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MIXED MONOTONE ITERATIVE TECHNIQUE FOR FRACTIONAL IMPULSIVE EVOLUTION EQUATIONS

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Abstract. In this paper, we deal with the existence of the mild solutions to the fractional impulsive evolution equations. By definitions of the lower and upper quasi-solutions and technique of mixed monotone iterative, we get several existence results. The results are new and extend previously known results. An example illustrates the main results.

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1. Introduction

It is well known that fractional differential equation is one of the most valuable tools in modeling of many phenomena in various fields, such as physics, chemistry, aerodynamics, etc(see[8,23]). Recently, many researchers pay more attention to fractional evolution equations because they are applied more widely than the ordinary differential equations. There has been a significant theoretical development in fractional evolution equations(see[1,3,4,10–21,24]).

As for impulsive differential equations, they are used to describe the dynamics of real processes and phenomena in which sudden, discontinuous jumps occurs, such as shocks, harvesting or natural disasters, and so on. The theory of impulsive differential equations have been emerging as an important area of investigation. Particularly, the fractional impulsive evolution equations are more useful and valuable because of its widely used in control, mechanics, electrical engineering, biological, and so on.

In [21], Mu discussed the existence of the mild solutions of the following fractional evolution equations in an ordered Banach space X by using the monotone iterative

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technique.

$$\begin{cases} {}^cD^\alpha u(t) + Au(t) = f(t,u(t)), & \text{for } t \in I, \\ u(0) = x_0 \in X, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, $I = [0,T]$, $A : D(A) \subset X \to X$ be a closed linear operator, $-A$ is the infinitesimal generator of an analytic semigroup of uniformly bounded linear operators $T(t)(t \geq 0)$, and $f : I \times X \to X$ is continuous.

In this paper, we use the monotone iterative technique of mixed monotone operator to discuss the existence of the mild solutions of the following fractional impulsive evolution equations in an ordered Banach space E

$$\begin{cases} {}^cD^\alpha u(t) + Au(t) = f(t,u(t),u(t)), 0 < \alpha < 1,\ t \in J', \\ \Delta\, u\,|_{t=t_k} = I_k(u(t_k),u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha u(t)$ denotes a Caputo fractional derivative of $u(t)$, $J = [0,T]$, $J' = J\backslash\{t_1,t_2,\ldots,t_m\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ in E, $f \in C(J \times E \times E, E)$, $I_k \in C(E \times E, E)$, $u_0 \in E$, $\Delta\, u(t)\,|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and the left-hand limit of the function $u(t)$ at $t = t_k$, respectively.

By applying the operators semigroups theory and the method of mixed monotone iterative, we get the existence of mild solutions for the problem (1.1). The results are new and are the extension of [21]. Moreover, we also discuss the existence of mild solutions for the problem (1.1) under the situation that the coupled lower and upper mild quasi-solutions of problem (1.1) do not exist.

The rest of this paper is organized as follows: In Section 2, we present some useful and necessary definitions, preliminary results and notations that will be used to prove our main results. In Section 3, under suitable assumptions, we use the mixed monotone iterative technique to show the existence of the mild solutions of (1.1). Finally, in Section 4, we give an example to illustrate our main results.

2. PRELIMINARY CONSIDERATIONS

Suppose that $(X, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset X$, i.e.,$x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. We denote θ be the zero element of X. Recall that a non-empty closed convex set $P \subset X$ is a cone if it satisfies (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Moreover, P is called normal if there exists a constant $N > 0$ such that, for all x, $y \in X$, $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$. In this case, N is called the normality constant of P.

Let E be an ordered Banach space with the norm $\| \cdot \|$ and partial order $\leq$, whose positive cone $P = \{x \in E \mid x \geq \theta\}$ is normal with normal constant N. Let $PC(J,E) = \{u : J \to E \mid u(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$ and $u(t_k^+)$ exists, $k = 1,2,\ldots,m\}$. $PC(J,E)$ is a Banach space with the norm $\| u \|_{PC} = \sup_{t \in J} \| u(t) \|$.

Let $C(J,E)$ denote the Banach space of all continuous E-value functions on J with the norm $\| u \|_C = \max_{t \in J} \| u(t) \|$, denoted by Y. Then Y is an ordered Banach space by the normal cone $P_C = \{u \in Y \mid u \geq \theta,\ t \in J\}$. We use E_1 to denote the Banach space $D(A)$ with the graph norm $\| \cdot \|_1 = \| \cdot \| + \| A \cdot \|$.

