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COMPLEX B-SPLINE COLLOCATION METHOD FOR SOLVING WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

M. RAMEZANI, H. JAFARI, S.J. JOHNSTON, AND D. BALEANU

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Abstract. In this paper we propose a new collocation type method for solving Volterra integral equations of the second kind with weakly singular kernels. In this method we use the complex B-spline basics in collocation method for solving Volterra integral. We compare the results obtained by this method with exact solution. A few numerical examples are presented to demonstrate the effectiveness of the proposed method.

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1. INTRODUCTION

In this paper we consider the Volterra integral equation with the second kind weakly singular kernel, namely

$$u(t) = f(t) + \int_0^t k(t,s)u(s)\,ds, \qquad t \in (a,b], \tag{1.1}$$

where $k(t,s)$ and $f(t)$ are known and $u(t)$ is unknown. The function $k(t,s)$ is called a polar kernel if

$$k(t,s) = \frac{g(t,s)}{(t-s)^{\alpha}}, \quad \alpha \in (0,1),$$

where g is bounded on s, $g(t,t) \neq 0$ and for all

$$t,s \in C[a,b]; g(t,s) \in C([a,b]\times[a,b]).$$

We rewrite the equation (1.1) in the following operator form:

$$(I-K)u = f,$$

where the operator K is assumed to be compact on a Banach Space X to X.

During the past few decades, this equation has been used to study various problems of mathematical chemistry and physics, such as reactions including stereology, heat conduction with mixed boundary conditions [10], crystal growth, electrochemistry, super fluidity and the radiation of heat [7], electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics [12, 13, 27], the particle transport problems of astrophysics, potential theory and Dirichlet problem, electrostatic and radiative heat transfer problems and in some engineering fields [29, 30], astronomy, optics, computational electromagnetic, quantum mechanics, seismology image processing [1, 6].

In addition a method with exponential order convergence rate has been developed by Riley [23] for Volterra integral equations of the form

$$u(t) - \int_a^t (t-s)^{p-1} k(t,s) u(s) ds = f(t), \qquad a \leq t \leq b, \tag{1.2}$$

where the kernel is also assumed to be weakly singular and the solution u is generally not differentiable at $t = a$. In [16], the equation (1.2) has been solved with fractional B-spline basics.

In most of the cases, it is difficult to obtain analytical solution of integral equations, therefore many numerical methods such as collocation method with different basics [2,19,20], orthogonal bases and wavelets [17,21], Galerkin methods have been developed to solve equation (1.1) [4–6, 14].

Recently, many different basic functions have been utilized to estimate the result of integral equations, such as modified quadrature [24], optimal homotopy asymptotic method [15], Tau approximate method [18].

Spline functions are very efficient and useful in signal processing, mathematical and computer graphics [8, 9, 22, 25, 26]. In [3], Blu and Unser gave an extension of B-splines to fractional orders and later in [11], Forster *et. al.* gave an extension to complex power.

In this paper, we solve equation (1.1) by using complex B-spline to obtain approximate solution. The paper is organized as follows: In Section 2, we recall some basic definitions and theorems of complex B-splines and its properties. Section 3 is devoted to the solution of weakly singular integral equation of second kind using collocation methods with complex B-spline basics. In Section 4, by considering numerical examples reported in our work, the accuracy of the proposed scheme is demonstrated.

2. Complex B-splines

In this section we state some definitions and theorems [11, 20] that will be used later in our work.

Definition 1. The inner product $\int \overline{f(x)} g(x) dx$ between two complex L^2 functions f, g is denoted by (f, g), and the associated Euclidean norm is written as $\|.\|_2$.

Definition 2. The Riemann zeta function is defined as $\xi(s) = \sum_{n \geq 1} n^{-s}$ for all real $s > 1$.

