A NOTE ON LATTICE VARIANT OF THRESHOLDNESS OF BOOLEAN FUNCTIONS

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Abstract. Lattice induced threshold function is a Boolean function determined by a particular linear combination of lattice elements. We prove that every isotone Boolean function is a lattice induced threshold function and vice versa. We give the generalization of this result to Boolean functions on a k-element set.

1. INTRODUCTION

„Threshold functions provide a simple but fundamental model for many questions investigated in electrical engineering, artificial intelligence, game theory and many other areas." (Quotation from [3].) In [10], modeling neurons and political decisions are also mentioned, as application of classical threshold functions. A classical **threshold function** is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that there exist real numbers $w_1, \ldots, w_n, t$, fulfilling

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} w_i \cdot x_i \geq t, \tag{1.1}$$

where w_i is called the **weight** of x_i, for $i = 1, 2, \ldots, n$ and t is a constant called the **threshold value**. In this paper we define a new but related notion: the so called *lattice induced threshold function* and we investigate its properties.

1.1. *Historical background*

For the historical as well as for the basic mathematical background we recommend the books [3, 5, 10, 11]. For some algebraic aspects of classical threshold functions, we list relevant algebraic papers from several areas of algebra. In [1] the authors reveal the connection between classical threshold functions and fundamental ideals of group-rings. Paper [8] determines the invariance groups of threshold functions. The topic of monotone Boolean functions has constantly been investigated. For the recent results see e.g. [2, 6, 9]. Paper [7] proves that classical threshold functions cannot be characterized by a finite set of standard or generalized constraints.

1.2. *Motivation*

Isotone Boolean functions constitute a clone; threshold functions are not closed under superposition, see Theorem 9.2 in [3], so they do not constitute a clone. It is easy to see that threshold functions with positive weights and a threshold value are isotone. However, an isotone Boolean function is not necessarily threshold, e.g. $f = x \cdot y \vee w \cdot z$ is isotone, but not a threshold function. To see this, it is enough to consider its invariance group, which is the following:

$$D8 = \{(), (1324), (12)(34), (1423), (12), (34), (13)(24), (14)(23)\},$$

however by [8] the invariance group of any threshold function is a direct product of symmetric groups.

Therefore, our aim is to generalize threshold functions in the framework of lattice valued functions, and then, in particular, to obtain a characterization of all isotone Boolean functions and to represent them by particular linear combinations.

1.3. *Outline*

In Preliminaries, we present properties of lattice valued functions, Boolean functions and classical threshold functions.

In Section 3 we define threshold functions induced by complete lattices; these are Boolean functions determined by a particular linear combination of lattice elements. We prove that every isotone Boolean function is a lattice induced threshold function and vice versa. In Section 4 we give a generalization of results for isotone functions on k-element set. In Section 5 we present some examples and further observations.

2. PRELIMINARIES

2.1. *Order, lattices*

Our basic notion is a partially ordered set, **ordered set, poset**, denoted by $(P, \leq)$, where $\leq$ is an order on a set P. If a poset is a **lattice**, then it is denoted by $(L, \leq)$, with the meet and the join of $a, b \in L$ being $a \wedge b$ and $a \vee b$ respectively. The bottom and the top of a poset, if they exist, are 0 and 1, respectively. If $(P, \leq)$ is a poset,

then we denote by $\bigwedge M$ and $\bigvee M$ the meet and the join of $M \subseteq P$ respectively, if they exist. An **up-set** (**semi-filter**) on a poset $(P, \leq)$ is a subset F of P fulfilling the property: if $x \in F$ and $x \leq y$, then $y \in F$.

We deal with **complete lattices** and **free distributive lattices with n generators**. We also use finite **Boolean lattices**, represented by all n-tuples of 0 and 1, ordered componentwise, and denoted by $(\{0, 1\}^n, \leq)$.

2.2. *Boolean functions; threshold functions*

A **Boolean function** is a mapping $f : \{0, 1\}^n \to \{0, 1\}, n \in \mathbb{N}$.

The domain $\{0, 1\}^n$ of a Boolean function is usually ordered componentwise, with respect to the natural order $0 \leq 1 : (x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ if and only if for every $i \in \{1, \ldots, n\}$, $x_i \leq y_i$.

As it is known, the poset $(\{0, 1\}^n, \leq)$ is a Boolean lattice. Moreover, every finite Boolean lattice with n atoms is isomorphic to this one.

A Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is **isotone** (order preserving, positive, as in [3]), if for every $x, y \in \{0, 1\}^n$, from $x \leq y$, it follows that $f(x) \leq f(y)$.

The following is easy to check.

Lemma 1. *The set $F \subseteq \{0, 1\}^n$ is an up-set on $(\{0, 1\}^n, \leq)$ if and only if a Boolean function f defined by*

$$f(x) = 1 \ \text{ if and only if } \ x \in F$$

is isotone.

As mentioned in Introduction, a **threshold function** is a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ such that there exist real numbers $w_1, \ldots, w_n, t$, fulfilling

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^n w_i \cdot x_i \geq t,$$

where w_i are called the **weights** of x_i, for $i = 1, 2, \ldots, n$ and t is a constant called the **threshold value**.

3. THRESHOLD FUNCTIONS INDUCED BY COMPLETE LATTICES

In this section we introduce threshold functions induced by complete lattices, and we use them for investigating isotone Boolean functions and their representation.

We deal with Boolean functions over the Boolean lattice $(\{0, 1\}^n, \leq)$, and in addition we use a complete lattice L in which the bottom and the top are (also) denoted by 0 and 1 respectively; however, it is clear from the context whether 0 (1) is a component in some $(x_1, \ldots, x_n) \in \{0, 1\}^n$, or it is from L.

For $x \in \{0, 1\}$, and $w \in L$, we define a mapping $L \times \{0, 1\}$ into L denoted by "$\cdot$", as follows:

$$w \cdot x := \begin{cases} w, & \text{if} \quad x = 1 \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{3.1}$$

A function $f : \{0, 1\}^n \to \{0, 1\}$ is a **lattice induced threshold function**, if there is a complete lattice L and $w_1, \ldots, w_n, t \in L$, such that

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t. \tag{3.2}$$

Proposition 1. *Every lattice induced threshold function is isotone.*

Proof. Let L be a complete lattice, let $w_1, \ldots, w_n, t \in L$, and $f : \{0, 1\}^n \to \{0, 1\}$ a lattice induced threshold function.

Let $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$. Then, for every i, we have $w_i \cdot x_i \leq w_i \cdot y_i$, by the definition (3.1). Hence,

$$\bigvee_{i=1}^{n} (w_i \cdot x_i) \leq \bigvee_{i=1}^{n} (w_i \cdot y_i).$$

Therefore, if $f(x_1, \ldots, x_n) = 1$, then

$$\bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t, \text{ and hence } \bigvee_{i=1}^{n} (w_i \cdot y_i) \geq t.$$

This implies $f(y_1, \ldots, y_n) = 1$ and we obtain

$$f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n),$$

which proves that f is an isotone function. $\square$

Theorem 1. *Every isotone Boolean function is a lattice induced threshold function.*

Proof. We prove that for every $n \in \mathbb{N}$, there is a lattice L such that every isotone Boolean function is a lattice induced threshold function over L.

Let $n \in \mathbb{N}$. We take L to be a free distributive lattice with n generators $w_1, w_2, \ldots, w_n$ (the join and meet of empty set of generators are also counted here, as the bottom and the top of L, respectively). Recall that every element in a free distributive lattice can be uniquely represented in a "conjunctive normal form" by means of generators (i.e., every element is a meet of elements of the type $\bigvee_{i \in J} w_j$, where $J \subseteq \{1, \ldots, n\}$,) see e.g., [4].

Therefore, for $x, y \in L$, if $x = \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j$ and $y = \bigwedge_{k=1}^{l} \bigvee_{s \in J_k} w_s$, then $x \leq y$ if and only if for every $u \in \{1, \ldots, l\}$ there is $k \in \{1, \ldots, p\}$ such that $I_k \subseteq J_u$. (*)

Let $f : \{0,1\}^n \to \{0,1\}$ be an isotone Boolean function. Let F be the corresponding up-set on $\{0,1\}^n$ (according to Lemma 1). Further, let $m_1,\ldots,m_p$ be minimal elements of this up-set. Let $I_1,\ldots,I_p$ be subsets of $\{1,2,\ldots,n\}$, i.e., sets of indices, such that $i \in I_k$ if and only if i-th coordinate of m_k is equal to 1.

