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ON THE SIZE OF DIOPHANTINE M -TUPLES FOR LINEAR POLYNOMIALS

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Abstract. In this paper we prove that there does not exist a set with more than 16 nonzero polynomials in $\mathbb{K}[X]$, where \mathbb{K} is any field of characteristic 0, such that the product of any two of them increased by a linear polynomial $n \in \mathbb{K}[X]$ is a square of a polynomial from $\mathbb{K}[X]$.

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1. INTRODUCTION

Diophantus of Alexandria [1] was the first who studied the problem of finding sets with the property that the product of any two of its distinct elements increased by 1 is a perfect square. Such a set consisting of m elements is therefore called a Diophantine m -tuple. Diophantus found the first Diophantine quadruple consisting of rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$, while the first Diophantine quadruple of integers, the set $\{1, 3, 8, 120\}$, was found by Fermat. In the case of rational numbers no upper bound for the size of such sets is known. In integer case which is the most studied, Dujella [4] proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples. The folklore conjecture is that there does not exist a Diophantine quintuple over the integers.

Many generalizations of this problem were also considered, for example by adding a fixed integer n instead of 1, looking at k th powers instead of squares, or considering the problem over other domains than \mathbb{Z} or \mathbb{Q} .

Definition 1. Let $m \geq 2$, $k \geq 2$ and let R be a commutative ring with 1. Let $n \in R$ be a nonzero element and let $\{a_1, \dots, a_m\}$ be a set of m distinct nonzero elements from R such that $a_i a_j + n$ is a k th power of an element of R for $1 \leq i < j \leq m$. The set $\{a_1, \dots, a_m\}$ is called a k th power Diophantine m -tuple with the property $D(n)$ or simply a k th power $D(n)$ - m -tuple in R .

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The first such set, a second power $D(256)$ -quadruple $\{1, 33, 68, 105\}$, was found by Diophantus [1]. It is interesting to find upper bounds for the number of elements of such sets. Dujella [2, 3] found such bounds for the integer case and for $k = 2$. For other similar results see [5, 6, 9, 12].

The first polynomial variant of the above problem was studied by Jones [13, 14] for $R = \mathbb{Z}[X]$, $k = 2$ and $n = 1$. In this case, Dujella and Fuchs [6] proved that there does not exist a second power Diophantine quintuple. Dujella and Luca [11] considered the case $n = 1$, $k \geq 3$ and $R = \mathbb{K}[X]$, where \mathbb{K} is an algebraically closed field of characteristic 0. Using many results from [11], Dujella and Jurasić [9] proved that there does not exist a second power Diophantine 8-tuple in $\mathbb{K}[X]$ for $n = 1$. There were also considered other variants of such a polynomial problem. Dujella and Fuchs, jointly with Tichy [7] and later with Walsh [8], considered the case $R = \mathbb{Z}[X]$, $k = 2$ and n is a linear polynomial from $\mathbb{Z}[X]$. They proved that in this case $m \leq 12$. Jurasić [15] proved that $m \leq 98$ for n a quadratic polynomial in $\mathbb{Z}[X]$.

We will consider the case $R = \mathbb{K}[X]$, where \mathbb{K} is any field of characteristic 0, $k = 2$ and n is a linear polynomial from $\mathbb{K}[X]$. Without loss of generality we assume that \mathbb{K} is algebraically closed. If we omit the condition that \mathbb{K} is a field of characteristic 0, then we could not obtain some results where the factorisation of a polynomial is considered. For brevity, instead of second power $D(n)$ - m -tuple in $\mathbb{K}[X]$, from now on we shall refer to a polynomial $D(n)$ - m -tuple. Observe that, at most one polynomial a_i for $i \in \{1, \dots, m\}$ in such a polynomial $D(n)$ - m -tuple is constant. Otherwise, we would have two different constants a and b for which $ab + n = r^2$, where $r \in \mathbb{K}[X]$. This is not possible, because then $\deg(n) = 1 = 2\deg(r)$.

An improper k th power $D(n)$ - m -tuple in R is an m -tuple with the property from Definition 1, but with relaxed condition that its elements need not be distinct and need not be nonzero. For linear n , we cannot have 0 in an improper polynomial $D(n)$ - m -tuple. Let us assume that there exists a non-constant polynomial a such that $a^2 + n = r^2$, for some $r \in \mathbb{K}[X]$. Then $\deg(a) = \deg(r) \geq 1$ and $\deg(n) = \deg(r - a) + \deg(r + a)$. This is not possible if $\deg(a) \geq 2$ but, for example, we have $(X + 3)^2 - 4X - 8 = (X + 1)^2$. So, in an improper polynomial $D(n)$ - m -tuple we can have infinitely many equal linear polynomials. In the rest of the paper we consider only proper $D(n)$ - m -tuples, described in Definition 1. We have the following theorem.

Theorem 1. *There are at most 16 elements in a polynomial $D(n)$ - m -tuple for a linear polynomial n , i.e.*

$$m \leq 16.$$

In order to prove Theorem 1, we follow the strategy used in [7, 8] for linear n and in [15] for quadratic n . In those papers the ring $\mathbb{Z}[X]$ was considered, using the relation " $<$ " between its elements. Instead of that, in $\mathbb{K}[X]$ we have to use the relation " \leq " between the degrees of its elements. The paper is organized as follows. In Section 2

we estimate the number of polynomials with given degree k in a polynomial $D(n)$ - m -tuple and consider separate cases depending on k . In Section 3, we adapt the gap principle for the degrees of the elements of a polynomial $D(n)$ -quadruple, proved in [7] for the ring $\mathbb{Z}[X]$, to $\mathbb{K}[X]$. Using the bounds from Section 2 and combining the gap principle with an upper bound for the degree of the largest element in a polynomial $D(n)$ -quadruple, obtained in [8] and also valid in $\mathbb{K}[X]$, in Section 4 we give the proof of Theorem 1.