Definition 3. The basic functions for Schoenberg's polynomial splines with uniform knots [3, 20] are defined as

$$\beta^n(x) = \frac{1}{n!}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(x-j)_+^n \quad x \in \mathbb{R}, n \in \mathbb{N},$$

where

$$(x-j)_+^n = \begin{cases} (x-j)^n & \textit{if} \quad x > j, \\ 0 & \textit{if} \quad x \le j. \end{cases}$$

Definition 4. x_+^z denotes the truncated power function of complex degree z with knot zero:

$$x_+^z = \begin{cases} x^{\Re z} e^{i\Im z \ln x}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition 5. The complex B-spline β^z of complex degree z is defined in $L^2(\mathbb{R})$ via its Plancherel transform as

$$\hat{\beta}^z(\omega) = (\frac{1-e^{-i\omega}}{i\omega})^{z+1}, \tag{2.1}$$

where $z = \alpha + i\gamma$ with parameters $\alpha, \gamma \in \mathbb{R}$ and $\alpha > \frac{-1}{2}$.

Theorem 1 (cf. [11])**.** *The complex B-spline β^z is well-defined, uniformly continuous and belongs to the space $L^2(\mathbb{R})$.*

Theorem 2 (cf. [11])**.** *The time domain representation of the complex B-spline β^z is given by*

$$\beta^z(x) = \frac{1}{\Gamma(z+1)}\sum_{k\ge 0}(-1)^k\binom{z+1}{k}(x-k)_+^z. \tag{2.2}$$

This equation is valid pointwise for all $x \in \mathbb{R}$ and $L^2(\mathbb{R})$.

The complex B-splines generate dyadic multiresolution analysis; *i.e.* they generate a sequence of spaces:

$$\{0\} \subset ... \subset V_{-1} \subset V_0 \subset V_1 \subset ... \subset L^2(\mathbb{R})$$

with the following properties:

(1) $\cap_i V_i = \{0\}$ and $\overline{\cup_i V_i} = L^2(\mathbb{R})$,
(2) $f(\bullet) \in V_i$ if and only if $f(2^{-i}\bullet) \in V_0$,
(3) $f(\bullet) \in V_0$ if and only if $f(\bullet - k) \in V_0$ for all $k \in \mathbb{Z}$,
(4) there exists a function $\varphi \in V_0$, called a scaling function, such that $\varphi(\bullet - k)_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_0.

V_i is the complex B-spline of order $z \in \mathbb{C}$ with knot points $k.2^i, k \in \mathbb{Z}$.

Theorem 3 (cf. [11]). *Let $Re\, z > 0$. Then the spaces*

$$V_i = \overline{span\{\beta^z(\frac{x-2^i k}{2^i})\}^{L^2(\mathbb{R})}} \quad ,i \in \mathbb{Z} \tag{2.3}$$

form a dyadic multiresolution analysis with scaling function β^z.

The complex B-spline spaces at scale a is defined as

$$S_a^z = \{s_a : \exists c \in \ell^2, s_a(x) = \sum_{k\in\mathbb{Z}} c(k)\beta^z(\frac{x}{a}-k)\}. \tag{2.4}$$

Then given an arbitrary function $f \in L^2(\mathbb{R})$, we determine its least-squares approximation in S_a^z by applying the following orthogonal projection operator

$$P_a f = \sum_{k\in\mathbb{Z}} (f, \frac{1}{a}\tilde{\beta}^z(\frac{\bullet}{a}-k))\beta^z(\frac{\bullet}{a}-k). \tag{2.5}$$

This defines a projector because the functions β^z and $\tilde{\beta}^z$ are biorthonormal [11], where $\tilde{\beta}^z \in S_a^z$ is the dual B-spline whose Fourier transform is

$$\hat{\tilde{\beta}}^z(\bullet) = \frac{\hat{\beta}^z(\bullet)}{\sqrt{\sum_{k\in\mathbb{Z}} |\hat{\beta}^z(\bullet+2\pi k)|^2}}. \tag{2.6}$$

Theorem 4 (cf. [28]). *The complex B-splines have a fractional order of approximation $\alpha+1$. Specifically, the least-squares approximation error is bounded by:*

$$\forall f \in W_2^{\alpha+1}, \|f - P_a f\|_2 \le \frac{\sqrt{2\xi(\alpha+2)-\frac{1}{2}}}{\pi^{\alpha+1}} \|D^{\alpha+1}\|_2 a^{\alpha+1}. \tag{2.7}$$

