_2} \binom{m_2}{k_{r-1}} (x_{r-1} t)^{k_{r-1}} w_2^{m_2-k_{r-1}}}{m_2!} \\ \times \dots \frac{\sum_{k_2=0}^{m_{r-1}} \binom{m_{r-1}}{k_2} (x_2 t)^{k_2} w_{r-1}^{m_{r-1}-k_2}}{m_{r-1}!}$$

and

$$\Psi(x_1, \dots, x_r) \\ = \sum_{k_1, m_1, \dots, m_{r-1}=0}^{\infty} f(m_1 + \dots + m_{r-1} + Mk_1) \\ \times A_{m_1+m_2+\dots+m_{r-1}+Mk_1, m_2+\dots+m_{r-1}+Mk_1, k_1, m_{r-1}+Mk_1, \dots, m_3+\dots+m_{r-1}+Mk_1} \\ \times \frac{(x_1 t^M)^{k_1}}{k_1!} \frac{(x_r t + w_1)^{m_1}}{m_1!} \frac{(x_{r-1} t + w_2)^{m_2}}{m_2!} \dots \frac{(x_2 t + w_{r-1})^{m_{r-1}}}{m_{r-1}!}.$$

Whence the result. \square

Corollary 1. Let the polynomials $S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, x_2, \dots, x_r)$ be defined by equation (2.1). Suppose also that $r-1$ -variable polynomials $P_{m_2}^M(x_1, \dots, x_{r-1})$ are defined by

$$P_{m_2}^M(x_1, \dots, x_{r-1}) \tag{2.3} \\ = \sum_{m_3=0}^{m_2} \dots \sum_{m_{r-1}=0}^{m_{r-2}} \sum_{k_1=0}^{\lfloor \frac{m_{r-1}}{M} \rfloor} A_{m_1+m_2, m_2, k_1, m_{r-1}, \dots, m_3}$$

$$\times \frac{(x_1)^{k_1}}{k_1!} \frac{(x_2)^{m_{r-1}-Mk_1}}{(m_{r-1}-Mk_1)!} \frac{(x_3)^{m_{r-2}-m_{r-1}}}{(m_{r-2}-m_{r-1})!} \cdots \frac{(x_{r-1})^{m_2-m_3}}{(m_2-m_3)!}.$$

Then, for a suitably bounded sequence $\{f(n)\}_{n=0}^\infty$, the following family of two-sided linear generating relations holds true between the r -variable polynomials $S_n^{m_1, m_2, \dots, m_{r-1}, M}(x_1, \dots, x_r)$ and $(r-1)$ -variable polynomials $P_{m_2}^M(x_1, \dots, x_{r-1})$:

$$\begin{aligned} & \sum_{m_1, \dots, m_{r-1}, n=0}^\infty f(m_1 + \dots + m_{r-1} + n) S_n^{m_1, \dots, m_{r-1}, M}(x_1, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \cdots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \\ &= \sum_{m_1, m_2=0}^\infty f(m_1 + m_2) \frac{(x_r t + w_1)^{m_1}}{m_1!} P_{m_2}^M(x_1 t^M, x_2 t + w_{r-1}, \dots, x_{r-1} t + w_2) \end{aligned} \tag{2.4}$$

provided that each member of the series identity (2.4) exists.

Proof. If we set $m_{r-1} \rightarrow m_{r-1} - Mk_1, m_{r-2} \rightarrow m_{r-2} - m_{r-1}, \dots, m_2 \rightarrow m_2 - m_3$ in the right side of the equation (2.2) then

$$\begin{aligned} & \sum_{m_1, \dots, m_{r-1}, n=0}^\infty f(m_1 + \dots + m_{r-1} + n) S_n^{m_1, \dots, m_{r-1}, M}(x_1, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \cdots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} t^n \\ &= \sum_{m_1, m_2=0}^\infty f(m_1 + m_2) \frac{(x_r t + w_1)^{m_1}}{m_1!} \\ & \times \sum_{m_3=0}^{m_2} \cdots \sum_{m_{r-1}=0}^{m_{r-2}} \sum_{k_1=0}^{\lfloor \frac{m_{r-1}}{M} \rfloor} A_{m_1+m_2, m_2, k_1, m_{r-1}, \dots, m_3} \frac{(x_1 t^M)^{k_1}}{k_1!} \\ & \times \frac{(x_2 t + w_{r-1})^{m_{r-1}-Mk_1}}{(m_{r-1}-Mk_1)!} \frac{(x_3 t + w_{r-2})^{m_{r-2}-m_{r-1}}}{(m_{r-2}-m_{r-1})!} \cdots \frac{(x_{r-1} t + w_2)^{m_2-m_3}}{(m_2-m_3)!} \\ &= \sum_{m_1, m_2=0}^\infty f(m_1 + m_2) \frac{(x_r t + w_1)^{m_1}}{m_1!} P_{m_2}^M(x_1 t^M, x_2 t + w_{r-1}, \dots, x_{r-1} t + w_2). \end{aligned}$$

□

By setting $t = -\frac{w_1}{x_r}$ in (2.4), we get the following result.

