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# SOME IDENTITIES ON CONDITIONAL SEQUENCES BY USING MATRIX METHOD

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*Abstract.* In this paper, we consider the Fibonacci conditional sequence $\{f_n\}$ and the Lucas conditional sequence $\{l_n\}$. We derive some properties of Fibonacci and Lucas conditional sequences by using the matrix method. Our results are elegant as the results for ordinary Fibonacci and Lucas sequences.

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## 1. INTRODUCTION

The Fibonacci numbers $F_n$ are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$, where

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The sequence $L_n$ of Lucas numbers, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. These numbers are famous for possessing wonderful properties, see also [5] and [8] for additional references and history.

This sequences has been generalized in various ways. Horadam [3] has considered a generalized sequence $\{W_n\}$ defined by

$$W_n = pW_{n-1} - qW_{n-2}, \quad n \geq 2$$

with the initial conditions $W_0 = a$ and $W_1 = b$, where $a, b, p, q$ are arbitrary integers. If we take $a = 0$, $b = 1$ in $\{W_n\}$, we get the generalized Fibonacci sequence and if we take $a = 2$, $b = p$ in $\{W_n\}$, we get the generalized Lucas sequence. In [1], the author use matrix techniques to give proofs of well-known properties of the generalized Fibonacci and Lucas sequences.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In [7], author introduced a new generalization of the Fibonacci sequence $\{f_n\}$, defined by for $n \geq 2$

$$f_n = \begin{cases} a_1 f_{n-1} + b_1 f_{n-2}, & \text{if } n \text{ is even} \\ a_2 f_{n-1} + b_2 f_{n-2}, & \text{if } n \text{ is odd} \end{cases} \tag{1.1}$$

with initial values $f_0 = 0$ and $f_1 = 1$, where $a_1$, $a_2$, $b_1$, $b_2$ are nonzero numbers. If we take $a_1 = a_2 = b_1 = b_2 = 1$ in $\{f_n\}$, we get the classical Fibonacci sequence. Here we call $\{f_n\}$, the Fibonacci conditional sequence. We can easily define the Lucas conditional sequence $\{l_n\}$ with the initial conditions $l_0 = 2$ and $l_1 = a_2$ similar to (1.1). In [9], in the case of $b_1 = b_2$, the author gave some well known identities for $\{f_n\}$ such as Cassini's, Catalan's identities, etc. Without the case of $b_1 = b_2$, Gelin-Cesaro identity and Catalan identity for the even indices of $\{f_n\}$ are given in [7]. Also, in [4], for the case of $b_1 = b_2 = 1$ author defined a generalization of Lucas sequence and derived several identities involving the generalized Fibonacci and Lucas sequences. All of these proofs are based on the generating function.

In this paper, we take the non-zero numbers $a_1, a_2, b_1, b_2$ completely arbitrary and we derive some properties of Fibonacci and Lucas conditional sequences by using the matrix method. For Lucas conditional sequences, we just need the case of $b_1 = b_2$. We collect our main results in Theorem 1, 2, 3 and 4 which are elegant as the results for ordinary Fibonacci and Lucas sequences. We also obtain some more new results (4.10)-(4.13) and (4.15).

We need the following results which can be obtained by taking $r = 2$ in [6, Theorem 5,6,9].

For $n \geq 4$,

$$f_n = Af_{n-2} - Bf_{n-4} \tag{1.2}$$

where $A := a_1 a_2 + b_1 + b_2$ and $B := b_1 b_2$. The same result for $\{l_n\}$ also holds.

The generating function of the sequence $\{f_n\}$ is

$$F(x) = \frac{x + a_1 x^2 - b_1 x^3}{1 - Ax^2 + Bx^4} \tag{1.3}$$

and the generating function of the sequence $\{l_n\}$ is

$$L(x) = \frac{2 + a_2 x - (a_1 a_2 + 2b_2)x^2 + b_1 a_2 x^3}{1 - Ax^2 + Bx^4}. \tag{1.4}$$

