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MODULES THAT HAVE A WEAK SUPPLEMENT IN EVERY EXTENSION

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Abstract. We say that over an arbitrary ring a module M has *the property* (WE) (respectively, (WEE)) if M has a weak supplement (respectively, ample weak supplements) in every extension. In this paper, we provide various properties of modules with these properties. We show that a module M has the property (WEE) iff every submodule of M has the property (WE). A ring R is left perfect iff every left R-module has the property (WE) iff every left R-module has the property (WEE). A ring R is semilocal iff every left R-module has a weak supplement in every extension with small radical. We also study modules that have a weak supplement(respectively, ample weak supplements) in every coatomic extension, namely *the property* (WE^*)(respectively, (WEE^*)).

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unital left R-modules, unless otherwise stated. Let M be an R-module. The notation $U \leq M$ means that U is a submodule of M. A submodule U of M is called *small* in M, denoted as $U << M$, if $M \neq U + L$ for every proper submodule L of M. By $Rad(M)$ we denote the intersection of all maximal submodules of M, equivalently the sum of all small submodules of M (see [14]). A module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$.

As a proper generalization of direct summands of a module, the notion of supplement submodules is defined. For U, V submodules of a module M, V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V \ll V$. Then, it is natural to introduce a generalization of supplement submodules by [14, Section 19.3.(2)]. A submodule V of M is called a *weak supplement* of U in M if $U + V = M$ and $U \cap V \ll M$. A module M is called *weakly supplemented* if every submodule of M has a weak supplement in M (see [9], [14] and [17]). A submodule U of M has *ample (weak) supplements* in M

if, whenever $M = U + L$, L contains a (weak) supplement of U in M. Under given definitions, we clearly have the following implication on submodules:

$$\text{direct summand} \Longrightarrow \text{supplement} \Longrightarrow \text{weak supplement}$$

Let R be a ring and M be an R-module. An R-module N is called an *extension* of M provided $M \subseteq N$. A module M is said to be *injective* if it is a direct summand in its every extension N.

Modules that have a supplement (resp. ample supplements) in every extension, i.e. modules with *the property* (E) (resp. (EE)), was first introduced by H. Zöschinger in [16], as a generalization of injective modules. The author determined in the same paper the structure of modules with these properties.

Adapting his concepts, we introduce the properties (WE) and (WEE) as a generalization of the properties (E) and (EE) in Section 2. We call a module that has *the property* (WE) (resp. (WEE)) if it has a weak supplement (resp. ample weak supplements) in every extension. Moreover in this section, we show that a module M has the property (WEE) if and only if every submodule of M has the property (WE). This gives us that every submodule of a module with the property (WEE) is weakly supplemented. We prove that the property (WE) is inherited by direct summands. In Corollary 2, we obtain that if a ring R is left hereditary, then every factor module of an R-module with the property (WE) has the property (WE). Thanks to Lemma 3.3 of Zöschinger's paper [16], we directly say that over a complete local dedekind domain R, an R-module M has the property (WE) if and only if M has the property (E). We also give new characterizations of left perfect rings via the modules with the properties (WE) and (WEE).

Let R be a ring and M be an R-module. R. Alizade et al. [1] say a submodule U of M *cofinite* in M if the factor module $\frac{M}{U}$ is finitely generated. In [5], H. Çalışıcı and E. Türkmen called an extension N of M *cofinite extension* if M is cofinite in N. Following [5], the authors studied modules that have a supplement (resp. ample supplements) in every cofinite extension, namely *the property* (CE)*(resp.* (CEE)), as a generalization of the property (E) (resp. (EE)). In addition, they showed in [5, Theorem 2.12] that a ring R is semiperfect if and only if every left R-module has the property (CE).

In [15], a module M is said to be *coatomic* if $Rad(\frac{M}{K}) = \frac{M}{K}$ implies that $K = M$ for some submodule K of M, that is, every radical factor module of M is zero. M is coatomic if and only if every proper submodule of M is contained in a maximal submodule of M. Note that semisimple modules are coatomic.