2. SETS WITH POLYNOMIALS OF EQUAL DEGREE

Let L_k be the number of polynomials of degree k from $\mathbb{K}[X]$ in a polynomial $D(n)$ - m -tuple for linear n . The first step which leads us to the proof of Theorem 1 is to estimate the numbers L_k for $k \geq 0$. We already proved that

$$L_0 \leq 1.$$

The following lemma, which is [7, Lemma 1], plays the key role in our proofs. It is proved for polynomials with integer coefficients, but the proof is obtained using only algebraic manipulations so it holds in $\mathbb{K}[X]$ also.

Lemma 1. *Let $\{a, b, c\}$ be a polynomial $D(n)$ -triple in $\mathbb{K}[X]$ and let*

$$ab + n = r^2, \quad ac + n = s^2, \quad bc + n = t^2, \tag{2.1}$$

for some $r, s, t \in \mathbb{K}[X]$. Then there exist polynomials $e, u, v, w \in \mathbb{K}[X]$ such that

$$ae + n^2 = u^2, \quad be + n^2 = v^2, \quad ce + n^2 = w^2. \tag{2.2}$$

More precisely,

$$e = n(a + b + c) + 2abc - 2rst, \tag{2.3}$$

$$u = at - rs, \quad v = bs - rt, \quad w = cr - st \tag{2.4}$$

and

$$c = a + b + \frac{e}{n} + \frac{2}{n^2}(abe + ruv). \tag{2.5}$$

We also define

$$\bar{e} = n(a + b + c) + 2abc + 2rst \tag{2.6}$$

and we get

$$e \cdot \bar{e} = n^2(c - a - b - 2r)(c - a - b + 2r). \tag{2.7}$$

Also, from (2.3), using (2.1) and (2.4), we get

$$e = n(a + b - c) + 2rw, \tag{2.8}$$

$e = n(a - b + c) + 2sv$ and $e = n(-a + b + c) + 2tu$.

Let $\deg(n) = 1$ and $\deg(a) = \deg(b) = \deg(c) = k \geq 1$. From (2.1) we obtain $\deg(r) = \deg(s) = \deg(t) = k \geq 1$. Let A, B, C, R, S, T be the leading coefficients

of the polynomials a, b, c, r, s, t , respectively. Then, from (2.1), we have $AB = R^2$, $AC = S^2$ and $BC = T^2$ so $ABC = \pm RST$. Let us consider both cases.

1.) If $ABC = RST$, from (2.6) we have $\deg(\bar{e}) = 3k$ and then, from (2.7),

$$\deg(e) \leq 2 - k. \quad (2.9)$$

2.) If $ABC = -RST$, from (2.3) we conclude

$$\deg(e) = 3k, \quad (2.10)$$

and then, from (2.7), $\deg(\bar{e}) \leq 2 - k$.

In order to bound the number L_k for $k \geq 1$, we are interested to find the number of possible c -s, for fixed a and b , such that (2.1) holds. The first step is finding all possible e -s from Lemma 1. The trivial situation in 1.) is $e = 0$. Then, from (2.7), we obtain

$$c_{\pm} = a + b \pm 2r. \quad (2.11)$$

Analogue situation in 2.) is $\bar{e} = 0$. Beside those trivial situations we suppose that $\deg(e) \geq 0$ and $\deg(\bar{e}) \geq 0$.

Polynomial $D(n)$ -triples can be classified as regular or irregular, depending on whether they satisfy the condition given in the next definition (see e.g. [10]).

Definition 2. A $D(n)$ -triple $\{a, b, c\}$ is called regular if it satisfies the condition

$$(c - b - a)^2 = 4(ab + n). \quad (2.12)$$

Observe that any permutation of a, b, c leaves equation (2.12) invariant. Also, from (2.12), using (2.1), we get (2.11) and we have

$$ac_{\pm} + n = (a \pm r)^2, \quad bc_{\pm} + n = (b \pm r)^2. \quad (2.13)$$

Moreover, a semi-regular $D(n)$ -quadruple is one which contains a regular triple, and a twice semi-regular $D(n)$ -quadruple is one that contains two regular triples.

Let us consider a polynomial $D(n)$ -triple $\{a, b, c\}$ in the two following lemmas. We will use the divisibility of polynomials in $\mathbb{K}[X]$. A polynomial p divides a polynomial q if there exists a polynomial m from $\mathbb{K}[X]$ such that $q = mp$. This situation will be denoted by $p|q$.

Lemma 2. *At most one element of a polynomial $D(n)$ -triple $\{a, b, c\}$ is divisible by n .*

Proof. Without loss of generality, let a and b be divisible by n . Then, from (2.1), it follows that $n|r$. Hence, $n^2|n$, a contradiction. \square

In the following lemma, we adapt the important result from [8] for $\mathbb{K}[X]$.

Lemma 3. *Let $\{a, b\}$ be a polynomial $D(n)$ -pair with $\deg(a) = \deg(b) = k \geq 1$ and such that $ab + n = r^2$, where $r \in \mathbb{K}[X]$. Let $ae + n^2 = u^2$, $be + n^2 = v^2$, where $u, v, e \in \mathbb{K}[X]$ and $\deg(e) \geq 0$. Then for each such e there is at most one $c \in \mathbb{K}[X]$, with $\deg(c) = k$, such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple.*

Proof. Suppose that $\{a, b\}$ is a polynomial $D(n)$ -pair such that $ab + n = r^2$, where $r \in \mathbb{K}[X]$. From $ae + n^2 = u^2$ and $be + n^2 = v^2$, where $u, v \in \mathbb{K}[X]$, and from the fact that \mathbb{K} is an algebraically closed field, we conclude that there are at most two u -s and at most two v -s. Namely, we have $\pm u$ and $\pm v$. Then, from (2.5), we obtain two possible c -s:

$$c_{\pm} = a + b + \frac{e}{n} + \frac{2}{n^2}(abe \pm ruv). \tag{2.14}$$

From this, we get

$$c_+ \cdot c_- = b^2 + a(a - 2b) + \frac{e^2}{n^2} - \frac{2ae}{n} - \frac{2be}{n} - 4n. \tag{2.15}$$

