



ON GENERALIZED STATISTICAL CONVERGENCE OF SEQUENCES OF SETS OF ORDER α

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Abstract. In this paper we define generalized statistical convergence for sequences of sets of order α , $0 < \alpha \leq 1$ in sense of Wijsman and study some basic properties of this concept.

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1. INTRODUCTION

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. We shall define Wijsman λ -statistical convergence for sequences of sets of order α , ($0 < \alpha \leq 1$) and establish some basic results regarding the notions Wijsman λ -statistical convergence and Wijsman statistical convergence for sequences of sets of α , ($0 < \alpha \leq 1$).

The idea of statistical convergence first appeared, under the name of "almost convergence" in the first edition Zygmund [25] of celebrated monograph of Zygmund [26]. Later, this idea was introduced by Fast [8] and Steinhaus [22] and later was introduced by Schoenberg [21], also independently by Buck [1] and studied various authors (see [5, 9, 20]). Mursaleen [17], introduced the notion λ -statistical convergence for real sequences. For more details on λ -statistical convergence we refer to [2] and many others. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers.

A real or complex number sequence $x = (x_k)$ is said to be *statistically convergent* to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S - \lim x = L$ or $x_k \rightarrow L(S)$ and S denotes the set of all statistically convergent sequences.

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ) -summable to number L (see [15]) if $t_n(x) \rightarrow L$ as $n \rightarrow \infty$. If $\lambda_n = n$, then (V, λ) -summability reduces to $(C, 1)$ -summability.

Mursaleen [17] defined λ -statistically convergent sequence as follows. A sequence $x = (x_k)$ is said to be λ -statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0.$$

Let S_λ denotes the set of all λ -statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S .

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset $A \subset X$, the distance from x to A is defined by

$$d(x, A) = \inf_{y \in A} \rho(x, y).$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ ($k \in \mathbb{N}$), we say that the sequence (A_k) is Wijsman convergent (see [23, 24]) to A if $\lim_k d(x, A_k) = d(x, A)$ for each $x \in X$. In this case we write $W - \lim A_k = A$. The concepts of Wijsman statistical convergence and boundedness for the sequence (A_k) were given by Nuray and Rhoades [18] as follows. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$ ($k \in \mathbb{N}$), we say that the sequence (A_k) is Wijsman statistical convergent to A if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim_k A_k = A$ or $A_k \rightarrow A(S^W)$. The sequence (A_k) is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. The set of all bounded sequences of sets denoted by L_∞ . For details on Wijsman statistical convergent we refer to [11–14] and many others.

2. WIJSMAN λ -STATISTICAL CONVERGENCE

In this section, we will define Wijsman λ -statistical convergence of sequences of sets and will give the relations between Wijsman statistical and Wijsman λ -statistical convergence of sequences of sets.

Definition 1. Let (X, ρ) be a metric space and $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and $I_n = [n - \lambda_n + 1, n]$. For any non-empty closed subsets $A, A_k \subset X (k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman λ -statistical convergent to A if the sequence $(d(x, A_k))$ is λ -statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\lambda^W - \lim_k A_k = A$ or $A_k \rightarrow A (B_\lambda^W)$.

Example 1. Let $X = \mathbb{R}^2$ and the sequence (A_k) is defined as follows:

$$A_k = \begin{cases} \{(x, y) : x^2 + (y - 1)^2 = k^{-1}\}, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, k \\ & \text{is square integer} \\ \{(0, 0)\}, & \text{otherwise} \end{cases}.$$

Then the sequence (A_k) is Wijsman λ -statistical convergent to $A = \{(0, 0)\}$ since

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |d(x, A_k) - d(x, \{(0, 0)\})| \geq \varepsilon\}| = 0.$$

But it is not Wijsman convergent.