For the threshold $t \in L$ associated to the given function f we take

$$t = \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j.$$

Now, we prove that

$$f(x_1,\ldots,x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t. \qquad (3.3)$$

Indeed, from $f(x_1,\ldots,x_n) = 1$, it follows that there is a minimal element m_l in the corresponding up-set, such that $(x_1,\ldots,x_n) \geq m_l$. Hence,

$$\bigvee_{i=1}^{n} (w_i \cdot x_i) \geq \bigvee_{j \in I_l} w_j \geq \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j = t.$$

Now we suppose that

$$\bigvee_{i=1}^{n} (w_i \cdot x_i) \geq \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j$$

for an ordered n-tuple $(x_1,\ldots,x_n)$. Let $I \subseteq \{1,\ldots n\}$ be the set of indices such that $x_i = 1$ if and only if $i \in I$. We prove that there is $s \in \{1,\ldots p\}$ such that $I_s \subseteq I$. This follows directly from the above mentioned (*) property of the free distributive lattice with n generators $w_1,\ldots,w_n$.

Now, it follows that $(x_1,\ldots,x_n) \geq (y_1,\ldots,y_n)$, where $y_i = 1$ if and only if $i \in I_s$. Therefore,

$$\bigvee_{i=1}^{n} (w_i \cdot y_i) \geq t, \text{ and } f(y_1,\ldots,y_n) = 1.$$

This finally implies $f(x_1,\ldots,x_n) = 1$. $\qquad \square$

Remark 1. In the previous proposition it is proved not only that every isotone n-ary Boolean function is a lattice induced threshold function, but also that the corresponding lattice in each case can be the free distributive lattice with n generators. $\square$

Remark 2. If we want to induce **all** isotone function with **one** distributive lattice, then clearly no lattice with smaller cardinality than the free distributive lattice with n generators is appropriate, since different elements of the free distributive lattice provide different isotone functions. $\square$

Here we investigate lattice valued functions on finite Boolean lattices.

3.1. *Representation of isotone lattice valued Boolean functions by cuts*

A function $f : \{0,1\}^n \to L$, where L is a complete lattice, is a **lattice valued** (**L-valued**) **Boolean function**.

For $f : \{0,1\}^n \to L$ and $p \in L$, a **cut set (cut)** f_p is a subset of $\{0,1\}^n$:

$$f_p = \{x \in \{0,1\}^n \mid f(x) \geq p\}.$$

A characteristic function of the cut f_p is an ordinary Boolean function and we denote it by the same symbol f_p.

The following lemma is straightforward.

Lemma 2. *Let B be a Boolean lattice and $\mu : B \to L$ an L-valued Boolean function. Then μ is isotone if and only if all the cuts of μ are up-sets on B.*

In this part, we represent isotone L-valued functions on B in the framework of cut sets and lattice induced threshold functions.

Our main tool here is a particular L-valued Boolean function, defined as follows.

Let $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$, let L_D a free distributive lattice with n generators $w_1, \ldots, w_n$ and $\overline{\beta} : B \to L_D$, an L_D-valued function on B defined in the following way: for $x = (x_1, \ldots, x_n) \in B$

$$\overline{\beta}(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i), \tag{3.4}$$

where the function "$\cdot$" is defined by (3.1).

By the definition, $\overline{\beta}$ is uniquely (up to a permutation of generators w_i) determined by a finite Boolean lattice $B = (\{0,1\}^n, \leq)$, i.e., by a positive integer n.

A connection of isotone functions and cuts is presented in Lemma 2. Next we prove that in this framework, $\overline{\beta}$ can be considered as the main representative of all isotone L-valued functions on B.

Theorem 2. *Every up-set of a finite Boolean lattice $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$, is a cut of $\overline{\beta}$.*

Proof. Let $B = (\{0,1\}^n, \leq)$ be a finite Boolean lattice. By Theorem 1, every isotone Boolean function with n variables is a lattice induced threshold function over a free distributive lattice L_D with n generators. By Lemma 1, every up-set on B is (as a characteristic function) an isotone Boolean function $f : \{0,1\}^n \to \{0,1\}$. This means that there are elements $w_1, \ldots, w_n \in L_D$ (generators), such that

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t, \text{ for some } t \in L_D.$$

Thereby, by (3.4),

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \overline{\beta}(x) \geq t \text{ if and only if } x \in \overline{\beta}_t.$$

Hence the up-set whose characteristic function is f, coincides with the cut $\overline{\beta}_t$ of $\overline{\beta}$. $\qquad\square$

By Lemma 2, we have the following obvious consequence of Theorem 2.

