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ALGEBRAIC CONE B-METRIC SPACES AND ITS EQUIVALENCE

HAMIDREZA RAHIMI, GHASEM SOLEIMANI RAD, AND STOJAN RADENOVIĆ

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Abstract. In the present work, we introduce the notion of algebraic cone b-metric space, which is a generalization of algebraic cone metric space. Then we prove that for every complete algebraic b-metric space there exists a correspondent isomorphic complete usual (associated) b-metric space via two approach (nonlinear scalarization function and Minkowski functional).

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1. Introduction and preliminaries

Consistent with Niknam et al. [15, 23], Du [9] and Nikolskij [16], the following definitions and results will be needed in the sequel.

Let Y be a real vector space and K be a convex subset of Y. A point $x \in K$ is said to be an algebraic interior point of K if for each $v \in Y$ there exists $\epsilon > 0$ such that $x + tv \in K$, for all $t \in [0, \epsilon]$. This definition is equivalent to the following statement:

A point x is called an algebraic interior point of the convex set $K \subseteq Y$ if $x \in K$ and for each $v \in Y$ there exists $\epsilon > 0$ such that $[x, x + \epsilon v] \subset K$, where $[x, x + \epsilon v] = \{\lambda x + (1-\lambda)(x + \epsilon v) : \forall \lambda \in [0,1]\}$. The set of all algebraic interior points of K is called algebraic interior and is denoted by $aint\ K$. Also, K is called algebraically open if $K = aint\ K$.

Let Y be vector space with the zero vector θ. A proper nonempty and convex subset K of E is called an algebraic cone if $K + K \subset K$, $\lambda K \subset K$ for $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$. Given a algebraic cone $K \subset E$, a partial ordering $\preceq_a$ with respect to K is defined by $x \preceq_a y \Leftrightarrow y - x \in K$. We shall write $x \prec_a y$ to mean $x \preceq_a y$ and $x \neq y$. Also, we write $x \ll_a y$ if and only if $y - x \in aint\ K$, where $aint\ K$ is the algebraic interior of K. Also, Y is said to be Archimedean if for each $x, y \in Y$ there exists $n \in \mathbb{N}$ such that $x \preceq_a ny$.

Lemma 1 ([15, 23]). *Let Y be a real vector space and K be an algebraic cone in Y with non-empty algebraic interior.*

(i) $K + aint\ K \subset aint\ K$;

(ii) *$\alpha aint\ K \subset aint\ K$, for each $\alpha > 0$;*
(iii) *For any $x, y, z \in X$, $x \preceq_a y$ and $y \ll_a z$ implies that $x \ll_a z$.*

Definition 1 ([15,23])**.** Let X be a nonempty set and (Y, K) be an algebraic cone space with $aint\ K \neq \varnothing$. Suppose that a vector valued function $d_a : X \times X \to Y$ satisfies the following conditions:
$(ACM1)$ $\theta \preceq_a d_a(x,y)$ for all $x, y \in X$ and $d_a(x,y) = \theta$ if and only if $x = y$;
$(ACM2)$ $d_a(x,y) = d_a(y,x)$ for all $x, y \in X$;
$(ACM3)$ $d_a(x,z) \preceq_a d_a(x,y) + d_a(y,z)$ for all $x, y, z \in X$.
Then d_a is called an algebraic cone metric and (X, d_a) is called an algebraic cone metric space.

Ordered normed spaces and cones have many applications in applied mathematics and optimization theory [8,19,24]. Fixed point theory in K-metric and K-normed spaces was developed in the mid-20th century ([17], see also [2,19,24]). The main idea consists in using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Huang and Zhang [10] reintroduced such spaces under the name of cone metric spaces and obtained some fixed point results (see also [1,11, 18,20–22] and references contained therein).

On the other hand, topological vector space-valued cone metric space (or tvs-cone metric space) introduced by Du [9] as a generalization of the Banach-valued cone metric space. Actually, Du has shown that some of fixed point results in cone metric spaces can be obtained in an easier way, using the so-called nonlinear scalarization function. Also, in 2011, Kadelburg et al. [13] have shown that the same can be obtained even more easily using Minkowski functionals in topological vector spaces. Their approach is even easier than that of Du [9]. The nonlinear scalarization function [6,9] $\xi_e : E \to \mathbb{R}$ is defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}$$

for all $y \in E$, where $e \in int\ K$ is fixed. Consider real vector spaces Y instead of topological vector spaces E. Thus, we have the following definition.