Let R be a ring and M, N be R-modules. N is called a *coatomic extension* of M in case $M \subseteq N$ and $\frac{N}{M}$ is coatomic. In [11], B. N. Türkmen studied on modules that have a supplement (resp. ample supplements) in every coatomic extension and termed these modules E^**-modules* (resp. EE^**-modules*). Since finitely generated modules are coatomic, E^*-modules (resp. EE^*-modules) have the property (CE) (resp. (CEE)).

In Section 3, we also call a module that has *the property* (WE^*) (resp. (WEE^*)) if it has a weak supplement (resp. ample weak supplements) in every coatomic extension. We prove that over a left V-ring R, every left R-module with (WE^*) is injective. In addition, we give also a characterization of semilocal rings via the modules that have a weak supplement in every extension with small radical. Finally, we give an example of modules that have a weak supplement in every extension with small radical but not have the property (CEE).

2. Modules with the properties (WE) and (WEE)

It is shown in [16, Lemma 1.3.(a)] that direct summands of modules with the property (E) have the property (E). Now we give an analogue of this fact for the modules with the property (WE).

Proposition 1. *Let M be a module. If M has the property (WE), then every direct summand of M has the property (WE).*

Proof. Let M_1 be a direct summand of M. Then there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let N be any extension of M_1. Let N' be the external direct sum $N \oplus M_2$ and $\vartheta : M \to N'$ be the canonical embedding. Then $M \cong \vartheta(M)$ has the property (WE). Hence, there exists a submodule V of N' such that $N' = \vartheta(M) + V$ and $\vartheta(M) \cap V \ll N'$. By the projection $\pi : N' \to N$, we have that $M_1 + \pi(V) = N$. Also since $Ker(\pi) \subseteq \vartheta(M)$, $\pi(\vartheta(M) \cap V) = \pi(\vartheta(M)) \cap \pi(V) = M_1 \cap \pi(V) \ll N$. Hence $\pi(V)$ is a weak supplement of M_1 in N. □

Proposition 2. *A module M has the property (WEE) if and only if every submodule of M has the property (WE).*

Proof. Suppose that every submodule of M has the property (WE). For any extension N of M, let $N = M + K$ for some submodule K of N. Since $M \cap K$ has the property (WE), there exists a submodule L of K such that $(M \cap K) + L = K$ and $(M \cap K) \cap L = M \cap L \ll K$. Note that $N = M + K = M + ((M \cap K) + L)) = M + L$. It follows that L is a weak supplement of M in N.

Conversely, let M be a module with the property (WEE) and M_1 be any submodule of M. For any extension N of M_1, let $F = \frac{M \oplus N}{H}$, where the submodule H is the set of all elements $(m', -m')$ of $M \oplus N$ with $m' \in M_1$ and let $\gamma : M \to F$ via $\gamma(m) = (m, 0) + H$, $\psi : N \to F$ via $\psi(n) = (0, n) + H$ for all $m \in M, n \in N$. For inclusion homomorphisms $\iota_1 : M_1 \to N$ and $\iota_2 : M_1 \to M$, we can draw the following pushout:

$$\begin{array}{ccc} M_1 & \xrightarrow{\iota_1} & N \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \psi} \\ M & \xrightarrow{\gamma} & F \end{array}$$

It is clear that $F = Im(\gamma) + Im(\psi)$. Since γ is monomorphism, by assumption, $Im(\gamma)$ has the property (WEE). It means that $Im(\gamma)$ has a weak supplement V in F such that $V \leq Im(\psi)$, i.e. $F = Im(\gamma) + V$ and $Im(\gamma) \cap V \ll F$. Then we obtain that $N = \psi^{-1}(Im(\gamma)) + \psi^{-1}(V) = M_1 + \psi^{-1}(V)$ and $M_1 \cap \psi^{-1}(V) \ll N$. Hence $\psi^{-1}(V)$ is a weak supplement of M_1 in N. □

Corollary 1. *Every submodule of a module with the property (WEE) is weakly supplemented.*

Lemma 1. *Every simple submodule S of a module M is either a direct summand of M or small in M.*

Proof. Suppose that S is not small in M, then there exists a proper submodule K of M such that $S + K = M$. Since S is simple and $K \neq M$, $S \cap K = 0$. Thus $M = S \oplus K$. □

Let R be a ring and M be an R-module. M is called *local* if the sum of all proper submodules of M is a proper submodule of M. R is called a *local ring* if ${}_RR$ (or R_R) is a local module.