Corollary 2. Under the hypotheses of Corollary 1, we get the following generating relation

$$\sum_{m_1, \dots, m_{r-1}, n=0}^\infty f(m_1 + \dots + m_{r-1} + n)$$

$$\begin{aligned} & \times S_n^{m_1, \dots, m_{r-1}, M}(x_1, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \left(-\frac{w_1}{x_r}\right)^n \\ & = \sum_{m_2=0}^{\infty} f(m_2) P_{m_2}^M \left(x_1 \left(-\frac{w_1}{x_r}\right)^M, x_2 \left(-\frac{w_1}{x_r}\right) + w_{r-1}, \dots, x_{r-1} \left(-\frac{w_1}{x_r}\right) + w_2\right). \end{aligned}$$

Remark 3. Choosing $M = 1$ and

$$A_{m,n,k_2,k_3,\dots,k_{r-1}} = (\alpha_1)_{k_2} (\alpha_2)_{k_3-k_2} \dots (\alpha_{r-1})_{n-k_{r-1}} (\alpha_r)_{m-n} \quad (2.5)$$

in (2.3) we observe that

$$\begin{aligned} & A_{m_1+m_2, m_2, k_1, m_{r-1}, m_{r-2}, \dots, m_4, m_3} \\ & = (\alpha_1)_{k_1} (\alpha_2)_{m_{r-1}-k_1} (\alpha_3)_{m_{r-2}-m_{r-1}} \dots (\alpha_{r-1})_{m_2-m_3} (\alpha_r)_{m_1} \end{aligned}$$

then

$$P_n^1(x_1, \dots, x_{r-1}) = (\alpha_r)_{m_1} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1, x_2, \dots, x_{r-1}) \quad (2.6)$$

where $g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1, x_2, \dots, x_{r-1})$ is the Lagrange polynomials of $(r-1)$ -variables.

Hence, upon setting $M = 1$ and considering (2.5) and (2.7) in Corollary 2, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} f(n) g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})} \left(x_1 \left(-\frac{w_1}{x_r}\right), x_2 \left(-\frac{w_1}{x_r}\right) + w_{r-1}, \dots, x_{r-1} \left(-\frac{w_1}{x_r}\right) + w_2\right) \\ & = \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} f(m_1 + \dots + m_{r-1} + n) (\alpha_2)_{m_{r-1}} (\alpha_3)_{m_{r-2}} \dots (\alpha_{r-1})_{m_2} (\alpha_r)_{m_1} \\ & \times g_n^{(\alpha_1, \alpha_2+m_{r-1}, \alpha_3+m_{r-2}, \dots, \alpha_r+m_1)}(x_1, x_2, \dots, x_r) \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \left(-\frac{w_1}{x_r}\right)^n. \end{aligned} \quad (2.7)$$

Let choose $f(n) = 1$ in (2.7) and consider Remark 1 then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})} \left(x_1 \left(-\frac{w_1}{x_r}\right), x_2 \left(-\frac{w_1}{x_r}\right) + w_{r-1}, \dots, x_{r-1} \left(-\frac{w_1}{x_r}\right) + w_2\right) \quad (2.8) \\ & = \sum_{m_1, \dots, m_{r-1}, n=0}^{\infty} \sum_{k_{r-1}=0}^n \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} (\alpha_2)_{m_{r-1}} (\alpha_3)_{m_{r-2}} \dots (\alpha_{r-1})_{m_2} (\alpha_r)_{m_1} \\ & \times (\alpha_1)_{k_1} (\alpha_2 + m_{r-1})_{k_2-k_1} \dots (\alpha_{r-1} + m_2)_{k_{r-1}-k_{r-2}} (\alpha_r + m_1)_{n-k_{r-1}} \\ & \times \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2-k_1}}{(k_2-k_1)!} \dots \frac{x_{r-1}^{k_{r-1}-k_{r-2}}}{(k_{r-1}-k_{r-2})!} \frac{x_r^{n-k_{r-1}}}{(n-k_{r-1})!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \left(-\frac{w_1}{x_r}\right)^n. \end{aligned}$$

