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ON EQUIVALENCE OF TWO INTEGRABILITY METHODS

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Abstract. In this paper, we introduce the concept of $| R, p |_k, k \geq 1$ integrability of improper integrals and by this definition we prove a theorem, that generalizes a theorem of Orhan [3].

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1. INTRODUCTION

Throughout this paper we assume that f is a real valued function which is continuous on $[0,\infty)$ and $s(x) = \int_0^x f(t)dt$. By $\sigma(x)$, we denote the Cesàro mean of $s(x)$. The integral $\int_0^\infty f(t)dt$ is said to be integrable $| C, 1 |_k, k \geq 1$, in the sense of Flett [2], if

$$\int_0^\infty x^{k-1} | \sigma'(x) |^k dx = \int_0^\infty \frac{| v(x) |^k}{x} dx \tag{1.1}$$

is convergent. Here, $v(x) = \frac{1}{x}\int_0^x tf(t)dt$ is called a generator of the integral $\int_0^\infty f(t)dt$. Let p be a real valued, non-decreasing function on $[0,\infty)$ such that

$$P(x) = \int_0^x p(t)dt, p(x) \neq 0, p(0) = 0.$$

The Riesz mean of $s(x)$ is defined by

$$\sigma_p(x) = \frac{1}{P(x)} \int_0^x p(t)s(t)dt.$$

We say that the integral $\int_0^\infty f(t)dt$ is integrable $| R, p |_k, k \geq 1$, if

$$\int_0^\infty x^{k-1} | \sigma_p'(x) |^k dx \tag{1.2}$$

is convergent. In the special case if we take $p(x) = 1$ for all values of x, then $| R, p |_k$ integrability reduces to $| C, 1 |_k$ integrability of improper integrals.
Given any functions f, g, it is customary to write $g(x) = O(f(x))$, if there exist η

and N, for every $x > N, \mid \frac{g(x)}{f(x)} \mid \leq \eta$.
The difference between $s(x)$ and its nth weighted mean $\sigma_p(x)$, which is called the weighted Kronecker identity, is given by the identity

$$s(x) - \sigma_p(x) = v_p(x), \tag{1.3}$$

where

$$v_p(x) = \frac{1}{P(x)} \int_0^x P(u) f(u) du.$$

We note that if we take $p(x) = 1$, for all values of x then we have the following identity(see [1])

$$s(x) - \sigma(x) = v(x).$$

Since

$$\sigma_p'(x) = \frac{p(x)}{P(x)} v_p(x),$$

condition (1.3) can be rewritten as

$$s(x) = v_p(x) + \int_0^x \frac{p(u)}{P(u)} v_p(u) du. \tag{1.4}$$

In view of the identity (1.4), the function $v_p(x)$ is called the generator function of $s(x)$.
Condition (1.1) can also be written as

$$\int_0^\infty x^{k-1} \left(\frac{p(x)}{P(x)} \right)^k \mid v_p(x) \mid^k dx \tag{1.5}$$

is convergent. We note that for infinite series, an analogous definition was introduced by Orhan [3]. Using this definition, Orhan [3] proved the following theorem dealing with $\mid R, p_n \mid_k$ and $\mid R, q_n \mid_k$ summability methods.

Theorem 1. *The $\mid R, p_n \mid_k, (k \geq 1)$ summability implies the $\mid R, q_n \mid_k, (k \geq 1)$ summability provided that*

$$nq_n = O(Q_n), \tag{1.6}$$

$$P_n = O(np_n), \tag{1.7}$$

$$Q_n = O(nq_n). \tag{1.8}$$

2. Main result

The aim of this paper is to state Orhan's theorem for $\mid R,p\mid_k$ and $\mid R,q\mid_k$ integrability of improper integrals. Now we shall prove the following theorem.

Theorem 2. *Let p and q be real valued, non-decreasing functions on $[0,\infty)$ such that as $x\to\infty$*

$$xq(x)=O(Q(x)), \tag{2.1}$$

$$P(x)=O(xp(x)), \tag{2.2}$$

$$Q(x)=O(xq(x)). \tag{2.3}$$

If $\int_0^\infty f(t)dt$ is integrable $\mid R,p\mid_k$, then it is also integrable $\mid R,q\mid_k, k\geq 1$.

3. Proof of the Theorem

Let $\sigma_p(x)$ and $\sigma_q(x)$ be the functions of (R,p) and (R,q) means of the integral $\int_0^\infty f(t)dt$. Since $\int_0^\infty f(t)dt$ is integrable $\mid R,p\mid_k$, we can write

$$\int_0^\infty x^{k-1}\left(\frac{p(x)}{P(x)}\right)^k \mid v_p(x)\mid^k dx$$

is convergent. Differentiating the equation (1.4), we have

$$f(x)=v_p'(x)+\frac{p(x)}{P(x)}v_p(x).$$

By definition, we obtain

$$\sigma_q(x)=\frac{1}{Q(x)}\int_0^x q(t)s(t)dt=\frac{1}{Q(x)}\int_0^x (Q(x)-Q(t))f(t)dt$$

and

$$\begin{aligned}\sigma_q'(x)&=\frac{q(x)}{Q^2(x)}\int_0^x Q(t)f(t)dt\\ &=\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\left[v_p'(t)+\frac{p(t)}{P(t)}v_p(t)\right]dt\\ &=\frac{q(x)}{Q^2(x)}\int_0^x Q(t)v_p'(t)dt+\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt.\end{aligned}$$