Proposition 3. *Local modules have the property (WE).*

Proof. Let S be a module and N be any extension of S. If S is small in N, N is a weak supplement of S in N. Suppose that S is not small in N. Then there is a proper submodule S' of N such that $S + S' = N$. From Lemma 1, if S is simple, S' is a direct summand of N. If S is local, $S \cap S'$ is small in S. In both cases, S' is a weak supplement of S in N. □

Let M be a module and U be a submodule of M. If the factor module $\frac{M}{U}$ has the property (WE), M does not need to have the property (WE). For example, for the ring $R = \mathbb{Z}$, the R-module $M = \frac{2\mathbb{Z}}{6\mathbb{Z}}$ has a weak supplement in every extension because it is simple. But $2\mathbb{Z}$ does not have a weak supplement in its extension $\mathbb{Z}$. Now we show that the statement mentioned above is true under a special condition.

Proposition 4. *Let M be a module and U be a submodule of M. If $U \ll M$ and the factor module $\frac{M}{U}$ has the property (WE), M has the property (WE).*

Proof. Let N be any extension of M. Since $\frac{M}{U}$ has the property (WE), there exists a submodule $\frac{V}{U}$ of $\frac{N}{U}$ such that $\frac{M}{U} + \frac{V}{U} = \frac{N}{U}$ and $\frac{M \cap V}{U} \ll \frac{N}{U}$. Note that $M + V = N$. Suppose that $M \cap V + S = N$ for a submodule S of N. Then we obtain $\frac{M \cap V}{U} + \frac{S+U}{U} = \frac{N}{U}$. Since $\frac{M \cap V}{U} \ll \frac{N}{U}$, we have that $\frac{S+U}{U} = \frac{N}{U}$. By hypothesis, it follows that $N = S + U = S$. Hence $M \cap V \ll N$. □

For a module M, we will denote by $Soc(M)$ the sum of all simple submodules of M. Note that $Soc(M)$ is the largest semisimple submodule of M.

Remark 1. Let M be a finitely generated semisimple module. Then M is artinian. Since artinian modules have the property (E), it has the property (WE). Note that here the condition "finitely generated" is necessary. For example, consider the left $\mathbb{Z}$ -module $M = \prod_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$, where Ω is the set of all prime numbers. Then, the semisimple module $Soc(M) = \bigoplus_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$. By [3, Lemma 2.9], there exists a submodule N of M such that $\frac{N}{Soc(M)} \cong \mathbb{Q}$. If $Soc(M)$ has a weak supplement K in N, we have $N = Soc(M) \oplus K$ since $Rad(M) = 0$. Therefore, K is injective and so $K = Rad(K) \subseteq Rad(M) = 0$, a contradiction.

In [7] a ring R is said to be a *left V-ring* if every simple left R-module is injective. It is well known that a ring R is a left V-ring if and only if $Rad(M) = 0$ for every left R-module M. A ring R is called *left hereditary* if every left ideal of R is projective. R is a left hereditary ring if and only if every factor module of an injective left R-module is injective [14, Section 39.16].

The next example shows that every factor module of a module with the property (WE) does not need to have the property (WE). Firstly we need the following lemma.

Lemma 2. *Let R be a left V-ring. An R-module M has the property* (WE) *if and only if M is injective.*

Proof. Let M has the property (WE) and N be any extension of M. Then M has a weak supplement V in N. We have $M + V = N$, $M \cap V \ll N$. Hence $M \cap V \leqslant Rad(N)$. Since $Rad(N) = 0$, we have $N = M \oplus V$.

