ANALYSIS OF CHAOTIC AND HYPERCHAOTIC CONSERVATIVE COMPLEX NONLINEAR SYSTEMS

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Abstract. The aim of this paper is to introduce and analyze chaotic and hyperchaotic conservative complex nonlinear systems. These systems appear in several branches of applied sciences. Lyapunov exponents are calculated to observe chaotic and hyperchaotic behaviors. The wide range of systems parameters at which chaotic and hyperchaotic solutions exist is calculated. The modules for complex variables are computed and plotted. The projections of chaotic and hyperchaotic solutions are shown in 3- spaces and 2-planes. The systems of this paper leave rooms for further studies in the near future, e.g. control and several types of synchronization of the solutions of these systems.

1. INTRODUCTION

In recent years chaotic and hyperchaotic dissipative complex nonlinear systems have been proposed and studied in the literature [1–16, 18]. However, there also are many interesting cases of conservative complex nonlinear systems have not yet been as actively explored. As examples, of dissipative systems we state here the chaotic complex Lorenz, Chen and Lü systems, and hyperchaotic complex Lorenz, Chen and Lü systems. More recently, simple conservative, autonomous, second-order chaotic complex variable systems are studied [17]. Conservative nonlinear systems with real variables are introduced and studied in the literature and references therein [15, 17, 19–21]. Complex systems appear in several areas of applied sciences, for example, high-energy accelerators, rotor dynamics, elastic pendulums with two degrees of freedom, robots and beams and plates under static and dynamics loadings [6]. Fore more details we refer the reader to the review paper [6] and references therein. Complex Lorenz, Chen and Lü systems many be used to describe and simulate physics of detuned lasers and thermal convection of liquid flows [1, 2, 13]. A nonlinear system is defined as a chaotic system if it has one positive Lyapunov exponent. It is called hyperchaotic system or hyperchaotic of order m ($m \geq 2$) if m of its Lyapunov exponents are positive. A nonlinear dynamical system is called conservative system if the

sum of its Lyapunov exponents is zero. For complex nonlinear dynamical systems, the observable or measurable physical quantities usually are modules. For example, in alternating current machine, the current amplitude is represented by module [18] so the module is more important that of real part and imaginary part.

In this paper, we introduce and study the basic properties of n-dimensional (nD) chaotic and hyperchaotic conservative complex nonlinear systems, and give examples on which we can illustrate our investigations. Consider the n-dimensional chaotic (or hyperchaotic) conservative complex nonlinear systems:

$$\dot{z} = Az + Bf(z), \tag{1.1}$$

where, $z = (z_1, z_2, ..., z_n)^T$ is a state complex vector with $z_i = u_{2i-1} + ju_{2i}$, $i = 1, 2, ..., n$, T denotes transpose, and $j = \sqrt{-1}$, $A, B \in C^{n \times n}$ are real (or complex) matrices of system parameters and $f = (f_1, f_2, ..., f_n)^T$ is a vector of nonlinear complex functions, dots represent derivative with respect to t. The module of z_i can be defined as $M_{z_i} = \sqrt{u_{2i-1}^2 + u_{2i}^2}$, $i = 1, 2, ..., n$.

For example, we propose a conservative chaotic complex system with $n = 2$ as:

$$\dot{z_1} = \alpha z_2, \tag{1.2}$$

$$\dot{z_2} = \beta \bar{z_1}^2 - \gamma z_1^2 \bar{z_1}, \tag{1.3}$$

where α, β and γ are positive real parameters. The function $\bar{z}$ means the complex conjugate of z. This system has three terms with two nonlinear terms in the right hand side. If system (1.3) contains only two terms, it becomes nonchaotic one as we checked numerically.

For hyperchaotic complex conservative systems, we introduce the following example (n=3):

$$\begin{aligned}
\dot{z_1} &= z_2, \\
\dot{z_2} &= -\mu z_1 + z_2 z_3, \\
\dot{z_3} &= 1 - z_2^2,
\end{aligned} \tag{1.4}$$

where μ is a positive real parameter. The counter real part of system (2.1) with $\mu = 1$ is a chaotic system as shown in Ref. [19].

