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ON TAUBERIAN REMAINDER THEOREMS FOR CESÀRO SUMMABILITY METHOD OF NONINTEGER ORDER

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Abstract. In this paper, we prove some Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$.

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1. INTRODUCTION

Let A_n^α be defined by the generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$, $(|x|<1)$, where $\alpha > -1$. For a real sequence $u = (u_n)$, the Cesàro means of the sequence (u_n) of noninteger order α are defined by

$$\sigma_n^{(\alpha)}(u) = \frac{1}{A_n^\alpha} \sum_{j=0}^{n} A_{n-j}^{\alpha-1} u_j .$$

We say that a sequence (u_n) is (C,α) summable to a finite number s, where $\alpha > -1$ if

$$\lim_{n\to\infty} \sigma_n^{(\alpha)}(u) = s, \tag{1.1}$$

and we write $u_n \to s\ (C,\alpha)$. We denote the backward difference of (u_n), by $\Delta u_n = u_n - u_{n-1}$, with $\Delta u_0 = u_0$. We define $\tau_n(u) = n\Delta u_n \quad (n = 0,1,2,..)$ and indicate $\tau_n^{(\alpha)}(u)$ as (C,α) mean of $(\tau_n(u))$.

Note that if taking $\alpha = k$ where k is a nonnegative integer, then we obtain the (C,k) summability method and for $\alpha = 0$, the $(C,0)$ summability is ordinary convergence.

The (C,α) summability method is regular, more generally, if a sequence (u_n) is (C,α) summable to s, where $\alpha > -1$ and $\beta \geq \alpha$ for α,β, then (u_n) is also (C,β) summable to s. However, the converse is not always true. The converse of this statement is valid under some conditions called Tauberian conditions. Any theorem which states that convergence of a sequence follows from a summability method and some Tauberian condition(s) is said to be a Tauberian theorem. Recently, a number of authors such as Estrada and Vindas [4, 5], Natarajan [15], Çanak et al. [2], Erdem and Çanak [3], Çanak and Erdem [1] have investigated Tauberian theorems for several summability methods.

For a sequence (u_n) and for each integer $m \geq 1$,

$$(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1}u_n), \tag{1.2}$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$.

For $\alpha > -1$, the identity

$$\tau_n^{(\alpha)}(u) = n\Delta\sigma_n^{(\alpha)}(u) \tag{1.3}$$

was proved by Kogbetliantz [9]. Note that $\tau_n^{(0)}(u) = \tau_n(u)$.

The identity

$$\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u) = \frac{1}{\alpha+1}\tau_n^{(\alpha+1)}(u) \tag{1.4}$$

is used in the various steps of proofs (see [10]).

Çanak et al. [2] represent the identity

$$n\Delta\tau_n^{\alpha+1} = (\alpha+1)(\tau_n^{\alpha} - \tau_n^{\alpha+1}), \tag{1.5}$$

for $\alpha > -1$.

Erdem and Çanak [3] prove that for $\alpha > -1$ and any integer $k \geq 1$

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k}(-1)^{j+1}A_k^{(j)}(\alpha)n\Delta\tau_n^{(\alpha+j)}(u), \tag{1.6}$$

where $A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha)$, $\quad a_k^{(0)}(\alpha) = 0$ and

$$a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha+i) \sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k \\ r < s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}).$$

2. Tauberian remainder theorems

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence (u_n) is called bounded with the rapidity (λ_n) (in short λ-bounded) if

$$\lambda_n(u_n - s) = O(1),$$

with $\lim_{n\to\infty} u_n = s$. Let

$$m^{\lambda} = \{u = (u_n) | \lim_{n\to\infty} u_n = s \quad \text{and} \quad \lambda_n(u_n - s) = O(1)\}. \tag{2.1}$$

A sequence (u_n) is called λ-bounded by the (C, α) method of summability if

$$\lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1), \tag{2.2}$$

with $\lim_{n\to\infty} \sigma_n^{(\alpha)}(u) = s$. Shortly, we write $u \in ((C, \alpha), m^{\lambda})$.

G. Kangro [7] introduced the concepts of Tauberian remainder theorems using summability with given rapidity λ. G. Kangro [8] and Tammeraid [16, 17] proved some Tauberian remainder theorems for several summability method, such as Riesz, Cesàro, Hölder and Euler-Knopp methods. Recently, various authors have represented some Tauberian remainder theorems (see [12, 13]). In [18], Tammeraid proved some Tauberian remainder theorems in which the (C,α) summability method is used. Tauberian remainder theorems have also been studied by many authors via the Fourier integral method. [6, 11]

Meronen and Tammeraid [14] proved the following Tauberian remainder theorems:

Theorem 1. *Let the condition*

$$\lambda_n \tau_n^{(1)}(u) = O(1)$$

be satisfied. If $u \in ((C,1), m^{\lambda})$, *then* $u \in m^{\lambda}$.