Conversely, let M be injective and N be any extension of M. Then there exists a submodule K of N such that $N = M \oplus K$. Hence K is a weak supplement of M in N. □

Example 1. Let R be the product of the family $\{F_i\}_{i\in I}$, where each F_i is a field for an infinite index set I. The ring R is a commutative Von Neumann regular but not hereditary [10, Example 2.15]. Then by [14, Section 23.5], R is a left V-ring. R is injective from [8, Corollary, 3.11.B]. By Lemma 2, the left R-module ${}_RR$ has the property (WE). Since R is not hereditary, there is at least one factor module of R which is not injective. This factor module does not have the property (WE) by using Lemma 2.

Next we prove that under proper conditions a factor module of a module with the property (WE) has the property (WE).

Proposition 5. *Let $K \subseteq M \subseteq L$ be modules with $\frac{L}{K}$ injective. If M has the property* (WE), *then $\frac{M}{K}$ has the property* (WE).

Proof. Let N be any extension of $\frac{M}{K}$. Since $\frac{L}{K}$ is injective, by [10, Lemma 2.16] we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \hookrightarrow & M & \xrightarrow{\sigma} & M/K & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

Since h is monomorphism and M has the property (WE), $M \cong Im(h)$ has a weak supplement V in P, that is, $Im(h) + V = P$ and $Im(h) \cap V \ll P$. We claim that $g(V)$ is a weak supplement of $\frac{M}{K}$ in N.
$N = g(P) = g(h(M)) + g(V) = (f\sigma)(M) + g(V) = \frac{M}{K} + g(V)$ and
$\frac{M}{K} \cap g(V) = f(\sigma(M)) \cap g(V) = g[h(M) \cap V] \ll g(P)$. Hence $\frac{M}{K} \cap g(V) \ll N$.
□

Corollary 2. *If R is a left hereditary ring and M is an R-module with the property (WE), then every factor module of M has the property (WE).*

If a module M has a supplement in its injective envelope, M need not to have a weak supplement in every extension. For example, for the ring $R = \mathbb{Z}$, the R-module $M = 2\mathbb{Z}$ has a supplement in its injective envelope $\mathbb{Q}$. But $M = 2\mathbb{Z}$ does not have a weak supplement in its extension $\mathbb{Z}$. Now we prove that over a local dedekind domain, a module M has a supplement in its injective envelope if and only if M has a weak supplement in every extension.

Lemma 3. *Let R be a local dedekind domain and M be an R-module. The following statements are equivalent:*

(1) *M has a supplement in its injective envelope.*
(2) *M has the property (WE).*
(3) *M is an E^*-module.*

Proof. It is clear by [16, Lemma 3.3]. □

Proposition 6. *Let R be a complete local dedekind domain and M be an R-module. M has the property (WE) if and only if M has the property (E).*

Proof. Let M has the property (WE) and N be any extension of M. Since M has the property (WE), there exists a submodule X of N such that $M + X = N$, $M \cap X \ll N$. By [16, Section 3, Corollary 5], there exists a supplement V of M in N with $V \subset X$. Hence M has the property (E). □

Proposition 7. *Let R be a non-local dedekind domain and M be a semisimple R-module. Then, the following three statements are equivalent:*

(1) *M has the property (WE).*
(2) *M has the property (E).*
(3) *M is of the form $K \oplus \prod_p A_p$, where K is injective and A_p is a bounded p-primary module for every prime element $p \in R$.*

Proof. (1) $\Longleftrightarrow$ (2) It follows from [12, Proposition 2.1].
(2) $\Longleftrightarrow$ (3) By [16, Theorem 5.6]. □

It is known from [14, Section 43.9] that a ring R is left perfect if and only if every left R-module has the property (E). The next theorem gives new characterizations of left perfect rings via their modules which have the property (WE).