By using (1.3) and (1.4), the Binet's formulas for the sequence $\{f_n\}$ and $\{l_n\}$ are given;

$$f_{2m} = a_1 \frac{\alpha^m - \beta^m}{\alpha - \beta} \tag{1.5}$$

$$f_{2m+1} = (a_1 a_2 + b_2) \frac{\alpha^m - \beta^m}{\alpha - \beta} - (b_1 b_2) \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \tag{1.6}$$

$$l_{2m} = (a_1a_2 + 2b_1)\frac{\alpha^m - \beta^m}{\alpha - \beta} - 2(b_1b_2)\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \tag{1.7}$$

$$l_{2m+1} = \left(a_1a_2^2 + 2b_1a_2 + a_2b_2\right)\frac{\alpha^m - \beta^m}{\alpha - \beta} - (b_1b_2a_2)\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \tag{1.8}$$

where $\alpha = \frac{A+\sqrt{A^2-4B}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4B}}{2}$ that is, $\alpha$ and $\beta$ are the roots of the polynomial $p(z) = z^2 - Az + B$.

## 2. Fibonacci Case

It is known that for the matrix $Q := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. See [2] for a detailed history of the $Q$ matrix.

Similar to this well-known result, setting $S := \begin{pmatrix} A & -B \\ 1 & 0 \end{pmatrix}$,we also have

$$S^n = \frac{1}{a_1}\begin{pmatrix} f_{2(n+1)} & -Bf_{2n} \\ f_{2n} & -Bf_{2(n-1)} \end{pmatrix} \tag{2.1}$$

which can be easily proven by induction and by the help of (1.2). It should be noted that $S$ is invertible. We use the matrix identity (2.1) to get the following theorem easily.

**Theorem 1.** *The sequence $\{f_n\}$ satisfy*

$$-a_1^2B^{m-1} = f_{2(m-1)}f_{2(m+1)} - f_{2m}^2 \tag{2.2}$$

$$a_1B^m f_{2(n-m)} = f_{2n}f_{2(m+1)} - f_{2m}f_{2(n+1)} \tag{2.3}$$

$$a_1 f_{2n} = f_{2m}f_{2(n-m+1)} - Bf_{2(n-m)}f_{2(m-1)} \tag{2.4}$$

$$a_1 f_{2(n+m+1)} = f_{2(n+1)}f_{2(m+1)} - Bf_{2n}f_{2m} \tag{2.5}$$

$$a_1 f_{2(n+m)} = f_{2(n+1)}f_{2m} - Bf_{2n}f_{2(m-1)} \tag{2.6}$$

$$a_1 f_{2(n+m-1)} = f_{2n}f_{2m} - Bf_{2(n-1)}f_{2(m-1)} \tag{2.7}$$

*for any positive integers $m$ and $n$.*

*Proof.* By taking the determinant of both sides of (2.1), we get the result (2.2) which is *Cassini's identity* for the even indices of the sequence $\{f_n\}$.

Since S is invertible,

$$S^{-n} = \frac{1}{a_1B^n}\begin{pmatrix} -Bf_{2(n-1)} & Bf_{2n} \\ -f_{2n} & f_{2(n+1)} \end{pmatrix}.$$

Considering $S^{n-m} = S^nS^{-m}$ and by using the matrix identity (2.1), we get;

$$\frac{1}{a_1}\begin{pmatrix} f_{2(n-m+1)} & -Bf_{2(n-m)} \\ f_{2(n-m)} & -Bf_{2(n-m-1)} \end{pmatrix}$$

$$= \frac{-1}{a_1^2 B^{m-1}} \begin{pmatrix} f_{2(n+1)} f_{2(m-1)} - f_{2n} f_{2m} & f_{2n} f_{2(m+1)} - f_{2m} f_{2(n+1)} \\ f_{2(m-1)} f_{2n} - f_{2(n-1)} f_{2m} & f_{2(n-1)} f_{2(m+1)} - f_{2n} f_{2m} \end{pmatrix}$$

and by equating the $(1,2)$ entries of the matrices on both sides, we get the result (2.3), which corresponds to the *d'Ocagne's identity* for the even indices of $\{f_n\}$.