The paper is organized as follows: In the next Section the complex behavior of the chaotic conservative complex nonlinear system (1.3) is studied. Based on the calculation of Lyapunov exponents, we calculated numerically the rang of parameter values of the system (1.3) at which chaotic solution exist. This system is chaotic for a wide range of the system parameters. In Section 3, we constructed different examples of chaotic conservative complex nonlinear systems, and their Lyapunov exponents are calculated. The hyperchaotic conservative complex nonlinear system (2.1) has been studied in Section 4. Finally, concluding remarks were given in the last Section.

2. THE CHAOTIC CONSERVATIVE COMPLEX SYSTEM (1.3)

In this section, we study the basic properties of the proposed chaotic conservative complex nonlinear system (1.3).

The real version of (1.3) which is a 4D real chaotic autonomous conservative system is:

$$
\begin{aligned}
\dot{u}_1 &= \alpha u_3, \\
\dot{u}_2 &= \alpha u_4, \\
\dot{u}_3 &= \beta(u_1^2 - u_2^2) - \gamma(u_1^3 + u_1 u_2^2), \\
\dot{u}_4 &= -\beta(2u_1 u_2) - \gamma(u_2^3 + u_2 u_1^2).
\end{aligned}
\tag{2.1}
$$

System (1.3) is conservative since

$$
\frac{\partial \dot{u}_1}{\partial u_1} + \frac{\partial \dot{u}_2}{\partial u_2} + \frac{\partial \dot{u}_3}{\partial u_3} + \frac{\partial \dot{u}_4}{\partial u_4} = 0.
\tag{2.2}
$$

(2.1) Lyapunov exponents of system (1.3)

In this subsection we calculate Lyapunov exponents of system (1.3). Based on these exponents, we show that our system (1.3) is a chaotic one.

System (2.2) in vector notation can be written as:

$$
\dot{U}(t) = h(U(t); \zeta),
\tag{2.3}
$$

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ is the state space vector, $h = [h_1, h_2, h_3, h_4]^T$, ζ is a set of parameters and $[...]^T$ denoting transpose. The equations for small deviations δU from the trajectory $U(t)$ are:

$$
\delta \dot{U}(t) = L_{ij}(U(t); \zeta)\delta U, \quad i, j = 1, 2, 3, 4
\tag{2.4}
$$

where $L_{i,j} = \dfrac{\partial h_i}{\partial u_j}$ is the Jacobian matrix of the form:

$$
L_{i,j} = \begin{pmatrix}
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & \alpha \\
2\beta u_1 - \gamma(3u_1^2 + u_2^2) & -2\beta u_2 - 2\gamma u_1 u_2 & 0 & 0 \\
-2\beta u_2 - 2\gamma u_1 u_2 & -2\beta u_1 - \gamma(3u_2^2 + u_1^2) & 0 & 0
\end{pmatrix}.
\tag{2.5}
$$

The Lyapunov exponents λ_i of the system is defined by:

$$
\lambda_i = \lim_{t \longrightarrow \infty} \frac{1}{t} \log \frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|}.
\tag{2.6}
$$

To find λ_i, Eqs. (2.3) and (2.4) must be numerically solved simultaneously. Runge-Kutta method of order 4 is used to calculate λ_i.

For the choice $\alpha = 10$, $\beta = 3$, and $\gamma = 2$ and the initial conditions $t_0 = 0$, $u_1(0) = 0.5$, $u_2(0) = 0.5$, $u_3(0) = 1$, and $u_4(0) = 0.0$, we calculate the Lyapunov exponents which are: $\lambda_1 = 2.1004$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$ and $\lambda_4 = -2.1004$.

This means that our system (1.3) for this choice of α, β, and γ has chaotic behavior since one of its Lyapunov exponents λ_1 is positive and conservative system since the sum of its Lyapunov exponents is zero ($\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$). It is clear that from the calculation of Lyapunov exponents that λ_1 equals λ_4 with negative sign.

(2.2) The range of system parameters of system (1.3)

In this subsection we calculate the range of the parameters α, β and γ at which chaotic behavior exist based on the calculations of Lyapunov exponents (2.6). The Lyapunov exponents versus the system parameters are plotted in Figure 1 using the initial conditions $t_0 = 0$, $u_1(0) = 0.5$, $u_2(0) = 0.5$, $u_3(0) = 1$, and $u_4(0) = 0$.

(i) Fix $\beta = 3$, $\gamma = 2$, and vary α:

In Figure 1a we plotted the corresponding Lyapunov exponents λ_i, $i = 1, 2, ..., 4$ of system (1.3) versus α. It is clear that from this Figure when $\alpha \in (0, 50]$ system (1.3) has chaotic solutions.

