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ASYMPTOTIC INTEGRATION OF SINGULARLY PERTURBED LINEAR SYSTEMS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. The paper deals with problem on asymptotic solutions to a system of singular perturbed linear differential-algebraic equations. Case of multiple roots of a characteristic equation is studied. Technique of constructing the asymptotic solutions is developed.

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1. Introduction

Linear differential-algebraic equations (DAEs) with constant coefficients

$$A\frac{dx}{dt} = Bx + f(t) \tag{1.1}$$

were well studied in the middle of the past century. In particular, Luzin [12] and Gantmacher [7] have found necessary and sufficient conditions for solvability of DAEs, and they have proposed approaches of constructing their solutions. It should be note that Gantmacher's algorithm [7] for constructing particular solutions of DAEs (1.1) was based on the idea of reduction of a pencil $B - \lambda A$ to Kronecker normal form. At the same time similar technique could not be used to DAEs with variable coefficients

$$A(t)\frac{dx}{dt} = B(t)x + f(t), \tag{1.2}$$

in general case, since its application as a rule causes changing the Kronecker's form of a pencil $B(t) - \lambda A(t)$ of system (1.2).

To solve the problem Campbell has proposed the notion of standard canonical form of system (1.2) that was represented in the following form [3]

$$\begin{pmatrix} I_{n-s} & 0 \\ 0 & N_s(t) \end{pmatrix} \frac{dx}{dt} = \begin{pmatrix} M(t) & 0 \\ 0 & I_s \end{pmatrix} x + h(t), \tag{1.3}$$

where I_s and I_{n-s} are identity matrices of orders s and $n-s$, respectively, and $N_s(t)$ is a nilpotent lower (or upper) triangular matrix.

Note that in particular case, when matrix $N_s(t)$ is additionally constant, system (1.3) is called the strong standard canonical form of system (1.2).

Later Petzold and Gear have found sufficient conditions of reduction of system (1.2) to its Kronecker's form [8, 13]. It allows to find the general solution of system (1.2) and to study then Cauchy problem, boundary value problems, and others [1, 4, 5, 14].

Another way to deal with DAEs is to decouple them by means of canonical projectors. Using a concept of the tractability index, the effective numerical methods for solving DAEs were developed in the papers by Gear and Petzold [8], Griepentrog and März [9], Brenan, Campbell and Petzold [2].

Since 1980s, there is considered singular perturbed DAEs [1, 14, 19]

$$\varepsilon A(t,\varepsilon)\frac{dx}{dt} = B(t,\varepsilon)x,\ t \in [0;T],\ \varepsilon \in (0;\varepsilon_0], \tag{1.4}$$

where $A(t,\varepsilon)$, $B(t,\varepsilon)$ are $n \times n$–matrices with real or complex-valued elements, possessing uniform asymptotic expansions of the following form

$$A(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k A_k(t),\ B(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k B_k(t),\ \varepsilon \to 0,\ t \in [0;T],$$

with infinitely differentiable coefficients, and ε is a small parameter.

In particular, Starun [19] has found as many linearly independent asymptotic solutions of system (1.4) as there are roots of the corresponding characteristic equation

$$det(B_0(t) - \lambda A_0(t)) = 0. \tag{1.5}$$

It follows that if $\operatorname{rank} \mathrm{B}_0(\mathrm{t})$ is equal to the degree of the polynomial $det(B_0(t) - \lambda A_0(t))$ then a formal fundamental matrix solution of system (1.4) can be constructed [1, 14]. Generalizing Starun's ideas Yakovets has shown that under certain conditions system (1.4) has two types of formal solutions corresponding to finite or infinite elementary divisors of the pencil $B_0(t) - \lambda A_0(t)$ [14]. Moreover, their linear combination is formal general solution of system (1.4).

It is well known [14], in case of multiple spectrum of the main pencil $B_0(t) - \lambda A_0(t)$ asymptotic expansions of solutions of system (1.4) should be constructed in some fractional powers of small parameter ε. Moreover, value of fractional powers of small parameter depends on the structure of the perturbing matrices $A_k(t)$, $B_k(t)$, $k \in N$. However, in the case of multiple eigenvalues of $B_0(t) - \lambda A_0(t)$ the technique of constructing asymptotic expansions is tedious and quite complicated. That is why we propose in present paper a modified approach for finding formal asymptotic solutions to system (1.4). Namely, our technique is based on transformation of the system with multiple spectrum of the main pencil of matrices into system whose main pencil of matrices has a simple spectrum [17]. On our opinion, the proposed approach is more rational one than others.

2. Formal solutions

Assume that the following conditions are satisfied:

(i) the pencil of matrices $B_0(t)-\lambda A_0(t)$ is regular for all $t \in [0;T]$;

(ii) the pencil $B_0(t)-\lambda A_0(t)$ has one eigenvalue $\lambda_0(t)$, two finite elementary divisors $(\lambda-\lambda_0(t))^{p_1}$, $(\lambda-\lambda_0(t))^{p_2}$, $2 \le p_1 < p_2$, and two infinite elementary divisors of multiplicity q_1 and q_2, $2 \le q_1 < q_2$; furthermore $p_1+p_2+q_1+q_2 = n$.

Then there exist the nonsingular of matrices $P(t,\varepsilon)$, $Q(t,\varepsilon) \in C^{\infty}([0;T]\times[0;\varepsilon_0])$ such that

$$P(t,\varepsilon)A(t,\varepsilon)Q(t,\varepsilon) = H(t,\varepsilon) \equiv diag\{N_q(t,\varepsilon), I_p(t,\varepsilon)\}, \tag{2.1}$$

$$P(t,\varepsilon)B(t,\varepsilon)Q(t,\varepsilon) = \Omega(t,\varepsilon) \equiv diag\{I_q(t,\varepsilon), W_p(t,\varepsilon)\}, \tag{2.2}$$

where $I_q(t,0) = I_q$, $I_p(t,0) = I_p$, I_q and I_p are the identity matrices of orders q $(q = q_1+q_2)$ and p $(p = p_1+p_2)$, respectively;

$N_q(t,0) = N_q \equiv diag\{N_{q_1}, N_{q_2}\}$, $W_p(t,0) = diag\{W_{p_1}(t), W_{p_2}(t)\}$, $p = p_1+p_2$, N_{q_i} is the square matrix of order q_i such that

$$N_{q_i} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \; i = 1,2,$$

and $W_{p_i}(t) = \lambda_0(t)I_{p_i} + N_{p_i}$, $i = 1,2$ [15, 16, 18].