Definition 2. Let (X, ρ) be a metric space. Then for any non-empty closed subsets $A, A_k \subset X (k \in \mathbb{N})$, we say that the sequence (A_k) is Wijsman λ -Cesáro summable to A if

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| = 0.$$

In this case, we write $w_\lambda^W - \lim_k A_k = A$ or $A_k \rightarrow A (w_\lambda^W)$ and

$$w_\lambda^W = \left\{ (A_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| = 0 \right\}.$$

If $\lambda_n = n$, then Wijsman λ -Cesáro summable becomes Wijsman-Cesáro summable, i.e.

$$w^W = \left\{ (A_k) : \lim_n \frac{1}{n} \sum_{k \in \mathbb{N}} |d(x, A_k) - d(x, A)| = 0 \right\}.$$

Theorem 1. Let (X, ρ) be a metric space and $A, A_k \subset X$ ($k \in \mathbb{N}$) be non-empty closed subsets of X . Then

- (a) $w_\lambda^W \subset S_\lambda^W$ and the inclusion is proper.
- (b) Let $(A_k) \in L_\infty$, then $S_\lambda^W \subset w_\lambda^W$.
- (c) $S_\lambda^W \cap L_\infty = w_\lambda^W \cap L_\infty$.

Proof. (a) Let $\varepsilon > 0$ and $(A_k) \in w_\lambda^W$. Then we can write

$$\begin{aligned} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| &\geq \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)| \\ &\geq \varepsilon |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \end{aligned}$$

which gives the result. To show that the inclusion is strict, we define the sequence (A_k) as follows:

$$A_k = \begin{cases} \{k\}, & \text{if } n - \lceil \sqrt{\lambda_n} \rceil + 1 \leq k \leq n; \\ \{0\}, & \text{otherwise} \end{cases}$$

It is clear that $(A_k) \notin L_\infty$ and for $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}| = \lim_n \frac{1}{\lambda_n} \lceil \sqrt{\lambda_n} \rceil = 0.$$

So $(A_k) \in S_\lambda^W$, but

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |d(x, A_k) - d(x, \{0\})| = \lim_n \frac{1}{\lambda_n} \frac{(\lceil \sqrt{\lambda_n} \rceil)(\lceil \sqrt{\lambda_n} \rceil + 1)}{2} = \frac{1}{2} \neq 0.$$

Therefore $(A_k) \notin w_\lambda^W$. This completes the proof of (a).

(b) Suppose that $(A_k) \in S_\lambda^W$ and $(A_k) \in L_\infty$, say $|d(x, A_k) - d(x, A)| \leq M$ for each $x \in X$ and for all $k \in \mathbb{N}$. Given $\varepsilon > 0$, we get

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| \\ &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)| \\ &\quad + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} |d(x, A_k) - d(x, A)| \\ &\leq \frac{M}{\lambda_n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| + \varepsilon \end{aligned}$$

from which the result follows.

(c) It follows from (a) and (b). □

If we let $\lambda_n = n$ in Theorem 1, then we have the following corollary.

Corollary 1. *Let (X, ρ) be a metric space and $A, A_k \subset X$ ($k \in \mathbb{N}$) be non-empty closed subsets of X . Then*

- (a) $w^W \subset S^W$ and the inclusion is proper.
- (b) Let $(A_k) \in L_\infty$, then $S^W \subset w^W$.
- (c) $S^W \cap L_\infty = w^W \cap L_\infty$.

Theorem 2. $S^W \subset S_\lambda^W$ if and only if $\liminf \frac{\lambda_n}{n} > 0$.

Proof. Suppose that $\liminf \frac{\lambda_n}{n} > 0$. For given $\varepsilon > 0$, we have

$$\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \supset \{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}.$$

Therefore

$$\begin{aligned} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| &\geq \frac{1}{n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &\geq \frac{\lambda_n}{n} \cdot \frac{1}{\lambda_n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using $\liminf \frac{\lambda_n}{n} > 0$, we get the desired result.

Conversely, suppose that $\liminf_n \frac{\lambda_n}{n} = 0$. Then we can select a subsequence $(n(i))_{i=1}^\infty$ such that

$$\frac{\lambda_{n(i)}}{n(i)} < \frac{1}{i}.$$

We define a sequence (A_k) as follows:

$$A_k = \begin{cases} \{1\}, & \text{if } n(i) - \lceil \sqrt{\lambda_{n(i)}} \rceil + 1 \leq k \leq n(i), i = 1, 2, 3, \dots; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then (A_k) is Wijsman-statistically convergent, so $(A_k) \in S^W$. But $(A_k) \notin w_\lambda^W$. Therefore Theorem 1 (b) implies that $(A_k) \notin S_\lambda^W$. This completes the proof. □

Theorem 3. $S_\lambda^W \subset s^W$ if $\liminf \frac{\lambda_n}{n} = 1$.