(ii) Fix $\alpha = 10$, $\gamma = 2$, and vary β:

One can concludes from Figure 1b, that system (1.3) exhibits chaotic solutions for $\beta \in [3, 5.8]$, $[7.3, 7.45]$, and $[7.8, 11]$, where λ_1 is positive in these intervals .

(iii) Fix $\alpha = 10$, $\beta = 3$, and vary γ:

As we did for α, β, we plotted λ_i, versus γ in Figure 1c and we see that (1.3) has chaotic solutions for $\gamma \in [0.56, 0.98]$, $[1.54, 10]$.

To show the chaotic solutions of system (1.3), we plotted them in different projections in 3-spaces as shown in Figure 2 and 2-planes as in Figure 3. In Figure 3g, we plotted the module M_{z_1}-M_{z_2} plane where $M_{z_1} = \sqrt{u_1^2 + u_2^2}$, and $M_{z_2} = \sqrt{u_3^2 + u_4^2}$.

3. DIFFERENT EXAMPLES OF CHAOTIC COMPLEX SYSTEMS

In this section we can construct different examples of the chaotic conservative complex nonlinear systems. We present here, three examples and calculate their Lyapunov exponents with the same initial conditions of Figure 1 as:

$$\dot{z_1} = \alpha z_2,$$
$$\dot{z_2} = \bar{z_1}^2 - z_1^2 \bar{z_1}. \tag{3.1}$$

For the case $\alpha = 10$, the Lyapunov exponents for system (3.1) are:
$\lambda_1 = 0.5713$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$, $\lambda_4 = -0.5713$.

$$\dot{z_1} = \alpha z_2,$$
$$\dot{z_2} = \beta \bar{z_1}^2 - z_1^2 \bar{z_1}. \tag{3.2}$$

For the case $\alpha = 10$, $\beta = 2$, the Lyapunov exponents for system (3.2) are:
$\lambda_1 = 1.8522$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$, $\lambda_4 = -1.8522$.

$$\dot{z_1} = \alpha z_2,$$
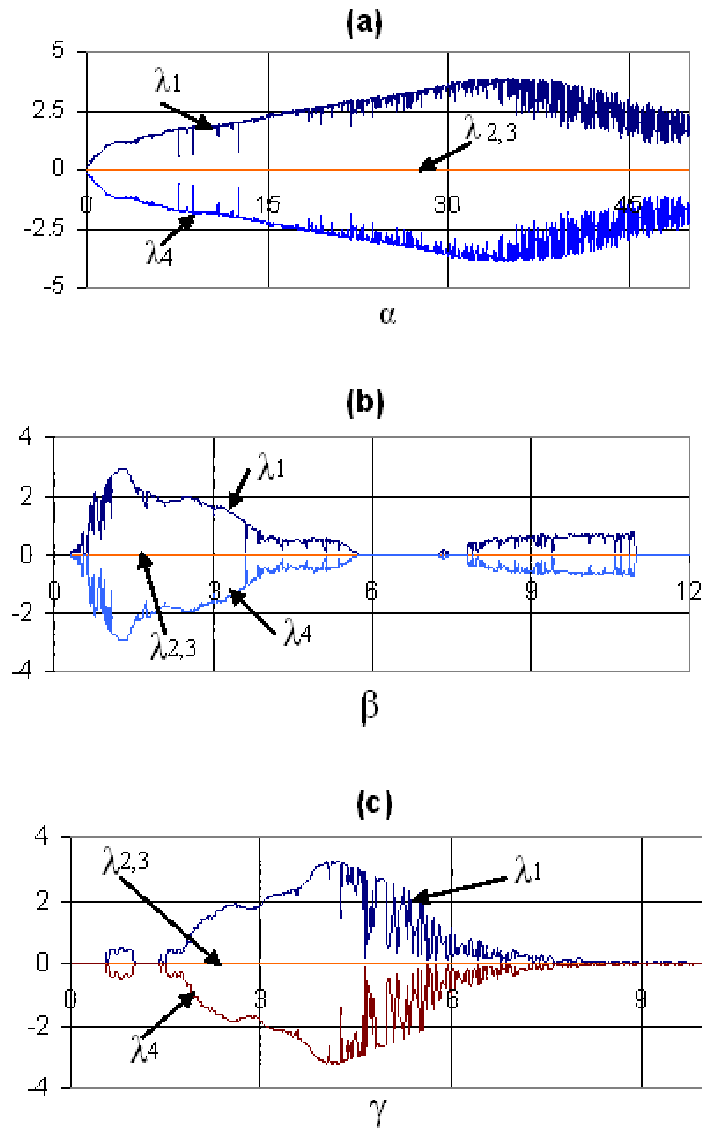$$\dot{z_2} = \bar{z_1}^2 - \beta z_1^2 \bar{z_1}. \tag{3.3}$$