3. The complex B-spline collocation method

To solve approximately the integral equation equation (1.1), we assume that K is compact on a Banach space X to X. We choose a finite dimensional family of functions $\tilde{u}(x)$ which is close to the exact solution $u(x)$. In practice, we choose a sequence of dimensional subspaces $X_n \subset X, n \ge 1$, with X_n having dimension d_n. Let X_n have a basis $\{\varphi_1, ..., \varphi_d\}$ with $d \equiv d_n$ for notational simplicity. We seek $u_n(x) \in X_n$, which can be written as

$$u_n(x) = \sum_{j=0}^{d} c_j \varphi_j(x), \quad x \in D. \tag{3.1}$$

This is substituted into equation (1.1), and coefficients $\{c_1, ..., c_d\}$ are determined by forcing the equation to be exact in some sense. For later use, we introduce

$$r_n(x) = u_n(x) - \int_D k(x,s) u_n(s) ds - f(x),$$

$$= \sum_{j=1}^{d} c_j \{\varphi_j(x) - \int_D k(x,s)\varphi_j(s)ds\} - f(x), \quad x \in D. \tag{3.2}$$

We pick distinct node $x_1, \cdots, x_d \in D$, and require

$$r_n(x_i) = 0, \quad i = 1, \cdots, d. \tag{3.3}$$

This leads to determining $\{c_1, ..., c_d\}$ as the solution of the linear system

$$\sum_{j=1}^{d} c_j \{u(x_i) - \int_D k(x_i,s)u(s)ds\} = f(x_i), \quad i = 1, \cdots, d. \tag{3.4}$$

In following we show that this method can be used to solve equation (1.1), In this regard we give the following Lemma 1 and Theorem 5.

Lemma 1 (cf. [2]). *Let X be a Banach space and P_n be a family of bounded projections on X with*

$$P_n u \longrightarrow u \quad as \quad n \longrightarrow \infty, u \in X$$

and $K : X \longrightarrow X$ be compact. Then

$$\|K - P_n K\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

Theorem 5 (cf. [11]). *If $G \in \mathbb{R}$ be an integral equation with a weakly singular kernel then it is a compact operator on $C(G)$, where $C(G)$ is space of continuous real or complex valued functions on compact subsets $G \in \mathbb{R}$.*

Theorem 6. *Equation (1.1) can be solved with collocation method by using complex B-spline basis.*

Proof. If we introduce equation (3.1) to projection operator P_n that maps X onto X_n, define $P_n u(x)$ to be that element of X_n that interpolates X at the nodes $\{x_1, ..., x_d\}$. This means writing

$$P_n u(x) = \sum_{j=1}^{d} c_j \varphi_j(x)$$

with the coefficients $\{c_j\}$ determined by solving the linear system

$$\sum_{j=1}^{d} c_j \varphi_j(x_i) = u(x_i), \quad i = 1, \cdots, d$$

Then this linear system has a unique solution if

$$det[\varphi_j(x_j)] \neq 0.$$

From Theorem 1, complex B-spline basis belong to $L^2(\mathbb{R})$ and with the help Theorem 2 this method is convergent. Then in view of Lemma 1 and Theorem 5 we can use collocation method for these type of integral equations. □

Now we can use collocation methods for solving weakly singular integral equation of second kind with complex B-spline basis.

In equation (1.1), let $X = L^2(\mathbb{R})$ and $V_n = X_n$. Then if $u(x) \in L^2(\mathbb{R})$ and $u_n(x) \in V_n$, where

$$u_n(x) = \sum_{j\in\mathbb{Z}} c(j)\beta^z(2^n x - j), \quad j \in \mathbb{Z},$$

with $0 \leq t \leq b$ and $n \in \mathbb{N}$ then we have

$$u_n^{2^n}(x) = \sum_{j=1-2^n}^{b} c(j)\beta^z(2^n x - j), \quad b \in \mathbb{R}, \tag{3.5}$$

with nodes $x_i = \frac{bi}{2^n}$. Then

$$r_n^{2^n}(x_i) = \sum_{j=1-2^n}^{b} c_j\{u(x_i) - \int_D k(x_i,s)u(s)ds\} - f(x_i) = 0 \quad i = 0,\cdots,b. \tag{3.6}$$

We define the absolute error

$$E_n^{2^n}(u(x)) = \|u(x) - u_n^{2^n}(x)\|_2 = \left(\int_0^b |u(x) - u_n^{2^n}(x)|^2\right)^{\frac{1}{2}}, \tag{3.7}$$

and note that when $n \to \infty$ and $d \to \infty$ then $u_n^{2^n}(x) \to u(x)$.