Corollary 1. *Lattice valued function $\overline{\beta}$ defined by (3.4) is isotone. Moreover, the collection of cuts of every L-valued isotone function on B (for any L) is contained in the collection of cuts of $\overline{\beta}$.*

4. GENERALIZATION: ISOTONE FUNCTIONS ON THE k-ELEMENT SET

In this part we investigate isotone functions on a k-element set $\{1, 2, \ldots, k\}$. Generalizing previously presented techniques, we introduce multi-value threshold functions induced by complete lattices and use them for representing the mentioned isotone functions.

Let L be a complete lattice with the bottom element 0 and top element 1. For $x \in \{1, \ldots, k\}$, and $w^1, \ldots, w^k \in L$, we define a mapping

$$\{w^1, \ldots, w^k\} \times \{1, \ldots, k\} \to L$$

denoted by "$\cdot$", as follows:

$$w^j \cdot x := \begin{cases} w^j, & \text{if} \qquad j \leq x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Now, $f : \{1, \ldots, k\}^n \to \{1, \ldots, k\}$ is a **lattice induced multi-value threshold function**, if there is a complete lattice L and $w_1^1, \ldots, w_1^k, \ldots, w_n^1, \ldots, w_n^k, t_{i_1}, \ldots, t_{i_p} \in L$, where $i_1, \ldots, i_p \in \{1, \ldots k\}$ and from $c < d$ it follows that $i_c < i_d$ [1], such that for $r \in \{1, \ldots, k\}$,

$$f(x_1, \ldots, x_n) = r \text{ if and only if}$$

$$\bigvee_{i=1}^n \bigvee_{j=1}^k (w_i^j \cdot x_i) \in \uparrow t_r, \text{ for } r = i_p \text{ and}$$
$$\bigvee_{i=1}^n \bigvee_{j=1}^k (w_i^j \cdot x_i) \in \uparrow t_r \setminus \left(\bigcup_{h > r} \uparrow t_h \right), \text{ for } r < i_p.$$

Proposition 2. *Every lattice induced multi-value threshold function is isotone.*

Proof. Let L be a complete lattice and $w_1^1, \ldots, w_1^k, \ldots, w_n^1, \ldots, w_n^k, t_{i_1}, \ldots, t_{i_p} \in L$, where $i_1, \ldots, i_p \in \{1, \ldots k\}$
such that for $r \in \{1, \ldots, k\}$,

$$f(x_1, \ldots, x_n) = r \text{ if and only if}$$

$$\bigvee_{i=1}^n \bigvee_{j=1}^k (w_i^j \cdot x_i) \in \uparrow t_r, \text{ for } r = i_p \text{ and}$$
$$\bigvee_{i=1}^n \bigvee_{j=1}^k (w_i^j \cdot x_i) \in \uparrow t_r \setminus \left(\bigcup_{h > r} \uparrow t_h \right), \text{ for } r < i_p$$

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[1]Hence, i_p is the greatest element of $i_1, \ldots, i_p$

is a lattice induced multi-value threshold function.

Let $(x_1, x_2, \ldots, x_n) \le (y_1, y_2, \ldots, y_n)$. Then, for every $i \in \{1, \ldots, k\}$

$$\bigvee_{j=1}^{k} (w_i^j \cdot x_i) \le \bigvee_{j=1}^{k} (w_i^j \cdot y_i)$$

by the definition (4.1) of the mapping "$\cdot$". Hence,

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \le \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot y_i).$$

Therefore, if $f(x_1, \ldots, x_n) = r$, then

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \ge t_r \quad \text{and hence} \quad \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot y_i) \ge t_r.$$

This implies $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot y_i) \in \left(\uparrow t_r \setminus \left(\bigcup_{h>r} \uparrow t_h\right)\right) \bigcup \left(\bigcup_{h>r} \uparrow t_h\right)$, hence, either $f(y_1, \ldots, y_n) = r$ or $f(y_1, \ldots, y_n) > r$, in either case $f(x_1, \ldots, x_n) \le f(y_1, \ldots, y_n)$. $\quad\square$

Theorem 3. *Every isotone function on the k-element set is a lattice induced multi-value threshold function.*

Proof. We prove that for any natural number n, and $k > 2$, there is a lattice L, such that every n-ary isotone function on the k-element set is a lattice induced multi-value threshold function.

This lattice L is a free distributive lattice with $k \cdot n$ generators $w_1^1, \ldots, w_1^k$, $\ldots, w_n^1, \ldots, w_n^k$. In some instances we denote the set of generators by $\{w_1^1, \ldots, w_1^k, \ldots, w_n^1, \ldots, w_n^k\}$ and in other by $\{w_j \mid j \in \{1, \ldots, k \cdot n\}\}$.