If $c_+ \cdot c_- \in \mathbb{K}[X]$, then $\frac{e(e-2n(a+b))}{n^2} \in \mathbb{K}[X]$. From this, we conclude that $n|e$. Then, from (2.2), we get that $n|u$, $n|v$, and further $n^2|ae$ and $n^2|be$. If $n^2 \nmid e$, then $n|a$ and $n|b$, which is in contradiction with Lemma 2. Hence, $n^2|e$. In that case (2.9) is not possible and we must have (2.10). Then, from (2.15), we obtain that

$$\deg(c_+) + \deg(c_-) = \deg\left(\frac{e^2}{n^2}\right) = 6k - 2.$$

Finally, we conclude that one of the polynomials c_{\pm} has a degree $5k - 2$ if the other of them has a degree k . □

Remark 1. From the proof of Lemma 3 we have that if $n|e$, then $n^2|e$. A polynomial $D(16X + 9)$ -triple

$$\{X, 16X + 8, 36X + 20\}, \tag{2.16}$$

from [7], for which $e = 33X + 18$ is an example for which (2.9) holds. But in this case $c_+ \cdot c_- \notin \mathbb{K}[X]$.

2.1. Linear polynomials

Let us prove the following proposition.

Proposition 1. $L_1 \leq 7$.

For the proof of Proposition 1 we use the results from the previous lemmas and the results stated below. Let us fix $a, b \in \mathbb{K}[X]$ such that $\deg(a) = \deg(b) = 1$. We are looking at the extensions of $\{a, b\}$ to a polynomial $D(n)$ -triple $\{a, b, c\}$ with $\deg(c) = 1$ and then at the corresponding $e \in \mathbb{K}[X]$ defined by (2.3). From (2.9), we have the first possibility that $\deg(e) \leq 1$ and from (2.10) we have the second possibility that $\deg(e) = 3$. Therefore, we have to consider the possibilities that $\deg(e) \in \{0, 1, 3\}$.

Lemma 4. For a fixed polynomial $D(n)$ -pair $\{a, b\}$ with $\deg(a) = \deg(b) = 1$ there is at most one c with $\deg(c) = 1$ such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple and such that the corresponding e , defined by (2.3), is from $\mathbb{K} \setminus \{0\}$.

Proof. By Lemma 1, there is $u \in \mathbb{K}[X]$ such that $ae + n^2 = u^2$ and $\deg(u) = 1$. Since $a = A(X - \phi)$, where $\phi \in \mathbb{K}$ and $A \in \mathbb{K} \setminus \{0\}$, we assume without loss of generality that $u - n = \varepsilon_1(X - \phi)$, $u + n = \varepsilon_2$, where $\varepsilon_1, \varepsilon_2 \in \mathbb{K} \setminus \{0\}$ and $\varepsilon_1 \varepsilon_2 = Ae$. This implies that

$$2n = -\varepsilon_1 X + \varepsilon_2 + \varepsilon_1 \phi. \quad (2.17)$$

Assume that, for fixed a and b , two distinct e -s exist, namely e and f , where $f \in \mathbb{K} \setminus \{0\}$ such that Lemma 1 holds. Therefore, $af + n^2 = u_1^2$, where $u_1 \in \mathbb{K}[X]$ and $\deg(u_1) = 1$. We conclude that

$$\begin{aligned} u_1 - n &= \varphi_1(X - \phi), \\ u_1 + n &= \varphi_2, \end{aligned} \quad (2.18)$$

or

$$\begin{aligned} u_1 - n &= \varphi_1, \\ u_1 + n &= \varphi_2(X - \phi), \end{aligned} \quad (2.19)$$

where $\varphi_1 \varphi_2 = Af$ and $\varphi_1, \varphi_2 \in \mathbb{K} \setminus \{0\}$. Let us first consider the case (2.18). We get $2n = -\varphi_1 X + \varphi_2 + \varphi_1 \phi$. Hence, from (2.17), $-\varepsilon_1 = -\varphi_1$ and $\varepsilon_2 + \varepsilon_1 \phi = \varphi_2 + \varphi_1 \phi$. So, we have $\varepsilon_1 = \varphi_1$ and $\varepsilon_2 - \varphi_2 = \phi(-\varepsilon_1 + \varphi_1)$, from which it follows that $\varepsilon_2 = \varphi_2$. Therefore, $e = f$.

Assume now that (2.19) holds. Then, $2n = \varphi_2 X - \varphi_2 \phi - \varphi_1$. Therefore, $-\varepsilon_1 = \varphi_2$ and $\varepsilon_2 = -\varphi_1$. This yields $e = f$. Hence, for fixed a and b , there is at most one $e \in \mathbb{K} \setminus \{0\}$. For that e , from Lemma 3, there is at most one $c \in \mathbb{K}[X]$, with $\deg(c) = 1$, such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple. \square

Remark 2. Using the proof of Lemma 4, we get a polynomial $D(-2X + \frac{1}{2})$ -triple

$$\left\{ X + \frac{\sqrt{2} - 10}{49}, 8X + \frac{64\sqrt{2} + 46}{49}, X(9 - 4\sqrt{2}) \right\},$$

for which $e = \frac{36 + 16\sqrt{2}}{49}$. Hence, the situation described in Lemma 4 is possible.

Lemma 5. *For a fixed polynomial $D(n)$ -pair $\{a, b\}$ with $\deg(a) = \deg(b) = 1$ there are at most two c -s with $\deg(c) = 1$ such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple and such that the corresponding e , defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e) = 1$.*

Proof. From (2.2), we have

$$c = \frac{(w - n)(w + n)}{e}. \quad (2.20)$$

Therefore, $w \pm n = \lambda e$ where $\lambda \in \mathbb{K} \setminus \{0\}$, for at least one of the signs \pm . Inserting that into (2.20), we get

$$c = \lambda(\lambda e \mp 2n). \quad (2.21)$$

Then, from (2.8), it follows that $e(1 - 2r\lambda) = n(a + b - c \mp 2r)$. From Remark 1, we know that if $n|e$ then $n^2|e$, so $n \nmid e$. Therefore,

$$1 - 2r\lambda = \lambda_1 n, \tag{2.22}$$

$$\lambda_1 e = a + b - c \mp 2r, \tag{2.23}$$

where $\lambda, \lambda_1 \in \mathbb{K} \setminus \{0\}$.