Now, we recall some properties of the measure of noncompactness which will be used later. Let $\mu(\cdot)$ denote the Kuratowski measure of noncompactness of bounded set. For more details of the definition and properties of the measure of noncompactness, see [2].

Next,Let us recall the basic definitions and propertiy of fractional calculus(for more details, see [8, 23]):

Definition 1. For $\alpha > 0$, the integral

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

is called the Riemann-Liouville fractional integral of order α.

Definition 2. For a function $f(t)$, the Caputo derivative of order α can be written as

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n-1 < \alpha \leq n$.

Theorem 1. *Let $n-1 < \alpha \leq n$ and $f(t) \in C^n[0,T]$, then we have the following equality*

$$I^{\alpha}(^{c}D^{\alpha} f(t)) = f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{\Gamma(i+1)} t^i.$$

Guo[5, 6] intoduced the definition of a mixed monotone operator:

Definition 3. $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y. i.e., $u_i, v_i (i = 1,2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1,v_1) \leq A(u_2,v_2)$. Element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

Heinz[7] proved the following result:

Theorem 2. *Let $B = \{u_n\} \subset PC(J,E)$ be a bounded and countable set, then $\mu(B(t))$ is Lebesgue integral on J, and*

$$\mu\left(\left\{\int_J u_n(t) dt \mid n = 1,2,\ldots\right\}\right) \leq 2 \int_J \mu(B(t)) dt.$$

We make a frequent use of the following result due to Ye[25]:

Theorem 3. *Suppose that $b \geq 0$, $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ and suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b\int_0^t (t-s)^{\alpha-1}u(s)ds$$

on this interval, then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]ds.$$

3. Main results

In this section, we use the mixed monotone iterative technique to discuss the existence of the mild solutions of the problem (1.1). Consider the following linear fractional impulsive evolution equation in E:

$$\begin{cases} {}^cD^{\alpha}u(t) + Au(t) = h(t), \quad 0 < \alpha < 1,\ t \in J', \\ \Delta\, u\,|_{t=t_k} = y_k, \quad k = 1,2,\ldots,m, \\ u(0) = u_0 \in E, \end{cases} \tag{3.1}$$

We also quote the following results of [24]:

Definition 4. For each $h \in L^p(J,E)(p > \frac{1}{\alpha})$, $y_k \in E, k = 1,2,\ldots,m$, a function $u \in PC(J,E)$ is called a mild solution of the problem (3.1), if the following integral equations are satisfied.

$$u(t) = \begin{cases} S_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)h(s)ds, t \in [0,t_1], \\ S_{\alpha}(t)u_0 + S_{\alpha}(t-t_1)y_1 + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)h(s)ds, \quad t \in (t_1,t_2], \\ \vdots \\ S_{\alpha}(t)u_0 + \sum_{i=1}^m S_{\alpha}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)h(s)ds, \\ \qquad t \in (t_m,b], \end{cases}$$

where

$$S_{\alpha}(t) = \int_0^{\infty} \xi_{\alpha}(\theta)T(t^{\alpha}\theta)d\theta, \qquad T_{\alpha}(t) = \alpha\int_0^{\infty} \theta\xi_{\alpha}(\theta)T(t^{\alpha}\theta)d\theta,$$

and

$$\xi_{\alpha}(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_{\alpha}(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi_{\alpha}(\theta) = \frac{1}{\pi}\Sigma_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0, \;\; \theta \in (0,\infty) \; and \; \int_0^\infty \xi_\alpha(\theta) d\theta = 1.$$

Theorem 4. *For a uniformly bounded C_0-semigroup $T(t)(t \geq 0)$ (i.e. $\sup_{t\in[0,\infty)} \| T(t) \| \leq \overline{M}$), we have that for any fixed $t \geq 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators, i.e.,*

$$\| S_\alpha(t) \|_E \leq \overline{M}, \;\; and \;\; \| T_\alpha(t) \|_E \leq \frac{\overline{M}}{\Gamma(\alpha)}.$$

The following definition is given by Li[9]:

Definition 5. A C_0-semigroup $T(t)(t \geq 0)$ in E is said to be positive, if order inequality $T(t)u \geq \theta$ holds for every $u \geq \theta$, $u \in E$ and $t \geq 0$.

We introduce the mild quasi-solutions of problem (1.1).