Proof. Since $\lim_n \frac{\lambda_n}{n} = 1$, then for $\varepsilon > 0$, we observe that

$$\begin{aligned} &\frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &\leq \frac{1}{n} |\{k \leq n - \lambda_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &\quad + \frac{1}{n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &\leq \frac{n - \lambda_n}{n} + \frac{1}{n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \end{aligned}$$

$$= \frac{n - \lambda_n}{n} + \frac{\lambda_n}{n} \frac{1}{\lambda_n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|.$$

This implies that (A_k) Wijsman-statistically convergent, if (A_k) is Wijsman λ -statistically convergent. Thus $S_\lambda^W \subset S^W$. \square

Remark 1. Since $\lim_n \frac{\lambda_n}{n} = 1$, this implies that $\liminf_n \frac{\lambda_n}{n} > 0$, then from Theorem 3, we have $S^W \subset S_\lambda^W$. Hence $S_\lambda^W = S^W$.

3. WIJSMAN λ -STATISTICAL CONVERGENCE OF ORDER α

The notion of order statistical convergence was introduced by Gadjiev and Orhan [10] and after that statistical convergence of order α studied by Çolak [3], λ -statistical convergence of order α studied by Çolak and Bektaş [4], λ -statistical convergence of order α of sequence of functions studied by Et et al., [6, 7] and many authors.

In this section, we define the concept of $S_{\lambda, \alpha}^W$ -statistical convergence and establish the relationship of S_λ^W with w_λ^W . Also we introduce the notion of S_λ^W -statistical convergence of order α of real number sequences and obtain some inclusion relations between the set of S^W -statistical convergence of order α .

Definition 3. Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n$, $\lambda_1 = 1$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ and $I_n = [n - \lambda_n + 1, n]$. We denote the family of such sequences $\lambda = (\lambda_n)$ by Λ . For any non-empty closed subsets $A, A_k \subset X$ ($k \in \mathbb{N}$), we say that the sequence (A_k) is Wijsman λ -statistical convergent to A of order α or $S_{\lambda, \alpha}^W$ -convergent to A , if for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_n \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $S_{\lambda, \alpha}^W - \lim A_k = A$ or $A_k \rightarrow A(S_{\lambda, \alpha}^W)$. If $\lambda_n = n$ for all $n \in \mathbb{N}$, we obtain the sequence space S_α^W such that

$$S_\alpha^W = \left\{ (A_k) : \lim_n \frac{1}{n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0 \right\}$$

and if $\lambda_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$ we obtain the sequence space S^W such that

$$S^W = \left\{ (A_k) : \lim_n \frac{1}{n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0 \right\}.$$

Definition 4. Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and $\lambda = (\lambda_n) \in \Lambda$. For any non-empty closed subsets $A, A_k \subset X$ ($k \in \mathbb{N}$), we say that the sequence (A_k) is Wijsman λ -Cesaro summable to A of order α or $w_{\lambda, \alpha}^W$ -convergent to A , if for each $x \in X$

$$\lim_n \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $w_{\lambda, \alpha}^W - \lim A_k = A$ or $A_k \rightarrow A(w_{\lambda, \alpha}^W)$. If $\lambda_n = n$ for all $n \in \mathbb{N}$, we obtain the sequence space w_{α}^W such that

$$w_{\alpha}^W = \left\{ (A_k) : \lim_n \frac{1}{n^{\alpha}} \sum_{k \in I_n} |d(x, A_k) - d(x, A)| = 0 \right\}$$

and if $\lambda_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$ we obtain the sequence space w^W such that

$$w^W = \left\{ (A_k) : \lim_n \frac{1}{n} \sum_k |d(x, A_k) - d(x, A)| = 0 = 0 \right\}.$$

Theorem 4. *Let (X, ρ) be a metric space and $\alpha \in (0, 1]$. If $S_{\lambda, \alpha}^W - \lim A_k = A$ then A is unique.*