Definition 2 ([15,23])**.** Let Y be a real vector space, K be an algebraic cone in Y and $e \in aint\ K$. The nonlinear scalarization functio $\xi_e : Y \to \mathbb{R}$ is defined as follow:

$$\xi_e(y) = \inf M_{e,y},$$

where

$$M_{e,y} = \{r \in \mathbb{R} : y \in re - K\}.$$

Lemma 2 ([15,23])**.** *For each $e \in aint\ K$, $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

(i) *$\xi_e(y) < r$ if and only if $y \in re - aint\ K$;*
(ii) *$\xi_e(.)$ is positively homogeneous on Y;*
(iii) *if $y_1 \in y_2 + K$ (indeed, $y_2 \preceq_a y_1$), then $\xi_e(y_2) \leq \xi_e(y_1)$;*

(iv) $\xi_e(y_1+y_2) \leq \xi_e(y_1)+\xi_e(y_2)$ *for all* $y_1, y_2 \in Y$;
(v) $\xi_e(y) \geq 0$ *and if* $y \in aint\ K$, *then* $\xi_e(y) > 0$.

Also, the notion of a b-metric space was introduced by Bakhtin [4] and Czerwik [7] as a generalization of metric space.

Definition 3. Let X be a nonempty set and $s \geq 1$ be a real number. Suppose the mapping $d_s : X \times X \to [0,\infty)$ satisfies
$(d1)$ $d_s(x,y) = 0$ if and only if $x = y$;
$(d2)$ $d_s(x,y) = d_s(y,x)$ for all $x, y \in X$;
$(d3)$ $d_s(x,z) \leq s(d_s(x,y)+d_s(y,z))$ for all $x, y, z \in X$.
(X, d_s) is called a b-metric space [4, 7] or metric type space [14].

Obviously, for $s = 1$, b-metric space (or type metric space) is a metric space. In the sequel of this section we suppose that K has the Archimedean property.

2. Main results

Definition 4. Let X be a nonempty set, (Y, K) be an algebraic cone space with $aint\ K \neq \varnothing$ and $s \geq 1$ be a given real number. Suppose that a vector valued function $d_a : X \times X \to Y$ satisfies the following conditions:
$(ACbM1)$ $\theta \preceq_a d_a(x,y)$ for all $x, y \in X$ and $d_a(x,y) = \theta$ if and only if $x = y$
$(ACbM2)$ $d_a(x,y) = d_a(y,x)$ for all $x, y \in X$;
$(ACbM3)$ $d_a(x,z) \preceq_a s[d_a(x,y)+d_a(y,z)]$ for all $x, y, z \in X$.
Then d_a is called an algebraic cone b-metric and (X, d_a) is called an algebraic cone b-metric space.

Definition 5. Let (X, d_a) be an algebraic cone b-metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then the following statements hold:

(i) $\{x_n\}$ algebraic b-cone converges to x if, for every $c \in Y$ with $\theta \ll_a c$ there exists an $n_0 \in \mathbb{N}$ such that $d_a(x_n, x) \ll_a c$ for all $n > n_0$. We denote this by $d_a - \lim_{n\to\infty} x_n = x$ or $x_n \to_{d_a} x$ as $n \to \infty$;
(ii) $\{x_n\}$ is called an algebraic b-cone Cauchy sequence if, for every $c \in Y$ with $\theta \ll_a c$ there exists an $n_0 \in \mathbb{N}$ such that $d_a(x_n, x_m) \ll_a c$ for all $m, n > n_0$;
(iii) (X, d_a) is complete algebraic cone b-metric space if every algebraic b-cone Cauchy sequence in X is convergent in X.

The following theorem is one of the main results in this paper.

Theorem 1. *Let* (X, d_a) *be an algebraic cone b-metric space and* $e \in aint\ K$. *Then* $d_s : X \times X \to [0,\infty)$ *defined by* $d_s = \xi_e \circ d_a$ *is a b-metric on* X.