Theorem 2. *Let the conditions*

$$\lambda_n \tau_n(u) = O(1),$$
$$\lambda_n n \Delta \tau_n^{(1)}(u) = O(1)$$

be satisfied. If $u \in ((C,1), m^{\lambda})$, *then* $u \in m^{\lambda}$.

The main purpose of this paper is to prove several Tauberian remainder theorems for Cesàro summability method of noninteger order $\alpha > -1$. Our main theorems improve Theorem 1 and Theorem 2 given by Meronen and Tammeraid [14].

3. A LEMMA

We require the following lemma to be used in the proofs of main theorems.

Lemma 1. *Let* $\alpha > -1$. *For any integer* $k \geq 2$,

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\tau_n^{(\alpha)}(u) - B_{1,1}\sigma_n^{(\alpha)}(u) + B_{1,1}\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k} \Big(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u) \Big),$$

where $B_{m,l} = (\alpha+m)(\alpha+l)(-1)^{m+1}A_k^{(m)}(\alpha)$ *and* $A_k^{(j)}(\alpha) = a_k^{(j-1)}(\alpha) + a_k^{(j)}(\alpha)$, $a_k^{(0)}(\alpha) = 0$ *and*

$$a_k^{(j)}(\alpha) = \Pi_{i=j+1}^{k}(\alpha+i) \sum_{\substack{j+1 \leq t_1, t_2, \ldots, t_{j-1} \leq k \\ r<s \Rightarrow t_r \leq t_s}} (\alpha+t_1)(\alpha+t_2)\ldots(\alpha+t_{j-1}).$$

Proof. From identity (1.6), we have

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = \sum_{j=1}^{k} (-1)^{j+1} A_k^{(j)}(\alpha) n\Delta\tau_n^{(\alpha+j)}(u)$$
$$= A_k^{(1)}(\alpha) n\Delta\tau_n^{(\alpha+1)}(u) + \sum_{j=2}^{k} (-1)^{j+1} A_k^{(j)}(\alpha) n\Delta\tau_n^{(\alpha+j)}(u).$$

It follows from identity (1.5) that

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1) A_k^{(1)}(\alpha)(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$
$$+ \sum_{j=2}^{k} (\alpha+j)(-1)^{j+1} A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u) - \tau_n^{(\alpha+j)}(u)).$$

By identity (1.4), we can write the above equation as

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u) = (\alpha+1) A_k^{(1)}(\alpha)\tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha)\left(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u)\right)$$
$$+ \sum_{j=2}^{k} (\alpha+j)(-1)^{j+1} A_k^{(j)}(\alpha)\left((\alpha+j-1)(\sigma_n^{(\alpha+j-2)}(u) - \sigma_n^{(\alpha+j-1)}(u))\right.$$
$$\left. -(\alpha+j)(\sigma_n^{(\alpha+j-1)}(u) - \sigma_n^{(\alpha+j)}(u))\right).$$

Therefore,

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u)$$
$$= (\alpha+1) A_k^{(1)}(\alpha)\tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha)}(u) + (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k} \left((\alpha+j)(\alpha+j-1)(-1)^{j+1} A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-2)}(u)\right.$$
$$- (\alpha+j)(\alpha+j-1)(-1)^{j+1} A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u)$$
$$\left. - (\alpha+j)^2(-1)^{j+1} A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u) + (\alpha+j)^2(-1)^{j+1} A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j)}(u)\right).$$

Hence, we have

$$(n\Delta)_k \tau_n^{(\alpha+k)}(u)$$
$$= (\alpha+1) A_k^{(1)}(\alpha)\tau_n^{(\alpha)}(u) - (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha)}(u) + (\alpha+1)^2 A_k^{(1)}(\alpha)\sigma_n^{(\alpha+1)}(u)$$
$$+ \sum_{j=2}^{k} \left((\alpha+j)(\alpha+j-1)(-1)^{j+1} A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-2)}(u)\right.$$
$$- (\alpha+j)(2\alpha+2j-1)(-1)^{j+1} A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j-1)}(u)$$

$$+ (\alpha+j)^2(-1)^{j+1}A_k^{(j)}(\alpha)\sigma_n^{(\alpha+j)}(u)\Big).$$

Taking $(\alpha+m)(\alpha+l)(-1)^{m+1}A_k^{(m)}(\alpha) = B_{m,l}$, we obtain

$$(n\Delta)_k\tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\tau_n^{(\alpha)}(u) - B_{1,1}\sigma_n^{(\alpha)}(u) + B_{1,1}\sigma_n^{(\alpha+1)}(u)$$
$$+\sum_{j=2}^{k}\Big(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u)\Big).$$