Suppose that for fixed a and b another e exists. We call it f , and we suppose that for such $f \in \mathbb{K}[X]$, where $\deg(f) = 1$, Lemma 1 holds. For a polynomial $D(n)$ -triple $\{a, b, c'\}$, where $\deg(c') = 1$, by Lemma 1 there is $w' \in \mathbb{K}[X]$ such that $c'f + n^2 = (w')^2$. Analogously as for e , it holds

$$1 - 2r\xi = \xi_1 n, \tag{2.24}$$

where $\xi_1 f = a + b - c' \mp 2r$, $w' \pm n = \xi f$ (for at least one of the signs \pm) and $\xi, \xi_1 \in \mathbb{K} \setminus \{0\}$.

From (2.22) and (2.24), we get $-2r(\lambda - \xi) = n(\lambda_1 - \xi_1)$. If $n|r$, we obtain a contradiction with (2.22). Therefore, $\lambda = \xi$ and $\lambda_1 = \xi_1$.

By inserting (2.21) into (2.23), we obtain

$$e = \frac{1}{\lambda^2 + \lambda_1} (a + b \mp 2r \pm 2n\lambda). \tag{2.25}$$

Analogously, we get

$$f = \frac{1}{\xi^2 + \xi_1} (a + b \mp 2r \pm 2n\xi). \tag{2.26}$$

Comparing (2.26) and (2.25), we conclude that there is at most one such $f \neq e$, namely one obtained for different combination of signs in (2.25) and (2.26).

Suppose that there is a third e , namely $h \in \mathbb{K}[X]$, where $\deg(h) = 1$, for which Lemma 1 holds. Analogously we conclude that $h = e$ or $h = f$. For each of polynomials e and f , by Lemma 3, we have at most one different linear polynomial c , namely c and c' , such that $\{a, b, c\}$ and $\{a, b, c'\}$ are polynomial $D(n)$ -triples. \square

Remark 3. The situation from the proof of Lemma 5 is possible. Examples for that are the polynomial $D(16X + 9)$ -triples (2.16) and $\{X, 16X + 8, 100X + 44\}$, for which $f = 273X + 126$. In this case $\lambda = \frac{2}{3}$ and $\lambda_1 = -\frac{1}{3}$. Also, by inserting X^2 instead of X in those examples, we get examples obtained in [15].

Let us now consider the last possibility, that (2.10) holds, i.e. $\deg(\bar{e}) \leq 1$. Since we assumed that $\deg(\bar{e}) \geq 0$, from (2.7) we conclude that $n|e$.

Lemma 6. *For a fixed polynomial $D(n)$ -pair $\{a, b\}$ with $\deg(a) = \deg(b) = 1$ there does not exist c with $\deg(c) = 1$ such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple and such that the corresponding e , defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e) = 3$.*

Proof. Assume that such an e exists. Since $n|e$ then $n^2|e$, by Remark 1. If we, for fixed a and b , have such a triple $\{a, b, c\}$, then $ce + n^2 = w^2$, for $w \in \mathbb{K}[X]$. We conclude that $\deg(w) = 2$ and $n|w$. Therefore, from the last equation, we get $ce_1 + 1 = w_1^2$, where $e_1, w_1 \in \mathbb{K}[X]$ and $\deg(e_1) = \deg(w_1) = 1$. We actually have two possibilities for the polynomial w_1 , namely $\pm w_1$.

By dividing (2.8) by n , we obtain

$$c = a + b - ne_1 \pm 2rw_1, \quad (2.27)$$

which is actually c_{\pm} given with (2.14), and one of the polynomials c_{\pm} must have a degree equal to 2. Since $\deg(c_{\pm}) + \deg(c_{\mp}) = 4$, we conclude that neither one of the polynomials c obtained in this way has degree equal to 1. \square

Now we can estimate the number L_1 .

Proof of Proposition 1. Let $a, b \in \mathbb{K}[X]$ be linear polynomials such that $ab + n = r^2$, with $r \in \mathbb{K}[X]$. We want to find the number of possible $D(n)$ -triples $\{a, b, c\}$, where $c \in \mathbb{K}[X]$ is also a linear polynomial.

For $e = 0$, from (2.11), we get as candidates for c at most two polynomials

$$c_{1,2} = c_{\pm} = a + b \pm 2r.$$

By Lemma 4, there can exist another c , which is one of the polynomials

$$c_3 = a + b + \frac{e_3}{n} + \frac{2}{n^2}(abe_3 \pm ru_3v_3),$$

with $e_3 \in \mathbb{K} \setminus \{0\}$, for which $ae_3 + n^2 = u_3^2$, $be_3 + n^2 = v_3^2$, where $u_3, v_3 \in \mathbb{K}[X]$. By Lemma 5, as a possible c we obtain at most one of the polynomials

$$c_i = a + b + \frac{e_i}{n} + \frac{2}{n^2}(abe_i \pm ru_iv_i)$$

for each $i = 4, 5$, with linear polynomials $e_i \in \mathbb{K}[X]$ and with $u_i, v_i \in \mathbb{K}[X]$ such that $ae_i + n^2 = u_i^2$, $be_i + n^2 = v_i^2$.

Since there are no other possibilities for e , we conclude that a polynomial $D(n)$ -tuple which contains polynomials a and b , and consists only of linear polynomials, has at most seven elements, namely, $\{a, b, c_1, c_2, c_3, c_4, c_5\}$. \square

Remark 4. The example $a = X - \frac{7}{12}$, $b = 4X$, $n = X + \frac{1}{9}$, $r = 2X - \frac{1}{3}$, $c_1 = c_+ = 9X - \frac{5}{4}$ and $c_2 = c_- = X + \frac{1}{12}$ shows that a twice semi-regular $D(n)$ -quadruple of the form $\{a, b, c_1, c_2\}$ is possible in the case of linear polynomials so we can not exclude this possibility in the proof of Proposition 1.

