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2-IRREDUCIBLE AND STRONGLY 2-IRREDUCIBLE IDEALS OF COMMUTATIVE RINGS

H. MOSTAFANASAB AND A. YOUSEFIAN DARANI

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Abstract. An ideal I of a commutative ring R is said to be *irreducible* if it cannot be written as the intersection of two larger ideals. A proper ideal I of a ring R is said to be *strongly irreducible* if for each ideals J, K of R, $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$. In this paper, we introduce the concepts of 2-irreducible and strongly 2-irreducible ideals which are generalizations of irreducible and strongly irreducible ideals, respectively. We say that a proper ideal I of a ring R is *2-irreducible* if for each ideals J, K and L of R, $I = J \cap K \cap L$ implies that either $I = J \cap K$ or $I = J \cap L$ or $I = K \cap L$. A proper ideal I of a ring R is called *strongly 2-irreducible* if for each ideals J, K and L of R, $J \cap K \cap L \subseteq I$ implies that either $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$.

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1. INTRODUCTION

Throughout this paper all rings are commutative with a nonzero identity. Recall that an ideal I of a commutative ring R is *irreducible* if $I = J \cap K$ for ideals J and K of R implies that either $I = J$ or $I = K$. A proper ideal I of a ring R is said to be *strongly irreducible* if for each ideals J, K of R, $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$ (see [3], [13]). Obviously a proper ideal I of a ring R is strongly irreducible if and only if for each $x, y \in R$, $Rx \cap Ry \subseteq I$ implies that $x \in I$ or $y \in I$. It is easy to see that any strongly irreducible ideal is an irreducible ideal. Now, we recall some definitions which are the motivation of our work. Badawi in [4] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of R to be a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It is shown that a proper ideal I of R is a 2-absorbing ideal if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$. In [9], Yousefian Darani and Puczyłowski studied the concept of 2-absorbing commutative semigroups. Anderson and Badawi [2] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) ideal if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, ..., a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \cdots I_{n+1}$ of R), then there are n of the a_i's (resp. n of the I_i's) whose product is in I. Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly n-absorbing ideal of R is also an n-absorbing ideal of R. The concept of 2-absorbing primary ideals, a generalization of primary ideals was introduced and investigated in [6]. A proper ideal I of a commutative ring R is called a *2-absorbing primary ideal* if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. We refer the readers to [5] for a specific kind of 2-absorbing ideals and to [19], [10], [11] for the module version of the above definitions. We define an ideal I of a ring R to be *2-irreducible* if whenever $I = J \cap K \cap L$ for ideals $I,\ J$ and K of R, then either $I = J \cap K$ or $I = J \cap L$ or $I = K \cap L$. Obviously, any irreducible ideal is a 2-irreducible ideal. Also, we say that a proper ideal I of a ring R is called *strongly 2-irreducible* if for each ideals $J,\ K$ and L of R, $J \cap K \cap L \subseteq I$ implies that $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$. Clearly, any strongly irreducible ideal is a strongly 2-irreducible ideal. In [8], [7] we can find the notion of 2-irreducible preradicals and its dual, the notion of co-2-irreducible preradicals. We call a proper ideal I of a ring R *singly strongly 2-irreducible* if for each $x, y, z \in R$, $Rx \cap Ry \cap Rz \subseteq I$ implies that $Rx \cap Ry \subseteq I$ or $Rx \cap Rz \subseteq I$ or $Ry \cap Rz \subseteq I$. It is trivial that any strongly 2-irreducible ideal is a singly strongly 2-irreducible ideal. A ring R is said to be an *arithmetical ring*, if for each ideals $I,\ J$ and K of R, $(I + J) \cap K = (I \cap K) + (J \cap K)$. This condition is equivalent to the condition that for each ideals $I,\ J$ and K of R, $(I \cap J) + K = (I + K) \cap (J + K)$, see [15]. In this paper we prove that, a nonzero ideal I of a principal ideal domain R is 2-irreducible if and only if I is strongly 2-irreducible if and only if I is 2-absorbing primary. It is shown that a proper ideal I of a ring R is strongly 2-irreducible if and only if for each $x, y, z \in R$, $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq I$ implies that $(Rx + Ry) \cap (Rx + Rz) \subseteq I$ or $(Rx + Ry) \cap (Ry + Rz) \subseteq I$ or $(Rx + Rz) \cap (Ry + Rz) \subseteq I$. A proper ideal I of a von Neumann regular ring R is 2-irreducible if and only if I is 2-absorbing if and only if for every idempotent elements e_1, e_2, e_3 of R, $e_1e_2e_3 \in I$ implies that either $e_1e_2 \in I$ or $e_1e_3 \in I$ or $e_2e_3 \in I$. If I is a 2-irreducible ideal of a Noetherian ring R, then I is a 2-absorbing primary ideal of R. Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with $1 \neq 0$. It is shown that a proper ideal J of R is a strongly 2-irreducible ideal of R if and only if either $J = I_1 \times R_2$ for some strongly 2-irreducible ideal I_1 of R_1 or $J = R_1 \times I_2$ for some strongly 2-irreducible ideal I_2 of R_2 or $J = I_1 \times I_2$ for some strongly irreducible ideal I_1 of R_1 and some strongly irreducible ideal I_2 of R_2. A proper ideal I of a unique factorization domain R is singly strongly 2-irreducible if and only if $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$, where p_i's are distinct prime elements of R and n_i's are natural numbers, implies that $p_r^{n_r} p_s^{n_s} \in I$, for some $1 \leq r, s \leq k$.

2. Basic properties of 2-irreducible and strongly 2-irreducible ideals

It is important to notice that when R is a domain, then R is an arithmetical ring if and only if R is a Prüfer domain. In particular, every Dedekind domain is an arithmetical domain.