Proof. Clearly, $d_s(x,y) = d_s(y,x)$ for all $x, y \in X$. By Lemma 2, we have $d_s(x,y) \geq 0$ for all $x, y \in X$. If $x = y$, then, by $(ACbM1)$, we have $d_a(x,y) = 0$. Conversely, if $d_s(x,y) = 0$, then, by Lemma 1 and $(ACbM1)$, we have $d_a(x,y) \in$

$K \cap (-K) = \{\theta\}$ for all $x, y \in X$ which implies that $d_a(x,y) = \theta$. Consequently, $x = y$. Also, by applying (ii), (iii) and (iv) of Lemma 2 and $(ACbM3)$, we have

$$\xi_e(d_a(x,z)) \leq s(\xi_e(d_a(x,y)) + \xi_e(d_s a(y,z)))$$

or

$$d_s(x,z) \leq s[d_s(x,y) + d_s(y,z)]$$

for all $x, y, z \in X$ with $s \geq 1$. Thus, the proof of the theorem is complete. □

Theorem 2. *Let (X, d_a) be an algebraic cone b-metric space, $\{x_n\}$ a sequence in X, $x \in X$ and $e \in a int\ K$. Set $d_s = \xi_e \circ d_a$. Then the following statements hold:*

(i) *$\{x_n\}$ converges to x in algebraic cone b-metric space (X, d_a) if and only if $d_s(x_n, x) \to 0$ as $n \to \infty$;*
(ii) *$\{x_n\}$ is a Cauchy sequence in algebraic cone b-metric space (X, d_a) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_s);*
(iii) *(X, d_a) is complete if and only if (X, d_s) is complete.*

Proof. Using a similar argument as in Niknam et al's works [15, 23], the reader can prove this theorem. □

The following theorem is a version for algebraic cone b-metric spaces of Banach contraction principle [5].

Theorem 3. *Let (X, d_a) be a complete algebraic cone b-metric space with $s \geq 1$ and $\lambda \in [0, 1/s)$. If $f : X \to X$ satisfies the contractive condition*

$$d_a(fx, fy) \preceq_a \lambda d_a(x,y),$$

for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{f_n x\}_{n \in \mathbb{N}}$ converges to the unique fixed point of f.

Proof. Set $d_s = \xi_e \circ d_a$. Theorem 1 implies that (X, d_a) is a b-metric spaces and Theorem 2 implies that the b-metric space (X, d_a) is complete. On the other hand, by applying Theorem 1 and Lemma 2, we conclude that

$$d_a(fx, fy) \preceq_a \lambda d_a(x,y) \Longrightarrow d_s(fx, fy) \leq \lambda d_s(x,y)$$

for all $x, y \in X$. Therefore, the conclusion follows from the Theorem 3.3 of Jovanović et al [12]. The proof is completed. □

Note that we just prove the Banach fixed point theorem in the setting of algebraic cone b-metric space can be easily derived from the existing result in the context of b-metric space. Using this approach, other fixed point results in algebraic cone b-metric spaces can be obtained from the existing result in b-metric space.

3. Other approach and application

Now, we obtain other procedure to obtain above results and Shamsi et al. [23].

Let V be an absolutely convex and absorbing subset of a tvs E, its Minkowski functional is defined by $q_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$ for $x \in E$. It is a semi-norm on E and $V \subset W$ implies that $q_W(x) \leq q_V(x)$ for $x \in E$. If V is an absolutely convex neighborhood of $\theta_E \in E$, then q_V is continuous and

$$\{x \in E : q_V(x) < 1\} = int\ V \subset V \subset \bar{V} = \{x \in E : q_V(x) \leq 1\}.$$

Similar to the case of scalarization method mentioned above, set real vector spaces Y instead of topological vector spaces E.

Now, let (Y,K) be an algebraic cone space and let $e \in aint\ K$. Then $[-e,e] = (K-e)\cap(e-K) = \{z \in E : -e \preceq z \preceq e\}$ is an absolutely convex neighborhood of θ; its Minkowski functional $q_{[-e,e]}$ will be denoted by q_e. Also, $aint\ [-e,e] = (aint\ K - e) \cap (e - aint\ K)$.

Theorem 4. *Let (X,d_a) be an algebraic cone b-metric space and $e \in aint\ K$. Let q_e be the corresponding Minkowski functional of $[-e,e]$. Then $d_s = q_e \circ d_a$ is a b-metric on X.*