Definition 6. Let $\lambda \geq 0$ be a constant, If functions $x_0,\; y_0 \in PC(J,E)$ satisfy

$$\begin{cases} {}^cD^\alpha x_0(t) + Ax_0(t) \leq f(t,x_0(t),y_0(t)) + \lambda(x_0(t)-y_0(t)), & t \in J', \\ \Delta\, x_0 \,|_{t=t_k} \leq I_k(x_0(t_k),y_0(t_k)), \quad k=1,2,\ldots,m, \\ x_0(0) \leq u_0, \end{cases}$$

$$\begin{cases} {}^cD^\alpha y_0(t) + Ay_0(t) \geq f(t,y_0(t),x_0(t)) + \lambda(y_0(t)-x_0(t)), t \in J', \\ \Delta\, y_0 \,|_{t=t_k} \geq I_k(y_0(t_k),x_0(t_k)), \quad k=1,2,\ldots,m, \\ y_0(0) \geq u_0, \end{cases}$$

we call $x_0,\; y_0$ coupled lower and upper mild quasi-solutions of problem (1.1). Moreover, change " $\leq$ ", " $\geq$ " into " $=$ ", we call $x_0,\; y_0$ coupled mild quasi-solutions of problem (1.1), if $x_0 = y_0 = u$, we call u a mild solution of problem (1.1).

Evidently, $PC(J,E)$ is also an ordered Banach space with the partial order $\leq$ reduced by the positive cone $P_1 = \{u \in PC(J,E) \mid u(t) \geq \theta,\; t \in J\}$. P_1 is also normal with the same normal constant N. For $x,\; y \in PC(J,E)$ with $x \leq y$, we use $[x,y]$ to denote the order interval $\{u \in PC(J,E) \mid x \leq u \leq y\}$ and $[x(t),y(t)]$ to denote the order interval $\{u \in E \mid x(t) \leq u(t) \leq y(t),\; t \in J\}$.

Theorem 5. *Let E be an ordered Banach space whose positive cone P is normal, $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in E, $f \in C(J\times E\times E,E)$, $I_k \in C(E\times E,E)$, $k=1,2,\ldots,m$. Assume that the problem (1.1) has coupled lower and upper mild quasi-solutions x_0 and y_0 such that $x_0 \leq y_0$ and suppose that the following conditions are satisfied:*
(H1) There exist constants $M > 0$ and $\lambda \geq 0$ such that

$$f(t,x_2,y_2) - f(t,x_1,y_1) \geq -M(x_2-x_1) - \lambda(y_1-y_2),$$

for any $t \in J$ and $x_0(t) \le x_1(t) \le x_2(t) \le y_0(t)$, $x_0(t) \le y_2(t) \le y_1(t) \le y_0(t)$.
(H2) The impulsive function I_k satisfies

$$I_k(x_1, y_1) \le I_k(x_2, y_2), \;\; k = 1, 2, \ldots, m,$$

for any $t \in J$ and $x_0(t) \le x_1(t) \le x_2(t) \le y_0(t)$, $x_0(t) \le y_2(t) \le y_1(t) \le y_0(t)$.
(H3) There exists a constant $M_1 > 0$ such that

$$\mu(\{f(t, x_n, y_n)\}) \le M_1(\mu(\{x_n\}) + \mu(\{y_n\})),$$

for any $t \in J$ and increasing monotone sequence $\{x_n\} \subset [x_0(t), y_0(t)]$ and decreasing monotone sequence $\{y_n\} \subset [x_0(t), y_0(t)]$.
Then (1.1) has minimal and maximal coupled mild solutions between x_0 and y_0.

Proof. For the C_0-semigroup $T(t)(t \ge 0)$,we know that there exist $\omega > 0$ and $\widetilde{M} \ge 1$ such that $\| T(t) \| \le \widetilde{M} e^{\omega t}$(see Theorem2.2 in [22]). Now let us take $M > \omega > 0$, it is easy to see that $-(A + MI)$ also generates a C_0-semigroup $S(t) = e^{-Mt}T(t)(t \ge 0)$ in E. $S(t)(t \ge 0)$ is positive because $T(t)(t \ge 0)$ is positive. Moreover, $\| S(t) \| = e^{-Mt} \| T(t) \| \le \widetilde{M} e^{-(M-\omega)t} \le \widetilde{M}$.