Using the Theorem 1 and Lemma 1, the relatively error is defined as

$$e_n = \frac{\max_{0\leq i\leq 2^n} |E_n^{2^n}(u(x_i))|}{\max_{0\leq x\leq b} |u(x)|}. \tag{3.8}$$

4. Illustrative examples

In order to show better the theoretical results of the previous sections, we now consider the numerical solution of the equation (1.1), with various choices of $f(x)$ for $x \in [0,1] = D$. By using equation (3.6), we obtain $\{c_1,...,c_d\}$. Then in view of (3.7) and (3.8) at several points of interval D we obtain the absolute and the relative errors.

Example 1. Let $b = 1$, $g(t,s) = ts$ and $f(x) = x(1-x) + \frac{16}{105}x^{\frac{7}{2}}(7-6x)$ with the exact solution $u(x) = x(1-x)$. Table 1 shows the absolute errors obtained by the knot points $x_i = \frac{i}{2^n}; i = 0,...,2^n$ with $z = 0.5 + i$.

In Figure 1, the horizontal axis represents the n index's V_n the vertical axis represents the relative error e_n is intentional, as can be seen, by increasing the index of n, the relative error decreases.

From Table 1 we see that the maximum error occurs at point $x = 1$. We now show the relative error for different interval of z this point $(x = 1)$ in Tables 3, 4 and 5.

TABLE 1. Absolute Errors

x	$E_0^{2^0}(u(x))$	$E_1^{2^1}(u(x))$	$E_2^{2^2}(u(x))$	$E_3^{2^3}(u(x))$	$E_4^{2^4}(u(x))$
0	0	0	0	0	0
0.25	0.180375	0.119014	0.0013166	0.0008720	0.0005532
0.5	0.1676936	0.126606	0.019329	0.013126	0.00670011
0.75	0.1676936	0.126606	0.019329	0.0131267	0.00670011
1	0.0720055	0.0578148	0.0425467	0.0263254	0.0111375

TABLE 2. Relatively Errors

n	0	1	2	3	4
e_n	0.837556	0.506424	0.1701868	0.10530168	0.04455

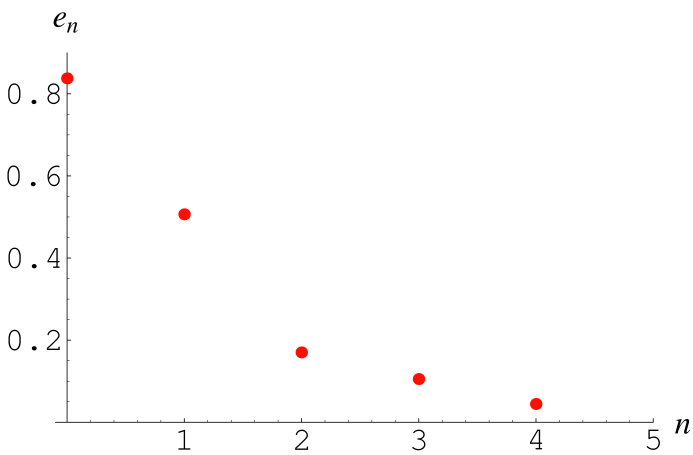

FIGURE 1. Points relative error in spaces V_n.

TABLE 3. The relative error for $|z| < 1$

z	0.1+0.01i	0.5+0.1i	0.2+0.9i
$E_4^{2^4}(u(x))$	0.00913271	0.00301154	0.01920683

TABLE 4. The relative error for $1 \leq |z| < 2$

z	1+i	1+0.5i	1+0.9i
$E_4^{2^4}(u(x))$	0.00745215	0.00248121	0.00842375

TABLE 5. The relative error for $2 \leq |z| < 3$

z	2+0.1i	2+0.5i	2+1i
$E_4^{2^4}(u(x))$	0.0054914	0.00949278	0.0301505

We note that when $1 \leq |z| < 2$, the error is minimum.