Thus, we conclude that Lemma 1 is true for each integer $k \geq 2$. □

4. Main results

In the main theorems, we prove some Tauberian remainder theorems to recover λ-bounded by the (C,α) summability of a sequence out of λ-bounded by the $(C,\alpha+j)$ summability for $j = 1,2$ and any integer $j = k$, and some suitable conditions. In special cases of main theorems, we obtain some classical type Tauberian remainder theorems for the $(C,1)$ summability method.

Theorem 3. *Let the conditions*

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) = O(1), \tag{4.1}$$

and

$$\lambda_n \tau_n^{(\alpha)}(u) = O(1) \tag{4.2}$$

be satisfied for $\alpha > -1$. If $u \in ((C,\alpha+1), m^\lambda)$, then $u \in ((C,\alpha), m^\lambda)$.

Proof. From identity (1.5), we have

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) = \lambda_n(\alpha+1)(\tau_n^{(\alpha)}(u) - \tau_n^{(\alpha+1)}(u))$$
$$= \lambda_n(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n(\alpha+1)\tau_n^{(\alpha+1)}(u).$$

From identity (1.4), we obtain

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u) = \lambda_n(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u) - \sigma_n^{(\alpha+1)}(u)).$$

Rewritten the above equation, we have

$$\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u) - s) = \lambda_n(\alpha+1)^2(\sigma_n^{(\alpha+1)}(u) - s)$$
$$+\lambda_n(\alpha+1)\tau_n^{(\alpha)}(u) - \lambda_n n\Delta\tau_n^{(\alpha+1)}(u).$$

Using (4.1) and (4.2), we get

$$\lambda_n(\alpha+1)^2(\sigma_n^{(\alpha)}(u) - s) = O(1) + O(1) + O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u) - s) = O(1)$. That means $u \in ((C,\alpha), m^\lambda)$. □

Notice that taking $\alpha = 0$, we obtain Theorem 2.

Proposition 1. *Let the conditions*

$$\lambda_n(n\Delta)_2\tau_n^{(\alpha+2)}(u) = O(1), \tag{4.3}$$

$$\lambda_n\tau_n^{(\alpha)}(u) = O(1), \tag{4.4}$$

and

$$\lambda_n(\sigma_n^{(\alpha+2)}(u)-s) = O(1) \tag{4.5}$$

be satisfied for $\alpha > -1$. *If* $u \in ((C,\alpha+1),m^{\lambda})$, *then* $u \in ((C,\alpha),m^{\lambda})$.

Proof. Taking $k = 2$ in Lemma 1, we have

$$\begin{aligned}
\lambda_n(n\Delta)_2\tau_n^{(\alpha+2)}(u) &= \lambda_n(\alpha+2)(\alpha+1)\left(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u)\right) \\
&\quad -\lambda_n(\alpha+2)^2\left(\tau_n^{(\alpha+1)}(u)-\tau_n^{(\alpha+2)}(u)\right) \\
&= \lambda_n(\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+2)(\alpha+1)\tau_n^{(\alpha+1)}(u) \\
&\quad -\lambda_n(\alpha+2)^2\tau_n^{(\alpha+1)}(u)+\lambda_n(\alpha+2)^2\tau_n^{(\alpha+2)}(u).
\end{aligned}$$

From identity (1.4), we get

$$\begin{aligned}
&\lambda_n(n\Delta)_2\tau_n^{(\alpha+2)}(u) \\
&= \lambda_n(\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+2)(\alpha+1)\left((\alpha+1)(\sigma_n^{(\alpha)}(u)-\sigma_n^{(\alpha+1)}(u))\right) \\
&\quad -\lambda_n(\alpha+2)^2\left((\alpha+1)(\sigma_n^{(\alpha)}(u)-\sigma_n^{(\alpha+1)}(u))\right) \\
&\quad +\lambda_n(\alpha+2)^2\left((\alpha+2)(\sigma_n^{(\alpha+1)}(u)-\sigma_n^{(\alpha+2)}(u))\right) \\
&= \lambda_n(\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)^2(\alpha+2)\sigma_n^{(\alpha)}(u) \\
&\quad +\lambda_n(\alpha+1)^2(\alpha+2)\sigma_n^{(\alpha+1)}(u)-\lambda_n(\alpha+2)^2(\alpha+1)\sigma_n^{(\alpha)}(u) \\
&\quad +\lambda_n(\alpha+2)^2(\alpha+1)\sigma_n^{(\alpha+1)}(u)+\lambda_n(\alpha+2)^3\sigma_n^{(\alpha+1)}(u)-\lambda_n(\alpha+2)^3\sigma_n^{(\alpha+2)}(u) \\
&= \lambda_n(\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)(\alpha+2)(2\alpha+3)\alpha_n^{(\alpha)}(u) \\
&\quad +\lambda_n((\alpha+2)^3+(\alpha+2)^2(\alpha+1)+(\alpha+1)^2(\alpha+2))\sigma_n^{(\alpha+1)}(u) \\
&\quad -\lambda_n(\alpha+2)^3\sigma_n^{(\alpha+2)}(u).
\end{aligned}$$