Next, let $\phi_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta) d\theta$, $\varphi_\alpha = \alpha \int_0^\infty \theta \xi_\alpha(\theta) S(t^\alpha \theta) d\theta$. According to Theorem 4 ,

$$\| \phi_\alpha(t) \| \le \widetilde{M}, \qquad \| \varphi_\alpha \| \le \frac{\widetilde{M}}{\Gamma(\alpha)}.$$

Define the operator $\Psi : [x_0, y_0] \times [x_0, y_0] \to PC(J, E)$ by

$$\Psi(x,y)(t) = \begin{cases} \phi_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} \varphi_\alpha(t-s)[f(s, x(s), y(s)) \\ \qquad + (M+\lambda)x(s) - \lambda y(s)]ds, & t \in [0, t_1], \\ \phi_\alpha(t)u_0 + \phi_\alpha(t-t_1)I_1(x(t_1), y(t_1)) + \int_0^t (t-s)^{\alpha-1} \varphi_\alpha(t-s) \\ \quad [f(s, x(s), y(s)) + (M+\lambda)x(s) - \lambda y(s)]ds, & t \in (t_1, t_2], \\ \vdots \\ \phi_\alpha(t)u_0 + \sum_{i=1}^m \phi_\alpha(t-t_i) I_i(x(t_i), y(t_i)) \\ + \int_0^t (t-s)^{\alpha-1} \varphi_\alpha(t-s) \\ \quad [f(s, x(s), y(s)) + (M+\lambda)x(s) - \lambda y(s)]ds, & t \in (t_m, T], \end{cases}$$

According to the Definition 4, we know that u is a mild solution of problem (1.1) if only if $u = \Psi(u, u)$.

Next we show that Ψ is a mixed monotone operator. For $x_0(t) \le x_1(t) \le x_2(t) \le y_0(t)$, $x_0(t) \le y_2(t) \le y_1(t) \le y_0(t)$, $t \in (t_k, t_{k+1}]$, from (H1), we can get that

$$f(t, x_1, y_1) + Mx_1 - \lambda y_1 \le f(t, x_2, y_2) + Mx_2 - \lambda y_2,$$

so

$$f(t, x_1, y_1) + (M+\lambda)x_1 - \lambda y_1 \le f(t, x_2, y_2) + (M+\lambda)x_2 - \lambda y_2.$$

From (H2), we have

$$I_k(x_1, y_1) \le I_k(x_2, y_2), \;\; k = 1, 2, \ldots, m.$$

Since $S(t)$ is a positive C_0-semigroup, so

$$\Psi(x_1,y_1)(t)\le\Psi(x_2,y_2)(t).$$

Ψ is a mixed monotone operator.

Then, we show that $\Psi:[x_0,y_0]\times[x_0,y_0]\to[x_0,y_0]$.

Let $h(t)={}^cD^\alpha x_0(t)+Ax_0(t)+Mx_0(t)$. From Definition 6, we get $h(t)\le f(t,x_0(t),y_0(t))+\lambda(x_0(t)-y_0(t))+Mx_0(t)$. According to Definition 4, for $t\in(t_k,t_{k+1}]$:

$$\begin{aligned}x_0(t)&=\phi_\alpha(t)x_0(0)+\sum_{i=1}^{k}\phi_\alpha(t-t_i)\,\Delta\,x_0\,|_{t=t_i}+\int_0^t(t-s)^{\alpha-1}\varphi_\alpha(t-s)h(s)ds\\ &\le\phi_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s,x_0(s),y_0(s))+(M+\lambda)x_0(s)-\lambda y_0(s)]ds\\ &\quad+\sum_{i=1}^{k}\phi_\alpha(t-t_i)I_i(x_0(t_i),y_0(t_i))\\ &\le\Psi(x_0,y_0)(t).\end{aligned}$$

So, $x_0(t)\le\Psi(x_0,y_0)(t)$. Similarly, we can get $\Psi(y_0,x_0)(t)\le y_0(t)$. That is to say $\Psi:[x_0,y_0]\times[x_0,y_0]\to[x_0,y_0]$ is a continuous mixed monotone operator.

Define two sequences $\{x_n\}$, $\{y_n\}$:

$$x_n=\Psi(x_{n-1},y_{n-1}),\quad y_n=\Psi(y_{n-1},x_{n-1}),\quad n=1,2,\ldots$$

Then from the mixed monotonicity of Ψ, we have:

$$x_0\le x_1\le x_2\le\cdots\le x_n\le\cdots\le y_n\le\cdots\le y_2\le y_1\le y_0.$$

Let $H=\{x_n\mid n=1,2,\ldots\}+\{y_n\mid n=1,2,\ldots\}$, $H_1=\{x_n\mid n=1,2,\ldots\}$, $H_2=\{y_n\mid n=1,2,\ldots\}$, $H_3=\{(x_{n-1},y_{n-1})\mid n=1,2,\ldots\}$, $H_4=\{(y_{n-1},x_{n-1})\mid n=1,2,\ldots\}$. Then we can get that $H_1(t)=\Psi(H_3(t))$, $H_2(t)=\Psi(H_4(t))$. Let $\Omega(t)=\mu(H(t))$, $t\in J$.