2.2. Quadratic polynomials

We intend to prove the following proposition.

Proposition 2. $L_2 \leq 4$.

The proof of Proposition 2 is based on the constructions from Lemmas 1- 3 and from the following lemmas which deal with a polynomial $D(n)$ -triple $\{a, b, c\}$, where $\deg(a) = \deg(b) = \deg(c) = 2$. First, we are looking for the possible e -s for fixed a and b . We have the following possible cases: $\deg(e) \leq 0$, which comes from (2.9), and $\deg(e) = 6$, from (2.10). Therefore, we consider the possibilities that $\deg(e) = 0$ and $\deg(e) = 6$.

Lemma 7. *For a fixed polynomial $D(n)$ -pair $\{a, b\}$ with $\deg(a) = \deg(b) = 2$ there is at most one c with $\deg(c) = 2$ such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple and such that the corresponding e , defined by (2.3), is from $\mathbb{K} \setminus \{0\}$.*

Proof. By Lemma 1, there is $u \in \mathbb{K}[X]$ such that $ae + n^2 = u^2$ and $\deg(u) \leq 1$. Since $a = A(X - \phi_1)(X - \phi_2)$, where $\phi_1, \phi_2 \in \mathbb{K}$ and $A \in \mathbb{K} \setminus \{0\}$, we assume that

$$\begin{aligned} u - n &= \varepsilon_1(X - \phi_1), \\ u + n &= \varepsilon_2(X - \phi_2), \end{aligned} \tag{2.28}$$

where $\varepsilon_1, \varepsilon_2 \in \mathbb{K} \setminus \{0\}$ and $\varepsilon_1\varepsilon_2 = Ae$. From that we conclude

$$2n = X(\varepsilon_2 - \varepsilon_1) + \varepsilon_1\phi_1 - \varepsilon_2\phi_2. \tag{2.29}$$

Let, for fixed a and b , two distinct e -s from $\mathbb{K} \setminus \{0\}$ exist, i.e. there is also $f \in \mathbb{K} \setminus \{0\}$ for which $af + n^2 = u_1^2$, where $u_1 \in \mathbb{K}[X]$ and $\deg(u_1) \leq 1$. We have

$$\begin{aligned} u_1 - n &= \varphi_1(X - \phi_1), \\ u_1 + n &= \varphi_2(X - \phi_2), \end{aligned} \tag{2.30}$$

or

$$\begin{aligned} u_1 - n &= \varphi_1(X - \phi_2), \\ u_1 + n &= \varphi_2(X - \phi_1), \end{aligned} \tag{2.31}$$

where $\varphi_1\varphi_2 = Af$ and $\varphi_1, \varphi_2 \in \mathbb{K} \setminus \{0\}$.

From (2.30), we get $2n = X(\varphi_2 - \varphi_1) + \varphi_1\phi_1 - \varphi_2\phi_2$. By comparing that with (2.29), we obtain $\phi_1(\varepsilon_1 - \varphi_1) = \phi_2(\varepsilon_2 - \varphi_2) = \phi_2(\varepsilon_1 - \varphi_1)$, from which it follows that $\phi_1 = \phi_2$ or $\varepsilon_1 = \varphi_1$. If $\phi_1 = \phi_2$ then, from (2.28), we conclude that $n^2|a$. Further, from (2.1), we get that $n|r$, and then, from (2.3), it follows that $n|e$, which is not possible. If $\varepsilon_1 = \varphi_1$, then $\varepsilon_2 = \varphi_2$, so $e = f$.

Assume now that (2.31) holds. Then, $2n = (\varphi_2 - \varphi_1)X - \varphi_2\phi_1 + \varphi_1\phi_2$. By comparing that with (2.29), we obtain $\phi_1(\varepsilon_1 + \varphi_2) = \phi_2(\varepsilon_2 + \varphi_1) = \phi_2(\varepsilon_1 + \varphi_2)$, from which $\phi_1 = \phi_2$ or $\varepsilon_1 = -\varphi_2$. For $\phi_1 = \phi_2$ we obtain a contradiction, as in the previous case. If $\varepsilon_1 = -\varphi_2$, then $\varepsilon_2 = -\varphi_1$, so $e = f$. Hence, for fixed a and b , there is at most one $e \in \mathbb{K} \setminus \{0\}$ for which Lemma 1 holds. For that e , from Lemma 3, there is at most one $c \in \mathbb{K}[X]$ with $\deg(c) = 2$, such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple. \square

We are left with the last possibility, that (2.10) holds, i.e. $\deg(\bar{e}) \leq 0$. Since we assumed that $\deg(\bar{e}) \geq 0$, from (2.7) we conclude that $n^2|e$.

Lemma 8. *For a fixed polynomial $D(n)$ -pair $\{a, b\}$ with $\deg(a) = \deg(b) = 2$ there does not exist c with $\deg(c) = 2$ such that $\{a, b, c\}$ is a polynomial $D(n)$ -triple and such that the corresponding e , defined by (2.3), is from $\mathbb{K}[X]$ and $\deg(e) = 6$.*

Proof. Assume, on the contrary that such an e exists. Then $ce + n^2 = w^2$ for some $w \in \mathbb{K}[X]$. We conclude that $\deg(w) = 4$ and $n|w$. By dividing the last equation by n^2 we get $ce_1 + 1 = w_1^2$, where $e_1, w_1 \in \mathbb{K}[X]$, $\deg(e_1) = 4$ and $\deg(w_1) = 3$. The polynomial w_1 is only determined up to the sign.

Analogously as in the proof of Lemma 6, we obtain (2.27), from which we conclude that one of the polynomials c_{\pm} has a degree equal to 5. Since $\deg(c_{\pm}) + \deg(c_{\mp}) = 10$, we conclude that neither one of polynomials $\pm w_1 c$ obtained this way has a degree equal to 2. \square

Before we estimate the number L_2 we will exclude one possibility, which exists in the case of linear polynomials.

