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ESTIMATE FOR INITIAL MACLAURIN COEFFICIENTS OF CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, estimates for second and third MacLaurin coefficients of certain subclasses of bi-univalent functions in the open unit disk defined by convolution are determined, and certain special cases are also indicated. The main result extends and improve a recent one obtained by Srivastava et al.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ be the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe one-quarter theorem [3] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains the disk with the center in the origin and the radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$, satisfying $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and

$$f\left(f^{-1}(w)\right) = w,\ |w| < r_0(f),\ r_0(f) \geq \frac{1}{4}.$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots,\ w \in f(\mathbb{D}). \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent*, if both f and f^{-1} are univalent in $\mathbb{D}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{D}$, and we denote by σ this class of bi-univalent functions.

In [8] the authors defined the classes of functions $\mathcal{P}_m(\beta)$ as follows: let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions P, normalized with $P(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} P(z) - \beta}{1 - \beta} \right| \mathrm{d}\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{D}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$, hence the class $\mathcal{P}_m$ represents the class of functions p analytic in $\mathbb{D}$, normalized with $p(0) = 1$, and having the representation

$$p(z) = \int_0^{2\pi} \frac{1 - ze^{it}}{1 + ze^{it}} \, \mathrm{d}\mu(t), \tag{1.2}$$

where μ is a real-valued function with bounded variation, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m, \; m \geq 2. \tag{1.3}$$

Clearly, $\mathcal{P} := \mathcal{P}_2$ is the well-known class of *Carathéodory functions*, i.e. the normalized functions with positive real part in the open unit disk $\mathbb{D}$.

Lewin [6] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the concept of bi-starlike functions and the bi-convex functions, and obtained estimates for the initial coefficients. Recently, Ali et al. [1], Srivastava et al. [9], Frasin and Aouf [4], Goyal and Goswami [5] and many others have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Motivated by work of Srivastava et al. [9], we introduce a new subclass of bi-univalent functions, as follows.

For the functions $f, \; h \in \mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad h(z) = z + \sum_{n=2}^{\infty} b_n z^n, \; z \in \mathbb{D},$$

we recall the *Hadamard (or convolution) product* of f and h, defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \; z \in \mathbb{D}.$$

Definition 1. For a given function $k \in \sigma$, a function $f \in \sigma$ is said to be in the class $\mathcal{BR}^k(m;\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied

$$\frac{(f * k)(z)}{z} \in \mathcal{P}_m(\beta),$$

$$\frac{(g*k)(w)}{w} \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$ and $z, w \in \mathbb{D}$.

Remark 1. Taking $k(z) = z/(1-z)^2$ and $m = 2$ in the Definition 1 we obtain the class $\mathcal{B}(\beta) := \mathcal{BR}^{z/(1-z)^2}(2;\beta)$ studied by Srivastava et al. [9, Definition 2].

Definition 2. For a given function $k \in \sigma$ and a number $\alpha \in \mathbb{C}$, a function $f \in \sigma$ is said to be in the class $\mathcal{BV}^k(m;\alpha,\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, if the following conditions are satisfied

$$(1-\alpha)\frac{z(f*k)'(z)}{(f*k)(z)} + \alpha\left(1 + \frac{z(f*k)''(z)}{(f*k)'(z)}\right) \in \mathcal{P}_m(\beta),$$
$$(1-\alpha)\frac{w(g*k)'(w)}{(g*k)(w)} + \alpha\left(1 + \frac{w(g*k)''(w)}{(g*k)'(w)}\right) \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$.

Remark 2. (i) Taking $\alpha = 0$ and $\alpha = 1$ in the above class $\mathcal{BV}^k(m;\alpha,\beta)$ we obtain the classes $\mathcal{S}_m^k(\beta) := \mathcal{BV}^k(m;0,\beta)$ and $\mathcal{C}_m^k(\beta) := \mathcal{BV}^k(m;1,\beta)$, respectively.

(ii) Moreover, if we take $k(z) = z/(1-z)$ and $m = 2$, the classes $\mathcal{S}_m^k(\beta)$ and $\mathcal{C}_m^k(\beta)$ reduces to the well-known classes of *bi-starlike* and *bi-convex functions*, respectively (see also [2]).

The object of the paper is to find estimates for the coefficients a_2 and a_3 for functions in the subclass $\mathcal{BR}^k(m;\beta)$ and $\mathcal{BV}^k(m;\alpha,\beta)$, and these bounds are obtained by employing the techniques used earlier by Srivastava et al. [9].

