Miskolc Mathematical Notes
Vol. 17 (2017), No. 2, pp. 749–759

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1267

GLOBAL RAINBOW DOMINATION IN GRAPHS

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Received 27 May, 2014

Abstract. For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\ldots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N(v)} f(u) = \{1,2,\ldots,k\}$ is fulfilled, where $N(v)$ is the neighborhood of v. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. A kRDF f is called a *global k-rainbow dominating function* (GkRDF) if f is also a kRDF of the complement $\overline{G}$ of G. The *global k-rainbow domination number* of G, denoted by $\gamma_{grk}(G)$, is the minimum weight of a GkRDF on G. In this paper, we initiate the study of the global k-rainbow domination number and we establish some sharp bounds for it.

2010 *Mathematics Subject Classification:* 05C69

Keywords: k-rainbow dominating function, k-rainbow domination number, global k-rainbow dominating function, global k-rainbow domination number

1. INTRODUCTION

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E). The order $|V|$ of G is denoted by $n = n(G)$. Denote by K_n the *complete graph*, by C_n the *cycle* and by P_n the *path* of order n, respectively. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. The *minimum* and *maximum* degrees of G are respectively denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$. A *leaf* of a graph is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v. Let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We use [12, 19] for terminology and notation which are not defined here.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S of

G is *global dominating set* of G if S is a dominating set both of G and $\overline{G}$. The *global domination number* $\gamma_g(G)$ of G is the minimum cardinality of a global dominating set. The global domination number was introduced independently by Brigham and Dutton [7] (the term factor domination number was used) and Sampathkumar [15] and has been studied by several authors (see for example [3, 20]). Since then some variants of the global domination parameter, such as connected (total) global domination, global minus domination, and global Roman domination, have been studied [4, 5, 10, 13].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\dots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N(v)} f(u) = \{1,2,\dots,k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [6] and has been studied by several authors (see for example [1, 2, 8, 9, 11, 14, 16–18]). A 2-rainbow dominating function (briefly, rainbow dominating function) $f : V \longrightarrow \mathcal{P}(\{1,2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$ and $V_{1,2} = \{v \in V \mid f(v) = \{1,2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

A kRDF f is called a *global k-rainbow dominating function* (GkRDF) if f is also a kRDF of the complement $\overline{G}$ of G. The *global k-rainbow domination number* of G, denoted by $\gamma_{grk}(G)$, is the minimum weight of a GkRDF on G. A $\gamma_{grk}(G)$-function is a GkRDF of G with weight $\gamma_{grk}(G)$. Since every global k-rainbow dominating function f of G is a kRDF of G and $\overline{G}$, and assigning 1 to the vertices with nonempty label under f is a global dominating set of G, and since assigning $\{1,2,\dots,k\}$ to the vertices of a global dominating set yields a GkRDF, we deduce that

$$\max\{\gamma_g(G), \gamma_{rk}(G), \gamma_{rk}(\overline{G})\} \le \gamma_{grk}(G) \le k\gamma_g(G). \tag{1.1}$$

We note that the global k-rainbow domination number can differ significantly from the k-rainbow domination number. For example, for $n \ge k+1$, $\gamma_{rk}(K_n) = k$ and $\gamma_{grk}(K_n) = n$.

Our purpose in this paper is to initiate the study of the global k-rainbow domination number in graphs. We study basic properties of the global k-rainbow domination number and we establish some bounds for it.

We make use of the following results in this paper.

Theorem A ([14])**.** *For any graph G of order n and maximum degree $\Delta(G) \ge 1$,* $\gamma_{rk}(G) \ge \dfrac{kn}{\Delta(G)+k}$.

Theorem B ([6]). *For $n \geq 1$,*

$$\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Theorem C ([6]). *For $n \geq 3$,*

$$\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

Theorem D ([1]). *If G is a graph of order n, then $\gamma_{rk}(G) \leq n - \Delta(G) + k - 1$.*

Theorem E ([9]). *Let G be a connected graph. If there is a path $v_3v_2v_1$ in G with $\deg(v_2) = 2$ and $\deg(v_1) = 1$, then G has a $\gamma_{r2}(G)$-function f such that $f(v_1) = \{1\}$, and $2 \in f(v_3)$.*

Since the function f defined by $f(v) = \{1\}$ for each $v \in V(G)$ is a GkRDF of a graph G, we have the first part of the following observation. The second part is easy to see and therefore its proof is omitted.

Observation 1. If G is a graph of order n, then $\gamma_{grk}(G) \leq n$. Furthermore, if $1 \leq n \leq 4$, then $\gamma_{grk}(G) = n$.

2. Graphs with $\gamma_{rk}(G) = \gamma_{grk}(G)$

In this section we provide some sufficient conditions for a graph to satisfy $\gamma_{rk}(G) = \gamma_{grk}(G)$.