Theorem 1. *Let R be a Dedekind domain and I be a nonzero proper ideal of R. The following conditions are equivalent:*

(1) *I is a strongly irreducible ideal;*
(2) *I is an irreducible ideal;*
(3) *I is a primary ideal;*
(4) *$I = Rp^n$ for some prime (irreducible) element p of R and some natural number n.*

Proof. See [13, Lemma 2.2(3)] and [18, p. 130, Exercise 36]. □

We recall from [1] that an integral domain R is called a GCD-domain if any two nonzero elements of R have a greatest common divisor (GCD), equivalently, any two nonzero elements of R have a least common multiple (LCM). Unique factorization domains (UFD's) are well-known examples of GCD-domains. Let R be a GCD-domain. The least common multiple of elements $x,\ y$ of R is denoted by $[x,y]$. Notice that for every elements $x,\ y \in R$, $Rx \cap Ry = R[x,y]$. Moreover, for every elements x,y,z of R, we have $[[x,y],z] = [x,[y,z]]$. So we denote $[[x,y],z]$ simply by $[x,y,z]$.

Recall that every principal ideal domain (PID) is a Dedekind domain.

Theorem 2. *Let R be a PID and I be a nonzero proper ideal of R. The following conditions are equivalent:*

(1) *I is a 2-irreducible ideal;*
(2) *I is a 2-absorbing primary ideal;*
(3) *Either $I = Rp^k$ for some prime (irreducible) element p of R and some natural number n, or $I = R(p_1^n p_2^m)$ for some distinct prime (irreducible) elements $p_1,\ p_2$ of R and some natural numbers $n,\ m$.*

Proof. (2)⇔(3) See [6, Corollary 2.12].
(1)⇒(3) Assume that $I = Ra$ where $0 \neq a \in R$. Let $a = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be a prime decomposition for a. We show that either $k = 1$ or $k = 2$. Suppose that $k > 2$. By [14, p. 141, Exercise 5], we have that $I = Rp_1^{n_1} \cap Rp_2^{n_2} \cap \cdots \cap Rp_k^{n_k}$. Now, since I is 2-irreducible, there exist $1 \leq i, j \leq k$ such that $I = Rp_i^{n_i} \cap Rp_j^{n_j}$, say $i = 1,\ j = 2$. Therefore we have $I = Rp_1^{n_1} \cap Rp_2^{n_2} \subseteq Rp_3^{n_3}$, which is a contradiction.
(3)⇒(1) If $I = Rp^k$ for some prime element p of R and some natural number n, then I is irreducible, by Theorem 1, and so I is 2-irreducible. Therefore, assume

that $I = R(p_1^n p_2^m)$ for some distinct prime elements $p_1,\ p_2$ of R and some natural numbers $n,\ m$. Let $I = Ra \cap Rb \cap Rc$ for some elements $a,\ b$ and c of R. Then $a,\ b$ and c divide $p_1^n p_2^m$, and so $a = p_1^{\alpha_1} p_2^{\alpha_2}$, $b = p_1^{\beta_1} p_2^{\beta_2}$ and $c = p_1^{\gamma_1} p_2^{\gamma_2}$ where $\alpha_i, \beta_i, \gamma_i$ are some nonnegative integers. On the other hand $I = Ra \cap Rb \cap Rc = R[a,b,c] = R(p_1^{\delta} p_2^{\varepsilon})$ in which $\delta = max\{\alpha_1, \beta_1, \gamma_1\}$ and $\varepsilon = max\{\alpha_2, \beta_2, \gamma_2\}$. We can assume without loss of generality that $\delta = \alpha_1$ and $\varepsilon = \beta_2$. So $I = R(p_1^{\alpha_1} p_2^{\beta_2}) = Ra \cap Rb$. Consequently, I is 2-irreducible. □

A commutative ring R is called a *von Neumann regular ring (or an absolutely flat ring)* if for any $a \in R$ there exists an $x \in R$ with $a^2 x = a$, equivalently, $I = I^2$ for every ideal I of R.

Remark 1. Notice that a commutative ring R is a von Neumann regular ring if and only if $IJ = I \cap J$ for any ideals $I,\ J$ of R, by [16, Lemma 1.2]. Therefore over a commutative von Neumann regular ring the two concepts of strongly 2-irreducible ideals and of 2-absorbing ideals are coincide.

Theorem 3. *Let I be a proper ideal of a ring R. Then the following conditions are equivalent:*

(1) *I is strongly 2-irreducible;*
(2) *For every elements x, y, z of R, $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq I$ implies that $(Rx + Ry) \cap (Rx + Rz) \subseteq I$ or $(Rx + Ry) \cap (Ry + Rz) \subseteq I$ or $(Rx + Rz) \cap (Ry + Rz) \subseteq I$.*

Proof. (1)⇒(2) There is nothing to prove.
(2)⇒(1) Suppose that $J,\ K$ and L are ideals of R such that neither $J \cap K \subseteq I$ nor $J \cap L \subseteq I$ nor $K \cap L \subseteq I$. Then there exist elements $x,\ y$ and z of R such that $x \in (J \cap K) \backslash I$ and $y \in (J \cap L) \backslash I$ and $z \in (K \cap L) \backslash I$. On the other hand $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq (Rx + Ry) \subseteq J$, $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq (Rx + Rz) \subseteq K$ and $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq (Ry + Rz) \subseteq L$. Hence $(Rx + Ry) \cap (Rx + Rz) \cap (Ry + Rz) \subseteq I$, and so by hypothesis either $(Rx + Ry) \cap (Rx + Rz) \subseteq I$ or $(Rx + Ry) \cap (Ry + Rz) \subseteq I$ or $(Rx + Rz) \cap (Ry + Rz) \subseteq I$. Therefore, either $x \in I$ or $y \in I$ or $z \in I$, which any of these cases has a contradiction. Consequently I is strongly 2-irreducible. □

A ring R is called a *Bézout ring* if every finitely generated ideal of R is principal. As an immediate consequence of Theorem 3 we have the next result:

Corollary 1. *Let I be a proper ideal of a Bézout ring R. Then the following conditions are equivalent:*

(1) *I is strongly 2-irreducible;*
(2) *I is singly strongly 2-irreducible;*

Now we can state the following open problem.