Proof. The proof is easy, so omitted. □

Theorem 5. *Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and non-empty closed subsets $A, B, A_k, B_k \subset X$ ($k \in \mathbb{N}$), then*

- (a) *If $S_{\lambda, \alpha}^W - \lim A_k = A$ and $c \in \mathbb{C}$, then $S_{\lambda, \alpha}^W - \lim cA_k = cA$;*
- (b) *If $S_{\lambda, \alpha}^W - \lim A_k = A$ and $S_{\lambda, \alpha}^W - \lim B_k = B$, then $S_{\lambda, \alpha}^W - \lim (A_k + B_k) = A + B$.*

Proof. (a) For $c = 0$, the result is trivial. Suppose that $c \neq 0$, then for every $\varepsilon > 0$ the result follows from the following inequality

$$\begin{aligned} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |d(cx, cA_k) - d(cx, cA)| \geq \varepsilon\}| \\ = \frac{1}{\lambda_n^{\alpha}} \left| \left\{ k \in I_n : |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{|c|} \right\} \right|. \end{aligned}$$

(b) For every $\varepsilon > 0$. The result follows from the following inequality.

$$\begin{aligned} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |d(x, A_k + B_k) - d(x, A + B)| \geq \varepsilon\}| \\ \leq \frac{1}{\lambda_n^{\alpha}} \left| \left\{ k \in I_n : |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{2} \right\} \right| \\ + \frac{1}{\lambda_n^{\alpha}} \left| \left\{ k \in I_n : |d(x, B_k) - d(x, B)| \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

□

If we take $\lambda_n = n$ in the above theorem, then we have

Corollary 2. *Let (X, ρ) be a metric space, $\alpha \in (0, 1]$ and non-empty closed subsets $A, B, A_k, B_k \subset X$ ($k \in \mathbb{N}$), then*

- (a) If $S_\alpha^W - \lim A_k = A$ and $c \in \mathbb{C}$, then $S_\alpha^W - \lim cA_k = cA$;
- (b) If $S_\alpha^W - \lim A_k = A$ and $S_\alpha^W - \lim B_k = B$, then $S_\alpha^W - \lim (A_k + B_k) = A + B$.

Theorem 6. If $0 < \alpha < \beta \leq 1$, then $S_{\lambda, \alpha}^W \subset S_{\lambda, \beta}^W$ and the inclusion is strict.

Proof. The proof of the result follows from the following inequality.

$$\begin{aligned} \frac{1}{\lambda_n^\beta} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ \geq \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

To prove the inclusion is strict, let λ be given and we consider a sequence (A_k) of non-empty closed subsets of (X, ρ) metric space defined by

$$\begin{aligned} A_k &= \begin{cases} \{(k, k)\}, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n; \\ \{(0, 0)\}, & \text{otherwise.} \end{cases} \\ &= \frac{1}{\lambda_n^\beta} |\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n\}| \leq \frac{\sqrt{\lambda_n}}{\lambda_n^\beta} \end{aligned}$$

Then we have $(A_k) \in S_{\lambda, \beta}^W$ for $\frac{1}{2} < \beta \leq 1$ but $(A_k) \notin S_{\lambda, \alpha}^W$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 7. If a sequence (A_k) of non-empty closed subsets of (X, ρ) metric space is $S_{\lambda, \alpha}^W$ -convergent to A , then it is S_λ^W -convergent to A for $0 < \alpha \leq 1$.

Proof. It is easy, so omitted. □

Theorem 8. Let (X, ρ) be a metric space and $\alpha \in (0, 1]$. Then $S_\alpha^W \subset S_{\lambda, \alpha}^W$ if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^\alpha}{n^\alpha} > 0.$$

Proof. If $A_k \rightarrow A (S_\alpha^W)$ then for every $\varepsilon > 0$ and for sufficiently large n we have

$$\begin{aligned} \frac{1}{n^\alpha} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ \geq \frac{1}{n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ \geq \frac{\lambda_n^\alpha}{n^\alpha} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using the given condition, we get $A_k \rightarrow A (S_{\lambda, \alpha}^W)$. This completes the proof of the theorem. □

Corollary 3. Let (X, ρ) be a metric space, $\lambda = (\lambda_n) \in \Lambda$ and $\alpha \in (0, 1]$. Then $S_{\lambda, \alpha}^W \subset S^W$.

