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# GENERALIZED TERRACED MATRICES

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*Abstract.* We know that every terraced matrix has the factorization $R_b = D_b C$, where $C$ is the Cesàro matrix and $D_b = diag\{(n+1)b_n\}$. In the present paper, we define the generalized terraced matrix by using the generalized Cesàro matrix in the expression above, and some properties of this matrix are given.

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## 1. INTRODUCTION

In [13], A.G. Siskakis gives the spectrum of the Cesàro matrix on $H^p$ by using the integral representation of the Cesàro operator.

Let $H(\mathbb{D})$ denotes the space of complex valued analytic functions on the unit disk $\mathbb{D}$, for $1 \leq p < \infty$, $H^p$ denotes the standard Hardy space on $\mathbb{D}$, and $\ell^p$ denotes the standard space of $p$-summable complex-valued sequences on the set of non-negative integers.

Suppose that $1 < p < \infty$ and $(b) = \{b_n\}_{n=0}^{\infty}$ is in $\ell^p$. Then the sequences

$$C(b) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} b_k \right\}_{n=0}^{\infty}$$

have $\ell^p$-norms satisfying

$$\|C(b)\|_p \leq \frac{p}{p-1} \|(b)\|_p$$

and the constant in this inequality is the best possible [4,6,7,10]. Thus $C$ is a bounded linear operator on $\ell^p$ for $1 < p < \infty$ with its norm equal to $p/(p-1)$.

If $f(z) = \sum_{k=0}^{\infty} b_k z^k$ is in $H^p$, let

$$C(f)(z) = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \sum_{k=0}^{n} b_k \right) z^n.$$

By computing Taylor series, we see that $C$ has the following integral representation: for $f \in H^p$,

$$C(f)(z) = \frac{1}{z}\int_0^z \frac{f(t)}{1-t}dt \tag{1.1}$$

In [19], Scott W. Young generalized Cesàro operator, by considering more general analytic functions instead of the function $1/(1-t)$ in equality (1.1), as follows.

**Definition 1.** Let $g$ be analytic on the unit disk. The operator $C_g : H^2 \to H^2$ defined by

$$C_g(f) := \frac{1}{z}\int_0^z f(t)\,g(t)\,dt \tag{1.2}$$

is called the generalized Cesàro operator with symbol $g$.

**Definition 2.** Let $I$ be an arc of the unit circle $\mathbb{T}$, and let $\varphi : \mathbb{T} \to \mathbb{C}$. Then, let $\varphi_I = \frac{1}{|I|}\int_I |\varphi|$, where $|I|$ denotes the arclength of $I$. $\varphi$ is said to be of bounded mean oscillation if

$$\|\varphi\|_* = \sup_{I\subset\mathbb{T}} \frac{1}{|I|}\int_I |\varphi - \varphi_I| < \infty.$$

We denote the set of all functions of bounded mean oscillation by $BMO$. If we endow $BMO$ with the norm $\|\varphi\|_{BMO} = \|\varphi\|_* + |\varphi(0)|$, then $BMO$ is a Banach space (see [5]).
We say that $g \in BMOA$ if $g \in H^2$ and $g\left(e^{i\theta}\right) \in BMO$.

**Definition 3.** Let $I$ be an arc of $\mathbb{T}$. We say that a function $\varphi : \mathbb{T} \to \mathbb{C}$ is of vanishing mean oscillation if

$$\lim_{\delta\to 0} \sup_{I\subset} \frac{1}{|I|}\int_I |\varphi - \varphi_I| = 0.$$

We denote the set of all functions of vanishing mean oscillation by $VMO$. $VMO$ is a closed subspace of $BMO$.
As with $BMOA$, we define $VMOA$ as the set of $g \in H^2$ such that $g\left(e^{i\theta}\right) \in VMO$. $VMOA$ is a closed subspace of $BMOA$ (see [5]).

**Definition 4.** A vector $x$ is a cyclic vector for a bounded operator $T$ on a Hilbert space $H$ if the set $\{p(T)x : p \text{ is polynomial}\}$ is dense in $H$. If $T$ has a cyclic vector, then $T$ is called a cyclic operator.