Now we show that $\Omega(t)\equiv 0$ for $t\in J$.

For $t\in[0,t_1]$, we have

$$\begin{aligned}\Omega(t)&=\mu(H(t))=\mu(H_1(t)+H_2(t))=\mu\left(\Psi(H_3(t))+\Psi(H_4(t))\right)\\ &=\mu\Bigg(\Bigg\{\phi_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s,x_{n-1}(s),y_{n-1}(s))\\ &\qquad+(M+\lambda)x_{n-1}(s)-\lambda y_{n-1}]ds+\phi_\alpha(t)u_0\\ &\quad+\int_0^t(t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s,y_{n-1}(s),x_{n-1}(s))+(M+\lambda)y_{n-1}(s)-\lambda x_{n-1}]ds\Bigg\}\Bigg)\end{aligned}$$

$$\leq \frac{2\widetilde{M}}{\Gamma(\alpha)} \int_0^t \mu(\{(t-s)^{\alpha-1}[f(s, x_{n-1}(s), y_{n-1}(s)) + f(s, y_{n-1}(s), x_{n-1}(s)) + M(x_{n-1}(s) + y_{n-1}(s))]\})ds$$

$$\leq \frac{2\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(2M_1 + M)(\mu(H_1(s)) + \mu(H_2(s)))ds$$

$$= \frac{2\widetilde{M}}{\Gamma(\alpha)}(2M_1 + M) \int_0^t (t-s)^{\alpha-1}\Omega(s)ds.$$

According to Theorem 3, we get $\Omega(t) \equiv 0$ for $t \in [0, t_1]$. Hence $\{x_n(t)\} + \{y_n(t)\}$ is precompact, so $\{x_n(t)\}$ and $\{y_n(t)\}$ are precompact for $t \in [0, t_1]$. In the same time we can get that $I_1(H_3(t_1))$ and $I_1(H_4(t_1))$ are precompact and $\mu(I_1(H_3(t_1))) = 0$, $\mu(I_1(H_4(t_1))) = 0$.

For $t \in (t_1, t_2]$,

$$\Omega(t) = \mu(H(t)) = \mu(H_1(t) + H_2(t)) = \mu(\Psi(H_3(t)) + \Psi(H_4(t)))$$

$$= \mu\Bigg(\Bigg\{\phi_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, x_{n-1}(s), y_{n-1}(s)) + (M+\lambda)x_{n-1}(s) - \lambda y_{n-1}]ds + \phi_\alpha(t-t_1)I_1(x_{n-1}(t_1), y_{n-1}(t_1)) + \phi_\alpha(t)u_0 + \phi_\alpha(t-t_1)I_1(y_{n-1}(t_1), x_{n-1}(t_1)) + \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, y_{n-1}(s), x_{n-1}(s)) + (M+\lambda)y_{n-1}(s) - \lambda x_{n-1}]ds\Bigg\}\Bigg)$$

$$\leq \frac{2\widetilde{M}}{\Gamma(\alpha)} \int_0^t \mu(\{(t-s)^{\alpha-1}[f(s, x_{n-1}(s), y_{n-1}(s)) + f(s, y_{n-1}(s), x_{n-1}(s)) + M(x_{n-1}(s) + y_{n-1}(s))]\})ds$$

$$\leq \frac{2\widetilde{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(2M_1 + M)(\mu(H_1(s)) + \mu(H_2(s)))ds$$

$$= \frac{2\widetilde{M}}{\Gamma(\alpha)}(2M_1 + M) \int_0^t (t-s)^{\alpha-1}\Omega(s)ds$$

$$= \frac{2\widetilde{M}}{\Gamma(\alpha)}(2M_1 + M) \int_{t_1}^t (t-s)^{\alpha-1}\Omega(s)ds.$$

According to Theorem 3, $\Omega(t) \equiv 0$ for $t \in (t_1, t_2]$. Continuing this process in each interval, we can prove that $\Omega(t) \equiv 0$ in J. Hence $\{x_n(t)\} + \{y_n(t)\}$ is precompact, so $\{x_n(t)\}$ and $\{y_n(t)\}$ are precompact. $\{x_n(t)\}$ is a increasing sequence and $\{y_n(t)\}$ is a decreasing sequence, then we can easily get that $\{x_n(t)\}$ and $\{y_n(t)\}$ are convergent.