Problem 1. *Let I be a singly strongly 2-irreducible ideal of a ring R. Is I a strongly 2-irreducible ideal of R?*

Proposition 1. *Let R be a ring. If I is a strongly 2-irreducible ideal of R, then I is a 2-irreducible ideal of R.*

Proof. Suppose that I is a strongly 2-irreducible ideal of R. Let J, K and L be ideals of R such that $I = J \cap K \cap L$. Since $J \cap K \cap L \subseteq I$, then either $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$. On the other hand $I \subseteq J \cap K$ and $I \subseteq J \cap L$ and $I \subseteq K \cap L$. Consequently, either $I = J \cap K$ or $I = J \cap L$ or $I = K \cap L$. Therefore I is 2-irreducible. □

Remark 2. It is easy to check that the zero ideal $I = \{0\}$ of a ring R is 2-irreducible if and only if I is strongly 2-irreducible.

Proposition 2. *Let I be a proper ideal of an arithmetical ring R. The following conditions are equivalent:*

(1) *I is a 2-irreducible ideal of R;*
(2) *I is a strongly 2-irreducible ideal of R;*
(3) *For every ideals I_1 ,I_2 and I_3 of R with $I \subseteq I_1$, $I_1 \cap I_2 \cap I_3 \subseteq I$ implies that $I_1 \cap I_2 \subseteq I$ or $I_1 \cap I_3 \subseteq I$ or $I_2 \cap I_3 \subseteq I$.*

Proof. (1)⇒(2) Assume that J, K and L are ideals of R such that $J \cap K \cap L \subseteq I$. Therefore $I = I + (J \cap K \cap L) = (I + J) \cap (I + K) \cap (I + L)$, since R is an arithmetical ring. So either $I = (I + J) \cap (I + K)$ or $I = (I + J) \cap (I + L)$ or $I = (I + K) \cap (I + L)$, and thus either $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$. Hence I is a strongly 2-irreducible ideal.
(2)⇒(3) is clear.
(3)⇒(2) Let J, K and L be ideals of R such that $J \cap K \cap L \subseteq I$. Set $I_1 := J + I$, $I_2 := K$ and $I_3 := L$. Since R is an arithmetical ring, then $I_1 \cap I_2 \cap I_3 = (J + I) \cap K \cap L = (J \cap K \cap L) + (I \cap K \cap L) \subseteq I$. Hence either $I_1 \cap I_2 \subseteq I$ or $I_1 \cap I_3 \subseteq I$ or $I_2 \cap I_3 \subseteq I$ which imply that either $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$, respectively. Consequently, I is a strongly 2-irreducible ideal of R.
(2)⇒(1) By Proposition 1. □

As an immediate consequence of Theorem 2 and Proposition 2 we have the next result.

Corollary 2. *Let R be a PID and I be a nonzero proper ideal of R. The following conditions are equivalent:*

(1) *I is a strongly 2-irreducible ideal;*
(2) *I is a 2-irreducible ideal;*
(3) *I is a 2-absorbing primary ideal;*
(4) *Either $I = Rp^k$ for some prime (irreducible) element p of R and some natural number n, or $I = R(p_1^n p_2^m)$ for some distinct prime (irreducible) elements p_1, p_2 of R and some natural numbers n, m.*

The following example shows that the concepts of strongly irreducible (irreducible) ideals and of strongly 2-irreducible (2-irreducible) ideals are different in general.

Example 1. Consider the ideal $6\mathbb{Z}$ of the ring $\mathbb{Z}$. By Corollary 2, $6\mathbb{Z} = (2.3)\mathbb{Z}$ is a strongly 2-irreducible (a 2-irreducible) ideal of $\mathbb{Z}$. But, Theorem 1 says that $6\mathbb{Z}$ is not a strongly irreducible (an irreducible) ideal of $\mathbb{Z}$.

It is well known that every von Neumann regular ring is a Bézout ring. By [15, p. 119], every Bézout ring is an arithmetical ring.

Corollary 3. *Let I be a proper ideal of a von Neumann regular ring R. The following conditions are equivalent:*

(1) *I is a 2-absorbing ideal of R;*
(2) *I is a 2-irreducible ideal of R;*
(3) *I is a strongly 2-irreducible ideal of R;*
(4) *I is a singly strongly 2-irreducible of R;*
(5) *For every idempotent elements e_1, e_2, e_3 of R, $e_1e_2e_3 \in I$ implies that either $e_1e_2 \in I$ or $e_1e_3 \in I$ or $e_2e_3 \in I$.*

Proof. (1)⇔(3) By Remark 1.
(2)⇔(3) By Proposition 2.
(3)⇔(4) By Corollary 1.
(1)⇒(5) is evident.
(5)⇒(3) The proof follows from Theorem 3 and the fact that any finitely generated ideal of a von Neumann regular ring R is generated by an idempotent element. □

Proposition 3. *Let I_1, I_2 be strongly irreducible ideals of a ring R. Then $I_1 \cap I_2$ is a strongly 2-irreducible ideal of R.*

Proof. Strightforward. □

Theorem 4. *Let R be a Noetherian ring. If I is a 2-irreducible ideal of R, then either I is irreducible or I is the intersection of exactly two irreducible ideals. The converse is true when R is also arithmetical.*

Proof. Assume that I is 2-irreducible. By [20, Proposition 4.33], I can be written as a finite irredundant irreducible decomposition $I = I_1 \cap I_2 \cap \cdots \cap I_k$. We show that either $k = 1$ or $k = 2$. If $k > 3$, then since I is 2-irreducible, $I = I_i \cap I_j$ for some $1 \leq i, j \leq k$, say $i = 1$ and $j = 2$. Therefore $I_1 \cap I_2 \subseteq I_3$, which is a contradiction. For the second atatement, let R be arithmetical, and I be the intersection of two irreducible ideals. Since R is arithmetical, every irreducible ideal is strongly irreducible, [13, Lemma 2.2(3)]. Now, apply Proposition 3 to see that I is strongly 2-irreducible, and so I is 2-irreducible. □

Corollary 4. *Let R be a Noetherian ring and I be a proper ideal of R. If I is 2-irreducible, then I is a 2-absorbing primary ideal of R.*

Proof. Assume that I is 2-irreducible. By the fact that every irreducible ideal of a Noetherian ring is primary and regarding Theorem 4, we have either I is a primary ideal or is the intersection of two primary ideals. It is clear that every primary ideal is 2-absorbing primary, also the intersection of two primary ideals is a 2-absorbing primary ideal, by [6, Theorem 2.4]. □

