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REFINED ALMOST DOUBLE DERIVATIONS AND LIE $*$-DOUBLE DERIVATIONS

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Abstract. In this paper, our approach allows to refine the results announced by Ebadian et al. [Results Math., 36 (2013), 409–423]. Namely, we reduce the distance between approximate and exact double derivations on Banach algebras and Lie C^*-algebras up to $\frac{1}{2^{n-1}}$ and $\frac{1}{2^{n-2}}$ for $n \geq 2$. Indeed, we give a correct utilization of fixed point theory in the sense of Diaz and Margolis [Bull. Amer. Math. Soc., 74 (1968), 305–309] concerning the stability of double derivations.

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1. INTRODUCTION

A classical question in the theory of functional equations is the following:

when is it true that a mapping which approximately satisfies a functional equation ξ must be somehow near to an exact solution of ξ?

In 1940, Ulam [7] gave a wide ranging talk and discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

Generally, the concept of stability for a functional equation comes up when the functional equation is replaced by an inequality which acts as a perturbation of that equation. The case of approximately additive functions was solved by D. Hyers [5] under certain assumptions. In 1950, Hyers' Theorem was generalized by Aoki [1] for additive mappings and independently, in 1978, by Rassias [6] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. For the history and various aspects of this theory we refer the reader to [3] and the references therein. Note that a functional equation ζ is stable if any function g satisfying the equation ζ approximately is near to true solution of ζ.

Recently, Ebadian et al. [3] used the fixed point alternative method to establish the Hyers–Ulam stability of double derivations on Banach algebras and Lie $*$-double derivations on Lie C^*-algebras associated with the following additive functional equation

$$\sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i=1,i\neq i_1,\dots,i_{n-k+1}}^{n} x_i-\sum_{r=1}^{n-k+1}x_{i_r}) + f(\sum_{i=1}^{n}x_i)=2^{n-1}f(x_1).$$

Throughout this paper following [3], we assume that $\mathcal{A}$ is a Banach algebra (Lie C^*-algebra). For given mapping $f:\mathcal{A}\to\mathcal{A}$, we define the difference operator $D_\mu f:\mathcal{A}^n\to\mathcal{A}$ by

$$D_\mu f(x_1,\dots,x_n):= \sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i=1,i\neq i_1,\dots,i_{n-k+1}}^{n} \mu x_i-\sum_{r=1}^{n-k+1}\mu x_{i_r}) + f(\sum_{i=1}^{n}\mu x_i)-2^{n-1}f(\mu x_1)$$

for all $\mu\in\{\lambda\in\mathbb{C}:|\lambda|=1\}$.

In this paper, we improve main results of [3] and reduce the distance between approximate and exact double derivations on Banach algebras and Lie C^*-algebras up to $\frac{1}{2^{n-1}}$ and $\frac{1}{2^{n-2}}$ for $n\geq 2$.

In section 2, we discuss on main results of [3] and improve some theorems and corollaries including Theorem 2.3, 2.5 and Corollary 2.4, 2.6. In section 3, we also refine some results of [3] including Theorem 3.2, 3.4 and Corollary 3.3, 3.5. Indeed, we are going to weaken their assumptions and giving a correct utilization of fixed point theory in the sense of Diaz and Margolis [2].

2. Almost double derivation

Throughout this section, we assume that $\mathcal{A}$ is a Banach algebra, $f(0)=g(0)=h(0)=0$, and for given mappings $f,g,h:\mathcal{A}\longrightarrow\mathcal{A}$, we define

$$C_{f,g,h}(a,b):=f(ab)-f(a)b-af(b)-g(a)h(b)-h(a)g(b)$$

for all $a,b\in\mathcal{A}$.

Definition 1. Let $\mathcal{A}$ be a Banach algebra and let $\delta,\varepsilon:\mathcal{A}\longrightarrow\mathcal{A}$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f:\mathcal{A}\longrightarrow\mathcal{A}$ is called a (δ,ε)-double derivation if

$$f(ab)=f(a)b+af(b)+\delta(a)\varepsilon(b)+\varepsilon(a)\delta(b)$$

for all $a, b \in \mathcal{A}$.