2. Main results

In order to prove our main result for the functions $f \in \mathcal{BR}^k(m;\beta)$, first we will prove the following lemma:

Lemma 1. *Let the function* $\Phi(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{D}$, *such that* $\Phi \in \mathcal{P}_m(\beta)$. *Then,*

$$|h_n| \leq m(1-\beta),\ n \geq 1.$$

Proof. From (1.2) and (1.3), like in [8] and [7], we can see that if $p \in \mathcal{P}_m$, then

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z), \tag{2.1}$$

where $p_1, p_2 \in \mathcal{P}$.

Further, if $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$, $z\in\mathbb{D}$, where $p_1(z)=1+\sum\limits_{n=1}^{\infty}p_n^{(1)}z^n$ and $p_2(z)=1+\sum\limits_{n=1}^{\infty}p_n^{(2)}z^n$ for all $z\in\mathbb{D}$, comparing the coefficients of both sides of (2.1) we get

$$p_n=\left(\frac{m}{4}+\frac{1}{2}\right)p_n^{(1)}-\left(\frac{m}{4}-\frac{1}{2}\right)p_n^{(2)},\ n\geq 1.$$

Since $p_1,p_2\in\mathcal{P}$, where $\mathcal{P}$ is the class of Carathéodory functions, it is well-known that $|p_n^{(1)}|\leq 2$ and $|p_n^{(2)}|\leq 2$ for all $n\geq 1$, and thus

$$|p_n|\leq\left(\frac{m}{4}+\frac{1}{2}\right)\left|p_n^{(1)}\right|+\left(\frac{m}{4}-\frac{1}{2}\right)\left|p_n^{(2)}\right|\leq$$
$$2\left(\frac{m}{4}+\frac{1}{2}\right)+2\left(\frac{m}{4}-\frac{1}{2}\right)=m,\ n\geq 1. \tag{2.2}$$

Now, the proof of this lemma is straight forward, if we write

$$\Phi(z)=(1-\beta)p(z)+\beta,\quad\text{where}\quad p(z)=1+\sum_{n=1}^{\infty}p_nz^n\in\mathcal{P}_m.$$

Then,

$$\Phi(z)=1+(1-\beta)\sum_{n=1}^{\infty}p_nz^n,\ z\in\mathbb{D},$$

which gives

$$h_n=(1-\beta)p_n,\ n\geq 1,$$

and using the inequality (2.2) we obtain the desired result. □

Theorem 1. *Let $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{BR}^k(m;\beta)$, where $k\in\sigma$ has the form $k(z)=z+\sum\limits_{n=2}^{\infty}k_nz^n$. If $k_2,k_3\neq 0$, then*

$$|a_2|\leq\min\left\{\sqrt{\frac{m(1-\beta)}{|k_3|}};\ \frac{m(1-\beta)}{|k_2|}\right\},\qquad |a_3|\leq\frac{m(1-\beta)}{|k_3|},$$

and

$$\left|2a_2^2-a_3\right|\leq\frac{m(1-\beta)}{|k_3|}.$$

Proof. Since $f\in\mathcal{BR}^k(m;\beta)$, from the Definition 1 we have

$$\frac{(f*k)(z)}{z}=p(z) \tag{2.3}$$

and

$$\frac{(g*k)(w)}{w} = q(w), \tag{2.4}$$

where $p,\, q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Using the fact that the functions p and q have the following Taylor expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots,\ z \in \mathbb{D}, \tag{2.5}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots,\ w \in \mathbb{D}, \tag{2.6}$$

and equating the coefficients in (2.3) and (2.4), from (1.1) we get

$$k_2 a_2 = p_1, \tag{2.7}$$

$$k_3 a_3 = p_2, \tag{2.8}$$

and

$$-k_2 a_2 = q_1,$$

$$k_3\left(2a_2^2 - a_3\right) = q_2. \tag{2.9}$$

Since $p,\, q \in \mathcal{P}_m(\beta)$, according to Lemma 1, the next inequalities hold:

$$|p_k| \le m(1-\beta),\ k \ge 1, \tag{2.10}$$

$$|q_k| \le m(1-\beta),\ k \ge 1, \tag{2.11}$$

and thus, from (2.8) and (2.9), by using the inequalities (2.10) and (2.11), we obtain

$$|a_2|^2 \le \frac{|q_2| + |p_2|}{2\,|k_3|} \le \frac{m(1-\beta)}{|k_3|},$$

which gives

$$|a_2| \le \sqrt{\frac{m(1-\beta)}{|k_3|}}. \tag{2.12}$$

From (2.7), by using (2.10) we obtain immediately that

$$|a_2| = \left|\frac{p_1}{k_2}\right| \le \frac{m(1-\beta)}{|k_2|},$$

and combining this with the inequality (2.12), the first inequality of the conclusion is proved.