We set $x(t,\varepsilon) = Q(t,\varepsilon)y(t,\varepsilon)$. Then system (1.4) can be written in the form

$$\varepsilon H(t,\varepsilon)\frac{dy}{dt} = C(t,\varepsilon)y, \tag{2.3}$$

where $C(t,\varepsilon) = \Omega(t,\varepsilon) - \varepsilon H(t,\varepsilon)Q^{-1}(t,\varepsilon)Q'(t,\varepsilon)$, $Q'(t,0) \equiv 0$, $t \in [0;T]$.

The transformation $y(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int\limits_0^t \lambda_0(t)dt\right) z(t,\varepsilon)$ changes system (2.3) into

$$\varepsilon H(t,\varepsilon)\frac{dz}{dt} = (C(t,\varepsilon) - \lambda_0(t)H(t,\varepsilon))z.$$

It should be noted that the pencils $\Omega(t,0) - \lambda_0(t)H(t,0) - \lambda H(t,0)$ and $diag\{I_q, N_p\} - \lambda\, diag\{N_q, I_p\}$ have the same Kronecker normal form [7]. Therefore, without loss of generality, we can assume that

$$\Omega(t,0) = diag\{I_q, N_p\}, \; N_p = diag\{N_{p_1}, N_{p_2}\}.$$

Let us define the matrices

$$H(t,\varepsilon) = \sum_{k\ge 0} \varepsilon^k H_k(t), \; C(t,\varepsilon) = \sum_{k\ge 0} \varepsilon^k C_k(t),$$

where $H_0(t) \equiv H_0 = diag\{N_q, I_p\}$, $C_0(t) \equiv C_0 = diag\{I_q, N_p\}$.

We seek an asymptotic solutions of system (2.3) in the following form:

$$y_i(t,\varepsilon) = u_i(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int\limits_{a_i}^{t}\lambda_i(t,\varepsilon)dt\right),\ i = \overline{1,n}, \tag{2.4}$$

where $u_i(t,\varepsilon)$ are n-dimensional vectors, $\lambda_i(t,\varepsilon)$ are scalar functions, and

$$u_i(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k \widetilde{u}_i^{(k)}(t,\varepsilon),\ \lambda_i(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k \widetilde{\lambda}_i^{(k)}(t,\varepsilon),\ i = \overline{1,n},$$

[14]. Constants a_i are defined below.

Substituting representation (2.4) into system (2.3), we get

$$(C_0 + \varepsilon C_1(t))u_i(t,\varepsilon) - (H_0 + \varepsilon H_1(t))u_i(t,\varepsilon)\lambda_i(t,\varepsilon) = \varepsilon H(t,\varepsilon)u_i'(t,\varepsilon) -$$
$$-\sum_{k\geq 2}\varepsilon^k C_k(t)u_i(t,\varepsilon) + \sum_{k\geq 2}\varepsilon^k H_k(t)u_i(t,\varepsilon)\lambda_i(t,\varepsilon), \tag{2.5}$$

where $C_1(t) = diag\{C_{1q}(t), C_{1p}(t)\}$, $H_1(t) = diag\{H_{1q}(t), H_{1p}(t)\}$; here, $C_{1q}(t)$, $H_{1q}(t)$ are square matrices of order q.

Let $K(t,\varepsilon)$ be defined by $K(t,\varepsilon) = diag\{I_q + \varepsilon C_{1q}(t), I_p + \varepsilon H_{1p}(t)\}$. Then

$$K^{-1}(t,\varepsilon) = \sum_{k\geq 0}\varepsilon^k M_k(t) \equiv diag\{I_q + \sum_{k\geq 1}\varepsilon^k M_{kq}(t), I_p + \sum_{k\geq 1}\varepsilon^k M_{kp}(t)\}$$

and $M_1(t) = diag\{-C_{1q}(t), -H_{1p}(t)\}$.

Multiplying both sides of relation (2.5) on the left by $K^{-1}(t,\varepsilon)$, we obtain

$$(C_0 + \varepsilon D_1(t))u_i(t,\varepsilon) - (H_0 + \varepsilon F_1(t))u_i(t,\varepsilon)\lambda_i(t,\varepsilon) = \varepsilon\sum_{k\geq 0}\varepsilon^k F_k(t)u_i'(t,\varepsilon) -$$
$$-\sum_{k\geq 2}\varepsilon^k D_k(t)u_i(t,\varepsilon) + \sum_{k\geq 2}\varepsilon^k F_k(t)u_i(t,\varepsilon)\lambda_i(t,\varepsilon),$$

where

$$D_1(t) = diag\{0, D_{1p}(t)\} \equiv diag\{0, C_{1p}(t) - H_{1p}(t)N_p\},$$
$$F_1(t) = diag\{F_{1q}(t), 0\} \equiv diag\{H_{1q}(t) - C_{1q}(t)N_q, 0\},$$
$$D_2(t) = C_2(t) + diag\{0, M_{2p}(t)N_p - H_{1p}(t)C_{1p}(t)\},$$
$$D_k(t) = \sum_{i=0}^{k-2}M_i(t)C_{k-i}(t) + diag\{0, M_{kp}(t)N_p + M_{k-1,p}C_{1p}(t)\},\ k\geq 3,$$
$$F_2(t) = H_2(t) + diag\{M_{2q}(t)N_q - C_{1q}(t)H_{1q}(t), 0\},$$
$$F_k(t) = \sum_{i=0}^{k-2}M_i(t)H_{k-i}(t) + diag\{M_{kq}(t)N_q + M_{k-1,q}H_{1q}(t), 0\},\ k\geq 3.$$

Equating the coefficients at the powers of ε in the following way, we get

$$(C_0+\varepsilon D_1(t)-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)(H_0+\varepsilon F_1(t)))\widetilde{u}_i^{(0)}(t,\varepsilon)=0, \tag{2.6}$$

$$(C_0+\varepsilon D_1(t)-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)(H_0+\varepsilon F_1(t)))\widetilde{u}_i^{(s)}(t,\varepsilon)=d_i^{(s)}(t,\varepsilon),\ s\in N, \tag{2.7}$$

where

$$d_i^{(s)}(t,\varepsilon)=\sum_{k=0}^{s}F_k(t)(\widetilde{u}_i^{(s-k-1)}(t,\varepsilon))'-\sum_{k=2}^{s}D_k(t)\widetilde{u}_i^{(s-k)}(t,\varepsilon)+$$

$$+\sum_{k=2}^{s}\sum_{j=0}^{k}F_k(t)\widetilde{u}_i^{(j)}(t,\varepsilon)\widetilde{\lambda}_i^{(s-k-j)}(t,\varepsilon)+(H_0+\varepsilon F_1(t))\sum_{k=0}^{s-1}\widetilde{u}_i^{(k)}(t,\varepsilon)\widetilde{\lambda}_i^{(s-k)}(t,\varepsilon).$$

According to the form of the matrices C_0 and H_0, we rewrite equation (2.6) as

$$(I_q-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)(N_q+\varepsilon F_{1q}(t)))\widetilde{u}_{i1}^{(0)}(t,\varepsilon)=0, \tag{2.8}$$

$$(N_p+\varepsilon D_{1p}(t)-\widetilde{\lambda}_i^{(0)}(t,\varepsilon)I_p)\widetilde{u}_{i2}^{(0)}(t,\varepsilon)=0, \tag{2.9}$$

where $\widetilde{u}_i^{(0)}(t,\varepsilon)=\begin{pmatrix}\widetilde{u}_{i1}^{(0)}(t,\varepsilon)\\ \widetilde{u}_{i2}^{(0)}(t,\varepsilon)\end{pmatrix}$.