FIGURE 1. Lyapunov exponents of system (1.3) for the choice $\alpha = 10$, $\beta = 3$, and $\gamma = 2$ and the initial conditions $t_0 = 0$, $u_1(0) = 0.5$, $u_2(0) = 0.5$, $u_3(0) = 1$, and $u_4(0) = 0.0$, (a) λ_1, λ_2, λ_3 and λ_4 versus α, (b) λ_1, λ_2, λ_3 and λ_4 versus β, (c) λ_1, λ_2, λ_3 and λ_4 versus γ.
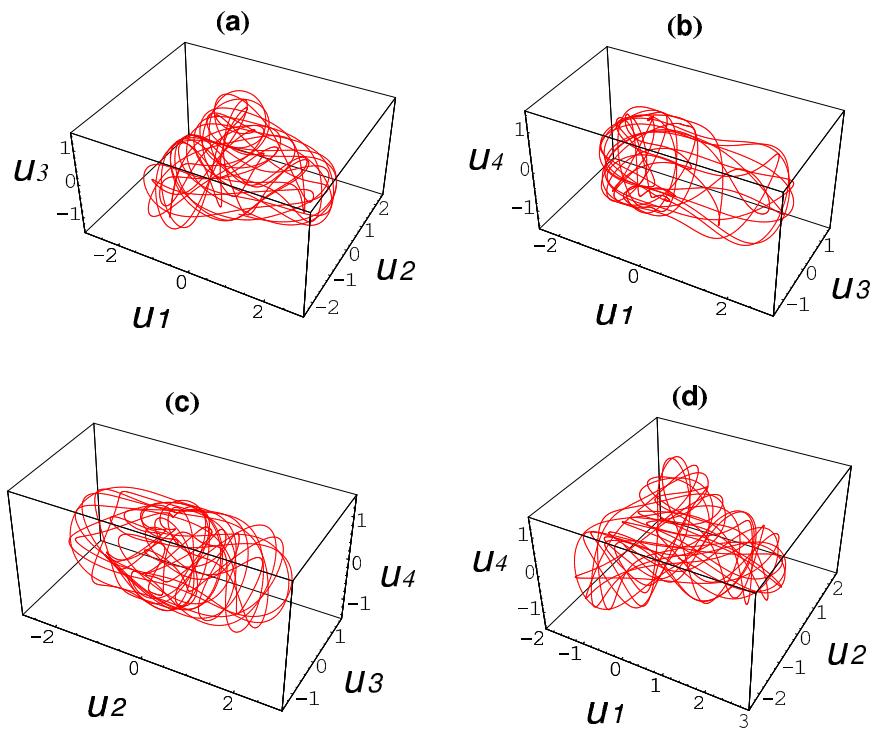
FIGURE 2. The chaotic solution of system (1.3) with the same initial conditions as in Figure 1 for the choice $\alpha = 10$, $\beta = 3$, and $\gamma = 2$. (a) (u_1, u_2, u_3) space, (b) (u_1, u_3, u_4) space, (c) (u_2, u_3, u_4) space, (d) (u_1, u_2, u_4) space.