$$x^*(t) = \lim_{n\to\infty} x_n(t), \quad y^*(t) = \lim_{n\to\infty} y_n(t), \qquad t \in J.$$

Evidently, x^* and y^* are bounded integrable in J. Since we have that $x_n(t) = \Psi(x_{n-1}, y_{n-1})(t)$ and $y_n(t) = \Psi(y_{n-1}, x_{n-1})(t)$, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we get

$$x^*(t) = \Psi(x^*, y^*)(t), \qquad y^*(t) = \Psi(y^*, x^*)(t),$$

and $x^*(t),\ y^*(t) \in PC(J, E)$, $x_0(t) \le x^*(t) \le y^*(t) \le y_0(t)$. By monotonicity of $\{x_n(t)\}$ and $\{y_n(t)\}$, $x^*(t)$ and $y^*(t)$ are the minimal and maximal coupled fixed points of A in $[x_0, y_0]$, respectively and they are the minimal and maximal coupled mild solutions of the problem (1.1) in $[x_0, y_0]$, respectively. □

Theorem 6. *Let E be an ordered Banach space whose positive cone P is normal, $-A$ generates a positive C_0-semigroup $T(t)(t \ge 0)$ in E, $f \in C(J \times E \times E, E)$, $I_k \in C(E \times E, E)$, $k = 1, 2, \ldots, m$. Assume that the problem* (1.1) *has coupled lower and upper mild quasi-solutions x_0 and y_0 such that $x_0 \le y_0$ and suppose that (H1), (H2) and (H4) are satisfied. Furthermore, we impose that: (H4) there exist constants $L_1 \ge 0$ and $L_2 \ge 0$ such that*

$$f(t, x_2, y_2) - f(t, x_1, y_1) \le L_1(x_2 - x_1) + L_2(y_1 - y_2),$$

for any $t \in J$ and $x_0(t) \le x_1(t) \le x_2(t) \le y_0(t)$, $x_0(t) \le y_2(t) \le y_1(t) \le y_0(t)$. Then (1.1) *has a unique mild solution between x_0 and y_0 .*

Proof. Firstly, we prove that (H1) and (H4) imply (H3). For $t \in J$, let $\{x_n\} \subset [x_0(t), y_0(t)]$ be an increasing monotone sequence and $\{y_n\} \subset [x_0(t), y_0(t)]$ be a decreasing monotone sequence. Let $m > n$, by (H1) and (H4), we have

$$\begin{aligned} 0 &\le (t, x_m, y_m) - f(t, x_n, y_n) + M(x_m - x_n) + \lambda(y_n - y_m) \\ &\le (M + L_1)(x_m - x_n) + (\lambda + L_2)(y_n - y_m). \end{aligned}$$

By the normality of cone P, we have

$$\begin{aligned} &\| f(t, x_m, y_m) - f(t, x_n, y_n) \| \\ &\quad \le N \| (M + L_1)(x_m - x_n) + (\lambda + L_2)(y_n - y_m) \| + M \| (x_m - x_n) \| \\ &\qquad + \lambda \| (y_n - y_m) \| \\ &\quad \le [N(M + L_1) + M] \| (x_m - x_n) \| + [N(M + L_2) + \lambda] \| (y_n - y_m) \| . \end{aligned}$$

By the definition of the measure of noncompactness, we have

$$\begin{aligned} \mu(\{f(t, x_n, y_n)\}) &\le [N(M + L_1) + M]\mu(\{x_n\}) + [N(\lambda + L_2) + \lambda]\mu(\{y_n\}) \\ &\le M_1(\mu(\{x_n\}) + \mu(\{y_n\})), \end{aligned}$$

where $M_1 = N(M + L_1 + \lambda + L_2) + M + \lambda$. So (H3) holds. Thus, by Theorem 5, the problem (1.1) has minimal and maximal coupled mild solutions $x^*(t)$ and $y^*(t)$ in $[x_0, y_0]$. Next we show that $x^*(t) \equiv y^*(t)$ in J.