Also, considering $S^{n-m} S^{m-1} = S^{n-1}$ and by using the matrix identity (2.1), we get;

$$\begin{pmatrix} f_{2(n-m+1)} f_{2m} - B f_{2(n-m)} f_{2(m-1)} & -B f_{2(m-1)} f_{2(n-m+1)} + B^2 f_{2(n-m)} f_{2(m-2)} \\ f_{2(n-m)} f_{2m} - B f_{2(n-m-1)} f_{2(m-1)} & -B f_{2(m-1)} f_{2(n-m)} + B^2 f_{2(n-m-1)} f_{2(m-2)} \end{pmatrix}$$
$$= a_1 \begin{pmatrix} f_{2n} & -B f_{2(n-1)} \\ f_{2(n-1)} & -B f_{2(n-2)} \end{pmatrix}.$$

By equating the $(1,1)$ entries of the matrices on both sides, we get the result (2.4) which corresponds to the *Convolution property* for the even indices of $\{f_n\}$.

Similarly, since $S^{n+m} = S^n S^m$ and by using the matrix identity (2.1), we get the equalities (2.5), (2.6) and (2.7). □

Special Cases:

(1) If we take $n = m$ in (2.5), we get

$$a_1 f_{2(2n+1)} = f_{2(n+1)}^2 - B f_{2n}^2. \tag{2.8}$$

(2) If we take $n = m$ in (2.6), we obtain the formula;

$$\begin{aligned} a_1 f_{4n} &= f_{2n} \left( f_{2(n+1)} - B f_{2(n-1)} \right) \\ &= f_{2n} \left( (b_2 - b_1) f_{2n} + a_1 l_{2n} \right). \end{aligned} \tag{2.9}$$

Note that for $2n = m$, if we take $a_1 = a_2 = b_1 = b_2 = 1$ in (2.9), it reduces to the well-known identity;

$$F_{2m} = F_m L_m.$$

We can get the Binet formula for the even indices of $\{f_n\}$ by using matrix method. For this, we assume $A^2 - 4B \neq 0$ to guarantee the existence of two distinct roots. Let $\alpha$ and $\beta$ denote the roots of $p(z) = z^2 - Az + B$ that is the characteristic polynomial of $S$.

**Theorem 2.** *The Binet formula of $\{f_{2n}\}$ is*

$$f_{2n} = a_1 \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

*Proof.* The eigenvalues and corresponding eigenvectors of the matrix $S$ are $\lambda_1 = \alpha$, $\lambda_2 = \beta$ and $\nu_1 = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$, $\nu_2 = \begin{pmatrix} \beta \\ 1 \end{pmatrix}$. Therefore; $P^{-1} S P = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $P = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}$. It follows $S^n = P \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} P^{-1}$. By equating corresponding entries of the matrices on both sides and by using the matrix identity (2.1), we obtain the result. □

## 3. LUCAS CASE

In this section, let $b_1 = b_2$. We also have a similar matrix identity for the conditional Lucas sequences $\{l_n\}$. Similar to the well-known result,

$$\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

we also have

$$\begin{pmatrix} l_{2(n+1)} & -Bl_{2n} \\ l_{2n} & -Bl_{2(n-1)} \end{pmatrix} = \begin{pmatrix} A & -2B \\ 2 & -A \end{pmatrix} \begin{pmatrix} A & -B \\ 1 & 0 \end{pmatrix}^n \tag{3.1}$$

which can be easily proven by induction and by the help of (1.2).

**Theorem 3.** *The sequence $\{l_n\}$ satisfy*

$$l_{2(n-1)}l_{2(n+1)} - l_{2n}^2 = B^{n-1}(\alpha-\beta)^2$$

*for any positive integer $n$.*

*Proof.* By taking the determinant of both sides of (3.1), we get

$$\left(l_{2(n-1)}l_{2(n+1)} - l_{2n}^2\right) = B^{n-1}\left(A^2-4B\right)$$

and since $A^2-4B = (\alpha-\beta)^2$, we get the result which is Cassini's identity for even indices of $\{l_n\}$. □

## 4. FIBONACCI-LUCAS CASE

Here, we also assume that $b_1 = b_2$ and $\Delta := A^2-4B \neq 0$. We can also use matrix techniques to prove some relations between the $\{f_n\}$ and $\{l_n\}$.

It is known that for the matrix $R := \frac{1}{2}\begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}$, $R^n = \frac{1}{2}\begin{pmatrix} L_n & 5F_n \\ F_n & L_n \end{pmatrix}$.