According to (2.8), from (2.10) we easily obtain

$$|a_3| = \left|\frac{p_2}{k_3}\right| \le \frac{m(1-\beta)}{|k_3|},$$

and from (2.9), by using (2.10) and (2.11) we finally deduce

$$\left|2a_2^2 - a_3\right| = \left|\frac{q_2}{k_3}\right| \le \frac{m(1-\beta)}{|k_3|},$$

which completes our proof. □

Setting $\beta = 0$ in Theorem 1 we get the following special case:

Corollary 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{BR}^k(m;0)$, *where* $k \in \sigma$ *has the form* $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. *If* $k_2, k_3 \neq 0$, *then*

$$|a_2| \le \min\left\{\sqrt{\frac{m}{|k_3|}};\ \frac{m}{|k_2|}\right\}, \quad |a_3| \le \frac{m}{|k_3|}, \quad \textit{and} \quad \left|2a_2^2 - a_3\right| \le \frac{m}{|k_3|}.$$

For $k(z) = z/(1-z)^2$ the above corollary reduces to the next result:

Example 1. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{BR}^{z/(1-z)^2}(m;0)$, then

$$|a_2| \le \sqrt{\frac{m}{3}}, \quad |a_3| \le \frac{m}{3}, \quad \text{and} \quad \left|2a_2^2 - a_3\right| \le \frac{m}{3}.$$

Taking $k(z) = z/(1-z)$ in Corollary 1, we get:

Example 2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{BR}^{z/(1-z)}(m;0)$, then

$$|a_2| \le \sqrt{m}, \quad |a_3| \le m, \quad \text{and} \quad \left|2a_2^2 - a_3\right| \le m.$$

If we put $k(z) = z/(1-z)^2$ in Theorem 1, we deduce the next corollary:

Corollary 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is in the class* $\mathcal{B}(\beta)$, *then*

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}, & \textit{if} \quad 0 \le \beta \le \frac{1}{3}, \\ 1-\beta, & \textit{if} \quad \frac{1}{3} < \beta < 1, \end{cases} \qquad |a_3| \le \frac{2(1-\beta)}{3},$$

and

$$\left|2a_2^2 - a_3\right| \le \frac{2(1-\beta)}{3}.$$

Remark 3. For the special case $\frac{1}{3} < \beta < 1$, the above first inequality, and the second one for all $0 \le \beta < 1$, improve the estimates given by Srivastava et al. in [9, Theorem 2].

Theorem 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{BV}^k(m;\alpha,\beta)$, *with* $\alpha \in \mathbb{C} \setminus \{-1\}$, *where* $k \in \sigma$ *has the form* $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$. *If* $k_2, k_3 \neq 0$ *and*

$$2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0,$$

then

$$|a_2| \le \min\left\{\sqrt{\frac{m(1-\beta)}{\left|2(1+2\alpha)k_3-(1+3\alpha)k_2^2\right|}};\frac{m(1-\beta)}{|1+\alpha|\,|k_2|}\right\},$$

and

$$|a_3| \le \min\left\{\frac{m(1-\beta)}{\left|2(1+2\alpha)k_3-(1+3\alpha)k_2^2\right|}+\frac{m(1-\beta)}{2\,|1+2\alpha|\,|k_3|};\right.$$
$$\frac{m(1-\beta)}{2\,|1+2\alpha|\,|k_3|}\left(1+\frac{m(1-\beta)\,|1+3\alpha|}{|1+\alpha|^2}\right);$$
$$\left.\frac{m(1-\beta)}{2\,|1+2\alpha|\,|k_3|}\left(1+\frac{m(1-\beta)\left|4(1+2\alpha)k_3-(1+3\alpha)k_2^2\right|}{|k_2|^2\,|1+\alpha|^2}\right)\right\},$$

whenever $\alpha \in \mathbb{C}\setminus\left\{-\frac{1}{2}\right\}$.

Proof. If $f \in \mathcal{BV}^k(m;\alpha,\beta)$, according to the Definition 2 we have

$$(1-\alpha)\frac{z(f*k)'(z)}{(f*k)(z)}+\alpha\left(1+\frac{z(f*k)''(z)}{(f*k)'(z)}\right)=p(z)$$

and

$$(1-\alpha)\frac{w(g*k)'(w)}{(g*k)(w)}+\alpha\left(1+\frac{w(g*k)''(w)}{(g*k)'(w)}\right)=q(w),$$

where $p,\ q \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$. Since

$$(1-\alpha)\frac{z(f*k)'(z)}{(f*k)(z)}+\alpha\left(1+\frac{z(f*k)''(z)}{(f*k)'(z)}\right)=$$
$$1+(1+\alpha)a_2k_2z+\left[2(1+2\alpha)a_3k_3-(1+3\alpha)a_2^2k_2^2\right]z^2+\ldots\ z\in\mathbb{D},$$

and according to (1.1)