Theorem 9. Let $\lambda = (\lambda_n) \in \Lambda$ and $\alpha \in (0, 1]$. Then $S^W \subset S_{\lambda, \alpha}^W$ if and only if

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^\alpha}{n} > 0. \tag{3.1}$$

Proof. Let the condition (3.1) holds and $(A_k) \in S^W$. For a given $\varepsilon > 0$ we have

$$\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \supset \{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}.$$

Then we have

$$\begin{aligned} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| &\geq \frac{1}{n} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &= \frac{\lambda_n^\alpha}{n} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

By taking limit as $n \rightarrow \infty$ and from relation (3.1) we have

$$A_k \rightarrow A(S^W) \Rightarrow A_k \rightarrow A(S_{\lambda, \alpha}^W).$$

Next we suppose that

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^\alpha}{n} = 0.$$

Then we can choose a subsequence (n_i) such that $\frac{\lambda_{n_i}^\alpha}{n_i} < \frac{1}{i}$. Define a sequence (A_k) as follows:

$$A_k = \begin{cases} \{(1, 1)\}, & \text{if } k \in I_{n_i}; \\ \{(0, 0)\}, & \text{otherwise.} \end{cases}$$

Then clearly $(A_k) \in S^W$ but $(A_k) \notin S_{\lambda, \alpha}^W$. Since $S_{\lambda, \alpha}^W \subset S_\lambda^W$, we have $(A_k) \notin S_{\lambda, \alpha}^W$, which is a contradiction. Hence the relation (3.1) holds. \square

Theorem 10. Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If

$$\liminf_{n \rightarrow \infty} \frac{\lambda_n^\alpha}{\mu_n^\beta}, \tag{3.2}$$

then $S_{\lambda, \beta}^W \subseteq S_{\lambda, \alpha}^W$.

Proof. Suppose that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and the condition (3.2) satisfied. Then $I_n \subset J_n$ and so that for $\varepsilon > 0$ we can write

$$\{k \in J_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \supset \{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}.$$

Then we have

$$\frac{1}{\mu_n^\beta} |\{k \in J_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|$$

$$\geq \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|,$$

for all $n \in \mathbb{N}$, where $J_n = [n - \mu_n + 1, n]$. Taking limit $n \rightarrow \infty$ in the last inequality and using (3.2), we have $S_{\lambda, \beta}^W \subseteq S_{\lambda, \alpha}^W$. \square

Corollary 4. Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.2) holds, then

- (a) $S_{\mu, \alpha}^W \subseteq S_{\lambda, \alpha}^W$ for $0 < \alpha \leq 1$,
- (b) $S_\mu^W \subseteq S_{\lambda, \alpha}^W$ for $0 < \alpha \leq 1$,
- (c) $S_\mu^W \subseteq S_\lambda^W$.

Theorem 11. Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If

$$\lim_{n \rightarrow \infty} \frac{\mu_n}{\lambda_n^\beta} = 1, \quad (3.3)$$

then $S_{\lambda, \alpha}^W \subseteq S_{\mu, \beta}^W$.

Proof. Let $S_{\lambda, \alpha}^W - \lim A_k = A$ and (3.3) be satisfied. Since $I_n \subset J_n$, for $\varepsilon > 0$ we can write

$$\begin{aligned} & \frac{1}{\mu_n^\beta} |\{k \in J_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ &= \frac{1}{\mu_n^\beta} |\{n - \mu_n + 1 \leq k \leq n - \lambda_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ & \quad + \frac{1}{\mu_n^\beta} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ & \leq \frac{\mu_n - \lambda_n}{\mu_n^\beta} + \frac{1}{\mu_n^\beta} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ & \leq \frac{\mu_n - \lambda_n^\beta}{\lambda_n^\beta} + \frac{1}{\mu_n^\beta} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\ & \leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1 \right) + \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|. \end{aligned}$$

Using the relation (3.3) and $S_{\lambda, \alpha}^W - \lim A_k = A$ the right-hand side of the above inequality tends to zero as $n \rightarrow \infty$. This implies that $S_{\lambda, \alpha}^W \subseteq S_{\mu, \beta}^W$. \square

Corollary 5. Let (X, ρ) be a metric space, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.3) holds, then

- (a) $S_{\lambda, \alpha}^W \subseteq S_{\mu, \alpha}^W$ for $0 < \alpha \leq 1$,

- (b) $S_{\lambda}^W \subseteq S_{\mu, \alpha}^W$ for $0 < \alpha \leq 1$,
- (c) $S_{\lambda}^W \subseteq S_{\mu}^W$.