Proposition 4. *Let R be a ring, and let P_1, P_2 and P_3 be pairwise comaximal prime ideals of R. Then $P_1P_2P_3$ is not a 2-irreducible ideal.*

Proof. The proof is easy. □

Corollary 5. *If R is a ring such that every proper ideal of R is 2-irreducible, then R has at most two maximal ideals.*

Theorem 5. *Let I be a radical ideal of a ring R, i.e., $I = \sqrt{I}$. The following conditions are equivalent:*

(1) *I is strongly 2-irreducible;*
(2) *I is 2-absorbing;*
(3) *I is 2-absorbing primary;*
(4) *I is either a prime ideal of R or is an intersection of exactly two prime ideals of R.*

Proof. (1)⇒(2) Assume that I is strongly 2-irreducible. Let J, K and L be ideals of R such that $JKL \subseteq I$. Then $J \cap K \cap L \subseteq \sqrt{J \cap K \cap L} \subseteq \sqrt{I} = I$. So, either $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$. Hence either $JK \subseteq I$ or $JL \subseteq I$ or $KL \subseteq I$. Consequently I is 2-absorbing.
(2)⇔(3) is obvious.
(2)⇒(4) If I is a 2-absorbing ideal, then either $\sqrt{I}$ is a prime ideal or is an intersection of exactly two prime ideals, [4, Theorem 2.4]. Now, we prove the claim by assumption that $I = \sqrt{I}$.
(4)⇒(1) By Proposition 3. □

Theorem 6. *Let $f : R \to S$ be a surjective homomorphism of commutative rings, and let I be an ideal of R containing $Ker(f)$. Then,*

(1) *If I is a strongly 2-irreducible ideal of R, then I^e is a strongly 2-irreducible ideal of S.*
(2) *I is a 2-irreducible ideal of R if and only if I^e is a 2-irreducible ideal of S.*

Proof. Since f is surjective, $J^{ce} = J$ for every ideal J of S. Moreover, $(K \cap L)^e = K^e \cap L^e$ and $K^{ec} = K$ for every ideals K, L of R which contain Ker(f).
(1) Suppose that I is a strongly 2-irreducible ideal of R. If $I^e = S$, then $I = I^{ec} = R$, which is a contradiction. Let J_1, J_2 and J_3 be ideals of S such that $J_1 \cap J_2 \cap J_3 \subseteq I^e$. Therefore $J_1^c \cap J_2^c \cap J_3^c \subseteq I^{ec} = I$. So, either $J_1^c \cap J_2^c \subseteq I$ or $J_1^c \cap J_3^c \subseteq I$ or $J_2^c \cap J_3^c \subseteq I$. Without loss of generality, we may assume that $J_1^c \cap J_2^c \subseteq I$. So, $J_1 \cap J_2 = (J_1 \cap J_2)^{ce} \subseteq I^e$. Hence I^e is strongly 2-irreducible.

(2) The necessity is similar to part (1). Conversely, let I^e be a strongly 2-irreducible ideal of S, and let $I_1,\ I_2$ and I_3 be ideals of R such that $I = I_1 \cap I_2 \cap I_3$. Then $I^e = I_1^e \cap I_2^e \cap I_3^e$. Hence, either $I^e = I_1^e \cap I_2^e$ or $I^e = I_1^e \cap I_3^e$ or $I^e = I_2^e \cap I_3^e$. We may assume that $I^e = I_1^e \cap I_2^e$. Therefore, $I = I^{ec} = I_1^{ec} \cap I_2^{ec} = I_1 \cap I_2$. Consequently, I is strongly 2-irreducible. □

Corollary 6. *Let $f : R \to S$ be a surjective homomorphism of commutative rings. There is a one-to-one correspondence between the 2-irreducible ideals of R which contain $Ker(f)$ and 2-irreducible ideals of S.*

Recall that a ring R is called a *Laskerian ring* if every proper ideal of R has a primary decomposition. Noetherian rings are some examples of Laskerian rings.

Let S be a multiplicatively closed subset of a ring R. In the next theorem, consider the natural homomorphism $f : R \to S^{-1}R$ defined by $f(x) = x/1$.

Theorem 7. *Let I be a proper ideal of a ring R and S be a multiplicatively closed set in R.*

(1) *If I is a strongly 2-irreducible ideal of $S^{-1}R$, then I^c is a strongly 2-irreducible ideal of R.*
(2) *If I is a primary strongly 2-irreducible ideal of R such that $I \cap S = \varnothing$, then I^e is a strongly 2-irreducible ideal of $S^{-1}R$.*
(3) *If I is a primary ideal of R such that I^e is a strongly 2-irreducible ideal of $S^{-1}R$, then I is a strongly 2-irreducible ideal of R.*
(4) *If R' is a faithfully flat extension ring of R and if IR' is a strongly 2-irreducible ideal of R', then I is a strongly 2-irreducible ideal of R.*
(5) *If I is strongly 2-irreducible and H is an ideal of R such that $H \subseteq I$, then I/H is a strongly 2-irreducible ideal of R/H.*
(6) *If R is a Laskerian ring, then every strongly 2-irreducible ideal is either a primary ideal or is the intersection of two primary ideals.*

Proof. (1) Assume that I is a strongly 2-irreducible ideal of $S^{-1}R$. Let $J,\ K$ and L be ideals of R such that $J \cap K \cap L \subseteq I^c$. Then $J^e \cap K^e \cap L^e \subseteq I^{ce} = I$. Hence either $J^e \cap K^e \subseteq I$ or $J^e \cap L^e \subseteq I$ or $K^e \cap L^e \subseteq I$ since I is strongly 2-irreducible. Therefore either $J \cap K \subseteq I^c$ or $J \cap L \subseteq I^c$ or $K \cap L \subseteq I^c$. Consequently I^c is a strongly 2-irreducible ideal of R.
(2) Suppose that I is a primary strongly 2-irreducible ideal such that $I \cap S = \varnothing$. Let $J,\ K$ and L be ideals of $S^{-1}R$ such that $J \cap K \cap L \subseteq I^e$. Since I is a primary ideal, then $J^c \cap K^c \cap L^c \subseteq I^{ec} = I$. Thus $J^c \cap K^c \subseteq I$ or $J^c \cap L^c \subseteq I$ or $K^c \cap L^c \subseteq I$. Hence $J \cap K \subseteq I^e$ or $J \cap L \subseteq I^e$ or $K \cap L \subseteq I^e$.
(3) Let I be a primary ideal of R, and let I^e be a strongly 2-irreducible ideal of $S^{-1}R$. By part (1), I^{ec} is strongly 2-irreducible. Since I is primary, we have $I^{ec} = I$, and thus we are done.
(4) Let $J,\ K$ and L be ideals of R such that $J \cap K \cap L \subseteq I$. Thus $JR' \cap KR' \cap LR' = (J \cap K \cap L)R' \subseteq IR'$, by [12, Lemma 9.9]. Since IR' is strongly 2-irreducible, then