Similar to this well-known result, setting $T := \frac{1}{2}\begin{pmatrix} A & \Delta \\ 1 & A \end{pmatrix}$, we also have

$$T^n = \frac{1}{2}\begin{pmatrix} l_{2n} & \Delta\frac{f_{2n}}{a_1} \\ \frac{f_{2n}}{a_1} & l_{2n} \end{pmatrix} \tag{4.1}$$

which can be easily proven by induction and by the help of (1.5) and (1.7).

**Theorem 4.** *The sequences $\{f_n\}$ and $\{l_n\}$ satisfy*

$$4B^n = l_{2n}^2 - \frac{\Delta}{a_1^2}f_{2n}^2 \tag{4.2}$$

$$2f_{2(m+n)} = f_{2m}l_{2n} + f_{2n}l_{2m} \tag{4.3}$$

$$2l_{2(m+n)} = l_{2m}l_{2n} + \frac{\Delta}{a_1^2} f_{2n} f_{2m} \tag{4.4}$$

$$2B^m f_{2(n-m)} = f_{2n}l_{2m} - l_{2n} f_{2m} \tag{4.5}$$

$$2B^m l_{2(n-m)} = l_{2n}l_{2m} - \frac{\Delta}{a_1^2} f_{2n} f_{2m} \tag{4.6}$$

$$l_{2n}l_{2m} = l_{2(n+m)} + B^m l_{2(n-m)} \tag{4.7}$$

$$f_{2n}l_{2m} = f_{2(n+m)} + B^m f_{2(n-m)} \tag{4.8}$$

*for any positive integers $m$ and $n$.*

*Proof.* By taking the determinant of both sides of (4.1), we get the result (4.2). Considering $T^{n+m} = T^n T^m$ and by using the matrix identity (4.1), we get;

$$\frac{1}{2}\begin{pmatrix} l_{2(n+m)} & \Delta \frac{f_{2(n+m)}}{a_1} \\ \frac{f_{2(n+m)}}{a_1} & l_{2(n+m)} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} l_{2n}l_{2m} + \frac{\Delta}{a_1^2} f_{2n} f_{2m} & \frac{\Delta}{a_1}\left(f_{2m}l_{2n} + f_{2n}l_{2m}\right) \\ \frac{1}{a_1}\left(f_{2n}l_{2m} + f_{2m}l_{2n}\right) & \frac{\Delta}{a_1^2} f_{2m} f_{2n} + l_{2n}l_{2m} \end{pmatrix}.$$

By equating the corresponding entries of the matrices on both sides, we get the identities (4.3) and (4.4).

Similarly, considering $T^{n-m} = T^n T^{-m} = T^n \left(T^m\right)^{-1}$ and by using the matrix identity (4.1), we get;

$$\frac{1}{2}\begin{pmatrix} l_{2(n-m)} & \frac{\Delta}{a_1} f_{2(n-m)} \\ \frac{1}{a_1} f_{2(n-m)} & l_{2(n-m)} \end{pmatrix} = \frac{1}{4B^m}\begin{pmatrix} l_{2n}l_{2m} - \frac{\Delta}{a_1^2} f_{2n} f_{2m} & \frac{-\Delta}{a_1}\left(l_{2n} f_{2m} - f_{2n}l_{2m}\right) \\ \frac{-1}{a_1}\left(l_{2n} f_{2m} - f_{2n}l_{2m}\right) & l_{2n}l_{2m} - \frac{\Delta}{a_1^2} f_{2n} f_{2m} \end{pmatrix}$$

By equating the corresponding entries of the matrices on both sides, we get the results (4.5) and (4.6).

Finally, consider the equation

$$T^{n+m} + B^m T^{n-m} = T^n T^m + B^m T^n \left(T^m\right)^{-1}$$

then we get (4.7) and (4.8). □

Up to now, we only consider the even indices of the conditional sequences. Now, we develop another matrix identity for both even and odd indices of Fibonacci and Lucas conditional sequences.

We know that there is a relation between continued fractions and $2\times 2$ matrices. By considering this relation and setting

$$C := \begin{pmatrix} a_2 & b_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_2 & a_2 b_1 \\ a_1 & b_1 \end{pmatrix}$$

then we have

$$C^n = \begin{pmatrix} f_{2n+1} & \frac{a_2 b_1}{a_1} f_{2n} \\ f_{2n} & \frac{b_1}{a_1}\left(f_{2n} - b_2 f_{2(n-1)}\right) \end{pmatrix} \tag{4.9}$$

by induction and also by the help of (1.1) and (1.2).