If $x, y \in L$, $x = \bigwedge_{u=1}^{p} \bigvee_{j \in I_u} w_j$ and $y = \bigwedge_{u=1}^{l} \bigvee_{s \in J_u} w_s$, then $x \le y$ if and only if for every $u \in \{1, \ldots, p\}$ there is $v \in \{1, \ldots, l\}$ such that $I_u \subseteq J_v$.

Let $f : \{1, \ldots, k\}^n \to \{1, \ldots, k\}$ be an isotone function on the k-element set. Let r be an element $1 \le r \le k$ and let F_r be the up-set corresponding to r, i.e. $F_r = f^{-1}(\uparrow r)$. Observe that in case no element is mapped by f to some $a \in \{1, \ldots, k\}$, then $F_a = F_{a+1}$. Further, let $m_{r,1}, \ldots, m_{r,p_r}$ be minimal elements of this up-set, in case $F_r \setminus \bigcup_{h>r} F_h$ is not empty. For $s \in \{1, \ldots, p_r\}$, let $m_{r,s} = (m_{1,r,s}, \ldots, m_{n,r,s})$.

For the thresholds $t_{i_1}, \ldots, t_{i_p} \in L$, where $i_1, \ldots, i_p \in \{1, \ldots k\}$ are associated to the given function f, we take

$$t_r = \bigwedge_{s=1}^{p_r} \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{i,r,s}) \quad \text{for } r \in \{1, \ldots, k\} \text{ for which } F_r \setminus \bigcup_{h>r} F_h \ne \varnothing.$$

Now we show that

$$f(x_1, \ldots, x_n) = r \text{ if and only if}$$

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \in \uparrow t_r, \text{ for } r = i_p \text{ and}$$

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \in \uparrow t_r \setminus \left(\bigcup_{h>r} \uparrow t_h \right), \text{ for } r < i_p$$

Suppose that $f(x_1, \ldots, x_n) = r$, for $r = i_p$. Indeed, from $f(x_1, \ldots, x_n) \geq r$, it follows that there is a minimal element m_{r,l_r} in the corresponding up-set, such that $(x_1, \ldots, x_n) \geq m_{r,l_r} = (m_{1,r,l_r}, \ldots, m_{n,r,l_r})$. Hence,

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \geq \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{1,r,l_r}) \geq \bigwedge_{s=1}^{p_r} \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{i,r,v_r}) = t_r.$$

On the other hand, let $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \in \uparrow t_r$, for $r = i_p$. Then, $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \geq \bigwedge_{s=1}^{p_r} \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{i,r,v_r}) = t_r$. By the definition of the order $\leq$ in the free distributive lattice, it follows that there is an m_{r,l_r}, such that $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \geq \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{1,r,l_r})$, and $(x_1, \ldots, x_n) \geq m_{r,l_r} = (m_{1,r,l_r}, \ldots, m_{n,r,l_r})$ and hence $f(x_1, \ldots, x_n) = r$.

Now, suppose that

$f(x_1, \ldots, x_n) = r$ for $r < i_p$. We have that $F_r \setminus \bigcup_{h>r} F_h \neq \varnothing$.

As above, from $f(x_1, \ldots, x_n) \geq r$, it follows that there is a minimal element m_{r,l_r} in the corresponding up-set, such that $(x_1, \ldots, x_n) \geq m_{r,l_r} = (m_{1,r,l_r}, \ldots, m_{n,r,l_r})$. Hence,

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \geq \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{1,r,l_r}) \geq \bigwedge_{s=1}^{p_r} \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{i,r,v_r}) = t_r$$

and

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \in \uparrow t_r \setminus \left(\bigcup_{h>r} \uparrow t_h \right).$$

Suppose now that $(x_1, \ldots, x_n)$ is an element such that

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \in \uparrow t_r \setminus \left(\bigcup_{h>r} \uparrow t_h \right).$$

Now $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \geq \bigwedge_{s=1}^{p_r} \bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot m_{i,r,v_r})$. By the arguments above, we have that there is a minimal element in the corresponding up-set, such that $(x_1, \ldots, x_n) \geq m_{r,l_r} = (m_{1,r,l_r}, \ldots, m_{n,r,l_r})$. Therefore, $f(x_1, \ldots, x_n) \geq r$. By the condition $\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} (w_i^j \cdot x_i) \notin \uparrow t_h$, for $h > r$ we obtain that $f(x_1, \ldots, x_n) \not\geq h$, for $h > r$, hence $f(x_1, \ldots, x_n) = r$.