An Orlicz function M is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$, $M(x) > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the *modulus function* and characterized by Ruckle [19]. An Orlicz function M is said to satisfy Δ_2 -condition for all values u , if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1. *An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.*

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{ for some } r > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p , which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

Definition 5. Let (X, ρ) be a metric space. Let M be an Orlicz function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\alpha \in (0, 1]$, $\lambda = (\lambda_n)$ be a sequence of positive reals, and for $\rho > 0$, now we define

$$w_{\lambda, \alpha}^W[M, p] = \left\{ (A_k) \in X : \lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M\left(\frac{|d(x, A_k) - d(x, A)|}{\rho}\right) \right]^{p_k} = 0, \right. \\ \left. \text{for some } A \text{ and for } x \in X \right\}.$$

If $M(x) = x$ and $p_k = p$ for all $k \in \mathbb{N}$ then we shall write $w_{\lambda, \alpha}^W[M, p] = w_{\lambda, \alpha}^W(p)$ and if $M(x) = x$ then we shall write $w_{\lambda, \alpha}^W[M, p] = w_{\lambda, \alpha}^W[p]$.

Theorem 12. *Let (p_k) be a bounded and $0 < \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Let $0 < \alpha \leq \beta \leq 1$, M be an Orlicz function and $\lambda = (\lambda_n)$ be a sequence of positive reals, then $w_{\lambda, \alpha}^W[M, p] \subset S_{\lambda, \beta}^W$.*

Proof. Let $(A_k) \in w_{\lambda, \alpha}^W [M, p]$. Let $\varepsilon > 0$ be given. As $\lambda_n^\alpha \leq \lambda_n^\beta$ for each n we can write

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\ &= \frac{1}{\lambda_n^\alpha} \left[\sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \right. \\ & \quad \left. + \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \right] \\ &\geq \frac{1}{\lambda_n^\beta} \left[\sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \right. \\ & \quad \left. + \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \right] \\ &\geq \frac{1}{\lambda_n^\beta} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^{p_k} \\ &\geq \frac{1}{\lambda_n^\beta} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} \min \left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H \right) \\ &\geq \frac{1}{\lambda_n^\beta} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \min \left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H \right), \end{aligned}$$

where $\varepsilon_1 = \frac{\varepsilon}{\rho}$. From the above inequality we have $(A_k) \in S_{\lambda, \alpha}^W$. □

Corollary 6. Let $0 < \alpha \leq 1$, M be an Orlicz function and $\lambda = (\lambda_n)$ be an element of Λ , then $w_{\lambda, \alpha}^W [M, p] \subset S_{\lambda, \alpha}^W$.

Theorem 13. Let M be an Orlicz function, (A_k) be a sequence in L_∞ , and $\lambda = (\lambda_n)$ be an element of Λ . If $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_n^\alpha} = 1$, then $S_{\lambda, \alpha}^W \subset w_{\lambda, \alpha}^W [M, p]$.

Proof. Suppose that (A_k) is in L_∞ and $S_{\lambda, \alpha}^W - \lim_k A_k = A$. As $(A_k) \in L_\infty$ there exists $K > 0$ such that $|d(x, A_k) - d(x, A)| \leq K$ for all k . For given $\varepsilon > 0$ we have

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\ &= \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\ & \quad + \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} \left[M \left(\frac{|d(x, A_k) - d(x, A)|}{\rho} \right) \right]^{p_k} \\ &\leq \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} \max \left\{ \left[M \left(\frac{K}{\rho} \right) \right]^h, \left[M \left(\frac{K}{\rho} \right) \right]^H \right\} \\ & \quad + \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^{p_k} \\ &\leq \max \left\{ \left[M \left(\frac{K}{\rho} \right) \right]^h, \left[M \left(\frac{K}{\rho} \right) \right]^H \right\} \frac{1}{\lambda_n^\alpha} |d(x, A_k) - d(x, A)| \geq \varepsilon| \\ & \quad + \frac{\lambda_n}{\lambda_n^\alpha} \max \left\{ \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^h, \left[M \left(\frac{\varepsilon}{\rho} \right) \right]^H \right\}. \end{aligned}$$