either $JR' \cap KR' \subseteq IR'$ or $JR' \cap LR' \subseteq IR'$ or $KR' \cap LR' \subseteq IR'$. Without loss of generality, assume that $JR' \cap KR' \subseteq IR'$. So, $(JR' \cap R) \cap (KR' \cap R) \subseteq IR' \cap R$. Hence $J \cap K \subseteq I$, by [17, Theorem 4.74]. Consequently I is strongly 2-irreducible.
(5) Let J, K and L be ideals of R containing H such that $(J/H) \cap (K/H) \cap (L/H) \subseteq I/H$. Hence $J \cap K \cap L \subseteq I$. Therefore, either $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$. Thus, $(J/H) \cap (K/H) \subseteq I/H$ or $(J/H) \cap (L/H) \subseteq I/H$ or $(K/H) \cap (L/H) \subseteq I/H$. Consequently, I/H is strongly 2-irreducible.
(6) Let I be a strongly 2-irreducible ideal and $\cap_{i=1}^{n} Q_i$ be a primary decomposition of I. Since $\cap_{i=1}^{n} Q_i \subseteq I$, then there are $1 \leq r, s \leq n$ such that $Q_r \cap Q_s \subseteq I = \cap_{i=1}^{n} Q_i \subseteq Q_r \cap Q_s$. □

Let S be a multiplicatively closed subset of a ring R. Set

$$C := \{I^c \mid I \text{ is an ideal of } R_S\}.$$

Corollary 7. *Let R be a ring and S be a multiplicatively closed subset of R. Then there is a one-to-one correspondence between the strongly 2-irreducible ideals of R_S and strongly 2-irreducible ideals of R contained in C which do not meet S.*

Proof. If I is a strongly 2-irreducible ideal of R_S, then evidently $I^c \neq R$, $I^c \in C$ and by Theorem 7(1), I^c is a strongly 2-irreducible ideal of R. Conversely, let I be a strongly 2-irreducible ideal of R, $I \cap S = \varnothing$ and $I \in C$. Since $I \cap S = \varnothing$, $I^e \neq R_S$. Let $J \cap K \cap L \subseteq I^e$ where J, K and L are ideals of R_S. Then $J^c \cap K^c \cap L^c = (J \cap K \cap L)^c \subseteq I^{ec}$. Now since $I \in C$, then $I^{ec} = I$. So $J^c \cap K^c \cap L^c \subseteq I$. Hence, either $J^c \cap K^c \subseteq I$ or $J^c \cap L^c \subseteq I$ or $K^c \cap L^c \subseteq I$. Then, either $J \cap K = (J \cap K)^{ce} \subseteq I^e$ or $J \cap L = (J \cap L)^{ce} \subseteq I^e$ or $K \cap L = (K \cap L)^{ce} \subseteq I^e$. Consequently, I^e is a strongly 2-irreducible ideal of R_S. □

Let n be a natural number. We say that I is an *n-primary ideal* of a ring R if I is the intersection of n primary ideals of R.

Proposition 5. *Let R be a ring. Then the following conditions are equivalent:*

(1) *Every n-primary ideal of R is a strongly 2-irreducible ideal;*
(2) *For any prime ideal P of R, every n-primary ideal of R_P is a strongly 2-irreducible ideal;*
(3) *For any maximal ideal m of R, every n-primary ideal of R_m is a strongly 2-irreducible ideal.*

Proof. (1)⇒(2) Let I be an n-primary ideal of R_P. We know that I^c is an n-primary ideal of R, $I^c \cap (R \backslash P) = \varnothing$, $I^c \in C$ and, by the assumption, I^c is a strongly 2-irreducible ideal of R. Now, by Corollary 7, $I = (I^c)_P$ is a strongly 2-irreducible ideal of R_P.
(2)⇒(3) is clear.
(3)⇒(1) Let I be an n-primary ideal of R and let m be a maximal ideal of R containing I . Then, I_m is an n-primary ideal of R_m and so, by our assumption, I_m is

a strongly 2-irreducible ideal of R_m. Now by Theorem 10(1), $(I_m)^c$ is a strongly 2-irreducible ideal of R, and since I is an n-primary ideal of R, $(I_m)^c = I$, that is, I is a strongly 2-irreducible ideal of R. □

Theorem 8. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings with $1 \neq 0$. Let J be a proper ideal of R. Then the following conditions are equivalent:*

(1) *J is a strongly 2-irreducible ideal of R;*
(2) *Either $J = I_1 \times R_2$ for some strongly 2-irreducible ideal I_1 of R_1 or $J = R_1 \times I_2$ for some strongly 2-irreducible ideal I_2 of R_2 or $J = I_1 \times I_2$ for some strongly irreducible ideal I_1 of R_1 and some strongly irreducible ideal I_2 of R_2.*