Rewritten the above equation, we have

$$\begin{aligned}
&\lambda_n(\alpha+1)(\alpha+2)(2\alpha+3)(\sigma_n^{(\alpha)}(u)-s) \\
&= -\lambda_n n\Delta\tau_n^{(\alpha+2)}(u)+\lambda_n(\alpha+2)(\alpha+1)\tau_n^{(\alpha)}(u)+\lambda_n((\alpha+2)^3+(\alpha+2)^2(\alpha+1) \\
&\quad +(\alpha+1)^2(\alpha+2)-s)\sigma_n^{(\alpha+1)}(u)-\lambda_n((\alpha+2)^3-s)\sigma_n^{(\alpha+2)}(u).
\end{aligned}$$

Using (4.3), (4.4) and (4.5), we get

$$\lambda_n(\alpha+1)(\alpha+2)(2\alpha+3)(\sigma_n^{(\alpha)}(u)-s) = O(1)+O(1)+O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s) = O(1)$. That means $u \in ((C,\alpha),m^{\lambda})$. □

Now, we represent a Tauberian remainder theorem which generalizes Theorem 3 and Proposition 1.

Theorem 4. *Let the conditions*

$$\lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u) = O(1), \tag{4.6}$$

$$\lambda_n\tau_n^{(\alpha)}(u) = O(1), \tag{4.7}$$

and

$$\lambda_n(\sigma_n^{(\alpha+j)}(u)-s) = O(1) \quad \textit{for} \quad 2 \le j \le k \tag{4.8}$$

be satisfied for $\alpha > -1$. *If* $u \in ((C,\alpha+1),m^{\lambda})$, *then* $u \in ((C,\alpha),m^{\lambda})$.

Proof. From Lemma 1 we have

$$\lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u) = B_{1,1-\alpha}\lambda_n\tau_n^{(\alpha)}(u) - B_{1,1}\lambda_n\sigma_n^{(\alpha)}(u) + B_{1,1}\lambda_n\sigma_n^{(\alpha+1)}(u)$$
$$+\lambda_n\sum_{j=2}^{k}\Big(B_{j,j-1}\sigma_n^{(\alpha+j-2)}(u) - 2B_{j,j-\frac{1}{2}}\sigma_n^{(\alpha+j-1)}(u) + B_{j,j}\sigma_n^{(\alpha+j)}(u)\Big),$$

Rewritten the above equation, we have

$$\begin{aligned} &B_{1,1}\lambda_n(\sigma_n^{(\alpha)}(u)-s)\\ &= B_{1,1-\alpha}\lambda_n\tau_n^{(\alpha)}(u) - \lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u) + B_{1,1}\lambda_n(\sigma_n^{(\alpha+1)}(u)-s)\\ &\quad + \lambda_n\sum_{j=2}^{k}B_{j,j-1}(\sigma_n^{(\alpha+j-2)}(u)-s) - \lambda_n\sum_{j=2}^{k}2B_{j,j-\frac{1}{2}}(\sigma_n^{(\alpha+j-1)}(u)-s)\\ &\quad + \lambda_n\sum_{j=2}^{k}B_{j,j}(\sigma_n^{(\alpha+j)}(u)-s). \end{aligned}$$

Using (4.6), (4.7) and (4.8), we get

$$B_{1,1}\lambda_n(\sigma_n^{(\alpha)}(u)-s) = O(1)+O(1)+O(1)+O(1)+O(1)+O(1) = O(1).$$

Therefore, $\lambda_n(\sigma_n^{(\alpha)}(u)-s) = O(1)$. That means $u \in ((C,\alpha),m^{\lambda})$. □

Theorem 5. *Let the condition*

$$\lambda_n\tau_n^{(\alpha+j+1)}(u) = O(1) \quad \textit{for} \quad 0 \le j \le k-1, \tag{4.9}$$

be satisfied for $\alpha > -1$. *If* $u \in ((C,\alpha+k),m^{\lambda})$, *then* $u \in ((C,\alpha),m^{\lambda})$.