Proposition 1. If G is a disconnected graph with at least two components of order at least k, then

$$\gamma_{grk}(G) = \gamma_{rk}(G).$$

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of G. Assume, without loss of generality, that $|V(G_i)| \geq k$ for $i = 1, 2$. Let f be a $\gamma_{rk}(G)$-function. Obviously, $\sum_{v \in V(G_i)} |f(v)| \geq k$ for $i = 1, 2$. If $f(x) = \varnothing$ for some $x \in V(G_i)$, then clearly $\bigcup_{v \in V(G_i)} f(v) = \{1, 2, \ldots, k\}$, otherwise we may assume $\bigcup_{v \in V(G_i)} f(v) = \{1, 2, \ldots, k\}$ for $i = 1, 2$ because $|V(G_1)| \geq k$ and $|V(G_2)| \geq k$. Then f is a GkRDF of G and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G)$. Now the result follows from (1.1). □

According to Proposition 1, if G is the disjoint union of two copies of the complete graph K_n ($n \geq k$), then $\gamma_{grk}(G) = \gamma_{rk}(G)$.

Proposition 2. If G is a disconnected graph with $r \geq 2$ components $G_1, G_2, \ldots,$ G_r of order at most $k - 1$ such that $\sum_{i=1}^{r} |V(G_i)| \geq k$, then

$$\gamma_{grk}(G) = \gamma_{rk}(G).$$

Proof. Assume that $\bigcup_{i=1}^{r} V(G_i) = \{v_1, v_2, \ldots, v_s\}$, and let f be a $\gamma_{rk}(G)$-function. Then clearly $f(v_i) \neq \varnothing$ for each i. Define $g : V(G) \longrightarrow \mathcal{P}(\{1, 2, \ldots, k\})$ by $g(v_i) = \{k - i - 1\}$ for $1 \leq i \leq k - 1$, $g(v_i) = \{1\}$ for $i = k, k+1, \ldots, s$ and $g(x) = f(x)$

for $x \in V(G) - \{v_1, v_2, \ldots, v_s\}$. Then obviously g is a GkRDF of G of weight $\omega(g) = \gamma_{rk}(G)$ and the proof is complete. □

According to Proposition 2, if G is the disjoint union of k copies of K_1 and a copy of the complete graph K_n $(n \geq k)$, then $\gamma_{grk}(G) = \gamma_{rk}(G)$.

Theorem 1. For any connected graph G with radius $\mathrm{rad}(G) \geq 4$, $\gamma_{gr2}(G) = \gamma_{r2}(G)$.

Proof. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(G)$-function such that $|V_{1,2}|$ is maximum. We show that f is a G2RDF of G. Suppose to the contrary that f is not a 2RDF of $\overline{G}$. Then there exists a vertex $v \in V_0$ such that $V_{1,2} \subseteq N(v)$ and either $V_1 \subseteq N(v)$ or $V_2 \subseteq N(v)$. Assume, without loss of generality, that $V_1 \subseteq N(v)$. Let u be an arbitrary vertex in $V(G)$. If $u \in V_1 \cup V_{1,2}$, then $d(u,v) = 1$. If $u \in V_0$, then u and v have a common neighbor in V_1 or $V_{1,2}$ implying that $d(u,v) \leq 2$. Let $u \in V_2$. If u has a neighbor in $V_1 \cup V_{1,2}$, then $d(u,v) \leq 2$ as above. If u has a neighbor w in V_0, then $d(u,v) \leq d(u,w) + d(w,v) \leq 3$. Otherwise, since G is connected, u has a neighbor x in V_2. Then the function g defined by $g(u) = \emptyset, g(x) = \{1,2\}$ and $g(y) = f(y)$ for $y \in V(G) - \{u,x\}$, is a $\gamma_{r2}(G)$-function which contradicts the choice of f. Thus f is a G2RDF of G and the proof is complete. □

Corollary 1. Let G be a connected graph of diameter $\mathrm{diam}(G) \geq 7$. Then

$$\gamma_{gr2}(G) = \gamma_{r2}(G).$$

The next results is an immediate consequence of Theorems B, C and 1.

Corollary 2. For $n \geq 8$,

$$\gamma_{gr2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Corollary 3. For $n \geq 8$,

$$\gamma_{gr2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

3. Bounds on the global k-rainbow domination number

In this section we present some sharp lower and upper bounds on $\gamma_{grk}(G)$.