Let Φ denote the right side of (2.8). Applying Lemma 1 into the right side of the equation (2.8) we get

$$\begin{aligned} \Phi = & \sum_{k_1, \dots, k_{r-1}, m_1, \dots, m_{r-1}, n=0}^{\infty} (\alpha_1)_{k_1} (\alpha_2 + m_{r-1})_{k_2} \dots (\alpha_{r-1} + m_2)_{k_{r-1}} \\ & \times (\alpha_r)_{m_1} (\alpha_{r-1})_{m_2} \dots (\alpha_3)_{m_{r-2}} (\alpha_2)_{m_{r-1}} (\alpha_r + m_1)_n \\ & \times \frac{((-w_1/x_r)x_1)^{k_1}}{k_1!} \frac{((-w_1/x_r)x_2)^{k_2}}{k_2!} \dots \frac{((-w_1/x_r)x_{r-1})^{k_{r-1}}}{k_{r-1}!} \frac{((-w_1/x_r)x_r)^n}{n!} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \end{aligned}$$

and

$$\begin{aligned} \Phi = & \sum_{m_1, \dots, m_{r-1}=0}^{\infty} (\alpha_r)_{m_1} (\alpha_{r-1})_{m_2} \dots (\alpha_3)_{m_{r-2}} (\alpha_2)_{m_{r-1}} \\ & \times (1 + (w_1/x_r)x_1)^{-\alpha_1} (1 + (w_1/x_r)x_2)^{-(\alpha_2+m_{r-1})} \dots (1 + (w_1/x_r)x_{r-1})^{-(\alpha_{r-1}+m_2)} \\ & \times (1 + (w_1/x_r)x_r)^{-(\alpha_r+m_1)} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \\ = & (1 + (w_1/x_r)x_1)^{-\alpha_1} (1 + (w_1/x_r)x_2)^{-\alpha_2} \dots (1 + (w_1/x_r)x_{r-1})^{-\alpha_{r-1}} (1 + (w_1/x_r)x_r)^{-\alpha_r} \\ & \times \sum_{m_1, \dots, m_{r-1}=0}^{\infty} (\alpha_r)_{m_1} (\alpha_{r-1})_{m_2} \dots (\alpha_3)_{m_{r-2}} (\alpha_2)_{m_{r-1}} \\ & \times (1 + (w_1/x_r)x_2)^{-m_{r-1}} \dots (1 + (w_1/x_r)x_{r-1})^{-m_2} (1 + (w_1/x_r)x_r)^{-m_1} \frac{w_1^{m_1}}{m_1!} \dots \frac{w_{r-1}^{m_{r-1}}}{m_{r-1}!} \\ = & (1 + (w_1/x_r)x_1)^{-\alpha_1} (1 + (w_1/x_r)x_2)^{-\alpha_2} \dots (1 + (w_1/x_r)x_{r-1})^{-\alpha_{r-1}} (1 + (w_1/x_r)x_r)^{-\alpha_r} \\ & \times (1 - \frac{w_1}{1 + (w_1/x_r)x_r})^{-\alpha_r} (1 - \frac{w_2}{1 + (w_1/x_r)x_{r-1}})^{-\alpha_{r-1}} \dots (1 - \frac{w_{r-1}}{1 + (w_1/x_r)x_2})^{-\alpha_2} \\ = & (1 + (w_1/x_r)x_1)^{-\alpha_1} (1 + (w_1/x_r)x_2 - w_{r-1})^{-\alpha_2} \dots (1 + (w_1/x_r)x_{r-1} - w_2)^{-\alpha_{r-1}} \\ & \times (1 + (w_1/x_r)x_r - w_1)^{-\alpha_r}. \end{aligned}$$

Corollary 3. *The generating relation*

$$\begin{aligned} & \sum_{n=0}^{\infty} g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})} (x_1(-w_1/x_r), x_2(-w_1/x_r) + w_{r-1}, \dots, x_{r-1}(-w_1/x_r) + w_2) \\ & = (1 + (w_1/x_r)x_1)^{-\alpha_1} (1 + (w_1/x_r)x_2 - w_{r-1})^{-\alpha_2} \dots (1 + (w_1/x_r)x_{r-1} - w_2)^{-\alpha_{r-1}} \end{aligned}$$

is given for $g_n^{(\alpha_1, \alpha_2, \dots, \alpha_{r-1})}(x_1, x_2, \dots, x_{r-1})$.

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