We can get the Binet formula for odd and even indices of $\{f_n\}$ by using this matrix identity. For this, again we assume $A^2-4B \neq 0$ to guarantee the existence of two distinct roots. Let $\alpha$ and $\beta$ denote the roots of $p(z) = z^2 - Az + B$ that is the characteristic polynomial of $C$. The eigenvalues and corresponding eigenvectors of the matrix $C$ are $\lambda_1 = \alpha$, $\lambda_2 = \beta$ and $v_1 = \begin{pmatrix} \frac{\alpha - b_1}{a_1} \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{\beta - b_1}{a_1} \\ 1 \end{pmatrix}$. Therefore; $P^{-1}CP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $P = \begin{pmatrix} \frac{\alpha - b_1}{a_1} & \frac{\beta - b_1}{a_1} \\ 1 & 1 \end{pmatrix}$. It follows $C^n = P \begin{pmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{pmatrix} P^{-1}$. By equating $(1,1)$ and $(2,1)$ entries of the matrices on both sides, we obtain the Binet formula for odd and even indices of $\{f_n\}$.

By taking the determinat of both sides of (4.9), we get

$$\frac{b_1}{a_1}\left(f_{2n+1}\left[f_{2n} - b_2 f_{2(n-1)}\right] - a_2 f_{2n}^2\right) = B^n. \tag{4.10}$$

For $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = m$; the above result reduces to the Cassini's identity for the ordinary Fibonacci sequence:

$$F_{m+1}F_{m-1} - F_m^2 = (-1)^m.$$

By using the matrix identity (4.9), if we consider $C^{n-m} = C^n C^{-m}$ and equating the corresponding entries of the matrices on both sides, we get the formula;

$$B^m f_{2(n-m)} = f_{2n} f_{2m+1} - f_{2m} f_{2n+1}. \tag{4.11}$$

For $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = r$, $2m = s$; the above result reduces to the d'Ocagne's identity for the ordinary Fibonacci sequence:

$$(-1)^s F_{r-s} = F_r F_{s+1} - F_s F_{r+1}.$$

And if we consider $C^{n+m} = C^n C^m$ and equating the corresponding entries of the matrices on both sides, we get the formulas;

$$f_{2(n+m)} = f_{2n} f_{2m+1} + f_{2m}\left(\frac{b_1}{a_1} f_{2n} - \frac{B}{a_1} f_{2(n-1)}\right) \tag{4.12}$$

$$f_{2(n+m)+1} = f_{2n+1} f_{2m+1} + \frac{a_2 b_1}{a_1} f_{2n} f_{2m}. \tag{4.13}$$

For $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = r$, $2m = s$; the above results reduce to the following well-known results

$$F_{r+s} = F_r F_{s+1} + F_s F_{r-1}$$
$$F_{r+s+1} = F_{r+1} F_{s+1} + F_r F_s.$$

We also have a similar matrix identity for both even and odd indices of $\{l_n\}$. For this, assume that $b_1 = b_2$. By induction and by the help of (1.1) and (1.2), we have

$$\begin{pmatrix} a_2 & 2\frac{a_2 b_1}{a_1} \\ 2 & -a_2 \end{pmatrix} \begin{pmatrix} a_1 a_2 + b_2 & a_2 b_1 \\ a_1 & b_1 \end{pmatrix}^n = \begin{pmatrix} l_{2n+1} & \frac{a_2 b_1}{a_1} l_{2n} \\ l_{2n} & \frac{b_1}{a_1}\left(l_{2n} - b_1 l_{2(n-1)}\right) \end{pmatrix} \tag{4.14}$$

and by taking the determinat of both sides of (4.14), we get

$$a_1 b_1 (l_{2n+1}[l_{2n} - b_1 l_{2(n-1)}] - a_2 l_{2n}^2) = -(\alpha - \beta)^2 B^n. \tag{4.15}$$

If we take $a_1 = a_2 = b_1 = b_2 = 1$ and $2n = m$ in (4.15), it reduces to the well-known identity for Lucas numbers:

$$L_{m+1} L_{m-1} - L_m^2 = (-1)^{m+1} 5.$$

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