Example 2. Let $b = 1$, $g(x,s) = 1$ and $f(x) = \frac{1}{2}\pi x + \sqrt{x}$ with the exact solution $u(x) = \sqrt{x}$ and $z = 0.5 + 0.5i$. Table 6 shows the absolute error obtained by the knot points $x_i = \frac{i}{2^n}, \quad i = 0, ..., 2^n$ with $z = 0.5 + 0.5i$.

TABLE 6. The absolute errors

x	$E_0^{2^0}(u(x))$	$E_1^{2^1}(u(x))$	$E_2^{2^2}(u(x))$	$E_3^{2^3}(u(x))$	$E_4^{2^4}(u(x))$
0	0	0	0	0	0
0.25	0.290563	0.126959	0.0413826	0.0199285	0.000803727
0.5	0.168809	0.0702198	0.0333131	0.0130709	0.0044490
0.75	0.0356813	0.0508203	0.00297352	0.00907127	0.00297352
1	0.1156	0.0540076	0.0205252	0.0067599	0.00221725

TABLE 7. The relatively error

m	0	1	2	3	4
e_m	0.290563	0.1269598	0.0413826	0.0199285	0.00803727

In Figure 2, the horizontal axis represents the n index's V_n and vertical axis represents the relative error e_n is intentional, as can be seen by increasing the index of n relative error decreases.

From Table 6 we see that the maximum error occurs at point $x = 0.5$. We now show the relative error for different interval of z this point($x = 0.5$) in Tables 8, 9 and 10.

TABLE 8. The relative error for $|z| < 1$

z	0.01+0.1i	0.5+0.01i	0.9+0.1i
$E_4^{2^4}(u(x))$	0.0156468	0.000083763	0.00115488

Here we note that when $|z| < 1$, the error is minimum.

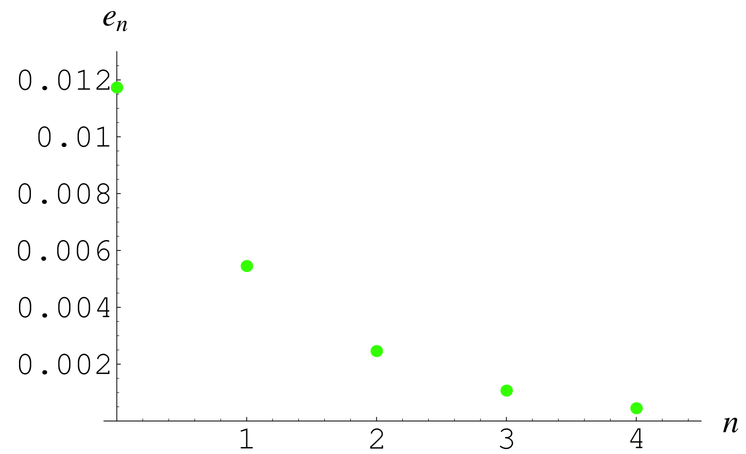

FIGURE 2. Points relative error in spaces V_n.

TABLE 9. The relative error for $1 \leq |z| < 2$

z	1+0.1i	0.5+i	0.9+i
$E_4^{2^4}(u(x))$	0.00112826	0.019533	0.00526669

TABLE 10. The relative error for $2 \leq |z| < 3$

z	1.5+i	2+0.1i	2+0.5i
$E_4^{2^4}(u(x))$	0.022596	0.0175306	0.0224039

Example 3. Let $a = -1$, $b = 1$, $g(x,s) = 1$ and $f(x) = \sqrt{x+1} + \frac{1}{9}(x+1)^{\frac{3}{2}}(3\ln(x+1)^2 - 16 + \ln(4096))$ with the exact solution $u(x) = \sqrt{x+1}$.
Table 11 shows the absolute error obtained by the knot points $x_i = \frac{i}{2^n} - 1, \quad i = 0, \ldots, 2^n$ with $z = 1 + 0.5i$.