Consider equation (2.8). The characteristic equation of $N_q+\varepsilon F_{1q}(t)$ has the form

$$w^q+\beta_1(t,\varepsilon)w^{q-1}+\ldots+\beta_{q-1}(t,\varepsilon)w+\beta_q(t,\varepsilon)=0, \tag{2.10}$$

where

$$\beta_j(t,\varepsilon)=O(\varepsilon),\ \varepsilon\to 0,\ j=\overline{1,q_2-1},$$

$$\beta_{q_2}(t,\varepsilon)=-h_{q,q_1+1}^{(1)}(t)\varepsilon+O(\varepsilon^2),\ \varepsilon\to 0,$$

$$\beta_j(t,\varepsilon)=O(\varepsilon^2),\ \varepsilon\to 0,\ i=\overline{q_2+1,q-1},$$

$$\beta_q(t,\varepsilon)=h_{q_1,1}^{(1)}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^2+O(\varepsilon^3),\ \varepsilon\to 0,$$

for all $t\in[0;T]$; here, $h_{ij}^{(1)}(t)$ is the element of the matrix $H_1(t)$.

Let the following condition be satisfied.

(iii) $h_{q_1,1}^{(1)}(t)\neq 0$, $h_{q,q_1+1}^{(1)}(t)\neq 0$, $h(t)\neq 0$, $t\in[0;T]$, where $h(t)=h_{q_1,1}^{(1)}(t)h_{q,q_1+1}^{(1)}(t)-h_{q_1,q_1+1}^{(1)}(t)h_{q1}^{(1)}(t)$.

We construct the characteristic polygon for the equation (2.10) [10,14]. Its vertices are the points $(q;0)$, $(q_1;1)$, and $(0;2)$.

Since $\operatorname{tg}\theta_1=\frac{1}{q_1}$, $\operatorname{tg}\theta_2=\frac{1}{q_2}$, then the solutions of equation (2.10) have the following form:

$$w_j(t,\varepsilon)=\sqrt[q_1]{h_{q_1,1}^{(1)}(t)}\varepsilon^{\frac{1}{q_1}}+O(\varepsilon^{\frac{2}{q_1}}),\ j=\overline{1,q_1},$$

$$w_j(t,\varepsilon)=\sqrt[q_2]{h_{q,q_1+1}^{(1)}(t)}\varepsilon^{\frac{1}{q_2}}+O(\varepsilon^{\frac{2}{q_2}}),\ j=\overline{q_1+1,q}.$$

Let $\varphi_j(t,\varepsilon)$, $j=\overline{1,q}$, be columns of the matrix $T_q(t,\varepsilon)$, where

$$T_q^{-1}(t,\varepsilon)(N_q+\varepsilon F_{1q}(t))T_q(t,\varepsilon)=\widetilde{W}_q(t,\varepsilon)\equiv diag\{w_1(t,\varepsilon),w_2(t,\varepsilon),...,w_q(t,\varepsilon)\}.$$

Then

$$(N_q+\varepsilon F_{1q}(t)-w_j(t,\varepsilon)I_q)\varphi_j(t,\varepsilon)=0,\ j=\overline{1,q}.$$

The components of the vectors $\varphi_j(t,\varepsilon)$, $j=\overline{1,q_1}$, and $\varphi_j(t,\varepsilon)$, $j=\overline{q_1+1,q}$ can be taken as the cofactors of the first and (q_1+1)th rows of the matrix $N_q+\varepsilon F_{1q}(t)-w_j(t,\varepsilon)I_q$, respectively. Hence it follows that

$$\varphi_j(t,\varepsilon)=\begin{pmatrix} a_j^{q_1-1}(t)h^{(1)}_{q,q_1+1}(t)+O(\varepsilon^{\frac{1}{q_1}}) \\ h(t)\varepsilon^{\frac{1}{q_1}}+O(\varepsilon^{\frac{2}{q_1}}) \\ a_j(t)h(t)\varepsilon^{\frac{2}{q_1}}+O(\varepsilon^{\frac{3}{q_1}}) \\ \cdots \\ a_j^{q_1-2}(t)h(t)\varepsilon^{\frac{q_1-1}{q_1}}+O(\varepsilon) \\ -a_j^{q_1-1}(t)h^{(1)}_{q1}(t)+O(\varepsilon^{\frac{1}{q_1}}) \\ -a_j^{q_1}(t)h^{(1)}_{q1}(t)\varepsilon^{\frac{1}{q_1}}+O(\varepsilon^{\frac{2}{q_1}}) \\ \cdots \\ -a_j^{2(q_1-1)}(t)h^{(1)}_{q1}(t)\varepsilon^{\frac{q_1-1}{q_1}}+O(\varepsilon) \\ O(\varepsilon) \\ \cdots \\ O(\varepsilon) \end{pmatrix},\ j=\overline{1,q_1},$$

$$\varphi_j(t,\varepsilon)=\begin{pmatrix} b_{j-q_1}^{q_2-1}(t)h_{q_1,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-q_1}{q_2}}+O(\varepsilon^{\frac{q_2-q_1+1}{q_2}}) \\ b_{j-q_1}^{q_2}(t)h_{q_1,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-q_1+1}{q_2}}+O(\varepsilon^{\frac{q_2-q_1+2}{q_2}}) \\ \dots \\ b_{j-q_1}^{q_2+q_1-2}(t)h_{q_1,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-1}{q_2}}+O(\varepsilon) \\ b_{j-q_1}^{q_1+q_2-1}(t)+O(\varepsilon^{\frac{1}{q_2}}) \\ b_{j-q_1}^{q_1}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^{\frac{1}{q_2}}+O(\varepsilon^{\frac{2}{q_2}}) \\ \dots \\ b_{j-q_1}^{q_1+q_2-2}(t)h_{q,q_1+1}^{(1)}(t)\varepsilon^{\frac{q_2-1}{q_2}}+O(\varepsilon) \end{pmatrix},\ j=\overline{q_1+1,q},$$