For the case $\alpha = 6$, $\beta = 3$, the Lyapunov exponents for system (3.3) are: $\lambda_1 = 0.3650$, $\lambda_2 = 0.0$, $\lambda_3 = 0.0$, $\lambda_4 = -0.3650$.
As we had for system (1.3), λ_1 and λ_4 are the same but with negative sign for our systems (3.1), (3.2) and (3.3). Therefor 2D complex systems may be not hyperchaotic ones. The systems of this section leave rooms for further investigations, for example,
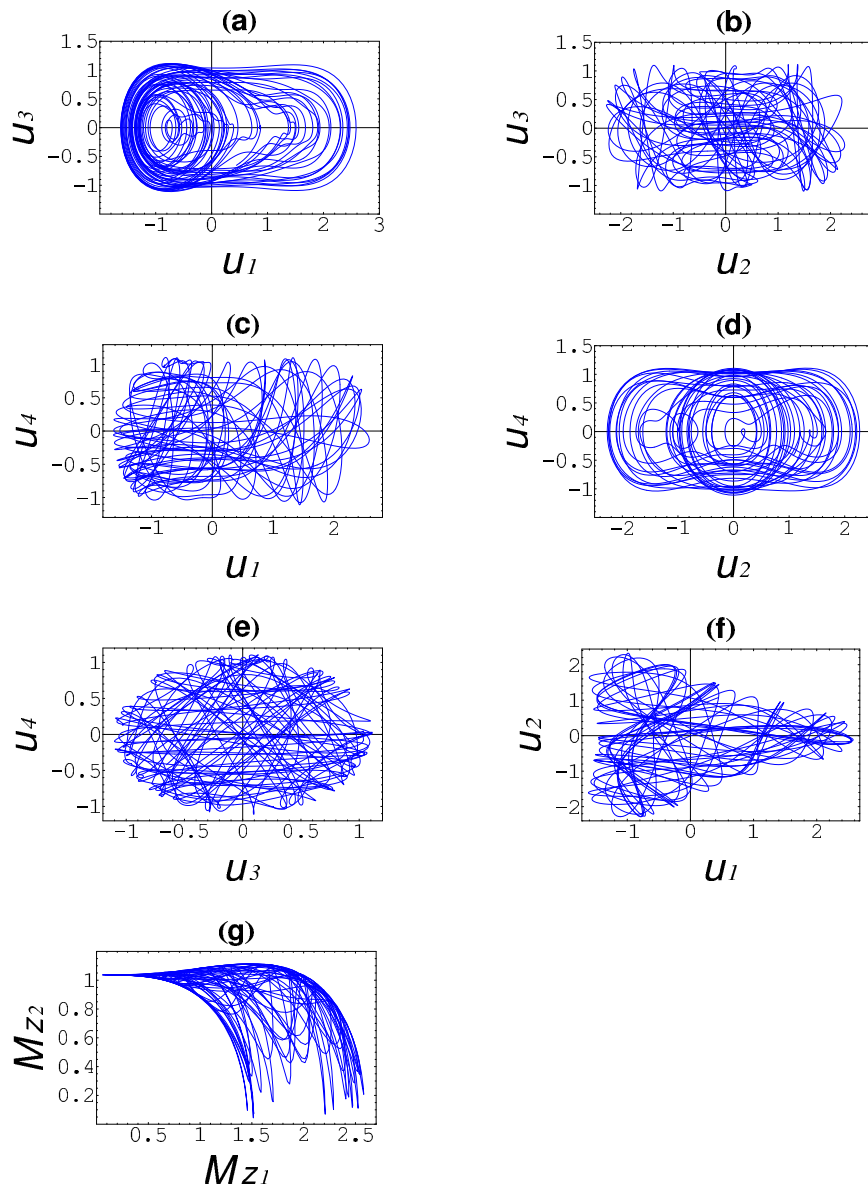
FIGURE 3. The chaotic solution of system (1.3) with the same initial conditions as in Figure 1 for $\alpha = 10$, $\beta = 3$, and $\gamma = 2$. (a) $u_1 - u_3$ plane, (b) $u_2 - u_3$ plane, (c) $u_1 - u_4$ plane, (d) $u_2 - u_4$ plane, (e) $u_3 - u_4$ plane, (f) $u_1 - u_2$ plane, (g) $M_{z_1} - M_{z_2}$ plane.

control and several types of synchronization (complete , modified projective and generalized synchronization) of solutions of these systems.

4. THE HYPERCHAOTIC CONSERVATIVE COMPLEX SYSTEM (2.1)

This section is devoted to analyze the hyperchaotic conservative complex nonlinear system (2.1).
The real version of (2.1) reads:

$$
\begin{aligned}
\dot{u}_1 &= u_3, \\
\dot{u}_2 &= u_4, \\
\dot{u}_3 &= -\mu u_1 + u_3 u_5 - u_4 u_6, \\
\dot{u}_4 &= -\mu u_2 + u_3 u_6 + u_4 u_5, \\
\dot{u}_5 &= 1 - (u_3^2 - u_4^2), \ \dot{u}_6 = -2u_3 u_4.
\end{aligned}
\tag{4.1}
$$

(4.1) Lyapunov exponents of system (2.1)