Proof. (1)⇒(2) Assume that J is a strongly 2-irreducible ideal of R. Then $J = I_1 \times I_2$ for some ideal I_1 of R_1 and some ideal I_2 of R_2. Suppose that $I_2 = R_2$. Since J is a proper ideal of R, $I_1 \neq R_1$. Let $R' = \frac{R}{\{0\}\times R_2}$. Then $J' = \frac{J}{\{0\}\times R_2}$ is a strongly 2-irreducible ideal of R' by Theorem 7(5). Since R' is ring-isomorphic to R_1 and $I_1 \simeq J'$, I_1 is a strongly 2-irreducible ideal of R_1. Suppose that $I_1 = R_1$. Since J is a proper ideal of R, $I_2 \neq R_2$. By a similar argument as in the previous case, I_2 is a strongly 2-irreducible ideal of R_2. Hence assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Suppose that I_1 is not a strongly irreducible ideal of R_1. Then there are $x,\ y \in R_1$ such that $R_1x \cap R_1y \subseteq I_1$ but neither $x \in I_1$ nor $y \in I_1$. Notice that $(R_1x \times R_2) \cap (R_1 \times \{0\}) \cap (R_1y \times R_2) = (R_1x \cap R_1y) \times \{0\} \subseteq J$, but neither $(R_1x \times R_2) \cap (R_1 \times \{0\}) = R_1x \times \{0\} \subseteq J$ nor $(R_1x \times R_2) \cap (R_1y \times R_2) = (R_1x \cap R_1y) \times R_2 \subseteq J$ nor $(R_1 \times \{0\}) \cap (R_1y \times R_2) = R_1y \times \{0\} \subseteq J$, which is a contradiction. Thus I_1 is a strongly irreducible ideal of R_1. Suppose that I_2 is not a strongly irreducible ideal of R_2. Then there are $z,\ w \in R_2$ such that $R_2z \cap R_2w \subseteq I_2$ but neither $z \in I_2$ nor $w \in I_2$. Notice that $(R_1 \times R_2z) \cap (\{0\} \times R_2) \cap (R_1 \times R_2w) = \{0\} \times (R_2z \cap R_2w) \subseteq J$, but neither $(R_1 \times R_2z) \cap (\{0\} \times R_2) = \{0\} \times R_2z \subseteq J$, nor $(R_1 \times R_2z) \cap (R_1 \times R_2w) = R_1 \times (R_2z \cap R_2w) \subseteq J$ nor $(\{0\} \times R_2) \cap (R_1 \times R_2w) = \{0\} \times R_2w \subseteq J$, which is a contradiction. Thus I_2 is a strongly irreducible ideal of R_2.
(2)⇒(1) If $J = I_1 \times R_2$ for some strongly 2-irreducible ideal I_1 of R_1 or $J = R_1 \times I_2$ for some strongly 2-irreducible ideal I_2 of R_2, then it is clear that J is a strongly 2-irreducible ideal of R. Hence assume that $J = I_1 \times I_2$ for some strongly irreducible ideal I_1 of R_1 and some strongly irreducible ideal I_2 of R_2. Then $I_1' = I_1 \times R_2$ and $I_2' = R_1 \times I_2$ are strongly irreducible ideals of R. Hence $I_1' \cap I_2' = I_1 \times I_2 = J$ is a strongly 2-irreducible ideal of R by Proposition 3. □

Theorem 9. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $2 \leq n < \infty$, and $R_1, R_2, ..., R_n$ are rings with $1 \neq 0$. Let J be a proper ideal of R. Then the following conditions are equivalent:*

(1) *J is a strongly 2-irreducible ideal of R.*
(2) *Either $J = \times_{t=1}^{n} I_t$ such that for some $k \in \{1, 2, ..., n\}$, I_k is a strongly 2-irreducible ideal of R_k, and $I_t = R_t$ for every $t \in \{1, 2, ..., n\}\backslash\{k\}$ or $J =$*

$\times_{t=1}^{n} I_t$ such that for some $k,m \in \{1,2,...,n\}$, I_k is a strongly irreducible ideal of R_k, I_m is a strongly irreducible ideal of R_m, and $I_t = R_t$ for every $t \in \{1,2,...,n\} \backslash \{k,m\}$.

Proof. We use induction on n. Assume that $n = 2$. Then the result is valid by Theorem 8. Thus let $3 \leq n < \infty$ and assume that the result is valid when $K = R_1 \times \cdots \times R_{n-1}$. We prove the result when $R = K \times R_n$. By Theorem 8, J is a strongly 2-irreducible ideal of R if and only if either $J = L \times R_n$ for some strongly 2-irreducible ideal L of K or $J = K \times L_n$ for some strongly 2-irreducible ideal L_n of R_n or $J = L \times L_n$ for some strongly irreducible ideal L of K and some strongly irreducible ideal L_n of R_n. Observe that a proper ideal Q of K is a strongly irreducible ideal of K if and only if $Q = \times_{t=1}^{n-1} I_t$ such that for some $k \in \{1,2,...,n-1\}$, I_k is a strongly irreducible ideal of R_k, and $I_t = R_t$ for every $t \in \{1,2,...,n-1\} \backslash \{k\}$. Thus the claim is now verified. □

Lemma 1. *Let R be a GCD-domain and I be a proper ideal of R. The following conditions are equivalent:*

(1) *I is a singly strongly 2-irreducible ideal;*
(2) *For every elements $x,y,z \in R$, $[x,y,z] \in I$ implies that $[x,y] \in I$ or $[x,z] \in I$ or $[y,z] \in I$.*

Proof. Since for every elements x, y of R we have $Rx \cap Ry = R[x,y]$, there is nothing to prove. □

Now we study singly strongly 2-irreducible ideals of a UFD.

Theorem 10. *Let R be a UFD, and let I be a proper ideal of R. Then the following conditions hold:*

(1) *I is singly strongly 2-irreducible if and only if for each elements x,y,z of R, $[x,y,z] \in I$ implies that either $[x,y] \in I$ or $[x,z] \in I$ or $[y,z] \in I$.*
(2) *I is singly strongly 2-irreducible if and only if $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$, where p_i's are distinct prime elements of R and n_i's are natural numbers, implies that $p_r^{n_r} p_s^{n_s} \in I$, for some $1 \leq r,s \leq k$.*
(3) *If I is a nonzero principal ideal, then I is singly strongly 2-irreducible if and only if the generator of I is a prime power or the product of two prime powers.*
(4) *Every singly strongly 2-irreducible ideal is a 2-absorbing primary ideal.*

Proof. (1) By Lemma 1.
(2) Suppose that I is singly strongly 2-irreducible and $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$ in which p_i's are distinct prime elements of R and n_i's are natural numbers. Then $[p_1^{n_1}, p_2^{n_2}, \ldots, p_k^{n_k}] = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$. Hence by part (1), there are $1 \leq r,s \leq k$ such that $[p_r^{n_r}, p_s^{n_s}] \in I$, i.e., $p_r^{n_r} p_s^{n_s} \in I$.

