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CAYLEY'S THEOREM AND ADAMS COMPLETION

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Abstract. Cayley's Theorem states that a permutation group of a group is isomorphic to the given group. We show that this permutation group is Adams completion of the group with respect to a suitably chosen set of morphisms in the category of groups and homomorphisms.

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1. INTRODUCTION

Many algebraic and geometrical constructions in algebraic and general topology can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to the chosen set of morphisms. The notion of generalized Adams completion arose from a general categorical completion process, suggested by Adams [1, 2]. Originally, this was considered for admissible categories and generalized homology (or cohomology) theories. The notion has been considered in a more general framework by Deleanu, Frei and Hilton [4] where an arbitrary category and an arbitrary set of morphisms of the category are considered.

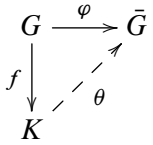
Let $\mathscr{C}$ be a category and S be a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions [6] of $\mathscr{C}$ with respect to S and $F : \mathscr{C} \to \mathscr{C}[S^{-1}]$ be the canonical functor. Let $\mathscr{S}$ denote the category of sets and functions. Then for a given object Y of $\mathscr{C}$, $\mathscr{C}[S^{-1}](-, Y) : \mathscr{C} \to \mathscr{S}$ defines a contravariant functor [6]. If this functor is representable by an object Y_S of $\mathscr{C}$, i.e., $\mathscr{C}[S^{-1}](-, Y) \cong \mathscr{C}(-, Y_S)$, then Y_S is called the *generalized Adams completion* [4] of Y with respect to the set of morphisms S or simply the S-*completion* of Y. Y_S is also referred as the *completion* of Y.

Let $\bar{S}$ be the set of all morphisms s in $\mathscr{C}$ such that $F(s)$ is an isomorphism in $\mathscr{C}[S^{-1}]$. Then $\bar{S}$ is called as the *saturation* of S. The set of morphisms S is said to be *saturated* [4] if $S = \bar{S}$.

2. Cayley's Theorem

Let G be a group. Then $\bar{G} = \{T_g : G \to G \mid T_g(x) = gx \text{ for all } x \in G,\ g \in G\}$ is a permutation group. Then by Cayley's Theorem [5] G is isomorphic to $\bar{G}$. The following result is easy to prove.

Theorem 1. *Let G be a group and $\bar{G}$ be a group of permutations. Then there is an isomorphism $\varphi : G \to \bar{G}$. If K is a group and $f : G \to K$ is an isomorphism, then there exists a unique isomorphism $\theta : K \to \bar{G}$ making the triangle commutative, i.e., $\theta f = \varphi$.*



Proof. Define $\theta : K \to \bar{G}$ by the rule $\theta(k) = \varphi f^{-1}(k)$ for all $k \in K$. Clearly, θ is well defined and is also a homomorphism. In order to show θ is injective, we have to show $\ker \theta = \{e_K\}$. Let $k \in \ker \theta$, i.e., $\theta(k) = \varphi f^{-1}(k) = e_{\bar{G}}$. So $f^{-1}(k) = e_G$, i.e., $k = e_K$, showing θ is injective. Next we show θ is surjective. $\theta(K) = \varphi f^{-1}(K) = \varphi(G) = \bar{G}$; so θ is surjective. Thus, θ is an isomorphism.

For any $g \in G$, $\theta f(g) = \varphi f^{-1}(f(g)) = \varphi(g)$. Thus $\theta f = \varphi$, i.e., the diagram is commutative. Next we show that θ is unique. Let there exist another $\theta' : K \to \bar{G}$ with $\theta' f = \varphi$. Then for any $k \in K$, $\theta(k) = \varphi f^{-1}(k) = \theta' f f^{-1}(k) = \theta'(k)$. Hence $\theta = \theta'$. This completes the proof of Theorem 1. □

3. The category $\mathscr{G}$

Let $\mathscr{U}$ be a fixed Grothendieck universe [6]. Let $\mathscr{G}$ denote the category of groups and homomorphisms. We assume that the underlying sets of the elements of $\mathscr{G}$ are elements of $\mathscr{U}$.

Let S be the set of all morphisms $s : P \to Q$ in $\mathscr{G}$ such that s is an isomorphism.

Theorem 2. *Let $s_i : P_i \to Q_i$ lie in S for each $i \in I$ where the index set I is an element of $\mathscr{U}$. Then $\underset{i\in I}{\vee} s_i : \underset{i\in I}{\vee} P_i \to \underset{i\in I}{\vee} Q_i$ lies in S.*

Proof. Coproducts in $\mathscr{G}$ are the free products. Define a map $s = \underset{i\in I}{\vee} s_i : P = \underset{i\in I}{\vee} P_i \to \underset{i\in I}{\vee} Q_i = Q$ by the rule $s(p_1 \cdots p_k) = \varphi(p_1) \cdots \varphi(p_k)$ where $\varphi(p_j) = s_i(p_j)$ if $p_j \in P_i$, $j = 1, \cdots, k$. Clearly, s is well defined and is also a homomorphism.