Based on Lyapunov exponents, we compute parameters values of our system (2.1) at which hyperchaotic solutions exist. The Jacobian matrix of our system (4.2) is of the form:

$$
L_{i,j} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-\mu & 0 & u_5 & -u_6 & u_3 & -u_4 \\
0 & -\mu & u_6 & u_5 & u_4 & u_3 \\
0 & 0 & -2u_3 & 2u_4 & 0 & 0 \\
0 & 0 & -2u_4 & -2u_3 & 0 & 0
\end{pmatrix}.
\tag{4.2}
$$

For the choice of $\mu = 1.5$, and the initial conditions $t_0 = 0$, $u_1(0) = 0$, $u_2(0) = 0$, $u_3(0) = 5$, $u_4(0) = 0$, $u_5(0) = 0$, $u_6(0) = 0$, we calculate the Lyapunov exponents as $\lambda_1 = 0.055$, $\lambda_2 = 0.055$, $\lambda_3 = 0.0$, $\lambda_4 = 0.0$, $\lambda_5 = -0.055$, $\lambda_6 = -0.055$. This means that our system (2.1) is hyperchaotic (λ_1, λ_2 are positive) and conservative system since the sum of its Lyapunov exponents is zero ($\lambda_1 + \lambda_2 + ... + \lambda_6 = 0$).
In Figure 4a we plotted the corresponding Lyapunov exponents λ_i, $i = 1, 2, ..., 6$ of system (2.1)versus μ using the initial conditions $t_0 = 0$, $u_1(0) = 0$, $u_2(0) = 0$, $u_3(0) = 5, u_4(0) = 0$, and $u_5(0) = 0$. It is clear that from Figure 4a, when $\mu \in (0.4, 1.82]$, $(1.85, 1.92]$ and $(2.58, 2.81]$ the system (2.1) has hyperchaotic solutions since $\lambda_1 = \lambda_2 > 0$, $\lambda_3 = \lambda_4 = 0$ and $\lambda_5 = \lambda_6 < 0$. In Figures 4b,4c,4d,4e we plotted hyperchaotic solutions in 3-spaces. Figure 5 contains hyperchaotic solutions in 3-spaces. The modules of the complex variables z_1, z_2 and z_3, are displayed in 2-planes in Figures 5d, 5e and 5f.
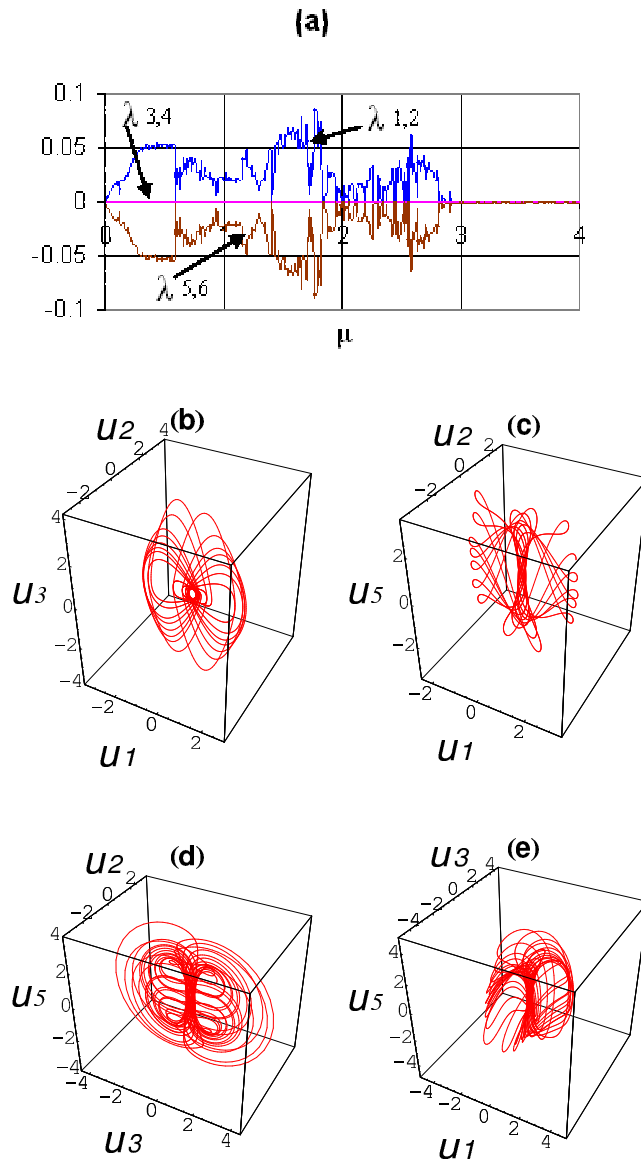