TABLE 11. The absolute errors

x	$E_0^{2^0}(u(x))$	$E_1^{2^1}(u(x))$	$E_2^{2^2}(u(x))$	$E_3^{2^3}(u(x))$	$E_4^{2^4}(u(x))$
-1	0	0	0	0	0
-0.5	0.016592	0.0056849	0.0026678	0.0012071	0.00052264
0	0.016534	0.0077106	0.0034793	0.0015144	0.00063639
0.5	0.007966	0.00063781	0.0030579	0.0013167	0.00060392
1	0.019279	0.00068	0.0004755	0.00002275	0.00001357

TABLE 12. The relative errors

n	0	1	2	3	4
e_n	0.0117325	0.0054522	0.0024602	0.0010708	0.00044999

In Figure 3, the horizontal axis represents the n index's V_n, axis represents the relative error e_n is intentional, as can be seen by increasing the index of n relative error decreases.

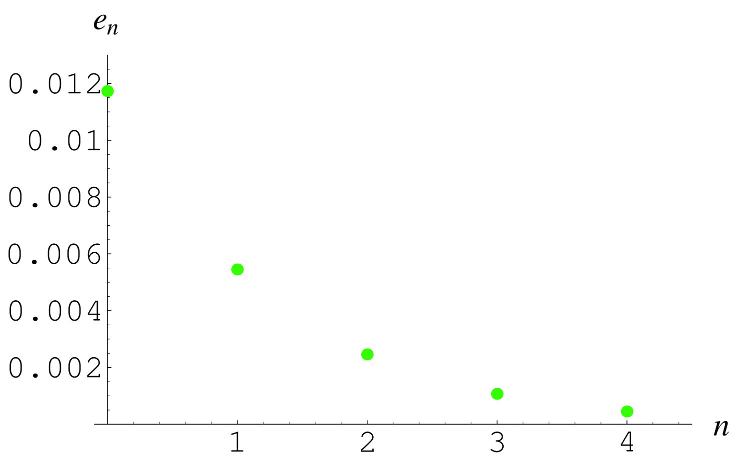

FIGURE 3. Points relative error in spaces V_n.

From Table 12 see that the maximum error occurs at point $x = 0.5$. Now show the relative error for different interval of z this point($x = 0.5$) in Tables 13, 14 and 15.

TABLE 13. The relative error for $|z| < 1$

z	0.1+0.1i	0.5+0.1i	0.2+0.9i
$E_4^{2^4}(u(x))$	0.080492	0.00864855	0.086332

TABLE 14. The relative error for $1 \leq |z| < 2$

z	0.1+i	1+0.5i	0.9+i
$E_4^{2^4}(u(x))$	0.100855	0.0415961	0.0482324

We see that when $|z| < 1$, the error is minimum.

TABLE 15. The relative error for $2 \leq |z| < 3$

z	1+i	2+0.5i	2+i
$E_4^{2^4}(u(x))$	0.0298086	0.0398539	0.0419062

5. CONCLUSION

In this paper, we proposed an efficient algorithm for solving Volterra integral equations of second kind with weakly singular kernels by collocation type method. We used complex B-spline basics as basic functions in the collocation method. This approach gives better solution with respect to ordinary B-spline basics function. We presented three numerical examples which demonstrated That our proposal method is very attractive. *Mathematica* has been used in this paper for computation.

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Authors' addresses

M. Ramezani

M. Ramezani, Young Researchers and Elite Club, Parand Branch, Islamic Azad University, Tehran, Iran

E-mail address: mr_63_90@yahoo.com

H. Jafari

Department of Mathematics, University of Mazandaran, Babolsar, Iran

Current address: Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, Johannesburg, South Africa

E-mail address: jafarh@unisa.ac.za

S.J. Johnston

S.J. Johnston, Department of Mathematical Sciences, University of South Africa, PO Box 392, UNISA 0003, Johannesburg, South Africa

E-mail address: johnssj@unisa.ac.za

D. Baleanu

D. Baleanu, Department of Mathematics and Computer Science, Çankaya University, Ankara, Turkey

Current address: Institute of Space Sciences, PO Box MG-23, R 76900, Magurele-Bucharest, Romania

E-mail address: dumitru@cankaya.edu.tr