Theorem 1. *For a ring R the following statements are equivalent:*

(1) *R is left perfect.*
(2) *Every left R-module is weakly supplemented.*
(3) *Every left R-module has the property (WE).*
(4) *$R^{(\mathbb{N})}$ is weakly supplemented.*
(5) *$R^{(\mathbb{N})}$ has the property (WEE).*
(6) *Every left R-module has the property (WEE).*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) is clear from [4, Theorem 1]. (3) $\Rightarrow$ (6) and (5) $\Rightarrow$ (4) follow from Proposition 2. (1) $\Rightarrow$ (3) follows from [14, Section 43.9]. (6) $\Rightarrow$ (5) is clear. □

The following definitions are given in the paper [6], and we recall them for the convenience of the reader:

By a *valuation ring* (also called a *chain ring*) we mean a commutative ring R whose ideals are totally ordered by inclusion. Equivalently, if $a, b \in R$, then either $a \in Rb$ or $b \in Ra$. A valuation ring that is a domain will be called a *valuation domain*. A valuation ring R is called *maximal* if ${}_RR$ is *linearly compact*, i.e., every family of cosets $\{a_i + L_i | i \in I\}$ with the finite intersection property has a non-empty intersection. Since linearly compact modules have ample supplements in every extension, a maximal ring R has the property (WEE).

The following example shows that a ring with the property (WEE) need not be left perfect, in general.

Example 2. Let R be the localization ring $\mathbb{Z}_{(p)}$ of the ring $\mathbb{Z}$ of integers at a prime ideal $p\mathbb{Z} \neq 0$. Then, the completion of $\mathbb{Z}_{(p)}$, the ring $J_{(p)}$ of p-adic integers, is a maximal valuation domain which is not field. Hence, $J_{(p)}$ has the property (WEE) but not perfect.

3. Modules with the properties (WE^*) and (WEE^*)

In this section, we study on modules with the property (WE^*) (resp. (WEE^*)), which have a weak supplement (resp. ample weak supplements) in every coatomic extension, as a generalization of modules with the property (WE) (resp. (WEE)). We prove that over a left V-ring R, every left R-module with the property (WE^*) is injective.

Proposition 8. *Let M be a module. If M has the property (WE^*), then every direct summand of M has the property (WE^*).*

Proof. Let M_1 be a direct summand of M and N be a coatomic extension of M_1. Then there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let N' be the external direct sum $N \oplus M_2$ and $\varphi : M \longrightarrow N'$ be the canonical embedding. Then $M \cong \varphi(M)$ has the property (WE^*). Note that $\frac{N}{M_1} \cong \frac{N \oplus M_2}{\varphi(M)} = \frac{N'}{\varphi(M)}$ is coatomic. Since $\varphi(M)$ has the property (WE^*), there exists a submodule V of N' such that $N' = \varphi(M) + V$ and $\varphi(M) \cap V \ll N'$. For the projection $\phi : N' \longrightarrow N$, we have that $M_1 + \phi(V) = N$. Also since $Ker(\phi) \subseteq \varphi(M)$, $\phi(\varphi(M) \cap V) \subseteq \phi(\varphi(M)) \cap \phi(V) = M_1 \cap \phi(V) \ll \phi(N') = N$. Hence $\phi(V)$ is a weak supplement of M_1 in N. □

Proposition 9. *A module M has the property (WEE^*) if and only if every submodule of M has the property (WE^*).*

Proof. Assume that every submodule of M has the property (WE^*). For a coatomic extension N of M, let $N = M + V$ for some submodule V of N. Then $\frac{N}{M} \cong \frac{V}{M \cap V}$ is coatomic and so V is a coatomic extension of $M \cap V$. Since $M \cap V$ has the property (WE^*), there exists a submodule K of V such that $V = M \cap V + K$ and $M \cap K \ll V$. Note that $N = M + V = M + (M \cap V + K) = M + K$. It follows that K is a weak supplement of M in N.