Therefore we have $(A_k) \in w_{\lambda, \alpha}^W [M, p]$. □

Theorem 14. Let $\lambda = (\lambda_n) \in \Lambda, 0 < \alpha \leq \beta \leq 1, p$ be a positive real number, then $w_{\lambda, \alpha}^W(p) \subseteq w_{\beta, \alpha}^W(p)$.

Proof. The proof is easy, so omitted. □

Corollary 7. Let $\lambda = (\lambda_n) \in \Lambda$ and p be a positive real number, then $w_{\lambda, \alpha}^W(p) \subseteq w_\lambda^W(p)$.

Theorem 15. Let $\lambda = (\lambda_n) \in \Lambda, 0 < \alpha \leq \beta \leq 1$ and p be a positive real number, then $w_{\lambda, \alpha}^W(p) \subseteq S_{\lambda, \beta}^W$.

Proof. Let $(A_k) \in w_{\lambda, \alpha}^W(p)$ and for $\varepsilon > 0$ we have

$$\sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p = \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)|^p$$

$$\begin{aligned}
 &+ \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} |d(x, A_k) - d(x, A)|^p \\
 \geq &\sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)|^p \\
 \geq &|\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \cdot \varepsilon^p.
 \end{aligned}$$

Therefore we have

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p \geq \frac{1}{\lambda_n^\beta} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \cdot \varepsilon^p.$$

The last inequality implies that $(A_k) \in S_{\lambda, \beta}^W$ if $(A_k) \in w_{\lambda, \alpha}^W(p)$. This completes the proof of the theorem. □

Theorem 16. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (3.3) holds, then $w_{\mu, \beta}^W(p) \subseteq w_{\lambda, \alpha}^W(p)$.

Proof. The proof is easy, so omitted. □

Corollary 8. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.3) holds, then

- (a) $w_{\mu, \alpha}^W(p) \subseteq w_{\lambda, \alpha}^W(p)$ for $0 < \alpha \leq 1$,
- (b) $w_{\mu}^W(p) \subseteq w_{\lambda}^W(p)$ for $0 < \alpha \leq 1$,
- (c) $w_{\mu}^W(p) \subseteq w_{\lambda}^W(p)$.

Theorem 17. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (3.3) holds, then $w_{\mu, \beta}^W(p) \subseteq S_{\lambda, \alpha}^W$.

Proof. Let $(A_k) \in w_{\mu, \beta}^W(p)$ and for $\varepsilon > 0$. Then we have

$$\begin{aligned}
 \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p &= \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)|^p \\
 &+ \sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| < \varepsilon}} |d(x, A_k) - d(x, A)|^p \\
 \geq &\sum_{\substack{k \in I_n \\ |d(x, A_k) - d(x, A)| \geq \varepsilon}} |d(x, A_k) - d(x, A)|^p \\
 \geq &|\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \cdot \varepsilon^p.
 \end{aligned}$$

Therefore we have

$$\frac{1}{\mu_n^\beta} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p \geq \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \cdot \varepsilon^p.$$

Since (3.3) holds and $(A_k) \in w_{\mu, \beta}^W(p)$. The last inequality implies that $(A_k) \in S_{\lambda, \alpha}^W$. This completes the proof of the theorem. \square

Corollary 9. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq 1$. If (3.3) holds, then

- (a) $w_{\mu, \alpha}^W(p) \subseteq S_{\lambda, \alpha}^W$,
- (b) $w_\mu^W(p) \subseteq S_{\lambda, \alpha}^W$,
- (c) $w_\mu^W(p) \subseteq S_\lambda^W$.

Theorem 18. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. If (3.3) holds, then $L_\infty \cap w_{\mu, \alpha}^W(p) \subseteq w_{\lambda, \beta}^W(p)$.