A fundamental result in fixed point theory is the following theorem. We recall the following theorem by Diaz and Margolis [2].

Theorem 1. *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$,*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J,*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$,*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

The following theorem is a refined version of [3, Theorem 2.3]:

Theorem 2. *Let $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings for which there exist functions $\varphi : \mathcal{A}^n \longrightarrow [0, \infty)$ and $\psi : \mathcal{A}^2 \longrightarrow [0, \infty)$ such that*

$$\lim_{m\to\infty} 2^m \varphi(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m}) = 0, \tag{2.1}$$

$$\lim_{m\to\infty} 4^m \psi(\frac{a}{2^m}, \frac{b}{2^m}) = 0, \tag{2.2}$$

$$||D_\mu f(x_1, ..., x_n)|| \leq \varphi(x_1, ..., x_n), \tag{2.3}$$

$$||C_{f,g,h}(a, b)|| \leq \psi(a, b) \tag{2.4}$$

for all $a, b, x_1, ..., x_2 \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, ..., x_n) \leq \frac{L}{2}\varphi(2x_1, ..., 2x_n)$ for all $x_1, ..., x_n \in \mathcal{A}$, then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$||f(x) - d(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x, x, 0, ..., 0), \tag{2.5}$$

$$||g(x) - \delta(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x, x, 0, ..., 0), \tag{2.6}$$

$$||h(x) - \varepsilon(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x, x, 0, ..., 0) \tag{2.7}$$

for all $x \in \mathcal{A}$, where $\alpha = 2^{n-1}$ and $n \geq 2$. Moreover, d is a (δ, ε)-double derivation on $\mathcal{A}$.

Proof. Put $\mu = 1, x_1 = x_2 = x$, and $x_3 = x_4 = ... = x_n = 0$ in (2.3) to reach

$$||\frac{\alpha}{2} f(2x) - \alpha f(x)\| \leq \varphi(x, , x, 0, ..., 0)$$

for all $x \in \mathcal{A}$ and so

$$||2f(\frac{x}{2}) - f(x)\| \leq \frac{2}{\alpha}\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0) \leq \frac{L}{\alpha}\varphi(x, , x, 0, ..., 0). \tag{2.8}$$

Define $F := \{f : \mathcal{A} \longrightarrow \mathcal{A}\}$. The metric defined on F by

$$\rho(f,g) := \inf\{c \in [0,\infty] : ||f(x) - g(x)|| \leq c\varphi(x,x,0,...,0), \forall x \in \mathcal{A}\}.$$

is a generalized metric and (F,ρ) is a generalized complete metric space. Consider the mapping $(Jf)(x) := 2f(\frac{x}{2})$ for all $f \in F$ and $x \in \mathcal{A}$. Use [4, Lemma 1.3] to see that J is a strictly contractive mapping with the Lipschitz constant L. It follows from (2.8) that $\rho(Jf,f) \leq \frac{L}{\alpha}$. Therefore according to Theorem 1, the sequence $\{J^m f\}$ converges to a fixed point d such that $d(x) = \lim_{m\to\infty} 2^m f(\frac{x}{2^m})$ and $d(2x) = 2d(x)$. Note that d is the unique fixed point of J and

$$\rho(d,f) \leq \frac{1}{1-L}\rho(Jf,f) \leq \frac{L}{\alpha(1-L)}.$$

This means that inequality (2.5) holds for all $x \in \mathcal{A}$. The proof of the linearity of d and also the rest of the proof is similar to that of [3, Theorem 2.3] and we omit it. □

The importance of our result becomes clear when we take

$$\varphi(x_1,...,x_n) = \theta_1 \sum_{i=1}^{n} ||x_i||^p, \ \psi(a,b) = \theta_2(||a||^q + ||b||^q).$$

In this situation, by choosing $L = 2^{1-p}$, we can get strong and close approximations of the functions f,g,h with linear mappings d,δ,ε, where d is a (δ,ε)-double derivation on $\mathcal{A}$. Thus, we improve [3, Corollary 2.4] up to $\frac{1}{2^{n-1}}$ as follows:

Corollary 1. *Let p,q,θ_1,θ_2 be non-negative real numbers with $p,q > 1$. Suppose that $f,g,h : \mathcal{A} \longrightarrow \mathcal{A}$ are mappings such that*

$$||D_\mu f(x_1,...,x_n)|| \leq \theta_1 \sum_{i=1}^{n} ||x_i||^p,$$

$$||C_{f,g,h}(a,b)|| \leq \theta_2(||a||^q + ||b||^q)$$

for all $a,b,x_1,...,x_n \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $d,\delta,\varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$||f(x) - d(x)|| \leq \frac{2\theta_1}{2^{n-1}(2^{p-1}-1)}||x||^p,$$

$$||g(x) - \delta(x)|| \leq \frac{2\theta_1}{2^{n-1}(2^{p-1}-1)}||x||^p,$$

$$||h(x) - \varepsilon(x)|| \leq \frac{2\theta_1}{2^{n-1}(2^{p-1}-1)}||x||^p$$

for all $x \in \mathcal{A}$. Moreover, d is a (δ,ε)-double derivation on $\mathcal{A}$.

In the following theorem we give an improved version of [3, Theorem 2.5]:

Theorem 3. *Suppose that $f,g,h:\mathcal{A}\longrightarrow\mathcal{A}$ are mappings satisfying (2.3) and (2.4) for which there exist functions $\varphi:\mathcal{A}^n\longrightarrow[0,\infty)$ and $\psi:\mathcal{A}^2\longrightarrow[0,\infty)$ such that*

$$\lim_{m\to\infty}\frac{1}{2^m}\varphi(2^m x_1,...,2^m x_n)=0, \tag{2.9}$$

$$\lim_{m\to\infty}\frac{1}{2^m}\psi(2^m a,2^m b)=0 \tag{2.10}$$

for all $a,b,x_1,...,x_2\in\mathcal{A}$. If there exists a constant $0<L<1$ such that $\varphi(x_1,...,x_n)\leq 2L\varphi(\frac{x_1}{2},...,\frac{x_n}{2})$ for all $x_1,...,x_n\in\mathcal{A}$, then there exist unique $\mathbb{C}$-linear mappings $d,\delta,\varepsilon:\mathcal{A}\to\mathcal{A}$ such that

$$||f(x)-d(x)||\leq\frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,...,0), \tag{2.11}$$

$$||g(x)-\delta(x)||\leq\frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,...,0), \tag{2.12}$$

$$||h(x)-\varepsilon(x)||\leq\frac{L}{\beta(1-L)}\varphi(\frac{x}{2},\frac{x}{2},0,...,0) \tag{2.13}$$

for all $x\in\mathcal{A}$, where $\beta=\frac{\alpha}{2}$ and $n\geq 2$. Moreover, d is a (δ,ε)-double derivation on $\mathcal{A}$.

Proof. It follows from (2.8) that

$$||\frac{1}{2}f(2x)-f(x)\|\leq\frac{1}{\alpha}\varphi(x,,x,0,...,0)\leq\frac{2L}{\alpha}\varphi(\frac{x}{2},\frac{x}{2},0,...,0) \tag{2.14}$$

for all $x\in\mathcal{A}$. Consider the generalized complete metric (F,ρ) with the generalized metric ρ defined by

$$\rho(f,g):=\inf\{c\in[0,\infty]:||f(x)-g(x)||\leq c\varphi(\frac{x}{2},\frac{x}{2},0,...,0),\forall x\in\mathcal{A}\}.$$