Lemma 9. *Let $a, b \in \mathbb{K}[X]$ such that $\deg(a) = \deg(b) = 2$ and $ab + n = r^2$. Then the set of the form $\{a, b, a + b + 2r, a + b - 2r\}$, which contains only quadratic polynomials, is not a polynomial $D(n)$ -quadruple.*

Proof. Assume that such a set is a polynomial $D(n)$ -quadruple. Let us consider the triples $\{a, a + b + 2r, a + b - 2r\}$ and $\{b, a + b + 2r, a + b - 2r\}$. If the first triple is regular, then, from (2.12), we have $(-4r - a)^2 = 4(a(a + b + 2r) + n)$. From that, $-4r - a = \pm 2(a + r)$ so $r|a$. Then, from (2.1), it follows that $r|n$, which is not possible, because $\deg(r) = 2$ and $\deg(n) = 1$. The second case is analogous. Therefore, neither one of those triples is regular.

By Lemma 7, for the pair $\{a + b + 2r, a + b - 2r\}$, there is at most one $e \in \mathbb{K} \setminus \{0\}$ such that Lemma 1 holds. Since no other e -s exist for that pair, it holds $a = b$ which is not possible. \square

Remark 5. The example $a = X^2 + X + \frac{17}{36}$, $b = X^2 + 2X - \frac{1}{9}$, $n = -\frac{2}{3}X + \frac{1}{18}$, $r = X^2 + \frac{3}{2}X + \frac{1}{18}$, $a + b + 2r = 4X^2 + 6X + \frac{17}{36}$ and $a + b - 2r = \frac{1}{4}$ shows that if we omit the condition in Lemma 9 that all polynomials in a set of the form $\{a, b, a + b + 2r, a + b - 2r\}$ are quadratic, then such a $D(n)$ -quadruple can exist.

Now we determine the upper bound for L_2 .

Proof of Proposition 2. Let $a, b \in \mathbb{K}[X]$ be quadratic polynomials such that $ab + n = r^2$, with $r \in \mathbb{K}[X]$. We look for the number of possible $D(n)$ -triples $\{a, b, c\}$, where $c \in \mathbb{K}[X]$ and c is also a quadratic polynomial.

By Lemma 7, there is at most one possibility for c , namely c_1 , for which $e \in \mathbb{K} \setminus \{0\}$. For $e = 0$ we obtain $c_{2,3} = a + b \pm 2r$. In Lemma 8, we excluded the last option

which comes from Lemma 1, those that $\deg(e) = 6$. Hence, the largest set which we can obtain is of the form $\{a, b, c_1, c_2, c_3\}$.

By Lemma 9, the set $\{a, b, c_2, c_3\}$ is not a polynomial $D(n)$ -quadruple. Therefore, the pair $\{a, b\}$ can be extended with at most 2 quadratic polynomials (c_1 and one of the polynomials $c_{2,3}$), so $L_2 \leq 4$. \square

2.3. *Polynomials of degree $k \geq 3$*

Now we are looking for the upper bound for the number of polynomials of degree $k \geq 3$ in a polynomial $D(n)$ -tuple.

Proposition 3. $L_k \leq 3$ for $k \geq 3$.

Let $a, b \in \mathbb{K}[X]$ be polynomials of degree $k \geq 3$ such that $ab + n = r^2$, with $r \in \mathbb{K}[X]$. We look for the number of possible c -s such that $\{a, b, c\}$, where $c \in \mathbb{K}[X]$ and $\deg(c) = k$ is a polynomial $D(n)$ -triple. First, we look for possible e -s such that Lemma 1 holds. From (2.9), $\deg(e) \leq -1$, and from (2.10), $\deg(\bar{e}) \leq -1$. Therefore, $e = 0$ or $\bar{e} = 0$. From (2.7), we obtain $c_{1,2} = a + b \pm 2r$. Hence, we have at most four elements $\{a, b, c_1, c_2\}$ in such a polynomial $D(n)$ -tuple, but we also have the following lemma.

Lemma 10. *Let $a, b \in \mathbb{K}[X]$ such that $\deg(a) = \deg(b) = k \geq 3$ and $ab + n = r^2$. Then the set of the form $\{a, b, a + b + 2r, a + b - 2r\}$, which contains only polynomials of degree $k \geq 3$, is not a polynomial $D(n)$ -quadruple.*

Proof. The first part of the proof is analogue as the proof of Lemma 9, except that here $\deg(r) = k \geq 3$. Since in the case of polynomials of degree $k \geq 3$ the only possible triples are regular ones, we proved the lemma. \square

Therefore, the pair $\{a, b\}$ can be extended with at most one polynomial of degree k , where $k \geq 3$, (namely, one of $c_{1,2}$), so we proved Proposition 3.

Example for this case is $D(X)$ -triple $\{X^3 - 1, X^3 + 2X^2 + X - 1, 4X^3 + 4X^2 + X - 4\}$, from [8].

3. GAP PRINCIPLE

We will prove a gap principle for the degrees of the elements in a polynomial $D(n)$ -quadruple. This result will be used in the proof of Theorem 1, together with the bounds from Section 2 and with the upper bound for the degree of the element in a polynomial $D(n)$ -quadruple [8, Lemma 1], given in the following lemma. This bound was obtained for polynomials with integer coefficients but it also holds in $\mathbb{K}[X]$ with slightly different assumption on the degrees of polynomials in quadruple.

Lemma 11. *Let $\{a, b, c, d\}$, where $\deg(a) \leq \deg(b) \leq \deg(c) \leq \deg(d)$, be a polynomial $D(n)$ -quadruple with $n \in \mathbb{K}[X]$. Then*

$$\deg(d) \leq 7\deg(a) + 11\deg(b) + 15\deg(c) + 14\deg(n) - 4.$$

The proof of this lemma is based on the theory of function fields, precisely it is obtained by using Mason's inequality [16].