For the converse, let $[x,y,z]\in I$ for some $x,y,z\in R\backslash\{0\}$. Assume that x, y and z have prime decompositions as below,

$$x=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s},$$
$$y=p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_k^{\gamma_k}r_1^{\delta_1}r_2^{\delta_2}\cdots r_u^{\delta_u},$$
$$z=p_1^{\varepsilon_1}p_2^{\varepsilon_2}\cdots p_{k'}^{\varepsilon_{k'}}q_1^{\lambda_1}q_2^{\lambda_2}\cdots q_{s'}^{\lambda_{s'}}r_1^{\mu_1}r_2^{\mu_2}\cdots r_{u'}^{\mu_{u'}}s_1^{\kappa_1}s_2^{\kappa_2}\cdots s_v^{\kappa_v},$$

in which $0\le k'\le k$, $0\le s'\le s$ and $0\le u'\le u$. Therefore,

$$[x,y,z]=p_1^{\nu_1}p_2^{\nu_2}\cdots p_{k'}^{\nu_{k'}}p_{k'+1}^{\omega_{k'+1}}\cdots p_k^{\omega_k}q_1^{\rho_1}q_2^{\rho_2}\cdots q_{s'}^{\rho_{s'}}$$
$$q_{s'+1}^{\beta_{s'+1}}\cdots q_s^{\beta_s}r_1^{\sigma_1}r_2^{\sigma_2}\cdots r_{u'}^{\sigma_{u'}}r_{u'+1}^{\delta_{u'+1}}\cdots r_u^{\delta_u}s_1^{\kappa_1}s_2^{\kappa_2}\cdots s_v^{\kappa_v}\in I,$$

where $\nu_i=max\{\alpha_i,\gamma_i,\varepsilon_i\}$ for every $1\le i\le k'$; $\omega_j=max\{\alpha_j,\gamma_j\}$ for every $k'<j\le k$; $\rho_i=max\{\beta_i,\lambda_i\}$ for every $1\le i\le s'$; $\sigma_i=max\{\delta_i,\mu_i\}$ for every $1\le i\le u'$. By part (2), we have twenty one cases. For example we investigate the following two cases. The other cases can be verified in a similar way.

Case 1. For some $1\le i,j\le k'$, $p_i^{\nu_i}p_j^{\nu_j}\in I$. If $\nu_i=\alpha_i$ and $\nu_j=\alpha_j$, then clearly $x\in I$ and so $[x,y]\in I$. If $\nu_i=\alpha_i$ and $\nu_j=\gamma_j$, then $p_i^{\alpha_i}p_j^{\gamma_j}\mid[x,y]$ and thus $[x,y]\in I$. If $\nu_i=\alpha_i$ and $\nu_j=\varepsilon_j$, then $p_i^{\alpha_i}p_j^{\varepsilon_j}\mid[x,z]$ and thus $[x,z]\in I$.

Case 2. Let $p_i^{\nu_i}p_j^{\omega_j}\in I$; for some $1\le i\le k'$ and $k'+1\le j\le k$. For $\nu_i=\alpha_i$, $\omega_j=\alpha_j$ we have $x\in I$ and so $[x,y]\in I$. For $\nu_i=\varepsilon_i$, $\omega_j=\gamma_j$ we have $[y,z]\in I$. Consequently I is singly strongly 2-irreducible, by part (1).

(3) Suppose that $I=Ra$ for some nonzero element $a\in R$. Assume that I is singly strongly 2-irreducible. Let $a=p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ be a prime decomposition for a such that $k>2$. By part (2) we have that $p_r^{n_r}p_s^{n_s}\in I$ for some $1\le r,s\le k$. Therefore $I=R(p_r^{n_r}p_s^{n_s})$.

Conversely, if a is a prime power, then I is strongly irreducible ideal, by [3, Theorem 2.2(3)]. Hence I is singly strongly 2-irreducible. Let $I=R(p^rq^s)$ for some prime elements p, q of R. Assume that for some distinct prime elements $q_1,q_2,...,q_k$ of R and natural numbers $m_1,m_2,...,m_k$, $q_1^{m_1}q_2^{m_2}\cdots q_k^{m_k}\in I=R(p^rq^s)$. Then $p^rq^s\mid q_1^{m_1}q_2^{m_2}\cdots q_k^{m_k}$. Hence there exists $1\le i\le k$ such that $p=q_i$ and $r\le m_i$, also there exists $1\le j\le k$ such that $q=q_j$ and $s\le m_j$. Then, since $p^rq^s\in I$, we have $q_i^{m_i}q_j^{m_j}\in I$. Now, by part (2), I is singly strongly 2-irreducible.

(4) Let I be singly strongly 2-irreducible and $xyz\in I$ for some $x,y,z\in R\backslash\{0\}$. Consider the following prime decompositions,

$$x=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s},$$
$$y=p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_k^{\gamma_k}r_1^{\delta_1}r_2^{\delta_2}\cdots r_u^{\delta_u},$$
$$z=p_1^{\varepsilon_1}p_2^{\varepsilon_2}\cdots p_{k'}^{\varepsilon_{k'}}q_1^{\lambda_1}q_2^{\lambda_2}\cdots q_{s'}^{\lambda_{s'}}r_1^{\mu_1}r_2^{\mu_2}\cdots r_{u'}^{\mu_{u'}}s_1^{\kappa_1}s_2^{\kappa_2}\cdots s_v^{\kappa_v},$$

in which $0\le k'\le k$, $0\le s'\le s$ and $0\le u'\le u$. By these representations we have,

$$xyz = p_1^{\alpha_1+\gamma_1+\varepsilon_1} p_2^{\alpha_2+\gamma_2+\varepsilon_2} \cdots p_{k'}^{\alpha_{k'}+\gamma_{k'}+\varepsilon_{k'}} p_{k'+1}^{\alpha_{k'+1}+\gamma_{k'+1}} \cdots p_k^{\alpha_k+\gamma_k} q_1^{\beta_1+\lambda_1} q_2^{\beta_2+\lambda_2} \cdots q_{s'}^{\beta_{s'}+\lambda_{s'}} q_{s'+1}^{\beta_{s'+1}} \cdots q_s^{\beta_s} r_1^{\delta_1+\mu_1} r_2^{\delta_2+\mu_2} \cdots r_{u'}^{\delta_{u'}+\mu_{u'}} r_{u'+1}^{\delta_{u'+1}} \cdots r_u^{\delta_u} s_1^{\kappa_1} s_2^{\kappa_2} \cdots s_v^{\kappa_v} \in I.$$