Proof. Suppose that $u \in ((C,\alpha+k),m^{\lambda})$. Taking $j = k-1$ in (4.9), it follows from the idendity

$$\tau_n^{(\alpha+k)}(u) = (\alpha+k)(\sigma_n^{(\alpha+k-1)}(u) - \sigma_n^{(\alpha+k)}(u))$$

that we obtain

$$\lambda_n(\alpha+k)(\sigma_n^{(\alpha+k-1)}(u)-s) = \lambda_n\tau_n^{(\alpha+k)}(u)+\lambda_n(\alpha+k)(\sigma_n^{\alpha+k}(u)-s)$$
$$= O(1)+O(1)=O(1)$$

then we obtain $\lambda_n(\sigma_n^{(\alpha+k-1)}-s)=O(1)$. Hence, that means

$$u\in((C,\alpha+k-1),m^{\lambda}).$$

From identity (1.4), we have

$$\tau_n^{(\alpha+k-1)}(u)=(\alpha+k-1)(\sigma_n^{(\alpha+k-2)}(u)-\sigma_n^{(\alpha+k-1)}(u)).$$

Taking $j=k-2$ in (4.9), we obtain

$$\lambda_n(\alpha+k-1)(\alpha_n^{(\alpha+k-2)}(u)-s)$$
$$=\lambda_n\tau_n^{(\alpha+k-1)}(u)+\lambda_n(\alpha+k-1)(\sigma_n^{(\alpha+k-1)}(u)-s)=O(1)+O(1)=O(1)$$

Therefore we have

$$u\in((C,\alpha+k-2),m^{\lambda}).$$

Continuing in this way, we obtain that

$$u\in((C,\alpha+1),m^{\lambda}).$$

Taking $j=0$ in (4.9), we obtain $\lambda_n\tau_n^{(\alpha+1)}=O(1)$. From identity (1.4), we have

$$\lambda_n(\alpha+1)(\sigma_n^{(\alpha)}(u)-s)=\lambda_n\tau_n^{(\alpha+1)}(u)+\lambda_n(\alpha+1)(\sigma_n^{(\alpha+1)}(u)-s)$$
$$=O(1)+O(1)=O(1)$$

This completes the proof. □

Theorem 6. *Let the condition*

$$\lambda_n(n\Delta)_j\tau_n^{(\alpha+j)}(u)=O(1)\quad \textit{for}\quad 0\le j\le k, \tag{4.10}$$

be satisfied for $\alpha>-1$. *If* $u\in((C,\alpha+k),m^{\lambda})$, *then* $u\in((C,\alpha),m^{\lambda})$.

Proof. By identity (1.6) for $k=1$, it follows

$$\lambda_n n\Delta\tau_n^{(\alpha+1)}(u)=\lambda_n(\alpha+1)(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u))$$
$$=\lambda_n(\alpha+1)\tau_n^{(\alpha)}(u)-\lambda_n(\alpha+1)\tau_n^{(\alpha+1)}(u).$$

Taking $j=0$ and $j=1$ in (4.10), we obtain

$$\lambda_n\tau_n^{(\alpha+1)}(u)=O(1)$$

From identity (1.6) for $k=2$, we get

$$\lambda_n(n\Delta)_2\tau_n^{(\alpha+2)}(u)=\lambda_n(\alpha+2)(\alpha+1)\left(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u)\right)$$
$$-\lambda_n(\alpha+2)^2\left(\tau_n^{(\alpha+1)}(u)-\tau_n^{(\alpha+2)}(u)\right).$$

Taking $j=0$ and $j=2$ in (4.10), we obtain

$$\lambda_n\tau_n^{(\alpha+2)}(u)=O(1)$$

Continuing in this way, by Lemma 1, we obtain

$$\lambda_n(n\Delta)_k\tau_n^{(\alpha+k)}(u)=(\alpha+1)A_k^{(1)}(\alpha)\lambda_n(\tau_n^{(\alpha)}(u)-\tau_n^{(\alpha+1)}(u))$$
$$+\lambda_n\sum_{j=2}^{k}(\alpha+j)(-1)^{j+1}A_k^{(j)}(\alpha)(\tau_n^{(\alpha+j-1)}(u)-\tau_n^{(\alpha+j)}(u)).$$

Taking $j=0$ and $j=k$ in (4.10), we obtain

$$\lambda_n\tau_n^{(\alpha+k)}(u)=O(1).$$

The conditions in Theorem 5 hold, the proof is completed. □

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