Now we will adjust the result from [7, Lemma 3], very similar as in the classical case for integers, to achieve the needed gap principle. We cannot use the gap principle from [7] and [8], because we do not have the relation " $<$ " between elements of $\mathbb{K}[X]$.

Lemma 12. *Let $\{a, b, c, d\}$, where $3 \leq \deg(a) \leq \deg(b) \leq \deg(c) \leq \deg(d)$, be a polynomial $D(n)$ -quadruple for linear $n \in \mathbb{K}[X]$. Then*

$$\deg(d) \geq \deg(b) + \deg(c) - 2.$$

Proof. Applying Lemma 1 to the polynomial $D(n)$ -triple $\{a, c, d\}$, we conclude that there exist $e, \bar{e} \in \mathbb{K}[X]$ such that, by (2.7), we have

$$e \cdot \bar{e} = n^2(d - a - c - 2s)(d - a - c + 2s). \quad (3.1)$$

From (3.1), for $e = 0$ or $\bar{e} = 0$, we get

$$d = a + c \pm 2s. \quad (3.2)$$

In this case $\deg(d) \leq \deg(c)$, so $\deg(d) = \deg(c)$.

Let $e, \bar{e} \in \mathbb{K}[X]$ be nonzero polynomials. Since $\deg(s) = \frac{\deg(a) + \deg(c)}{2} \leq \deg(c)$, from (3.1) we obtain

$$\deg(e) + \deg(\bar{e}) \leq 2 + 2\deg(d). \quad (3.3)$$

By (2.3) and (2.6), we conclude that the degree of one of the polynomials e and \bar{e} is equal to $\deg(a) + \deg(c) + \deg(d)$ and the degree of the other one is ≥ 0 . Hence, from (3.3) it follows that

$$\deg(d) \geq \deg(a) + \deg(c) - 2. \quad (3.4)$$

Analogously, applying Lemma 1 to the polynomial $D(n)$ -triple $\{b, c, d\}$, we obtain that either

$$d = b + c \pm 2t \quad (3.5)$$

or

$$\deg(d) \geq \deg(b) + \deg(c) - 2. \quad (3.6)$$

Assume that (3.5) holds. As for (3.2), we conclude that $\deg(d) = \deg(c)$. Therefore, (3.4) cannot hold at the same time. Otherwise, we would obtain $\deg(a) \leq 2$, a contradiction. Assume that (3.2) also holds. From (3.2) and (3.5) we obtain $a = b \pm 2s \pm 2t$. Therefore, if $\deg(a) < \deg(c)$ then $\deg(s) > \deg(a)$ and the polynomial on the right hand side of the previous equation has degree $> \deg(a)$ unless $\deg(a) = \deg(b)$ and $S = \pm T$. We conclude that

$$\deg(a) = \deg(b) = \deg(c) = \deg(d) \quad (3.7)$$

or

$$\deg(a) = \deg(b) < \deg(c) = \deg(d) \text{ and } S = \pm T. \quad (3.8)$$

If (3.7) holds, then we have a polynomial $D(n)$ -quadruple whose elements have degrees equal to k , where $k \geq 3$. This is not possible, according to Proposition 3. Therefore, it holds (3.8). Since $\{a, c, d\}$ and $\{b, c, d\}$ are both regular triples (i.e. $\{a, b, c, d\}$ is a twice semi-regular $D(n)$ -quadruple), by (2.13) we have $cd + n = (c \pm s)^2$ and $cd + n = (c \pm t)^2$, where the signs \pm in last two equations are the same as signs in (3.2) and (3.5), respectively. If $c \pm s = c \pm t$, then we obtain $s^2 = t^2$ so $a = b$, which is not possible. If $c \pm s = -c \mp t$, then $\pm s = -2c \mp t$. Since (3.8) holds, the degree of the polynomial on the left hand side of the previous equation is $< \deg(c)$ and the polynomial on the right hand side of that equation has a degree equal to $\deg(c)$. This is not possible.

Assume that (3.6) holds. Then (3.2) can not hold, because otherwise we would have $\deg(b) \leq 2$, a contradiction. But, from (3.6) we obtain (3.4), which shows that this situation is indeed possible. \square

4. PROOF OF THEOREM 1

Let $S = \{a_1, a_2, \dots, a_m\}$, where $\deg(a_1) \leq \deg(a_2) \leq \dots \leq \deg(a_m)$, be a polynomial $D(n)$ - m -tuple with $n \in \mathbb{K}[X]$ a linear polynomial. Since the product of each two elements from S increased by n is a square of a polynomial in $\mathbb{K}[X]$, it follows that if S contains a polynomial of degree ≥ 1 , then it contains only polynomials of even or only polynomials of odd degree. We proved that in S we have at most 1 nonzero constant. By Proposition 1, in S there are at most 7 linear polynomials. By Proposition 2, the number of quadratic polynomials in S is at most 4, and, by Proposition 3, in S there are at most 3 polynomials of degree k for every $k \geq 3$.

Assume that in S there is a polynomial of degree ≥ 1 . Let us first consider the case where the degrees of all polynomials in S are odd. We will combine the gaps between the degrees of the elements in S with the upper bound on the degree of the element of a $D(n)$ -quadruple. We assume that there are smallest possible gaps between the degrees of the elements in S . Then, there are possible the following bounds for the degrees:

$$\begin{aligned} \deg(a_1) \geq 1, \deg(a_2) \geq 1, \dots, \deg(a_7) \geq 1, \\ \deg(a_8) \geq 3, \deg(a_9) \geq 3, \deg(a_{10}) \geq 3. \end{aligned}$$

Applying Lemma 12 to the polynomial $D(n)$ -quadruple $\{a_8, a_9, a_{10}, a_{11}\}$ gives $\deg(a_{11}) \geq 4$ and, since this degree is odd, we conclude that

$$\deg(a_{11}) \geq 5.$$

If we continue in analogue way, we obtain

$$\begin{aligned} \deg(a_{12}) \geq 7, \quad \deg(a_{13}) \geq 11, \quad \deg(a_{14}) \geq 17, \quad \deg(a_{15}) \geq 27, \\ \deg(a_{16}) \geq 43, \quad \deg(a_{17}) \geq 69, \quad \deg(a_{18}) \geq 111, \quad \deg(a_{19}) \geq 179, \dots \end{aligned}$$