$\square$

Observe that in the case of isotone functions on a k-element sets, the analogue properties to those in Remarks 1 and 2 are satisfied.

5. SOME EXAMPLES AND OBSERVATIONS

In this part we discus whether we can get particular isotone Boolean functions using some lattices different than the free distributive ones.

Starting with our approach, it is possible to characterize isotone Boolean functions satisfying some relationships among components by a free distributive lattice with less than n generators and also by factor lattices of the free distributive lattices.

Without getting into detailed elaboration, in the sequel we briefly describe these characterizations.

1. Isotone Boolean functions $f : \{0,1\}^n \to \{0,1\}$ satisfying

$$f(x_1,...,1,...,0,...,x_n) = f(x_1,...,0,...,1,...,x_n) = f(x_1,...,1,...,1,...,x_n)$$

for all x_k for $k \in \{1,\ldots,n\} \setminus \{i,j\}$, $i \neq j$, where 1 and 0 are on i-th and j-th places, can be characterized by the free distributive lattice with $n-1$ generators (where $w_i = w_j$).

This can be generalized to the case with k components (out of n) among which at least one is 1; a characterization is obtained by the free distributive lattices with $n-k$ generators (with $w_{i_1} = \ldots = w_{i_k}$).

2. Isotone Boolean functions $f : \{0,1\}^n \to \{0,1\}$ satisfying

$$f(x_1,...,0,...,1,...,x_n) = f(x_1,...,1,...,1,...,x_n)$$

for all x_k with $k \in \{1,\ldots,n\} \setminus \{i,j\}$, $i \neq j$, where 0 and 1 (1 and 1), are on i-th and j-th places, respectively; a characterization is obtained by a factor lattice of free distributive lattice (corresponding to the relationship $w_i < w_j$).

Generalizations to isotone Boolean functions with k ordered components can be characterized using factor lattices of free distributive lattices with n generators (where $w_{i_1},\ldots,w_{i_k}$ constitute a corresponding poset).

3. Isotone Boolean functions $f : \{0,1\}^n \to \{0,1\}$ satisfying combinations of various relationships of the type 1. and 2. can be characterized by factor lattices of the free distributive lattices with n generators (which corresponds to distributive lattices freely generated by posets).

Furthermore, it is possible to characterize Boolean functions by various non -distributive lattices with smaller number of elements. We present here some examples.

Example 1. Let $f : \{0,1\}^3 \to \{0,1\}$ be a Boolean function, defined by $f(1,1,1) = f(1,1,0) = f(0,1,1) = f(1,0,1) = 1$ and $f(x,y,z) = 0$ otherwise. This Boolean function can be represented as a lattice-induced threshold function using a lattice M_3, where w_1, w_2, w_3 are atoms of the lattice and t is the top of the lattice.

Similarly, a representation can be obtained for all analogous Boolean functions $f : \{0,1\}^n \to \{0,1\}$ having value 1 for all $(x_1,...,x_n)$ with 2 and more coordinates with value 1, and having value 0 otherwise, by a lattice M_n, where w_1, ..., w_n are atoms of the lattice and t is the top of the lattice.

Example 2. Let $f : \{0,1\}^3 \rightarrow \{0,1\}$ be a Boolean function, defined by $f(0,0,0) = 0$ and $f(x,y,z) = 1$ otherwise. This Boolean function can be represented as a lattice-induced threshold function using a lattice $0 \oplus M_3$ (M_3 with an additional bottom element 0), where w_1, w_2, w_3 are co-atoms and t is the atom of the lattice. A similar representation is possible for all analogous Boolean functions $f : \{0,1\}^n \rightarrow \{0,1\}$.

By similar constructions, choosing suitable lattices and elements $w_1, \ldots, w_n$, t we can represent all particular Boolean functions. However, if we want to make a representation of all Boolean functions $f : \{0,1\}^n \rightarrow \{0,1\}$ by one lattice, then the smallest such lattice is the free Boolean lattice with n generators. Clearly, its number of elements coincides with the number of different Boolean functions.

6. CONCLUSION

To sum up, we have introduced a lattice induced threshold function, as a generalization of the class of classical threshold functions. Our motivation was to capture isotonicity of Boolean functions.

The obtained results could be further generalized in several directions. First of all, one could consider lattice functions instead of Boolean ones. Also, one might look for a characterization of the classical threshold functions within the class of lattice induced threshold functions. Furthermore, as mentioned above, with appropriately chosen lattices as co-domain, interesting classes of Boolean functions could be obtained.

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