FIGURE 4. Lyapunov exponents and the hyperchaotic solution of (2.1) for $\mu = 1.5$ and the initial conditions $t_0 = 0$, $u_1(0) = 0$, $u_2(0) = 0$, $u_3(0) = 5$, $u_4(0) = 0$, $u_5(0) = 0$, $u_6(0) = 0$. (a) λ_1, λ_2, λ_3, λ_4, λ_5, and λ_6 versus μ, (b) (u_1, u_2, u_3) space, (c) (u_1, u_2, u_5) space, (d) (u_3, u_2, u_5) space, (e) (u_1, u_3, u_5) space.
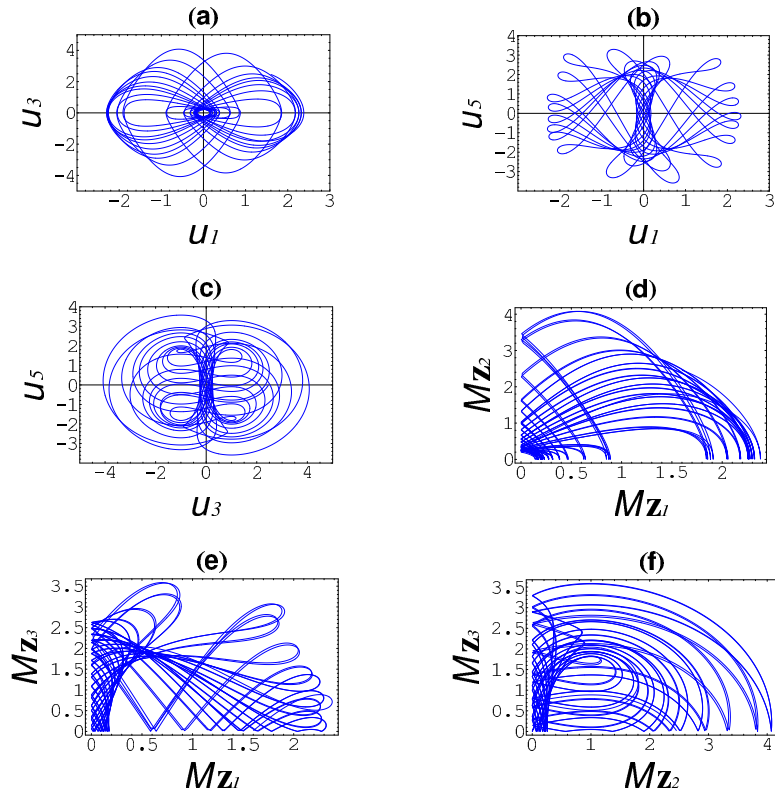
FIGURE 5. A hyperchaotic solution of system (2.1) with the same initial conditons as in Figure 4 and for $\mu = 1.5$: (a) $u_1 - u_3$ plane, (b) $u_1 - u_5$ plane, (c) $u_3 - u_5$ plane, (d) $M_{z_1} - M_{z_2}$ plane, (e) $M_{z_1} - M_{z_3}$ plane, (f) $M_{z_2} - M_{z_3}$ plane.

5. CONCLUSIONS

Our main goal in this paper was to introduce and analyze chaotic and hyperchaotic conservative complex nonlinear systems (1.3), (2.1), (3.1), (3.2) and (3.3). The sum of their Lyapunov exponents is zero. These systems have not yet been actively explored in the literature. We calculated numerically the values of the parameters at which chaotic and hyperchaotic solutions of system (1.3) and (2.1) exist in Figures 1 and 4a respectively. These solutions are exist for large values of system parameters, for example $\alpha \in (0.0, 50]$ and $\gamma \in (1.54, 10]$. The modules of the complex variables z_1, z_2 and z_3 of our systems (1.3) and (2.1) are displayed in Figure 3g, 5d, 5e, and 5f. Control and synchronization (e.g. complete synchronization, anti-synchronization, projective and modified projective synchronization and

lag synchronization) of chaotic and hyperchaotic solutions of our systems, need further investigations and will appear in the near future.

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