For $t \in [0, t_1]$, we have

$$\begin{aligned}
&0 \le y^*(t) - x^*(t) = \Psi(y^*, x^*)(t) - \Psi(x^*, y^*)(t) \\
&\le \phi_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, y^*(s), x^*(s)) + (M+\lambda)y^*(s) - \lambda x^*(s)]ds \\
&- \phi_\alpha(t)u_0 - \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, x^*(s), y^*(s)) + (M+\lambda)x^*(s) - \lambda y^*(s)]ds \\
&= \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, y^*(s), x^*(s)) - f(s, x^*(s), y^*(s)) \\
&\qquad + (M+2\lambda)(y^*(s) - x^*(s))]ds \\
&\le \frac{\widetilde{M}}{\Gamma(\alpha)}(L_1 + L_2 + M + 2\lambda)\int_0^t (t-s)^{\alpha-1}(y^*(s) - x^*(s))ds.
\end{aligned}$$

By Theorem 3, we obtain that $x^*(t) \equiv y^*(t)$ for $t \in [0, t_1]$. Particularly, $I_1(x^*(t_1), y^*(t_1)) = I_1(y^*(t_1), x^*(t_1))$. For $t \in (t_1, t_2]$, we have

$$\begin{aligned}
&0 \le y^*(t) - x^*(t) = \Psi(y^*, x^*)(t) - \Psi(x^*, y^*)(t) \\
&\le \phi_\alpha(t)u_0 + \phi_\alpha(t-t_1)I_1(y^*(t_1), x^*(t_1)) + \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, y^*(s), x^*(s)) \\
&\quad + (M+\lambda)y^*(s) - \lambda x^*(s)]ds - \phi_\alpha(t)u_0 - \phi_\alpha(t-t_1)I_1(x^*(t_1), y^*(t_1)) \\
&\quad - \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, x^*(s), y^*(s)) + (M+\lambda)x^*(s) - \lambda y^*(s)]ds \\
&= \int_0^t (t-s)^{\alpha-1}\varphi_\alpha(t-s)[f(s, y^*(s), x^*(s)) - f(s, x^*(s), y^*(s)) \\
&\qquad + (M+2\lambda)(y^*(s) - x^*(s))]ds \\
&\le \frac{\widetilde{M}}{\Gamma(\alpha)}(L_1 + L_2 + M + 2\lambda)\int_0^t (t-s)^{\alpha-1}(y^*(s) - x^*(s))ds.
\end{aligned}$$

By Theorem 3, we obtain that $x^*(t) \equiv y^*(t)$ for $t \in (t_1, t_2]$. Continuing this process in each interval, we can prove that $x^*(t) \equiv y^*(t)$ in J. So $x^*(t) \equiv y^*(t)$ is the unique mild solution of the problem (1.1) in $[x_0, y_0]$. □

In the following, we discuss the existence of mild solutions for the problem (1.1) under the situation that coupled lower and upper mild quasi-solutions of the problem (1.1) do not exist.

Theorem 7. *Let E be an ordered Banach space whose positive cone P is normal, $-A$ generates a positive C_0-semigroup $T(t)(t \ge 0)$ in E, $f \in C(J \times E \times E, E)$, $I_k \in C(E \times E, E)$, $k = 1, 2, \ldots, m$. Let (H1)-(H3) hold and assume that the following condition is satisfied:*
(H5) There exist $\rho \ge 0$, $h(t) \in PC(J, E)$, $h(t) \ge 0$, $y_k \in D(A)$, $y_k \ge 0$, $k = 1, 2, \ldots, m$,

such that

$$\rho x - h(t) \le f(t,-x,x) \le f(t,x,-x) \le \rho x + h(t),$$

$$-y_k \le I_k(-x,x) \le I_k(x,-x) \le y_k.$$

Then (1.1) *has minimal and maximal coupled mild solutions.*

Proof. Firstly, consider the following linear problem:

$$\begin{cases} {}^cD^\alpha u(t) + Au(t) - (\rho+2\lambda)u(t) = h(t), \quad 0<\alpha<1,\ t\in J', \\ \triangle\, u\,|_{t=t_k} = y_k, \quad k=1,2,\dots,m, \\ u(0)=u_0, \end{cases} \tag{3.2}$$

We know that $-(A-(\rho+2\lambda)I)$ generates a positive C_0-semigroup $S(t)=e^{(\rho+2\lambda)t}T(t)(t\ge 0)$ in E. From Definition 4, the linear problem (3.2) has a unique positive mild solution $\overline{u}\in PC(J,E)$. Let $x_0=-\overline{u}$, $y_0=\overline{u}$. By (H5), we have

$$\begin{cases} {}^cD^\alpha x_0(t) + Ax_0(t) = \rho x_0(t) - h(t) + 2\lambda x_0(t) \le f(t,x_0(t),y_0(t)) \\ +\lambda(x_0(t)-y_0(t)), \quad t\in J', \\ \triangle\, x_0\,|_{t=t_k} = -y_k \le I_k(x_0(t_k),y_0(t_k)), \quad k=1,2,\dots,m, \\ x_0(0) = -u_0 \le u_0, \end{cases}$$

and

$$\begin{cases} {}^cD^\alpha y_0(t) + Ay_0(t) = \rho y_0(t) + h(t) + 2\lambda y_0(t) \ge f(t,y_0(t),x_0(t)) \\ +\lambda(y_0(t)-x_0(t)), \quad t\in J', \\ \triangle\, y_0\,|_{t=t_k} = y_k \ge I_k(y_0(t_k),x_0(t_k)), \quad k=1,2,\dots,m, \\ y_0(0) \ge u_0, \end{cases}$$