where

$$a_j(t)=\sqrt[q_1]{|h_{q_1,1}^{(1)}(t)|}\left(\cos\frac{\arg h_{q_1,1}^{(1)}(t)+2\pi j}{q_1}+i\sin\frac{\arg h_{q_1,1}^{(1)}(t)+2\pi j}{q_1}\right),\ j=\overline{1,q_1},$$

$$b_j(t)=\sqrt[q_2]{|h_{q,q_1+1}^{(1)}(t)|}\left(\cos\frac{\arg h_{q,q_1+1}^{(1)}(t)+2\pi j}{q_2}+i\sin\frac{\arg h_{q,q_1+1}^{(1)}(t)+2\pi j}{q_2}\right),\ j=\overline{1,q_2}.$$

It is easy to see that the matrix $T_q(t,\varepsilon)$ is nonsingular. Indeed, let us define the matrix

$$T_q(t,\varepsilon)=\begin{pmatrix} T_1(t,\varepsilon) & T_2(t,\varepsilon) \\ T_3(t,\varepsilon) & T_4(t,\varepsilon) \end{pmatrix},$$

where $T_1(t,\varepsilon)$ is a square matrix of order q_1. Then

$$det\,T_q(t,\varepsilon)=det\,T_1(t,\varepsilon)\,det(T_4(t,\varepsilon)-T_3(t,\varepsilon)T_1^{-1}(t,\varepsilon)T_2(t,\varepsilon))=$$

$$=det V_1(t)\,det V_2(t)(h_{q,q_1+1}^{(1)}(t))^{q_2}h^{q_1-1}(t)\prod_{i=1}^{q_2}b_i^{q_1}(t)\varepsilon^{\frac{q-2}{2}}+O(\varepsilon^{\frac{q-2}{2}+\gamma}),\ \gamma>0,$$

where

$$V_1(t)=\begin{pmatrix} a_1^{q_1-1}(t) & a_2^{q_1-1}(t) & \dots & a_{q_1}^{q_1-1}(t) \\ 1 & 1 & \dots & 1 \\ a_1(t) & a_2(t) & \dots & a_{q_1}(t) \\ \dots & \dots & \dots & \dots \\ a_1^{q_1-2}(t) & a_2^{q_1-2}(t) & \dots & a_{q_1}^{q_1-2}(t) \end{pmatrix},$$

$$V_2(t)=\begin{pmatrix} b_1^{q_2-1}(t) & b_2^{q_2-1}(t) & \dots & b_{q_2}^{q_2-1}(t) \\ 1 & 1 & \dots & 1 \\ b_1(t) & b_2(t) & \dots & b_{q_2}(t) \\ \dots & \dots & \dots & \dots \\ b_1^{q_2-2}(t) & b_2^{q_2-2}(t) & \dots & b_{q_2}^{q_2-2}(t) \end{pmatrix},$$

because [7]

$$det\,T_1(t,\varepsilon) = det\,V_1(t)h^{(1)}_{q,q_1+1}(t)h^{q_1-1}(t)\varepsilon^{\frac{q_1-1}{2}} + O(\varepsilon^{\frac{q_1-1}{2}+\frac{1}{q_1}}).$$

Further, $det\,V_1(t)$ and $det\,V_2(t)$ are Vandermonde determinants up to a sign. Thus, $det\,V_1(t) \neq 0$, $det\,V_2(t) \neq 0$, $t \in [0;T]$, and $det\,T_q(t,\varepsilon) \neq 0$, $t \in [0;T]$.

Substituting the representation

$$\widetilde{u}^{(0)}_{i1}(t,\varepsilon) = T_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}(t,\varepsilon), \tag{2.11}$$

into equation (2.8), we get

$$(I_q - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))\widetilde{q}^{(0)}_{i1}(t,\varepsilon) = 0. \tag{2.12}$$

Therefore,

$$\widetilde{\lambda}^{(0)}_i(t,\varepsilon) = \frac{1}{w_i(t,\varepsilon)},$$

$$\{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_i = 1,\ \{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_j = 0,\ t \in [0;T],\ i \neq j,\ i,j = \overline{1,q},$$

where $\{\widetilde{q}^{(0)}_{i1}(t,\varepsilon)\}_j$ is the jth component of $\widetilde{q}^{(0)}_{i1}(t,\varepsilon)$.

Consider now equation (2.9). Suppose the following.

(iv) $c^{(1)}_{q+p_1,q+1}(t) \neq 0$, $c^{(1)}_{n,q+p_1+1}(t) \neq 0$, $c(t) \neq 0$, $t \in [0;T]$, where $c(t) = c^{(1)}_{q+p_1,q+1}(t)c^{(1)}_{n,q+p_1+1}(t) - c^{(1)}_{q+p_1,q+p_1+1}(t)c^{(1)}_{n,q+1}(t)$, $c^{(1)}_{ij}(t)$ is the element of the matrix $C_1(t)$.

Then the solutions of equation

$$det(N_p + \varepsilon D_{1p}(t) - wI_p) = 0 \tag{2.13}$$

have the following form:

$$w_j(t,\varepsilon) = \sqrt[p_1]{c^{(1)}_{q+p_1,q+1}(t)}\,\varepsilon^{\frac{1}{p_1}} + O(\varepsilon^{\frac{2}{p_1}}),\ j = \overline{q+1,q+p_1},$$

$$w_j(t,\varepsilon) = \sqrt[p_2]{c^{(1)}_{n,q+p_1+1}(t)}\,\varepsilon^{\frac{1}{p_2}} + O(\varepsilon^{\frac{2}{p_2}}),\ j = \overline{q+p_1+1,n}.$$

Let us $T_p(t,\varepsilon)$ be the square matrix of order p such that

$$T_p^{-1}(t,\varepsilon)(N_p + \varepsilon D_{1p}(t))T_p(t,\varepsilon) = \widetilde{W}_p(t,\varepsilon) \equiv$$
$$\equiv diag\{w_{q+1}(t,\varepsilon), w_{q+2}(t,\varepsilon), ..., w_n(t,\varepsilon)\}.$$

Substituting the representation

$$\widetilde{u}^{(0)}_{i2}(t,\varepsilon) = T_p(t,\varepsilon)\widetilde{q}^{(0)}_{i2}(t,\varepsilon) \tag{2.14}$$

into (2.9), we obtain

$$(\widetilde{W}_p(t,\varepsilon) - \widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{q}^{(0)}_{i2}(t,\varepsilon) = 0. \tag{2.15}$$