We will separate the cases depending on the number of linear polynomials in S . Assume first that in S we have at least three (and ≤ 7) linear polynomials. For

the quadruple $\{a_1, a_2, a_3, a_m\} \subseteq S$, where $\deg(a_1) = \deg(a_2) = \deg(a_3) = 1$ and $\deg(a_m) \geq 1$, applying Lemma 11, we get

$$\deg(a_m) \leq 7 + 11 + 15 + 14 - 4 = 43.$$

Hence, in this case

$$m \leq 16.$$

Assume next that in S we have two linear polynomials. Now we observe the quadruple $\{a_1, a_2, a_3, a_m\} \subseteq S$, where $\deg(a_1) = \deg(a_2) = 1$ and $\deg(a_3) = A$, for $A \geq 3$ an odd positive integer. As before, we have

$$\deg(a_m) \leq 7 + 11 + 15A + 14 - 4 = 15A + 28$$

and

$$\begin{aligned} \deg(a_4) &\geq A, & \deg(a_5) &\geq A, & \deg(a_6) &\geq 2A - 2, \dots, \\ \deg(a_{10}) &\geq 13A - 24, & \deg(a_{11}) &\geq 21A - 40, & \deg(a_{12}) &\geq 34A - 66, \\ \deg(a_{13}) &\geq 55A - 108, & \deg(a_{14}) &\geq 89A - 176, & \deg(a_{15}) &\geq 144A - 286, \dots, \end{aligned}$$

so we obtain $m \leq 13$.

Let in S we have one linear polynomial, i.e. $\deg(a_1) = 1, \deg(a_2) = A, \deg(a_3) = B$, where $3 \leq A \leq B$ and A, B are odd positive integers. We obtain

$$\deg(a_m) \leq 7 + 11A + 15B + 14 - 4 \leq 26B + 17.$$

Further,

$$\begin{aligned} \deg(a_4) &\geq B, & \deg(a_5) &\geq 2B - 2, & \deg(a_6) &\geq 3B - 4, \dots, \\ \deg(a_{12}) &\geq 55B - 108, & \deg(a_{13}) &\geq 89B - 176, & \deg(a_{14}) &\geq 144B - 286, \dots, \end{aligned}$$

so $m \leq 13$.

Finally, suppose that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$ where $3 \leq A \leq B \leq C$ and A, B, C are odd positive integers. We get

$$\deg(a_m) \leq 7A + 11B + 15C + 14 - 4 \leq 33C + 10$$

and

$$\begin{aligned} \deg(a_4) &\geq B + C - 2, & \deg(a_5) &\geq B + 2C - 4, \dots, \\ \deg(a_{12}) &\geq 34B + 55C - 176, & \deg(a_{13}) &\geq 55B + 89C - 286, \dots, \end{aligned}$$

then $m \leq 12$. Hence, if S contains only polynomials of odd degree then $m \leq 16$.

Let all polynomials in S have even degree. Now we have

$$\deg(a_1) \geq 0,$$

$$\deg(a_2) \geq 2, \deg(a_3) \geq 2, \deg(a_4) \geq 2, \deg(a_5) \geq 2 \text{ and}$$

$$\deg(a_6) \geq 4, \deg(a_7) \geq 4, \deg(a_8) \geq 4.$$

Applying Lemma 12 to the polynomial $D(n)$ -quadruple $\{a_6, a_7, a_8, a_9\}$ we get

$$\deg(a_9) \geq 6.$$

Analogously, it follows

$$\begin{aligned} \deg(a_{10}) \geq 8, \quad \deg(a_{11}) \geq 12, \quad \deg(a_{12}) \geq 18, \quad \deg(a_{13}) \geq 28, \\ \deg(a_{14}) \geq 44, \quad \deg(a_{15}) \geq 70, \quad \deg(a_{16}) \geq 112, \quad \deg(a_{17}) \geq 180, \dots \end{aligned}$$

Assume first that $\deg(a_1) = 0, \deg(a_2) = A, \deg(a_3) = B$ where $2 \leq A \leq B$ and A, B are even positive integers. If we apply Lemma 11 to a polynomial $D(n)$ -quadruple $\{a_1, a_2, a_3, a_m\}$, it follows that

$$\deg(a_m) \leq 0 + 11A + 15B + 14 - 4 \leq 26B + 10.$$

If $A = B = 2$, then $m \leq 14$. If $B \geq 4$, then

$$\begin{aligned} \deg(a_4) \geq B, \quad \deg(a_5) \geq B, \quad \deg(a_6) \geq 2B - 2, \dots, \\ \deg(a_{12}) \geq 34B - 66, \quad \deg(a_{13}) \geq 55B - 108, \quad \deg(a_{14}) \geq 89B - 176, \dots, \end{aligned}$$

so we obtain that $m \leq 13$.

Suppose finally that $\deg(a_1) = A, \deg(a_2) = B, \deg(a_3) = C$, where $2 \leq A \leq B \leq C$ and A, B, C are even positive integers. We have

$$\deg(a_m) \leq 7A + 11B + 15C + 14 - 4 \leq 33C + 10.$$

If $A = B = C = 2$ and

$$\begin{aligned} \deg(a_4) \geq C, \quad \deg(a_5) = D, \quad \deg(a_6) \geq D, \\ \deg(a_7) \geq D, \quad \deg(a_8) \geq 2D - 2, \quad \deg(a_9) \geq 3D - 4, \dots, \\ \deg(a_{14}) \geq 34D - 66, \quad \deg(a_{15}) \geq 55D - 108, \quad \deg(a_{16}) \geq 89D - 176, \dots, \end{aligned}$$

where $D \geq 4$ and D is even positive integer, then $m \leq 15$. Also, $m \leq 13$ if $C \geq 4$. We conclude that the set S has at most 15 polynomials of even degree. Therefore,

$$L \leq 16.$$

□

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