So $x_0(t)$ and $y_0(t)$ are coupled mild lower and upper solutions of (1.1). Hence, the conclusion follows from Theorem 5. □

4. Application

Consider the following fractional impulsive partial differential equation

$$\begin{cases} {}^cD^\alpha u - \triangle u = g(x,t,u,u), \quad 0<\alpha<1,\ t\in J', \\ u(t_k^+) - u(t_k^-) = J_k(u(x,t_k),u(x,t_k)), \quad k=1,2,\dots,m, \\ u\,|_{\partial\Omega} = 0, \\ u(x,0) = \sigma(x), \quad x\in\Omega, \end{cases} \tag{4.1}$$

where $\triangle$ is the Laplace operator, $J=[0,b]$, $0=t_0\le t_1\le\cdots\le t_m\le t_{m+1}=b$, $J'=J\setminus\{t_1,t_2,\dots,t_m\}$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. $g:\Omega\times J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, $J_k:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous.

Let $E = L^2(\Omega)$, $P = \{v \in L^2(\Omega) \mid v(x) \geq 0, a.e. x \in \Omega\}$, then E is a Banach space and P is a normal cone in E. Define the operator A as follow:

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Au = -\Delta u.$$

$-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in E.
In the following, we need the following assumptions:
(F1) There exist $\rho \geq 0$, $h \in PC(\Omega \times J)$, $h(x,t) \geq 0$, $y_k \in D(A)$, $y_k(x) \geq 0$, $k = 1,2,\ldots,m$, $\sigma \in D(A)$, $\sigma(x) \geq 0$ such that for any $u \geq 0 \in L^2(\Omega)$

$$\rho u - h(x,t) \leq g(x,t,-u,u) \leq g(x,t,u,-u) \leq \rho u + h(x,t), \qquad x \in \Omega,\ t \in J',$$

$$-y_k \leq J_k(-u,u) \leq J_k(u,-u) \leq y_k, \qquad x \in \Omega,\ k = 1,2,\ldots,m.$$

(F2) For $u_1 \leq u_2$, $v_2 \leq v_1$ such that

$$J_k(u_1(x,t_k), v_1(x,t_k)) \leq J_k(u_2(x,t_k), v_2(x,t_k)), \qquad x \in \Omega,\ k = 1,2,\ldots,m.$$

(F3) The partial derivative g'_u and g'_v are continuous and have upper bound.

Theorem 8. *Let (F1)-(F3) hold. Then the problem* (4.1) *has a unique mild solution.*

Proof. Let $f(t,u,u) = g(\cdot,t,u(\cdot),u(\cdot))$, $I_{u,u} = J_k(u(\cdot),u(\cdot))$. So the following linear problem:

$$\begin{cases} {}^cD^\alpha u - \triangle u - (\rho + 2\lambda)u = h(x,t), \quad 0 < \alpha < 1,\ t \in J', \\ u(t_k^+) - u(t_k^-) = y_k, \quad k = 1,2,\ldots,m, \\ u \mid_{\partial\Omega} = 0, \\ u(x,0) = \sigma(x), \quad x \in \Omega, \end{cases}$$

can be transformed into the following abstract problem:

$$\begin{cases} {}^cD^\alpha u(t) + Au(t) - (\rho + 2\lambda)u(t) = \overline{h}(t), \quad 0 < \alpha < 1,\ t \in J', \\ u(t_k^+) - u(t_k^-) = y_k, \quad k = 1,2,\ldots,m, \\ u(0) = \sigma, \end{cases}$$

where $\overline{h}(t) = h(\cdot,t)$. Use the same method as Theorem 7, we can prove that x_0 and y_0 are coupled mild lower and upper quasi-solutions of the problem (4.1). From assumptions (F2) and (F3), we can prove that (H1), (H2) and (H4) are satisfied. So by Theorem 6, the problem (4.1) has a unique mild solution. □

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