Thus

$$\widetilde{\lambda}^{(0)}_i(t,\varepsilon) = w_i(t,\varepsilon),$$

$$\{\widetilde{q}^{(0)}_{i2}(t,\varepsilon)\}_i = 1,\ \{\widetilde{q}^{(0)}_{i2}(t,\varepsilon)\}_j = 0,\ t\in[0;T],\ i\neq j,\ i,j=\overline{q+1,n},$$

and

$$\widetilde{q}^{(0)}_{i1}(t,\varepsilon) = 0,\ t\in[0;T],\ i=\overline{q+1,n},$$
$$\widetilde{q}^{(0)}_{i2}(t,\varepsilon) = 0,\ t\in[0;T],\ i=\overline{1,q}.$$

We rewrite equation (2.7) for $s=1$ as follows:

$$(I_q-\widetilde{\lambda}^{(0)}_i(t,\varepsilon)(N_q+\varepsilon F_{1q}(t)))\widetilde{u}^{(1)}_{i1}(t,\varepsilon) = d^{(1)}_{i1}(t,\varepsilon), \tag{2.16}$$

$$(N_p+\varepsilon D_{1p}(t)-\widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{u}^{(1)}_{i2}(t,\varepsilon) = d^{(1)}_{i2}(t,\varepsilon), \tag{2.17}$$

where $\widetilde{u}^{(1)}_i(t,\varepsilon) = \begin{pmatrix}\widetilde{u}^{(1)}_{i1}(t,\varepsilon)\\ \widetilde{u}^{(1)}_{i2}(t,\varepsilon)\end{pmatrix}$, and $\widetilde{d}^{(1)}_i(t,\varepsilon) = \begin{pmatrix}\widetilde{d}^{(1)}_{i1}(t,\varepsilon)\\ \widetilde{d}^{(1)}_{i2}(t,\varepsilon)\end{pmatrix}$.

The transformation $\widetilde{u}^{(1)}_{i1}(t,\varepsilon) = T_q(t,\varepsilon)\widetilde{q}^{(1)}_{i1}(t,\varepsilon)$, $\widetilde{u}^{(1)}_{i2}(t,\varepsilon) = T_p(t,\varepsilon)\widetilde{q}^{(1)}_{i2}(t,\varepsilon)$ changes (2.16), (2.17) into the equations

$$(I_q-\widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))\widetilde{q}^{(1)}_{i1}(t,\varepsilon) = g^{(1)}_{i1}(t,\varepsilon), \tag{2.18}$$

$$(\widetilde{W}_p(t,\varepsilon)-\widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)\widetilde{q}^{(1)}_{i2}(t,\varepsilon) = g^{(1)}_{i2}(t,\varepsilon), \tag{2.19}$$

where

$$g^{(1)}_{i1}(t,\varepsilon) = T_q^{-1}(t,\varepsilon)d^{(1)}_{i1}(t,\varepsilon) \equiv T_q^{-1}(t,\varepsilon)N_qT'_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1} + \widetilde{W}_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}\widetilde{\lambda}^{(1)}_i(t,\varepsilon),$$
$$g^{(1)}_{i2}(t,\varepsilon) = T_p^{-1}(t,\varepsilon)d^{(1)}_{i2}(t,\varepsilon) \equiv T_p^{-1}(t,\varepsilon)T'_p(t,\varepsilon)\widetilde{q}^{(0)}_{i2} + \widetilde{q}^{(0)}_{i2}\widetilde{\lambda}^{(1)}_i(t,\varepsilon).$$

Therefore,

$$\widetilde{\lambda}^{(1)}_i(t,\varepsilon) = -\frac{\{f^{(1)}_{i1}(t,\varepsilon)\}_i}{w_i(t,\varepsilon)},\ \{\widetilde{q}^{(1)}_{i1}(t,\varepsilon)\}_i \equiv 0,\ t\in[0;T],$$

$$\{\widetilde{q}^{(1)}_{i1}(t,\varepsilon)\}_j = \frac{\{g^{(1)}_{i1}(t,\varepsilon)\}_j w_i(t,\varepsilon)}{w_i(t,\varepsilon)-w_j(t,\varepsilon)},\ i\neq j,\ i,j=\overline{1,q},$$

$$\widetilde{q}^{(1)}_{i2}(t,\varepsilon) = (\widetilde{W}_p(t,\varepsilon)-\widetilde{\lambda}^{(0)}_i(t,\varepsilon)I_p)^{-1}g^{(1)}_{i2}(t,\varepsilon),\ i=\overline{1,q},$$

where $f^{(1)}_{i1}(t,\varepsilon) = T_q^{-1}(t,\varepsilon)N_qT'_q(t,\varepsilon)\widetilde{q}^{(0)}_{i1}$, and

$$\widetilde{\lambda}^{(1)}_i(t,\varepsilon) = -\{f^{(1)}_{i2}(t,\varepsilon)\}_i,\ \{\widetilde{q}^{(1)}_{i2}(t,\varepsilon)\}_i \equiv 0,\ t\in[0;T],$$

$$\{\widetilde{q}^{(1)}_{i2}(t,\varepsilon)\}_j = \frac{\{g^{(1)}_{i2}(t,\varepsilon)\}_j}{w_j(t,\varepsilon)-w_i(t,\varepsilon)},\ i\neq j,\ i,j=\overline{q+1,n},$$

$$\widetilde{q}^{(1)}_{i1}(t,\varepsilon) = (I_q-\widetilde{\lambda}^{(0)}_i(t,\varepsilon)\widetilde{W}_q(t,\varepsilon))^{-1}g^{(1)}_{i1}(t,\varepsilon),\ i=\overline{q+1,n},$$

where $f^{(1)}_{i2}(t,\varepsilon) = T_p^{-1}(t,\varepsilon)T'_p(t,\varepsilon)\widetilde{q}^{(0)}_{i2}$.

Let $T_q^{-1}(t,\varepsilon)$ be defined by

$$T_q^{-1}(t,\varepsilon) = \begin{pmatrix}V_1(t,\varepsilon) & V_2(t,\varepsilon)\\ V_3(t,\varepsilon) & V_4(t,\varepsilon)\end{pmatrix},$$

where $V_1(t,\varepsilon)$ is a square matrix of order q_1.