Proof. Let $(A_k) \in L_\infty \cap w_{\mu, \alpha}^W(p)$ and suppose that (3.3) holds. Since $(A_k) \in L_\infty$, there exists $K > 0$ such that $|d(x, A_k) - d(x, A)| \leq K$ for all k . Since $\lambda_n \leq \mu_n$ and $I_n \subset J_n$ for all $n \in \mathbb{N}$ we can write

$$\begin{aligned} & \frac{1}{\mu_n^\beta} \sum_{k \in J_n} |d(x, A_k) - d(x, A)|^p \\ &= \frac{1}{\mu_n^\beta} \sum_{k \in J_n - I_n} |d(x, A_k) - d(x, A)|^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p \\ &\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta} \right) K^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p \\ &\leq \left(\frac{\mu_n - \lambda_n^\beta}{\mu_n^\beta} \right) K^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p \\ &\leq \left(\frac{\mu_n - \lambda_n^\beta}{\lambda_n^\beta} \right) K^p + \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p \\ &\leq \left(\frac{\mu_n}{\lambda_n^\beta} - 1 \right) K^p + \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |d(x, A_k) - d(x, A)|^p. \end{aligned}$$

This implies that $(A_k) \in w_{\mu, \beta}^W(p)$. Hence $L_\infty \cap w_{\mu, \alpha}^W(p) \subseteq w_{\lambda, \beta}^W(p)$. \square

Corollary 10. Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in \mathbb{N}$. If (3.3) holds, then

- (a) $L_\infty \cap w_{\mu, \alpha}^W(p) \subseteq w_{\lambda, \beta}^W(p)$ for $0 < \alpha \leq 1$,

- (b) $L_\infty \cap w_{\lambda, \alpha}^W(p) \subseteq w_\mu^W(p)$ for $0 < \alpha \leq 1$,
- (c) $L_\infty \cap w_\lambda^W(p) \subseteq w_\mu^W(p)$.

Theorem 19. *Let M be an Orlicz function and if $\inf_k p_k > 0$, then limit of any sequence (A_k) in $w_{\lambda, \alpha}^W[M, p]$ is unique.*

Proof. Let $\lim_k p_k = s > 0$. Suppose that $A_k \rightarrow A_1 (w_{\lambda, \alpha}^W[M, p])$ and $A_k \rightarrow A_2 (w_{\lambda, \alpha}^W[M, p])$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_k) - d(x, A_1)|}{\rho} \right) \right]^{p_k} = 0,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_k) - d(x, A_2)|}{\rho} \right) \right]^{p_k} = 0.$$

Let $\rho = \max\{2\rho_1, 2\rho_2\}$. As M is nondecreasing and convex, we have

$$\begin{aligned} & \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_1) - d(x, A_2)|}{\rho} \right) \right]^{p_k} \\ & \leq \frac{D}{\lambda_n^\alpha} \sum_{k \in I_n} \frac{1}{2^{p_k}} \left(\left[M \left(\frac{|d(x, A_k) - d(x, A_1)|}{\rho} \right) \right]^{p_k p_k} \right. \\ & \quad \left. + \left[M \left(\frac{|d(x, A_k) - d(x, A_2)|}{\rho} \right) \right] \right) \end{aligned}$$

$$\begin{aligned} & \frac{D}{\lambda_n^\alpha} \sum_{k \in I_n} \left(\left[M \left(\frac{|d(x, A_k) - d(x, A_1)|}{\rho} \right) \right]^{p_k} \right. \\ & \quad \left. + \frac{D}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_k) - d(x, A_2)|}{\rho} \right) \right]^{p_k} \right) \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Therefore we get

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} \left[M \left(\frac{|d(x, A_1) - d(x, A_2)|}{\rho} \right) \right]^{p_k} = 0.$$

As $\lim_k p_k = s$, we have

$$\lim_{k \rightarrow \infty} \left[M \left(\frac{|d(x, A_1) - d(x, A_2)|}{\rho} \right) \right]^{p_k} = \left[M \left(\frac{|d(x, A_1) - d(x, A_2)|}{\rho} \right) \right]^s$$

and so $A_1 = A_2$. Hence the limit is unique. □

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