Define the mapping $(Jf)(x):=\frac{1}{2}f(2x)$ for all $f\in F$ and $x\in\mathcal{A}$. Apply [4, Lemma 1.2]) to find that J is a strictly contractive mapping with the Lipschitz constant L. It follows from (2.14) that $\rho(Jf,f)\leq\frac{2L}{\alpha}$. Applying Theorem 1, we get the sequence $\{J^m f\}$ converges to a unique fixed point d of J such that

$$\rho(d,f)\leq\frac{1}{1-L}\rho(Jf,f)\leq\frac{2L}{\alpha(1-L)}=\frac{L}{\beta(1-L)},$$

i. e., inequality (2.11) holds for all $x\in\mathcal{A}$. The rest of the proof is similar to that of [3, Theorem 2.3]. □

As we mentioned in Corollary 1, the importance of Theorem 3 becomes also clear when we put $L=2^{p-1}$ and

$$\varphi(x_1,...,x_n)=\theta_1+\theta_2\sum_{i=1}^{n}||x_i||^p,\ \psi(a,b)=\theta_1+\theta_2(||a||^q+||b||^q).$$

However, we can improve [3, Corollary 2.6] up to $\frac{1}{2^{n-2}}$ as follows:

Corollary 2. *Let p, q, θ_1, θ_2 be non-negative real numbers with $p, q \in (0,1)$. Suppose that $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ are mappings such that*

$$||D_\mu f(x_1, ..., x_n)|| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,$$

$$||C_{f,g,h}(a,b)|| \leq \theta_1 + \theta_2(||a||^q + ||b||^q)$$

for all $a, b, x_1, ..., x_n \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$||f(x) - d(x)|| \leq \frac{1}{2^{n-2}}\left(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p\right),$$

$$||g(x) - \delta(x)|| \leq \frac{1}{2^{n-2}}\left(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p\right),$$

$$||h(x) - \varepsilon(x)|| \leq \frac{1}{2^{n-2}}\left(\frac{\theta_1}{2^{1-p}-1} + \frac{\theta_2}{1-2^{p-1}}||x||^p\right)$$

for all $x \in \mathcal{A}$. Moreover, d is a (δ, ε)-double derivation on $\mathcal{A}$.

3. Almost Lie $*$-double derivation

A unital C^*-algebra $\mathcal{A}$, endowed with the Lie product $[x,y] = xy - yx$ on $\mathcal{A}$, is called a Lie C^*-algebra. In this section, we assume that $\mathcal{A}$ is a Lie C^*-algebra and $U(\mathcal{A}) = \{u \in \mathcal{A} : uu^* = u^*u = e\}$. For given mappings $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$, we let $f(0) = g(0) = h(0) = 0$ and define

$$J_{f,g,h}(a,b) := f([a,b]) - [f(a), b] - [a, f(b)] - [g(a), h(b)] - [h(a), g(b)]$$

for all $a, b \in \mathcal{A}$.

Definition 2. Let $\mathcal{A}$ be a Lie C^*-algebra and let $\delta, \varepsilon : \mathcal{A} \longrightarrow \mathcal{A}$ be $\mathbb{C}$-linear mappings. A $\mathbb{C}$-linear mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$ is called a Lie (δ, ε)-double derivation if

$$f([a,b]) = [f(a), b] + [a, f(b)] + [\delta(a), \varepsilon(b)] + [\varepsilon(a), \delta(b)]$$

for all $a, b \in \mathcal{A}$.

The presented results in this section are refinements of [3, Theorem 3.2, 3.4] and [3, Corollary 3.3, 3.5]:

Theorem 4. *Let $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ be mappings for which there exist functions $\varphi : \mathcal{A}^n \longrightarrow [0, \infty)$ and $\psi : \mathcal{A}^2 \longrightarrow [0, \infty)$ such that*

$$\lim_{m \to \infty} 2^m \varphi\left(\frac{x_1}{2^m}, ..., \frac{x_n}{2^m}\right) = 0, \tag{3.1}$$

$$\lim_{m \to \infty} 4^m \psi\left(\frac{a}{2^m}, \frac{b}{2^m}\right) = 0, \tag{3.2}$$