Using the Frobenius formula for the inverse of the block matrix[7], we find:

$$V_i(t,\varepsilon) = O\left(\varepsilon^{\delta_{2i}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}\right)\begin{pmatrix} O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \\ O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \\ \dots & \dots & \dots & \dots \\ O(1) & O(\varepsilon^{-\frac{1}{\gamma_i}}) & \dots & O(\varepsilon^{-\frac{\gamma_i-1}{\gamma_i}}) \end{pmatrix},$$

$t \in [0;T]$, $\varepsilon \to 0$; $\gamma_i = q_{\delta_{1i}} + q_{2\delta_{2i}} + q_{\delta_{3i}} + q_{2\delta_{4i}}$, $i = \overline{1,4}$; δ_{ij} is Kronecker delta.

It should be noted that the matrix $T_p^{-1}(t,\varepsilon)$ has the same form as $T_q^{-1}(t,\varepsilon)$.

Let the following condition be satisfied.

(v) $q_2 < p_1$.

Then

$$||\widetilde{u}_i^{(1)}(t,\varepsilon)|| = O(\varepsilon^{\frac{1}{q_2}}),\ |\widetilde{\lambda}_i^{(1)}(t,\varepsilon)| = O(1),\ i = \overline{1,q},$$

$$||\widetilde{u}_i^{(1)}(t,\varepsilon)|| = O(\varepsilon^{-\frac{1}{p_1}}),\ |\widetilde{\lambda}_i^{(1)}(t,\varepsilon)| = O(1),\ i = \overline{q+1,n},$$

$t \in [0;T]$, $\varepsilon \to 0$. In the same way we define the functions $\widetilde{u}_i^{(s)}(t,\varepsilon)$, $\widetilde{\lambda}_i^{(s)}(t,\varepsilon)$, $i = \overline{1,n}$, $s = 2,3....$ In addition,

$$\begin{aligned} ||\widetilde{u}_i^{(s)}(t,\varepsilon)|| &= O(\varepsilon^{-[\frac{s}{2}]}),\ i = \overline{1,n}, \\ |\widetilde{\lambda}_i^{(s)}(t,\varepsilon)| &= O(\varepsilon^{-[\frac{s}{2}]-\frac{1}{q_1}}),\ i = \overline{1,q}, \\ |\widetilde{\lambda}_i^{(s)}(t,\varepsilon)| &= O(\varepsilon^{-[\frac{s}{2}]-\frac{2}{p_2}(\frac{s}{2}-[\frac{s}{2}])+\frac{1}{p_2}}),\ i = \overline{q+1,n}, \end{aligned} \tag{2.20}$$

$s = 2,3,...,$ $t \in [0;T]$, $\varepsilon \to 0$, where $\left[\frac{s}{2}\right]$ is the integer part of $\frac{s}{2}$.

3. Asymptotic behavior of solutions

Assume $\operatorname{Re} w_i(t,\varepsilon) \neq 0$, $t \in [0;T]$, $i = \overline{1,n}$. Then $y_i^{(m)}(t,\varepsilon)$ can be defined by

$$y_i^{(m)}(t,\varepsilon) = u_i^{(m)}(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_{a_i}^{t}\lambda_i^{(m)}(t,\varepsilon)dt\right),\ i = \overline{1,n}, \tag{3.1}$$

where $u_i^{(m)}(t,\varepsilon) = \sum_{k=0}^{m}\varepsilon^k\widetilde{u}_i^{(k)}(t,\varepsilon)$, $\lambda_i^{(m)}(t,\varepsilon) = \sum_{k=0}^{m}\varepsilon^k\widetilde{\lambda}_i^{(k)}(t,\varepsilon)$, $i = \overline{1,n}$, and

$$a_i = \begin{cases} 0,\ \operatorname{Re}\widetilde{\lambda}_i^{(0)}(t,\varepsilon) < 0, \\ T,\ \operatorname{Re}\widetilde{\lambda}_i^{(0)}(t,\varepsilon) > 0,\ t \in [0;T],\ \varepsilon \in (0;\varepsilon_0]. \end{cases}$$

The change of variables $y_i(t,\varepsilon) = z_i(t,\varepsilon) + y_i^{(m)}(t,\varepsilon)$ transforms system (2.3) into

$$\varepsilon H(t,\varepsilon)\frac{dz_i}{dt} = C(t,\varepsilon)z_i + f(t,\varepsilon), \tag{3.2}$$

where $f(t,\varepsilon) = C(t,\varepsilon)y_i^{(m)}(t,\varepsilon) - \varepsilon H(t,\varepsilon)\frac{y_i^{(m)}(t,\varepsilon)}{dt}$. It is easy to show that the vector function $f(t,\varepsilon)$ is of order of magnitude $O(\varepsilon^{\alpha(m)})$, $\alpha(m) = m - \left[\frac{m}{2}\right] - \frac{1}{q_1}$.

Let us prove the existence of a solution of system (3.2) such that $z_i(a_i,\varepsilon) = 0$. Multiplying both sides of system (3.2) on the left by $K^{-1}(t,\varepsilon)$, we get

$$\varepsilon F(t,\varepsilon)\frac{dz_i}{dt} = D(t,\varepsilon)z_i + g(t,\varepsilon), \tag{3.3}$$

where

$$F(t,\varepsilon) = H_0 + \sum_{k\geq 1}\varepsilon^k F_k(t),\ D(t,\varepsilon) = C_0 + \sum_{k\geq 1}\varepsilon^k D_k(t),\ g(t,\varepsilon) = K^{-1}(t,\varepsilon)f(t,\varepsilon).$$

Let us define the matrix $T(t,\varepsilon) = diag\{T_q(t,\varepsilon), T_p(t,\varepsilon)\}$. Then

$$T^{-1}(t,\varepsilon)F(t,\varepsilon)T(t,\varepsilon) = diag\{\widetilde{W}_q(t,\varepsilon), I_p\} + \varepsilon L(t,\varepsilon),$$
$$T^{-1}(t,\varepsilon)D(t,\varepsilon)T(t,\varepsilon) = diag\{I_q, \widetilde{W}_p(t,\varepsilon)\} + \varepsilon S(t,\varepsilon),$$

where

$$L(t,\varepsilon) = diag\{L_q(t,\varepsilon), L_p(t,\varepsilon)\} \equiv$$
$$\begin{pmatrix} \varepsilon^{\frac{1}{q_1}}L_{11}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}L_{12}(t,\varepsilon) & 0 & 0 \\ \varepsilon^{\frac{1}{q_2}}L_{21}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}L_{22}(t,\varepsilon) & 0 & 0 \\ 0 & 0 & \varepsilon^{\frac{1}{p_1}}L_{33}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}L_{34}(t,\varepsilon) \\ 0 & 0 & \varepsilon^{\frac{1}{p_2}}L_{43}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}L_{44}(t,\varepsilon) \end{pmatrix},$$