We denote the spectrum of the linear operator $T$ by $\sigma(T)$. That is,

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ not invertible}\}.$$

Let $G(z) = \int_0^z g(w)\,dw$. Pommerenke [12] showed that $C_g$ is bounded on the Hilbert space $H^2$ if and only if $G \in BMOA$. Aleman and Siskakis [2] extended

Pommerenke's result to the Hardy spaces $H^p$ for all $p$ , $1 \leq p < \infty$, and showed that $C_g$ is compact on $H^p$ if and only if $G \in VMOA$.

Continuity of the Cesàro operator $C$ on the Hilbert space $H^2(\mathbb{D})$ is due to Hardy, Littlewood and Polya [7], and to Siskakis for the general Hardy and the unweighted Bergman space cases, [13, 14, 16]. In [15], Siskakis considered a class of generalized Cesàro operators associated with semigroups of weighted composition operators on $H^2(\mathbb{D})$, $1 \leq p < \infty$, characterized compactness within this class and identified the spectrum of the operators $C_g\big|_{H^p}$ for $g(z) = \frac{1+z}{1-z}$. He also raised question of the extent to which these operators were hyponormal or subnormal on $H^2(\mathbb{D})$. Brown, Halmos and Shields [3] and Kriete and Trutt [9] investigated these properties for the classical Cesàro operator. In [1] Albrecht, Miller and Neumann showed that $C_{(1+z)/(1-z)}$ is hyponormal on $H^2(\mathbb{D})$.

The matrix representation of $C_g$ in the standard basis $\{z^{n-1}\}_{n=1}^{\infty}$ of $H^2$ follows

$$C_g = \begin{pmatrix} a_0 & & & & \\ \frac{a_1}{2} & \frac{a_0}{2} & & & \\ \frac{a_2}{3} & \frac{a_1}{3} & \frac{a_0}{3} & & \\ \frac{a_3}{4} & \frac{a_2}{4} & \frac{a_1}{4} & \frac{a_0}{4} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.3}$$

where, $a_j$ are Taylor coefficients of $g(z)$, i.e. $\sum_{j=0}^{\infty} a_j z^j = g(z) \in H(\mathbb{D})$.

Given a sequence $\{b_n\}$ of scalars, the terraced matrix $R_b$ is the lower triangular matrix with constant row-segments

$$R_b = \begin{pmatrix} b_0 & & & & \\ b_1 & b_1 & & & \\ b_2 & b_2 & b_2 & & \\ b_3 & b_3 & b_3 & b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.4}$$

The Cesàro matrix is $R_{\{1/(n+1)\}}$ and more generally, if we take $b_n = n^{-z}$ we get the $z$-Cesàro matrix $C_z$.

In [11], G. Leibowitz gave the following relation between terraced matrix and Cesàro matrix $C$ .

If $D$ is the diagonal matrix $diag\{d_n\}$, then $DR_{\{b_n\}} = R_{\{d_n b_n\}}$. Hence every terraced matrix has the factorization $R_b = D_b C$, where $D_b = diag\{(n+1) b_n\}$; while if every $b_n \neq 0$, $C = \overline{D}_b R_b$ , where $\overline{D}_b = diag\left\{\frac{1}{(n+1) b_n}\right\}_{n=0}^{\infty}$.

In the present paper, we define the generalized terraced matrix by using the generalized Cesàro matrix and we show that the Cesàro matrix $C$, obtained when $b_n = 1/(n+1)$ and $g(z) = 1/(1-z)$ are taken in the generalized terraced matrix, is essentially the only generalized terraced matrix that is a Hausdorff matrix. That is, any generalized terraced matrix that is not a scalar multiple of $C$ is not a Hausdorff matrix. And we prove that every generalized terraced matrix commutes with an infinite matrix $B$, then $B$ is a scalar multiple of unit matrix. Also, we prove necessary and sufficient conditions related to normality and self-adjointedness of generalized terraced matrix.

**Definition 5.** Let $\{b_n\}$ be a scalar sequence and $g(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$. The matrix

$$R_b^g = \begin{pmatrix} a_0 b_0 & & & & \\ a_1 b_1 & a_0 b_1 & & & \\ a_2 b_2 & a_1 b_2 & a_0 b_2 & & \\ a_3 b_3 & a_2 b_3 & a_1 b_3 & a_0 b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.5}$$

is called the generalized terraced matrix with symbol $g$ on $H^2$.