Now, apply part (2). We investigate some cases that can be happened, the other cases similarly lead us to the claim that I is 2-absorbing primary. First, assume for some $1 \leq i, j \leq k'$, $p_i^{\alpha_i+\gamma_i+\varepsilon_i} p_j^{\alpha_j+\gamma_j+\varepsilon_j} \in I$. Choose a natural number n such that $n \geq max\{\frac{\alpha_i+\gamma_i}{\varepsilon_i}, \frac{\alpha_j+\gamma_j}{\varepsilon_j}\}$. With this choice we have $(n+1)\varepsilon_i \geq \alpha_i + \gamma_i + \varepsilon_i$ and $(n+1)\varepsilon_j \geq \alpha_j + \gamma_j + \varepsilon_j$, so $p_i^{(n+1)\varepsilon_i} p_j^{(n+1)\varepsilon_j} \in I$. Then $z^{n+1} \in I$, so $z \in \sqrt{I}$. The other one case; assume that for some $1 \leq i \leq k'$ and $k'+1 \leq j \leq k$, $p_i^{\alpha_i+\gamma_i+\varepsilon_i} p_j^{\alpha_j+\gamma_j} \in I$. Choose a natural number n such that $n \geq max\{\frac{\alpha_i+\varepsilon_i}{\gamma_i}, \frac{\alpha_j}{\gamma_j}\}$. With this choice we have $(n+1)\gamma_i \geq \alpha_i + \gamma_i + \varepsilon_i$ and $(n+1)\gamma_j \geq \alpha_j + \gamma_j$, thus $p_i^{(n+1)\gamma_i} p_j^{(n+1)\gamma_j} \in I$. Then $y^{n+1} \in I$, so $y \in \sqrt{I}$. Assume that $p_i^{\alpha_i+\gamma_i} s_j^{\kappa_j} \in I$, for some $k'+1 \leq i \leq k$ and some $1 \leq j \leq v$. Let n be a natural number where $n \geq \frac{\gamma_i}{\alpha_i}$, then $(n+1)\alpha_i \geq \alpha_i + \gamma_i$. Hence $p_i^{(n+1)\alpha_i} s_j^{(n+1)\kappa_j} \in I$ which shows that $xz \in \sqrt{I}$. Suppose that for some $s'+1 \leq i \leq s$ and $u'+1 \leq j \leq u$, $q_i^{\beta_i} r_j^{\delta_j} \in I$. Then, clearly $xy \in I$. □

Corollary 8. *Let R be a UFD.*

1. *Every principal ideal of R is a singly strongly 2-irreducible ideal if and only if it is a 2-absorbing primary ideal.*
2. *Every singly strongly 2-irreducible ideal of R can be generated by a set of elements of the forms p^n and $p_i^{n_i} p_j^{n_j}$ in which p, p_i, p_j are some prime elements of R and n, n_i, n_j are some natural numbers.*
3. *Every 2-absorbing ideal of R is a singly strongly 2-irreducible ideal.*

Proof. (1) Suppose that I is singly strongly 2-irreducible ideal. By Theorem 10(4), I is a 2-absorbing primary ideal. Conversely, let I be a nonzero 2-absorbing primary ideal. Let $I = Ra$, where $0 \neq a \in I$. Assume that $a = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be a prime decomposition for a. If $k > 2$, then since $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \in I$ and I is a 2-absorbing primary ideal, there exist a natural number n, and integers $1 \leq i, j \leq k$ such that $p_i^{nn_i} p_j^{nn_j} \in I$, say $i = 1$ and $j = 2$. Therefore $p_3 \mid p_1^{nn_1} p_2^{nn_2}$ which is a contradiction. Therefore $k = 1$ or 2, that is $I = Rp_1^{n_1}$ or $I = R(p_1^{n_1} p_2^{n_2})$, respectively. Hence by Theorem 10(3), I is singly strongly 2-irreducible.
(2) Let X be a generator set for a singly strongly 2-irreducible ideal of I, and let x be a nonzero element of X. Assume that $x = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be a prime decomposition for x such that $k \geq 2$. By Theorem 10(2), for some $1 \leq i, j \leq k$, we have $p_i^{n_i} p_j^{n_j} \in I$, and then $Rx \subseteq Rp_i^{n_i} p_j^{n_j} \subseteq I$. Consequently, I can be generated by a set of elements

of the forms p^n and $p_i^{n_i} p_j^{n_j}$.
(3) is a direct consequence of Theorem 10(2). □

The following example shows that in part (1) of Corollary 8 the condition that I is principal is necessary. Moreover, the converse of part (2) of this corollary need not be true.

Example 2. Let F be a field and $R = F[x, y, z]$, where $x,\ y$ and z are independent indeterminates. We know that R is a UFD. Suppose that $I = \langle x, y^2, z^2 \rangle$. Since $\sqrt{\langle x, y^2, z^2 \rangle} = \langle x, y, z \rangle$ is a maximal ideal of R, I is a primary ideal and so is a 2-absorbing primary ideal. Notice that $(x + y + z)yz \in I$, but neither $(x + y + z)y \in I$ nor $(x + y + z)z \in I$ nor $yz \in I$. Consequently, I is not singly strongly 2-irreducible, by Theorem 10(2).

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Authors' addresses

H. Mostafanasab

University of Mohaghegh Ardabili, Department of Mathematics and Applications, P. O. Box 179, Ardabil, Iran

E-mail address: h.mostafanasab@uma.ac.ir; h.mostafanasab@gmail.com

A. Yousefian Darani

University of Mohaghegh Ardabili, Department of Mathematics and Applications, P. O. Box 179, Ardabil, Iran

E-mail address: yousefian@uma.ac.ir, youseffian@gmail.com