$$S(t,\varepsilon) = \begin{pmatrix} \varepsilon^{\frac{1}{q_1}}S_{11}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}S_{12}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}S_{13}(t,\varepsilon) & \varepsilon^{\frac{1}{q_1}}S_{14}(t,\varepsilon) \\ \varepsilon^{\frac{1}{q_2}}S_{21}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}S_{22}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}S_{23}(t,\varepsilon) & \varepsilon^{\frac{1}{q_2}}S_{24}(t,\varepsilon) \\ \varepsilon^{\frac{1}{p_1}}S_{31}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}S_{32}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}S_{33}(t,\varepsilon) & \varepsilon^{\frac{1}{p_1}}S_{34}(t,\varepsilon) \\ \varepsilon^{\frac{1}{p_2}}S_{41}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}S_{42}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}S_{43}(t,\varepsilon) & \varepsilon^{\frac{1}{p_2}}S_{44}(t,\varepsilon) \end{pmatrix};$$

the diagonal blocks of $L(t,\varepsilon)$ and $S(t,\varepsilon)$ are square matrices of order q_1, q_2, p_1, p_2, respectively; $||L_{ij}(t,\varepsilon)|| = O(1)$, $||S_{ij}(t,\varepsilon)|| = O(1)$, $t \in [0;T]$, $\varepsilon \to 0$.

The transformation $z_i(t,\varepsilon) = T(t,\varepsilon)r_i(t,\varepsilon)$ changes system (3.3) into

$$\varepsilon\left(\begin{pmatrix} \widetilde{W}_q(t,\varepsilon) & 0 \\ 0 & I_p \end{pmatrix} + \varepsilon L(t,\varepsilon)\right)\frac{dr_i}{dt} = \left(\begin{pmatrix} I_q & 0 \\ 0 & \widetilde{W}_p(t,\varepsilon) \end{pmatrix} + \varepsilon S(t,\varepsilon) -\right.$$
$$\left. -\varepsilon\left(\begin{pmatrix} \widetilde{W}_q(t,\varepsilon) & 0 \\ 0 & I_p \end{pmatrix} + \varepsilon L(t,\varepsilon)\right)T^{-1}(t,\varepsilon)T'(t,\varepsilon)\right)r_i + T^{-1}(t,\varepsilon)g(t,\varepsilon)$$

i.e.

$$\varepsilon\frac{dr_{i1}}{dt} = \widetilde{W}_q^{-1}(t,\varepsilon)r_{i1} + \varepsilon(R_{11}(t,\varepsilon)r_{i1} + R_{12}(t,\varepsilon)r_{i2}) + h_1(t,\varepsilon), \tag{3.4}$$

$$\varepsilon\frac{dr_{i2}}{dt} = \widetilde{W}_p(t,\varepsilon)r_{i2} + \varepsilon(R_{21}(t,\varepsilon)r_{i1} + R_{22}(t,\varepsilon)r_{i2}) + h_2(t,\varepsilon), \tag{3.5}$$

where $r_i(t,\varepsilon) = \begin{pmatrix} r_{i1}(t,\varepsilon) \\ r_{i2}(t,\varepsilon) \end{pmatrix}$, $h(t,\varepsilon) = \begin{pmatrix} h_1(t,\varepsilon) \\ h_2(t,\varepsilon) \end{pmatrix}$,

$$h(t,\varepsilon) = \left(\begin{pmatrix} \widetilde{W}_q(t,\varepsilon) & 0 \\ 0 & I_p \end{pmatrix} + \varepsilon L(t,\varepsilon) \right)^{-1} T^{-1}(t,\varepsilon)g(t,\varepsilon),$$

$$R(t,\varepsilon) = \begin{pmatrix} R_{11}(t,\varepsilon) & R_{12}(t,\varepsilon) \\ R_{21}(t,\varepsilon) & R_{22}(t,\varepsilon) \end{pmatrix} \equiv \frac{1}{\varepsilon}\times$$

$$\times \begin{pmatrix} (\widetilde{W}_q(t,\varepsilon) + \varepsilon L_q(t,\varepsilon))^{-1} - \widetilde{W}_q^{-1}(t,\varepsilon) & 0 \\ 0 & ((I_p + \varepsilon L_p(t,\varepsilon))^{-1} - I_p)\widetilde{W}_p(t,\varepsilon) \end{pmatrix} +$$

$$+ \left(\begin{pmatrix} \widetilde{W}_q(t,\varepsilon) & 0 \\ 0 & I_p \end{pmatrix} + \varepsilon L(t,\varepsilon) \right)^{-1} S(t,\varepsilon) - T^{-1}(t,\varepsilon)T'(t,\varepsilon);$$

here, $R_{11}(t,\varepsilon)$ is a square matrix of order q. It is easy to see that $||h(t,\varepsilon)|| = O(\varepsilon^{\alpha(m)-1})$, $t \in [0;T]$, $\varepsilon \to 0$.

Assume that the following condition are satisfied.

(vi) $\operatorname{Re}\frac{1}{w_j(t,\varepsilon)} \le c_1\varepsilon^{l_1} < 0$, $-\frac{1}{q_1} \le l_1 \le 1 - \frac{1}{q_1}$, if $\operatorname{Re} w_j(t,\varepsilon) < 0$,
$\operatorname{Re}\frac{1}{w_j(t,\varepsilon)} \ge c_2\varepsilon^{l_2} > 0$, $-\frac{1}{q_1} \le l_2 \le 1 - \frac{1}{q_1}$, if $\operatorname{Re} w_j(t,\varepsilon) > 0$, $j = \overline{1,q}$.
$\operatorname{Re} w_j(t,\varepsilon) \le c_3\varepsilon^{l_3} < 0$, $\frac{1}{p_2} \le l_3 < 1$, if $\operatorname{Re} w_j(t,\varepsilon) < 0$,
$\operatorname{Re} w_j(t,\varepsilon) \ge c_4\varepsilon^{l_4} > 0$, $\frac{1}{p_2} \le l_4 < 1$, if $\operatorname{Re} w_j(t,\varepsilon) > 0$, $j = \overline{q+1,n}$, $t \in [0;T]$.

Then, without loss of generality, we can assume that

$$\widetilde{W}_q(t,\varepsilon) = diag\{\widetilde{W}_{q-}(t,\varepsilon), \widetilde{W}_{q+}(t,\varepsilon)\},\ \widetilde{W}_p(t,\varepsilon) = diag\{\widetilde{W}_{p-}(t,\varepsilon), \widetilde{W}_{p+}(t,\varepsilon)\}.$$

Here, $\widetilde{W}_{q-}(t,\varepsilon)$ and $\widetilde{W}_{q+}(t,\varepsilon)$ are the diagonal matrices whose eigenvalues are the eigenvalues of $\widetilde{W}_q(t,\varepsilon)$ with negative and positive real parts, respectively, and the matrix $\widetilde{W}_p(t,\varepsilon)$ has the same structure as $\widetilde{W}_q(t,\varepsilon)$.