The relation $R_b^g = D_b C_g$ is valid similar to the terraced matrix, where $D_b = diag\{(n+1) b_n\}_{n=0}^{\infty}$. We recall that $C = C_g$ for $g(z) = \frac{1}{1-z}$, since $g(z) = \sum_{k=0}^{\infty} z^k$, which fixes then $a_n = 1$ for all $n \in \mathbb{N}$. Thus, from (1.5) we get

$$R_b^g = \begin{pmatrix} b_0 & & & & \\ b_1 & b_1 & & & \\ b_2 & b_2 & b_2 & & \\ b_3 & b_3 & b_3 & b_3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = R_b$$

On the other hand $R_{\{1/(n+1)\}}^g = C_g$. Therefore this definition could be regarded as a two-way generalization of both terraced and Cesàro operators.

From (1.5) we can write

$$\left(R_b^g\right)_{nj} = \begin{cases} a_{n-j}b_n & , \quad n \geq j \\ 0 & , \quad n < j \end{cases} \tag{1.6}$$

and

$$\left[\left(R_b^g\right)^*\right]_{nj} = \begin{cases} \overline{a_{j-n}b_j} & , \quad j \geq n \\ 0 & , \quad j < n \end{cases} \tag{1.7}$$

## 2. Results

**Theorem 1.** *Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and $a_0 \neq 0 \neq a_1$. If $R_b^g$ commutes with $C_g$, then $R_b^g$ is a scalar multiple of $C_g$.*

*Proof.* We get by direct calculation

$$\left[R_b^g C_g\right]_{nj} = \begin{cases} b_n \sum_{k=0}^{n-j} \dfrac{a_k a_{n-k-j}}{k+j+1} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases}$$

and

$$\left[C_g R_b^g\right]_{nj} = \begin{cases} \dfrac{1}{n+1} \sum_{k=0}^{n-j} a_k a_{n-k-j} b_{k+j} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases}$$

If $R_b^g C_g = C_g R_b^g$ then equating the entries on the first subdiagonal,

$$\left[R_b^g C_g\right]_{n+1,n} = \left[C_g R_b^g\right]_{n+1,n};$$

this gives

$$a_0 a_1 b_{n+1} \left(\frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{a_0 a_1}{n+2}\left(b_n + b_{n+1}\right)$$

for all nonnegative integers $n$. From last equation we have

$$b_{n+1} = \frac{n+1}{n+2} b_n \tag{2.1}$$

From (2.1), we can prove by using strong induction that for every $n$,

$$b_n = \frac{1}{n+1} b_0$$

Hence, we have $R_b^g = R^g_{\left\{\frac{b_0}{n+1}\right\}} = b_0 R^g_{\left\{\frac{1}{n+1}\right\}} = b_0 C_g$. □

*Remark* 1. Proposition 2.1 of [11] is a special case of Theorem 1 with the case $g(t) = 1/(1-t)$.

**Theorem 2.** *If an infinite matrix $B$ commutes with all generalized terraced matrices, then $B$ is a scalar multiple of the identity matrix.*

*Proof.* If we consider $R_b^g$ with $g(t) = 1/(1-t)$, we obtain the Rhaly matrix $R_b$. Hence, the proof could be completed by Proposition 2.3 in [11]. □

**Theorem 3.** *Let $b_n \neq 0$ for each $n \in \mathbb{Z}^+$. The matrix $R_b^g$ is normal if and only if $g(z) = c$ for some $c \in \mathbb{C}$.*

*Proof.* We calculate $\left[\left(R_b^g\right)^*\left(R_b^g\right)\right]_{00}$ and $\left[\left(R_b^g\right)\left(R_b^g\right)^*\right]_{00}$ by matrix multiplication. We get

$$\left[\left(R_b^g\right)^*\left(R_b^g\right)\right]_{00} = \sum_{k=0}^{\infty}\left[\left(R_b^g\right)^*\right]_{0k}\left[R_b^g\right]_{k0} = \sum_{k=0}^{\infty}|a_k|^2|b_k|^2$$

and

$$\left[\left(R_b^g\right)\left(R_b^g\right)^*\right]_{00} = \sum_{k=0}^{\infty}\left[R_b^g\right]_{0k}\left[\left(R_b^g\right)^*\right]_{k0} = a_0b_0\overline{a_0b_0} = |a_0|^2|b_0|^2.$$