Proof. Clearly, $d_s(x,y) = d_s(y,x)$ for all $x,y \in X$ and $x = y$ implies that $d_s(x,y) = 0$. Also, since q_e is a semi-norm and d_a is an algebraic cone b-metric space, we have

$$q_e(d_a(x,z)) \leq s(q_e(d_a(x,y)) + q_e(a_a(y,z)))$$

or

$$d_s(x,z) \leq s[d_s(x,y) + d_s(y,z)]$$

for all $x,y,z \in X$ with $s \geq 1$. Now, we prove $d_s(x,y) = 0$ implies that $x = y$. Let $q_e \circ d_a(x,y) = 0$. Then $\inf\{\lambda > 0 : d_a(x,y) \in \lambda[-e,e]\} = 0$. Thus, there exists a sequence of positive scalars $\lambda_n \to 0$ such that $d_a(x,y) \in \lambda_n[-e,e]$. Suppose that $x \neq y$ (by contrary). Then, since $\theta_E \prec_a d_a \preceq_a \lambda_n e$, for each $c \in aint\ K$ there exists n_0 such that $d_a(x,y) \ll_a c$ for $n \geq n_0$. Since c is an arbitrary algebraic interior point of the cone K it follows that $d_a(x,y) = \theta$. This is a contradiction. Thus, the proof of the theorem is complete. □

The following consequences of Theorem 4 are evident.

Theorem 5. *Let (X,d_a) be an algebraic cone b-metric space, $\{x_n\}$ a sequence in X, $x \in X$ and $e \in aint\ K$. Set $d_s = q_e \circ d_a$. Then the following statements hold:*

(i) *$\{x_n\}$ converges to x in algebraic cone b-metric space (X,d_a) if and only if $d_s(x_n,x) \to 0$ as $n \to \infty$;*
(ii) *$\{x_n\}$ is a Cauchy sequence in algebraic cone b-metric space (X,d_a) if and only if $\{x_n\}$ is a Cauchy sequence in (X,d_s);*
(iii) *(X,d_a) is complete if and only if (X,d_s) is complete.*

Theorem 6. *Let (X, d_a) be a complete algebraic cone b-metric space with $s \geq 1$ and $\lambda \in [0, 1/s)$. If $f : X \to X$ satisfies the contractive condition*

$$d_a(fx, fy) \preceq_a \lambda d_a(x, y),$$

for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{f_n x\}_{n \in \mathbb{N}}$ converges to the unique fixed point of f.

Proof. Set $d_s = q_e \circ d_a$. Theorem 4 implies that (X, d_s) is a b-metric spaces and Theorem 5 implies that the b-metric space (X, d_s) is complete. On the other hand, by applying Theorem 4, we conclude that

$$d_a(fx, fy) \preceq_a \lambda d_a(x, y) \Longrightarrow d_s(fx, fy) \leq \lambda d_s(x, y)$$

for all $x, y \in X$. Therefore, the conclusion follows from the Theorem 3.3 of Jovanović et al [12]. This complete the proof. □

In Theorems 4, 5 and 6, consider $s = 1$. Then, we can obtain these results in the setting of algebraic cone metric spaces (as well as Niknam et al. [15,23] proved these results). Very recently, Akbari and Bagheri [3] proved several fixed point results in setting of algebraic cone metric spaces. Using our results in Section 2 and Section 3, then some results of Akbari and Bagheri [3] are not such actual.

As an application, we prove the equivalence between algebraic cone norm and usual norm.

Example 1. Let X be a vector space over F ($\mathbb{R}$ or $\mathbb{C}$) and $||.||_a : X \to E$ be a mapping that satisfies:

$(ACN1)$ $\theta \ll_a ||x||$ for all $x \in X \setminus \{\theta_X\}$ and $||x||_a = \theta$ if and only if $x = \theta_X$, where θ_X is the zero vector in X;

$(ACN2)$ $||\alpha x||_a = |a| ||x||_a$ for all $x \in X$ and $\alpha \in F$;

$(ACN3)$ $||x + y||_a \preceq_a ||x||_a + ||y||_a$.

Then, $||.||_a$ is called an algebraic cone norm on X and $(X, ||.||_a)$ is called an algebraic cone normed space [23]. Now, for all $e \in aint\ K$, $||.|| : X \to [0, \infty)$ defined by $||.|| = q_e \circ ||.||_a$ is a usual norm on X.

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References

[1] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *J. Math. Anal. Appl.*, vol. 341, pp. 416–420, 2008, doi: 10.1016/j.jmaa.2007.09.070.

[2] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed point theory and application*. Cambridge: Cambridge University Press, 2004.

[3] M. Akbari Tootkaboni and A. Bagheri Salec, "Algebraic cone metric spaces and fixed point theorems of contractive mappings," *Fixed Point Theory and Appl.*, vol. 2014:160, 2014, doi: 10.1186/1687-1812-2014-160.

[4] I. A. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Funct. Anal. Unianowsk Gos. Ped. Inst.*, vol. 30, pp. 26–37, 1989.