$$||D_\mu f(x_1, ..., x_n)|| \leq \varphi(x_1, ..., x_n), \tag{3.3}$$

$$||J_{f,g,h}(a,b)|| \leq \psi(a,b) \tag{3.4}$$

$$\max\{f(\frac{u^*}{2^k}) - f(\frac{u}{2^k})^*, g(\frac{u^*}{2^k}) - g(\frac{u}{2^k})^*, h(\frac{u^*}{2^k}) - h(\frac{u}{2^k})^*\} \leq \varphi(\frac{u}{2^k}, ..., \frac{u}{2^k}) \tag{3.5}$$

for all $a, b, x_1, ..., x_2 \in \mathcal{A}$, $k = 0, 1, 2, ...,$ $u \in U(A)$, *and* $\mu \in \mathbb{T}^1$. *If there exists a constant* $0 < L < 1$ *such that* $\varphi(x_1, ..., x_n) \leq \frac{L}{2}\varphi(2x_1, ..., 2x_n)$ *for all* $x_1, ..., x_n \in \mathcal{A}$, *then there exist unique* $\mathbb{C}$*-linear mappings* $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ *such that*

$$\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \varepsilon(x)||\} \leq \frac{L}{\alpha(1-L)}\varphi(x, x, 0, ..., 0)$$

for all $x \in \mathcal{A}$. *Moreover,* d *is a Lie* $*$-(δ, ε)*-double derivation on* $\mathcal{A}$.

Proof. Using the same methods as in the proof of [3, Theorem 2.3, 3.2], we can obtain the desired results. □

Corollary 3. *Let* p, q, θ_1, θ_2 *be non-negative real numbers with* $p, q > 1$. *Suppose that* $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ *are mappings such that*

$$||D_\mu f(x_1, ..., x_n)|| \leq \theta_1 \sum_{i=1}^{n} ||x_i||^p,$$

$$||J_{f,g,h}(u,b)|| \leq \theta_2(1 + ||b||^q)$$

$$\max\{f(\frac{u^*}{2^k}) - f(\frac{u}{2^k})^*, g(\frac{u^*}{2^k}) - g(\frac{u}{2^k})^*, h(\frac{u^*}{2^k}) - h(\frac{u}{2^k})^*\} \leq \frac{\theta_1 + \theta_2}{2^{kp}}$$

for all $a, b, x_1, ..., x_n \in \mathcal{A}$, $k = 0, 1, 2, ...,$ $u \in U(A)$, *and* $\mu \in \mathbb{T}^1$. *There exist unique* $\mathbb{C}$*-linear mappings* $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ *such that*

$$\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \varepsilon(x)||\} \leq \frac{2\theta_1}{2^{n-1}(2^{p-1} - 1)}||x||^p$$

for all $x \in \mathcal{A}$. *Moreover,* d *is a* (δ, ε)*-double derivation on* $\mathcal{A}$.

Proof. The results follows from above theorem by taking $L = 2^{1-p}$ and

$$\varphi(x_1, ..., x_n) = \theta_1 \sum_{i=1}^{n} ||x_i||^p, \ \psi(a,b) = \theta_2(1 + ||b||^q).$$

□

Theorem 5. *Suppose that* $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ *are mappings satisfying* (3.3) *and* (3.4) *for which there exist functions* $\varphi : \mathcal{A}^n \longrightarrow [0, \infty)$ *and* $\psi : \mathcal{A}^2 \longrightarrow [0, \infty)$ *such that*

$$\lim_{m \to \infty} \frac{1}{2^m}\varphi(2^m x_1, ..., 2^m x_n) = 0,$$

$$\lim_{m \to \infty} \frac{1}{4^m}\psi(2^m a, 2^m b) = 0$$

$$\max\{f(2^k u^*) - f(2^k u)^*, g(2^k u^*) - g(2^k u)^*, h(2^k u^*) - h(2^k u)^*\} \leq \varphi(2^k u, ...2^k u)$$

for all $a, b, x_1, ..., x_2 \in \mathcal{A}$, $k = 0, 1, 2, ...$, $u \in U(A)$, and $\mu \in \mathbb{T}^1$. If there exists a constant $0 < L < 1$ such that $\varphi(x_1, ..., x_n) \leq 2L\varphi(\frac{x_1}{2}, ..., \frac{x_n}{2})$ for all $x_1, ..., x_n \in \mathcal{A}$, then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \varepsilon(x)||\} \leq \frac{L}{\beta(1-L)}\varphi(\frac{x}{2}, \frac{x}{2}, 0, ..., 0)$$

for all $x \in \mathcal{A}$, where $\beta = \frac{\alpha}{2}$ and $n \geq 2$. Moreover, d is a Lie $*$-(δ, ε)-double derivation on $\mathcal{A}$.