Thus, systems (3.4), (3.5) can be written as

$$\varepsilon\frac{dr_{i11}}{dt} = \widetilde{W}_{q-}^{-1}(t,\varepsilon)r_{i11} + \varepsilon\left(\sum_{j=1}^{2}(R_{1j}(t,\varepsilon)r_{i1j} + R_{1,j+2}(t,\varepsilon)r_{i2j})\right) + h_{11}(t,\varepsilon), \tag{3.6}$$

$$\varepsilon\frac{dr_{i12}}{dt}=\widetilde{W}_{q+}^{-1}(t,\varepsilon)r_{i12}+\varepsilon\left(\sum_{j=1}^{2}(R_{2j}(t,\varepsilon)r_{i1j}+R_{2,j+2}(t,\varepsilon)r_{i2j})\right)+h_{12}(t,\varepsilon),\tag{3.7}$$

$$\varepsilon\frac{dr_{i21}}{dt}=\widetilde{W}_{p-}(t,\varepsilon)r_{i21}+\varepsilon\left(\sum_{j=1}^{2}(R_{3j}(t,\varepsilon)r_{i1j}+R_{3,j+2}(t,\varepsilon)r_{i2j})\right)+h_{21}(t,\varepsilon),\tag{3.8}$$

$$\varepsilon\frac{dr_{i22}}{dt}=\widetilde{W}_{p+}(t,\varepsilon)r_{i22}+\varepsilon\left(\sum_{j=1}^{2}(R_{4j}(t,\varepsilon)r_{i1j}+R_{4,j+2}(t,\varepsilon)r_{i2j})\right)+h_{22}(t,\varepsilon),\tag{3.9}$$

where

$$R(t,\varepsilon)=\begin{pmatrix}R_{11}(t,\varepsilon)&R_{12}(t,\varepsilon)&R_{13}(t,\varepsilon)&R_{14}(t,\varepsilon)\\R_{21}(t,\varepsilon)&R_{22}(t,\varepsilon)&R_{23}(t,\varepsilon)&R_{24}(t,\varepsilon)\\R_{31}(t,\varepsilon)&R_{32}(t,\varepsilon)&R_{33}(t,\varepsilon)&R_{34}(t,\varepsilon)\\R_{41}(t,\varepsilon)&R_{42}(t,\varepsilon)&R_{43}(t,\varepsilon)&R_{44}(t,\varepsilon)\end{pmatrix};$$

the dimensions of vectors r_{i11}, r_{i12}, r_{i21}, r_{i22} and h_{11}, h_{12}, h_{21}, h_{22} coincide with the orders of the matrices $\widetilde{W}_{q-}(t,\varepsilon)$, $\widetilde{W}_{q+}(t,\varepsilon)$, $\widetilde{W}_{p-}(t,\varepsilon)$, $\widetilde{W}_{p+}(t,\varepsilon)$, respectively.

Let us write a system of integral equations equivalent to system (3.6) – (3.9):

$$r_{i11}(t,\varepsilon)=\int_0^t Z_{11}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{1j}(\tau,\varepsilon)r_{i1j}+R_{1,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{11}(\tau,\varepsilon)\right)d\tau,\tag{3.10}$$

$$r_{i12}(t,\varepsilon)=\int_T^t Z_{12}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{2j}(\tau,\varepsilon)r_{i1j}+R_{2,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{12}(\tau,\varepsilon)\right)d\tau,\tag{3.11}$$

$$r_{i21}(t,\varepsilon)=\int_0^t Z_{21}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{3j}(\tau,\varepsilon)r_{i1j}+R_{3,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{21}(\tau,\varepsilon)\right)d\tau,\tag{3.12}$$

$$r_{i22}(t,\varepsilon)=\int_T^t Z_{22}(t,\tau,\varepsilon)\left(\sum_{j=1}^{2}(R_{4j}(\tau,\varepsilon)r_{i1j}+R_{4,j+2}(\tau,\varepsilon)r_{i2j})+\frac{1}{\varepsilon}h_{11}(\tau,\varepsilon)\right)d\tau,\tag{3.13}$$

where $Z_{11}(t,\tau,\varepsilon)$ and $Z_{12}(t,\tau,\varepsilon)$ are fundamental matrices solutions of the systems

$$\varepsilon\frac{dr_{i11}}{dt}=\widetilde{W}_{q-}^{-1}(t,\varepsilon)r_{i11},\ Z_{11}(\tau,\tau,\varepsilon)=I_k,$$

and

$$\varepsilon\frac{dr_{i12}}{dt} = \widetilde{W}_{q+}^{-1}(t,\varepsilon)r_{i12},\ Z_{12}(\tau,\tau,\varepsilon) = I_{q-k},$$

respectively; in the same way we define matrices $Z_{21}(t,\tau,\varepsilon)$ and $Z_{22}(t,\tau,\varepsilon)$.

Let us consider the mapping $\psi = Ar$ of the set

$$P = \{r(t,\varepsilon) \in C[0;T] : ||r(t,\varepsilon)|| \leq c\varepsilon^{\alpha(m)-2}\},$$

into itself given by system (3.10) – (3.13). The mapping $\psi = Ar$ is a contraction mapping. Thus, the operator equation $r = Ar$ (and consequently system (3.10) – (3.13) also) has one and only one solution.

Thus, the main result of the paper can be formulated as follows.

Theorem 1. *If $A_s(t)$, $B_s(t) \in C^{m+1}[0;T]$, $s \geq 0$, and the assumptions (i) – (vi) are satisfied, then there exist n linearly independent solutions $x_i = x_i(t,\varepsilon)$, $t \in [0;T]$, of system (1.4) such that*

$$||x_i(t,\varepsilon) - x_i^{(m)}(t,\varepsilon)|| = O(\varepsilon^{\alpha(m)-2}),\ \varepsilon \to 0,\ m \geq 5,$$

where $x_i^{(m)}(t,\varepsilon) = Q(t,\varepsilon)y_i^{(m)}(t,\varepsilon)$.

Remark 1. If the pencil $B_0(t) - \lambda A_0(t)$ has more than one distinct eigenvalue, then system (1.4) can be reduced to a set of systems of lower order in each of which the corresponding characteristic equation has only one eigenvalue [6, 10, 11].

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