Integrating by parts of the first statement, we have

$$\begin{aligned}\sigma_q'(x)&=\frac{q(x)}{Q^2(x)}\left[Q(x)v_p(x)-\int_0^x q(t)v_p(t)dt\right]+\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt\\ &=\frac{q(x)}{Q(x)}v_p(x)+\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt-\frac{q(x)}{Q^2(x)}\int_0^x q(t)v_p(t)dt\\ &=\sigma_{q,1}(x)+\sigma_{q,2}(x)+\sigma_{q,3}(x),\ say.\end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\int_0^m x^{k-1} \mid \sigma_{q,r}(x) \mid^k dx = O(1) \; as \; m \to \infty, \;\; for \; r = 1,2,3.$$

Using conditions (2.1) and (2.2), we have

$$\begin{aligned}
\int_0^m x^{k-1} \mid \sigma_{q,1}(x) \mid^k dx &= \int_0^m x^{k-1} \mid \frac{q(x)}{Q(x)} v_p(x) \mid^k dx \\
&= \int_0^m x^{k-1} \left(\frac{q(x)}{Q(x)}\right)^k \mid v_p(x) \mid^k dx \\
&= O(1) \int_0^m x^{k-1} \left(\frac{p(x)}{P(x)}\right)^k \mid v_p(x) \mid^k dx \\
&= O(1) \int_0^m x^{k-1} \mid \sigma'_p(x) \mid^k dx \\
&= O(1) \; as \; m \to \infty
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.
Applying Hölder's inequality with $k > 1$, we get

$$\begin{aligned}
&\int_0^m x^{k-1} \mid \sigma_{q,2}(x) \mid^k dx = \\
&= O(1) \int_0^m x^{k-1} \left(\frac{q(x)}{Q^2(x)}\right)^k \left(\int_0^x \frac{Q(t)p(t)}{P(t)} \mid v_p(t) \mid dt\right)^k dx \\
&= O(1) \int_0^m \frac{q(x)}{Q^{k+1}(x)} \left(\int_0^x \frac{Q(t)p(t)}{P(t)} \mid v_p(t) \mid dt\right)^k dx \\
&= O(1) \int_0^m \frac{q(x)}{Q^2(x)} \left(\int_0^x \left(\frac{Q(t)}{q(t)}\right)^k q(t) \left(\frac{p(t)}{P(t)}\right)^k \mid v_p(t) \mid^k dt\right) \\
&x \left(\frac{1}{Q(x)} \int_0^x q(t) dt\right)^{k-1} dx \\
&= O(1) \int_0^m t^k q(t) \left(\frac{p(t)}{P(t)}\right)^k \mid v_p(t) \mid^k dt \int_t^m \frac{q(x)}{Q^2(x)} dx \\
&= O(1) \int_0^m t^k \frac{q(t)}{Q(t)} \left(\frac{p(t)}{P(t)}\right)^k \mid v_p(t) \mid^k dt \\
&= O(1) \int_0^m t^{k-1} \left(\frac{p(t)}{P(t)}\right)^k \mid v_p(t) \mid^k dt \\
&= O(1) \int_0^m t^{k-1} \mid \sigma'_p(t) \mid^k dt
\end{aligned}$$

$$= O(1) \; as \; m \to \infty$$

by virtue of the hypotheses of Theorem 2.
Finally, again by Hölder's inequality with $k > 1$, we have

$$\begin{aligned}
\int_0^m x^{k-1} \mid \sigma_{q,3}(x) \mid^k dx &= O(1) \int_0^m x^{k-1} \left(\frac{q(x)}{Q^2(x)} \right)^k \left(\int_0^x q(t) \mid v_p(t) \mid^k dt \right)^k dx \\
&= O(1) \int_0^m \frac{q(x)}{Q^2(x)} \left(\int_0^x q(t) \mid v_p(t) \mid^k dt \right) \\
&\quad x \left(\frac{1}{Q(x)} \int_0^x q(t)dt \right)^{k-1} dx \\
&= O(1) \int_0^m q(t) \mid v_p(t) \mid^k dt \int_t^m \frac{q(x)}{Q^2(x)} dx \\
&= O(1) \int_0^m \frac{q(t)}{Q(t)} \mid v_p(t) \mid^k dt \\
&= O(1) \; as \; m \to \infty
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.
This completes the proof of the theorem.

Theorem 3. *Let p and q be real valued, non-decreasing functions on $[0,\infty)$ such that as $x \to \infty$*

$$xp(x) = O(P(x)), \tag{3.1}$$

$$Q(x) = O(xq(x)), \tag{3.2}$$

$$P(x) = O(xp(x)). \tag{3.3}$$

If $\int_0^\infty f(t)dt$ is integrable $\mid R, q \mid_k$, then it is also integrable $\mid R, p \mid_k, k \geq 1$.

Proof. In Theorem 2 if we take $p(x) = q(x)$ and $q(x) = p(x)$, then we get Theorem 3. □

Theorem 4. *Let p and q be real valued, non-decreasing functions on $[0,\infty)$ such that as $x \to \infty$*

$$xp(x) = O(P(x)), \tag{3.4}$$

$$P(x) = O(xp(x)), \tag{3.5}$$

$$xq(x) = O(Q(x)), \tag{3.6}$$

$$Q(x) = O(xq(x)). \tag{3.7}$$

Then the $\mid R, p \mid_k$ integrability of $\int_0^\infty f(t)dt$ is equivalent to the $\mid R, q \mid_k$ integrability of $\int_0^\infty f(t)dt$, where $k \geq 1$.

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