In order to show s is injective we have to show $\ker s = \{e_P\}$. Let $p = p_1 \cdots p_k \in \ker s$, i.e., $s(p_1 \cdots p_k) = e_Q = 1$; this implies that $\varphi(p_1) \cdots \varphi(p_k) = 1$ where $\varphi(p_j) = s_i(p_j) = \omega_i'(s_i(p_j))$ for $p_j \in P_i$, $j = 1, \cdots, k$ and $\omega_i' : Q_i \to Q$ defined by $\omega_i'(e_{Q_i}) = 1$ and $\omega_i'(b) = b$ for $b \in Q_i$, is a monomorphism for each $i \in I$. So each $\varphi(p_j) = s_i(p_j) = \omega_i'(s_i(p_j)) = 1 = \omega_i'(e_{Q_i})$ from which it follows that $s_i(p_j) = e_{Q_i}$, i.e.,

$p_j = e_{P_i}$ for $p_j \in P_i$ and $j = 1, \cdots, k$. Next $p_1 \cdots p_k = \eta(p_1) \cdots \eta(p_k)$ where $\eta(p_j) = \omega_i(p_j) = \omega_i(e_{P_i}) = 1$ for $p_j \in P_i$ and $\omega_i : P_i \to P$ is defined by $\omega_i(e_{P_i}) = 1$ and $\omega_i(a) = a$ for $a \in P_i$, is a monomorphism for each $i \in I$. So $p_1 \cdots p_k = 1 = e_P$. Hence s is injective.

Next take $q_1 \cdots q_k \in Q$ where $q_j \in Q_i$ for $i \in I$ and $j = 1, \cdots, k$. But $q_j = s_i(p_j)$ where $p_j \in P_i$. So $q_1 \cdots q_k = \varphi(p_1) \cdots \varphi(p_k)$ where $\varphi(p_j) = s_i(p_j)$ for $p_j \in P_i$. Hence $q_1 \cdots q_k = s(p_1 \cdots p_k)$, showing s is surjective. Therefore, $s : P \to Q$ is an isomorphism, i.e., $s = \underset{i \in I}{\vee} s_i$ lies in S. This completes the proof of Theorem 2. □

We will show that the set of morphisms S of the category $\mathcal{G}$ of groups and homomorphisms admits a calculus of left fractions [6].

Theorem 3. *S admits a calculus of left fractions.*

Proof. Since S consists of all isomorphisms in $\mathcal{G}$, clearly S is a closed family of morphisms of the category $\mathcal{G}$. We shall verify conditions (i) and (ii) of Theorem 1.3 ([4], p.67). Let $s : P \to Q$ and $t : Q \to R$ be two morphisms in $\mathcal{G}$. We show if $ts \in S$ and $s \in S$, then $t \in S$. Let $q \in \ker t$, i.e., $t(q) = e_R$. So $t(s(p)) = e_R$, $p \in P$. Since ts is an isomorphism we have $p = e_P$. So $q = s(e_P) = e_Q$, i.e., $\ker t = \{e_Q\}$, i.e., t is injective. Since $ts \in S$ and $s \in S$, we have $ts(P) = R$ and $s(P) = Q$. Then $t(Q) = t(s(P)) = R$. So t is surjective. Thus t is an isomorphism, i.e., $t \in S$. Hence condition (i) of Theorem 1.3 ([4], p.67) holds.

In order to prove condition (ii) of Theorem 1.3 ([4], p.67) consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \\ C & & \end{array}$$

in $\mathcal{G}$ with $s \in S$. We assert that the above diagram can be completed to a weak push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ C & \xrightarrow{g} & D \end{array}$$

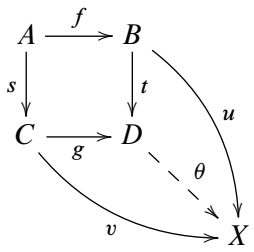
in $\mathcal{G}$ with $t \in S$. Let $D = (B * C)/N$, where N is a normal subgroup of $B * C$ generated by $\{f(a)s(a)^{-1} : a \in A\}$. Define $t : B \to D$ by the rule $t(b) = bN$ for all $b \in B$ and $g : C \to D$ by the rule $g(c) = cN$ for all $c \in C$. Clearly, the two maps are well defined and homomorphisms. For any $a \in A$, $tf(a) = f(a)N = s(a)N = gs(a)$, implies that $tf = gs$. Hence the diagram is commutative.