$$(1-\alpha)\frac{z(g*k)'(w)}{(g*k)(w)}+\alpha\left(1+\frac{z(g*k)''(w)}{(g*k)'(w)}\right)=1-(1+\alpha)a_2k_2w+$$
$$\left\{\left[4(1+2\alpha)k_3-(1+3\alpha)k_2^2\right]a_2^2-2(1+2\alpha)a_3k_3\right\}w^2+\ldots,\ w\in\mathbb{D},$$

from (2.5) and (2.6) combined with the above two expansion formulas, it follows that

$$(1+\alpha)a_2k_2=p_1, \tag{2.13}$$
$$2(1+2\alpha)a_3k_3-(1+3\alpha)a_2^2k_2^2=p_2, \tag{2.14}$$

and

$$-(1+\alpha)a_2k_2=q_1,$$
$$\left[4(1+2\alpha)k_3-(1+3\alpha)k_2^2\right]a_2^2-2(1+2\alpha)a_3k_3=q_2. \tag{2.15}$$

Now, from (2.14) and (2.15) we deduce that

$$a_2^2 = \frac{p_2 + q_2}{4(1+2\alpha)k_3 - 2(1+3\alpha)k_2^2}, \tag{2.16}$$

whenever $2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0$, and

$$4(1+2\alpha)k_3\left(a_3 - a_2^2\right) = p_2 - q_2.$$

Using (2.16) in the above relation, we obtain

$$a_3 = \frac{p_2 + q_2}{4(1+2\alpha)k_3 - 2(1+3\alpha)k_2^2} + \frac{p_2 - q_2}{4(1+2\alpha)k_3}, \tag{2.17}$$

$$\text{whenever} \quad 2(1+2\alpha)k_3 - (1+3\alpha)k_2^2 \neq 0,\ \alpha \in \mathbb{C} \setminus \left\{-\frac{1}{2}\right\}.$$

From (2.13) and (2.14) we get

$$a_3 = \frac{1}{2(1+2\alpha)k_3}\left[p_2 + \frac{1+3\alpha}{(1+\alpha)^2}p_1^2\right], \quad \text{for} \quad \alpha \in \mathbb{C} \setminus \left\{-1; -\frac{1}{2}\right\}, \tag{2.18}$$

while from (2.13) and (2.15) we deduce that

$$a_3 = \frac{1}{2(1+2\alpha)k_3}\left[-q_2 + \frac{4(1+2\alpha)k_3 - (1+3\alpha)k_2^2}{k_2^2(1+\alpha)^2}p_1^2\right], \tag{2.19}$$

for $\alpha \in \mathbb{C} \setminus \left\{-1; -\frac{1}{2}\right\}$.

Combining (2.13) and (2.16) for the computation of the upper-bound of $|a_2|$, and (2.17), (2.18) and (2.19) for the computation of $|a_3|$, by using Lemma 1 we easily find the estimates of our theorem. □

Taking $\alpha = 0$ and $\alpha = 1$ in Theorem 2 we obtain the following two special cases, respectively:

Corollary 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{S}_m^k(\beta)$*, where* $k \in \sigma$ *has the form* $k(z) = z + \sum_{n=2}^{\infty} k_n z^n$*. If* $k_2, k_3 \neq 0$ *and*

$$2k_3 - k_2^2 \neq 0,$$

then

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{|2k_3 - k_2^2|}}; \frac{m(1-\beta)}{|k_2|}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{m(1-\beta)}{|2k_3 - k_2^2|} + \frac{m(1-\beta)}{2|k_3|}; \frac{m(1-\beta)(1+m(1-\beta))}{2|k_3|};\right.$$

$$\frac{m(1-\beta)}{2|k_3|}\left(1+\frac{m(1-\beta)\left|4k_3-k_2^2\right|}{|k_2|^2}\right)\Bigg\}.$$

Corollary 4. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ *be in the class* $\mathcal{C}_m^k(\beta)$*, where* $k\in\sigma$ *has the form* $k(z)=z+\sum_{n=2}^{\infty}k_n z^n$*. If* $k_2, k_3\neq 0$ *and*

$$3k_3-2k_2^2\neq 0,$$

then

$$|a_2|\leq \min\left\{\sqrt{\frac{m(1-\beta)}{\left|6k_3-4k_2^2\right|}};\frac{m(1-\beta)}{2|k_2|}\right\},$$

and

$$|a_3|\leq \min\left\{\frac{m(1-\beta)}{\left|6k_3-4k_2^2\right|}+\frac{m(1-\beta)}{6|k_3|};\frac{m(1-\beta)\left(1+m(1-\beta)\right)}{6|k_3|};\right.$$

$$\left.\frac{m(1-\beta)}{6|k_3|}\left(1+\frac{m(1-\beta)\left|3k_3-k_2^2\right|}{|k_2|^2}\right)\right\}.$$

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