Since normality is defined to be $\left(R_b^g\right)^*\left(R_b^g\right) = \left(R_b^g\right)\left(R_b^g\right)^*$, we require that $\left[\left(R_b^g\right)^*\left(R_b^g\right)\right]_{00} = \left[\left(R_b^g\right)\left(R_b^g\right)^*\right]_{00}$. This implies that

$$|a_0|^2|b_0|^2 + \sum_{k=1}^{\infty}|a_k|^2|b_k|^2 = |a_0|^2|b_0|^2.$$

Hence, $\sum_{k=1}^{\infty}|a_k|^2|b_k|^2 = 0$. Since $b_k \neq 0$ for every $k \geq 1$, then $a_k = 0$ for every $k \geq 1$. Thus, $g(z) = \sum_{k=0}^{\infty}a_kz^k = a_0$. The converse direction is trivial since $g(z) = a_0$ implies that $R_b^g = diag\{a_0b_k\}_{k=1}^{\infty}$. □

**Corollary 1.** *Let $b_n \neq 0$ for each $n \in \mathbb{Z}^+$ and $b_0 \in \mathbb{R}$. $R_b^g$ is self-adjoint if and only if $g(z) = c$ for some $c \in \mathbb{R}$.*

*Proof.* From (1.6) and (1.7)

$$a_0b_0 = \overline{a_0b_0}\,,\ a_1b_1 = a_2b_2 = a_3b_3 = \cdots = 0$$

Since, $\forall n \in N$, $b_n \neq 0$, then

$$a_0 = \overline{a_0}\,,\ a_1 = a_2 = a_3 = \cdots = 0$$

Hence $a_0 \in \mathbb{R}$ and $g(z) = a_0 \in \mathbb{R}$. The other direction is obvious. □

**Theorem 4.** *Let $\forall n \in N$, $b_n > 0$ real number and $\{b_n\}$ be a strictly decreasing sequence. $\left(R_b^g\right)^*$ is cyclic for all $\int_0^z g(w)\,dw \in BMOA$.*

*Proof.* If $g(0) = 0$, then the result follows from [17], Theorem 2. If $g(0) \neq 0$, then the diagonal entries in (1.7) are distinct. Therefore, it is cyclic. See, for example, [8], Proposition 3.6. □

**Theorem 5.** *Let $g_\beta(z) := g(\beta z)$ with $|\beta| = 1$, then $R_b^{g_\beta}$ is unitarily equivalent to $R_b^g$.*

*Proof.* Define the map $U_\beta : H^2 \to H^2$ by $U_\beta(f)(z) = f(\beta z)$. It is easy to see that $U_\beta$ is unitary with $U_\beta^* = U_{\overline{\beta}}$. Now, to show the unitary equivalence, we must prove that $U_\beta^* R_b^{g_\beta} U_\beta = R_b^g$. The matrix representation of $U_\beta$ in the basis $\{z^{n-1}\}_{n=1}^\infty$ is the diagonal matrix $diag\{\beta^n\}$. Moreover, we know that $(U_\beta)^* = U_{\overline{\beta}} = (U_\beta)^{-1}$. Thus we have $U_\beta^* R_b^{g_\beta} U_\beta = R_b^g$ using these matrix representations and consequently $R_b^{g_\beta}$ is unitarily equivalent to $R_b^g$. □

**Corollary 2.** *Let $\mathbb{D}$ be a unit disk in the complex plane. If $\beta \in \partial(\mathbb{D})$ and $b_n > 0$ $\forall n \in \mathbb{N}$, then $\sigma\left(R_b^{1/(1-\beta z)}\right) = \sigma(R_b) = \{z : |z - L| \le L\} \cup S$ , where $L = \lim_{n\to\infty}(n+1)b_n$ and $0 \le L < +\infty$, $S = \{b_n : n = 0, 1, 2, \ldots\}$.*

*Proof.* This is immediate from the unitary equivalence and [18]. □

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