Conversely, let M be a module with the property (WEE^*) and let M_1 be any submodule of M. For a coatomic extension N of M_1, let $S = \frac{M \oplus N}{L}$, where the submodule L is the set of all elements $(m', -m')$ of $M \oplus N$ with $m' \in M_1$ and let $f : M \to S$ via $f(m) = (m, 0) + L$, $g : N \to S$ via $g(n) = (0, n) + L$ for all $m \in M, n \in N$. For the inclusion homomorphisms $\tau_1 : M_1 \to N$ and $\tau_2 : M_1 \to M$, we can draw the following pushout:

$$\begin{array}{ccc} M_1 & \xrightarrow{\tau_1} & N \\ \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & S \end{array}$$

It is clear that $S = Im(f) + Im(g)$. Now we define $\theta : S \to \frac{N}{M_1}$ by $\theta((m, n) + L) = n + M_1$ for all $(m, n) + L \in S$. Note that θ is an epimorphism and $Ker(\theta) = Im(f)$. It follows that $\frac{N}{M_1} \cong \frac{S}{Im(f)}$ is coatomic. Since f is monomorphism, by assumption, $Im(f)$ has the property (WEE^*). Then it follows immediately that $Im(f)$ has a weak supplement V in S such that $V \leq Im(g)$, i.e. $S = Im(f) + V$ and $Im(f) \cap V \ll S$. Then we obtain that $N = g^{-1}(Im(f)) + g^{-1}(V) = M_1 + g^{-1}(V)$ and $M_1 \cap g^{-1}(V) \ll N$. Hence $g^{-1}(V)$ is a weak supplement of M_1 in N. □

Recall from [2] a module M is called *cofinitely weak supplemented* if every cofinite submodule of M has a weak supplement in M. It is clear from Proposition 9 that if a module M has the property (WEE^*), then every maximal submodule of M has a weak supplement in M, equivalently M is cofinitely weak supplemented by [2, Theorem 2.16].

In [13], a module M is called *weakly radical supplemented (namely wrs-module)* if every submodule U of M with $Rad(M) \subseteq U$ has a weak supplement in M. A module M is called *semilocal* if $\frac{M}{Rad(M)}$ is semisimple. A ring R is semilocal if the left R-module ${}_R R$ is semilocal.

Corollary 3. *Let R be a semilocal ring and M be an R-module. If M has the property (WEE^*), then M is wrs-module.*

Proof. Let U be a submodule of M with $Rad(M) \subseteq U$. Since R is semilocal ring, it follows from [9, Theorem 3.5] that $\frac{M}{U}$ is semisimple as a factor module of the semisimple module $\frac{M}{Rad(M)}$. Hence $\frac{M}{U}$ is coatomic. By assumption and Proposition 9, U has a weak supplement in M. Hence M is a wrs-module. □

Proposition 10. *Over a left V-ring R, every left R-module with (WE^*) is injective.*

Proof. Let M be an R-module with (WE^*). Let N be any extension of M. Suppose that $Rad(\frac{N}{K}) = \frac{N}{K}$ for a submodule K of N. Since R is a left V-ring, $Rad(\frac{N}{K}) = 0$. Then it immediately follows that $N = K$. Hence N is coatomic. Then, by assumption, M has a weak supplement V in N, i.e. $N = M + V$ and $M \cap V \ll N$. Since R is a left V-ring, we obtain that $M \cap V \subseteq Rad(N) = 0$. This completes the proof. □

The next result can be directly obtained from Proposition 10 and Lemma 2.

Corollary 4. *Let R be a left V-ring and M be an R-module. The following statements are equivalent:*

(1) *M has the property (WE).*
(2) *M has the property (WE^*).*
(3) *M is injective.*

Now we shall give a characterization for semilocal rings via the modules that have a weak supplement in every extension with small radical.

Theorem 2. *For any ring R the following statements are equivalent:*

(1) *R is semilocal.*
(2) *Every left R-module with small radical is weakly supplemented.*
(3) *Every left R-module has a weak supplement in every extension with small radical.*

Proof. (1) ⇔ (2) follows from [9, Theorem 3.5].