Proof. The proof is similar to that of [3, Theorem 2.5, 3.2]. □

Corollary 4. *Let p, q, θ_1, θ_2 be non-negative real numbers with $p, q \in (0, 1)$. Suppose that $f, g, h : \mathcal{A} \longrightarrow \mathcal{A}$ are mappings such that*

$$||D_\mu f(x_1, ..., x_n)|| \leq \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p,$$

$$||J_{f,g,h}(u, b)|| \leq \theta_1 + \theta_2(1 + ||b||^q)$$

$$\max\{f(2^k u^*) - f(2^k u)^*, g(2^k u^*) - g(2^k u)^*, h(2^k u^*) - h(2^k u)^*\} \leq \frac{\theta_1 + \theta_2}{2^{kp}}$$

for all $a, b, x_1, ..., x_n \in \mathcal{A}$, $k = 0, 1, 2, ...$, $u \in U(A)$, and $\mu \in \mathbb{T}^1$. Then there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$\max\{||f(x) - d(x)||, ||g(x) - \delta(x)||, ||h(x) - \varepsilon(x)||\}$$
$$\leq \frac{1}{2^{n-2}}(\frac{\theta_1}{2^{1-p} - 1} + \frac{\theta_2}{1 - 2^{p-1}}||x||^p)$$

for all $x \in \mathcal{A}$. Moreover, d is a Lie $*$-(δ, ε)-double derivation on $\mathcal{A}$.

Proof. Apply above theorem by putting $L = 2^{p-1}$ and

$$\varphi(x_1, ..., x_n) = \theta_1 + \theta_2 \sum_{i=1}^{n} ||x_i||^p, \ \psi(a, b) = \theta_1 + \theta_2(1 + ||b||^q).$$

□

4. CONCLUSION

Our results can give the results proved by Ebadian et al. [3]. For instance, under the hypotheses of Theorem 2 we can conclude [3, Theorem 2.3], but not vice versa. In other words, if there exists a constant $0 < L < 1$ such that $\varphi(x_1, ..., x_n) \leq \frac{L}{2}\varphi(2x_1, ..., 2x_n)$ for all $x_1, ..., x_n \in \mathcal{A}$, then $\varphi(x_1, ..., x_n) \leq \frac{\alpha}{2}L\varphi(2x_1, ..., 2x_n)$, i.e, all of the hypotheses of [3, Theorem 2.3] hold. On the other hand, Theorem 2 says that there exist unique $\mathbb{C}$-linear mappings $d, \delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ such that

$$||f(x) - d(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x, x, 0, ..., 0),$$

$$||g(x)-\delta(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x,x,0,...,0),$$
$$||h(x)-\varepsilon(x)|| \leq \frac{L}{\alpha(1-L)}\varphi(x,x,0,...,0)$$

for all $x \in \mathcal{A}$, where $\alpha = 2^{n-1}$ and $n \geq 2$. Since $\frac{L}{\alpha(1-L)} \leq \frac{L}{1-L}$, we have

$$||f(x)-d(x)|| \leq \frac{L}{1-L}\varphi(x,x,0,...,0),$$
$$||g(x)-\delta(x)|| \leq \frac{L}{1-L}\varphi(x,x,0,...,0),$$
$$||h(x)-\varepsilon(x)|| \leq \frac{L}{1-L}\varphi(x,x,0,...,0),$$

which coincide with the results of [3, Theorem 2.3]. The same arguments can be applied for Theorem 2.5, 3.2, 3.4 and Corollary 2.4, 2.6, 3.3, 3.5 of [3].

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