Next we show $t \in S$, i.e., t is an isomorphism. Take $b \in \ker t$, i.e., $t(b) = e_D = N$; this implies $bN = N$, i.e., $b \in N$. Hence $b = f(a)s(a)^{-1} = f(a)s(a^{-1})$ for some $a \in A$. Then $b1 = f(a)s(a^{-1})$, implies that $b\delta_2(e_C) = f(a)\delta_2(s(a^{-1}))$ where $\delta_2 :$

$C \to B * C$, defined by $\delta_2(e_C) = 1$ and $\delta_2(c) = c$ for $c \in C$, is a monomorphism. Then $b = f(a)$, $\delta_2(e_C) = \delta_2(s(a^{-1}))$. As $\delta_2(e_C) = \delta_2(s(a^{-1}))$, we have $s(a^{-1}) = e_C$, giving $a = e_A$. Then $b = f(e_A) = e_B$, implies that $\ker t = \{e_B\}$, i.e., t is injective.

In orrder to show t is surjective, take an element $wN \in D$, where $w \in B * C$, and for $w \neq 1$, w can be uniquely written as $w = w_1 \cdots w_k$ where all factors are $\neq 1$ and two adjacent factors do not belong to the same group. Then $wN = w_1 \cdots w_k N = w_1 N \cdots w_k N = \varphi(w_1) \cdots \varphi(w_k)$ where $\varphi(w_i) = t(w_i)$ if $w_i \in B$ and $\varphi(w_i) = g(w_i)$ if $w_i \in C$. If $w_i \in C$, then $w_i = s(a_i)$ and $g(w_i) = g(s(a_i)) = gs(a_i) = tf(a_i)$. So $wN = t$(an element of B). t is surjective. Thus t is an isomorphism, i.e., $t \in S$.

Next let $u : B \to X$ and $v : C \to X$ in category $\mathscr{G}$ be such that $uf = vs$.



Define $\theta : D \to X$ by the rule $\theta(wN) = \varphi(w_1) \cdots \varphi(w_k)$, $w = w_1 \cdots w_k$ where $\varphi(w_i) = u(w_i)$ if $w_i \in B$ and $\varphi(w_i) = v(w_i)$ if $w_i \in C$. We can easily show that θ is well defined and also a homomorphism. Next we show that the two triangles are commutative. For any $b \in B$, $\theta t(b) = \theta(bN) = u(b)$ and for any $c \in C$, $\theta g(c) = \theta(cN) = v(c)$. So $\theta t = u$ and $\theta g = v$. This completes the proof of Theorem 3. □

Also from the above discussion we conclude the following.

Theorem 4. *The category $\mathscr{G}$ is cocomplete.*

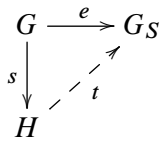
The proof of the following result is trivial.

Proposition 1. *S is saturated.*

All the conditions of Theorem 1 ([3], p.32) are satisfied for the category $\mathscr{G}$ and the set of morphisms S of $\mathscr{G}$. So from the Theorem 1.2 ([4], p.63), we have the following result.

Theorem 5. *Every object G of the category $\mathscr{G}$ has an Adams completion G_S with respect to the set of morphisms S. Furthermore, there exists a morphism $e : G \to G_S$ in S which is couniversal with respect to the morphisms in S : given a morphism $s : G \to H$ in S there exists a unique morphism $t : H \to G_S$ in S such that $ts = e$.*

In other words the following diagram is commutative :

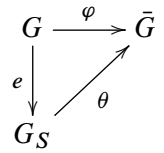


4. $\bar{G}$ AS ADAMS COMPLETION

We show that $\bar{G}$, a permutation group, as constructed in the proof of Cayley's Theorem for a group G, is the Adams completion G_S of the group G.

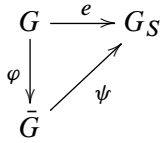
Theorem 6. $\bar{G} \cong G_S$.

Proof. Consider the following diagram :



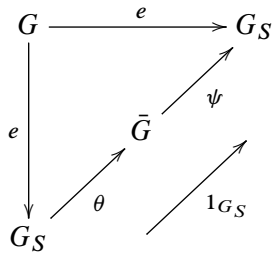
By Theorem 1, there exists a unique morphism $\theta : G_S \to \bar{G}$ in S such that $\theta e = \varphi$.

Next consider the following diagram :



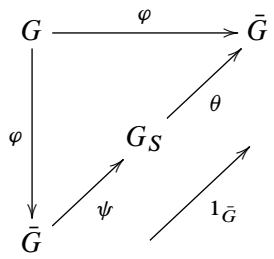
By Theorem 5, there exists a unique morphism $\psi : \bar{G} \to G_S$ in S such that $\psi\varphi = e$.

Consider the following diagram :



We have $\psi\theta e = \psi\varphi = e$. By the uniqueness condition of the couniversal property of e, we conclude $\psi\theta = 1_{G_S}$.

Next consider the following diagram :



We have $\theta\psi\varphi = \theta e = \varphi$. By the uniqueness condition of the couniversal property of φ, we conclude $\theta\psi = 1_{\bar{G}}$.

Thus $\bar{G} \cong G_S$. This completes the proof of Theorem 6. □

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