(2) ⇔ (3) M be a left R-module and N be an extension of M with small radical. By hypothesis, M has a weak supplement in N. Conversely, let M be an R-module with small radical and U be a submodule of M. By assumption, U has a weak supplement in M. □

Finally, we give an example of modules that have a weak supplement in every extension with small radical but not have the property (CEE).

Example 3. (see [14, Section 42.13, Exercise 4]). Let R be the following subring of the rational numbers:

$R = \{\frac{m}{n} | m, n \in \mathbb{Z}, (m, n) = 1$, 2 and 3 are not divisors of n $\}$

Since $\frac{R}{Rad(R)}$ is semisimple, the left R-module $_RR$ is a module which has a weak supplement in every extension with small radical by Theorem 2. Whereas, since R is not semiperfect, $_RR$ does not have the property (CEE) by [5, Theorem 2.12].

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REFERENCES

[1] R. Alizade, G. Bilhan, and P. Smith, "Modules whose maximal submodules have supplements," *Comm. Algebra*, vol. 29, no. 6, pp. 2389–2405, 2001, doi: 10.1081/AGB-100002396.

[2] R. Alizade and E. Büyükaşık, "Cofinitely weak supplemented modules," *Comm. Algebra*, vol. 31, no. 11, pp. 5377–5390, 2003.

[3] R. Alizade and E. Büyükaşık, "Extensions of weakly supplemented modules," *Math. Scand.*, vol. 103, no. 2, pp. 161–168, 2008.

[4] E. Büyükaşik and C. Lomp, "Rings whose modules are weakly supplemented are perfect. Applications to certain ring extensions," *Math. Scand.*, vol. 105, pp. 25–30, 2009.

[5] H. Çalışıcı and E. Türkmen, "Modules that have a supplement in every cofinite extension," *Georgian Math. J.*, vol. 19, no. 2, pp. 209–216, 2012, doi: 10.1515/gmj-2012-0018.

[6] L. Fuchs and S. L., *Modules over Non-Noetherian Domains*, ser. Math. Surveys Monographs. American Mathematical Society, Providence, 2000.

[7] S. K. Jain, A. K. Srivastava, and A. A. Tuganbaev, *Cyclic modules and the structure of rings*, ser. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2012. doi: 10.1093/acprof:oso/9780199664511.001.0001.

[8] T. Lam, *Lectures on Modules and Rings*, ser. Graduate texts in mathematics. Springer New York, 1999. doi: 10.1007/978-1-4612-0525-8.

[9] C. Lomp, "On semilocal modules and rings," *Comm. Algebra*, vol. 27, no. 4, pp. 1921–1935, 1999.

[10] S. Özdemir, "Rad-supplementing modules," *arXiv: 1210.2893v1 [math.RA]*, 10 Oct. 2012.

[11] B. N. Türkmen, "Modules that have a supplement in every coatomic extension," *Miskolc Mathematical Notes*, vol. 16, no. 1, pp. 543–551, 2015.

[12] B. N. Türkmen, "On generalizations of injective modules," *Publications de l'Institut Mathematique*, (accepted).

[13] B. N. Türkmen and E. Türkmen, "On a generalization of weakly supplemented modules," *Analele Stiintifice ale Universitatii "Al. I. Cuza" din Iasi, Matematica*, 2015, doi: 10.1515/aicu-2015-0012.

[14] R. Wisbauer, *Foundations of module and ring theory*, ser. Algebra, Logic and Applications. Gordon and Breach Science Publishers, Philadelphia, PA, 1991, vol. 3, a handbook for study and research.

[15] H. Zöschinger, "Komplementierte Moduln über Dedekindringen," *J. Algebra*, vol. 29, pp. 42–56, 1974, doi: 10.1016/0021-8693(74)90109-4.

[16] H. Zöschinger, “Moduln, die in jeder Erweiterung ein Komplement haben,” *Math. Scand.*, vol. 35, pp. 267–287, 1974.

[17] H. Zöschinger, “Invarianten wesentlicher Überdeckungen,” *Math. Annalen*, vol. 237, no. 3, pp. 193–202, 1978, doi: 10.1007/BF01420175.

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