[5] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fund. Math. J.*, vol. 3, pp. 133–181, 1922.

[6] G. Y. Chen, X. Q. Yang, and H. Yu, "A nonlinear scalarization function and generalized quasi-vector equilibrium problem," *J. Global Optim.*, vol. 32, no. 4, pp. 451–466, 2005, doi: 10.1007/s10898-003-2683-2.

[7] S. Czerwik, "Nonlinear set-valued contraction mappings in b-metric spaces," *Atti Semin. Mat. Fis. Univ. Modena*, vol. 46, no. 2, pp. 263–276, 1998.

[8] K. Deimling, *Nonlinear functional analysis*. Springer-Verlag, 1985.

[9] W. S. Du, "A note on cone metric fixed point theory and its equivalence," *Nonlinear Anal.*, vol. 72, no. 5, pp. 2259–2261, 2010, doi: 10.1016/j.na.2009.10.026.

[10] L. G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 332, pp. 1468–1476, 2007, doi: 10.1016/j.jmaa.2005.03.087.

[11] S. Janković, Z. Kadelburg, and S. Radenović, "On cone metric spaces; a survey," *Nonlinear Anal.*, vol. 74, no. 7, pp. 2591–2601, 2011, doi: 10.1016/j.na.2010.12.014.

[12] M. Jovanović, Z. Kadelburg, and S. Radenović, "Common fixed point results in metric-type spaces," *Fixed Point Theory and Appl.*, vol. 2010:978121, 2010, doi: 10.1155/2010/978121.

[13] Z. Kadelburg, S. Radenović, and V. Rakočević, "A note on the equivalence of some metric and cone metric fixed point results," *Appl. Math. Lett.*, vol. 24, no. 3, pp. 370–374, 2011, doi: 10.1016/j.aml.2010.10.030.

[14] M. A. Khamsi and N. Hussain, "KKM mappings in metric type spaces," *Nonlinear Anal.*, vol. 73, no. 9, pp. 3123–3129, 2010, doi: 10.1016/j.na.2010.06.084.

[15] A. Niknam, S. Shamsi Gamchi, and M. Janfada, "Some results on TVS-cone normed spaces and algebraic cone metric spaces," *Iranian J. Math. Sci. Inf.*, vol. 9, no. 1, pp. 71–80, 2014.

[16] N. K. Nikolskij, *Functional analysis I*. Springer-Verlag, 1992.

[17] A. I. Perov, "The Cauchy problem for systems of ordinary differential equations, approximate methods of solving differential equations," *Kiev. Naukova Dumka.*, pp. 115–134, 1964.

[18] H. Rahimi, B. E. Rhoades, S. Radenović, and G. Soleimani Rad, "Fixed and periodic point theorems for T-contractions on cone metric spaces," *Filomat.*, vol. 27, no. 5, pp. 881–888, 2013, doi: 10.2298/FIL1305881R.

[19] H. Rahimi and G. Soleimani Rad, *Fixed point theory in various spaces*. Germany: LAP LAMBERT Academic Publishing, 2013.

[20] H. Rahimi and G. Soleimani Rad, "Common fixed-point theorems and c-distance in ordered cone metric spaces," *Ukr. Math. J.*, vol. 65, no. 12, pp. 1845–1861, 2014.

[21] H. Rahimi and G. Soleimani Rad, "Fixed point theorems under c-distance in ordered cone metric space," *Int. J. Industrial Mathematics.*, vol. 6, no. 2, pp. 97–105, 2014.

[22] S. Rezapour and R. Hamlbarani, "Some note on the paper cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 345, pp. 719–724, 2008, doi: 10.1016/j.jmaa.2008.04.049.

[23] S. Shamsi Gamchi, M. Janfada, and A. Niknam, "On an equivalence of the topology of algebraic cone metric spaces and metric spaces," *arXiv*, vol. 2014, 2014.

[24] P. P. Zabrejko, "K-metric and K-normed linear spaces: Survey," *Collect. Math.*, vol. 48, no. 4-6, pp. 825–859, 1997.

Authors' addresses

Hamidreza Rahimi

Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, P. O. Box 13185/768, Tehran, Iran.

E-mail address: rahimi@iauctb.ac.ir

Ghasem Soleimani Rad

Young Researchers and Elite club, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

E-mail address: gha.soleimani.sci@iauctb.ac.ir; gh.soleimani2008@gmail.com

Stojan Radenović

Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

Current address: Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

E-mail address: stojan.radenovic@tdt.edu.vn