Proposition 3. For any integer $k \geq 2$ and any graph G of order $n \geq 2k$,

$$\gamma_{grk}(G) \geq 2k.$$

Proof. Let f be a $\gamma_{grk}(G)$-function, and let $V_0 = \{v \in V(G) \mid f(v) = \emptyset\}$. If $V_0 = \emptyset$, then $\gamma_{grk}(G) = n \geq 2k$. Let $V_0 \neq \emptyset$ and $v \in V_0$. Then $\bigcup_{x \in N_G(v)} f(x) = \{1,2,\ldots,k\}$ and $\bigcup_{x \in N_{\overline{G}}(v)} f(x) = \{1,2,\ldots,k\}$. Since $N_G(v) \cap N_{\overline{G}}(v) = \emptyset$, we obtain $\gamma_{grk}(G) = \omega(f) \geq 2k$, as desired. □

This bound is sharp for the disjoint union of two copies of the complete graph K_n $(n \geq k+1)$.

Proposition 4. For any graph G of order $n \geq 4$, $\gamma_{gr2}(G) = 4$ if and only if of G satisfies one of the following properties.
(i) $n = 4$,
(ii) there exist two vertices u and v in G such that $N(u) \cap N(v) = \emptyset$ and $N[u] \cup N[v] = V$,
(iii) there exist three distinct vertices u, v, w in G such that $N(u) \cap (N(v) \cup N(w)) = \emptyset$ and $N(u) \cup (N(v) \cap N(w)) = V - \{u, v, w\}$,
(iv) there exist four distinct vertices u, v, w, x in G such that $(N(u) \cap N(v)) \setminus \{w, x\} = \emptyset, (N(w) \cap N(x)) \setminus \{u, v\} = \emptyset$, $(N[u] \cup N[v]) \setminus \{w, x\} = V - \{w, x\}$ and $(N[w] \cup N[x]) \setminus \{u, v\} = V - \{u, v\}$.

Proof. If $n = 4$, then it is clear that $\gamma_{gr2}(G) = 4$. Let $n \geq 5$. If (ii) holds, then the function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1, 2\}$ and $f(z) = \emptyset$ for $z \in V(G) - \{u, v\}$, is a 2RDF of G and $\overline{G}$ which yields $\gamma_{gr2}(G) = 4$ by Proposition 3. If (iii) holds, then the function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $f(u) = \{1, 2\}, f(v) = \{1\}, f(w) = \{2\}$ and $f(z) = \emptyset$ for $z \in V(G) - \{u, v, w\}$, is a 2RDF of G and $\overline{G}$ which yields $\gamma_{gr2}(G) = 4$ again. Let (iv) hold. Then the function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1\}, f(w) = f(x) = \{2\}$ and $f(z) = \emptyset$ for $z \in V(G) - \{u, v, x, w\}$, is a 2RDF of G and $\overline{G}$. This implies that $\gamma_{gr2}(G) = 4$.

Conversely, Let $\gamma_{gr2}(G) = 4$ and let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{gr2}(G)$-function such that $|V_{1,2}|$ is maximum. We consider three cases.

Case 1. $|V_{1,2}| = 2$.
Let $V_{1,2} = \{u, v\}$. Then $V_0 = V(G) - \{u, v\}$. Since f is a G2RDF, each vertex in $w \in V(G) - \{u, v\}$ must be adjacent to a vertex in $\{u, v\}$ in both G and $\overline{G}$. It follows that $N[u] \cup N[v] = V$ and $N(u) \cap N(v) = \emptyset$, i.e. G satisfies (ii).

Case 2. $|V_{1,2}| = 1$.
Then $|V_1| = |V_2| = 1$. Let $V_{1,2} = \{u\}, V_1 = \{v\}$ and $V_2 = \{w\}$. Hence $V_0 = V(G) - \{u, v, w\}$. Every vertex of $w \in V(G) - \{u, v, w\}$ must be adjacent to u or both of v, w in G and $\overline{G}$ because f is a 2RDF of G and $\overline{G}$. This yields $N(u) \cap (N(v) \cup N(w)) = \emptyset$ and $N(u) \cup (N(v) \cap N(w)) = V - \{u, v, w\}$. Thus G satisfies (iii) in this case.

Case 3. $|V_{1,2}| = 0$.
If $V_0 = \emptyset$, then $V_1 \cup V_2 = V(G)$ which implies that $4 = \gamma_{gr2}(G) = |V_1 \cup V_2| = n$, i.e. G satisfies (i). Now assume that $V_0 \neq \emptyset$ and let $z \in V_0$. Since f is a 2RDF of G and $\overline{G}$, $\bigcup_{v \in N_G(z)} f(v) = \{1, 2\}$ and $\bigcup_{v \in N_{\overline{G}}(z)} f(v) = \{1, 2\}$. Assume that $u, w \in N_G(z)$ and $x, v \in N_{\overline{G}}(z)$ such that $f(u) = f(v) = \{1\}$ and $f(w) = f(x) = \{2\}$. Since f is a G2RDF, each vertex in $V(G) - \{u, v, w, x\}$ must be adjacent to a vertex in $\{u, v\}$ and a vertex in $\{w, x\}$ in G and $\overline{G}$. It follows that $(N(u) \cap N(v)) \setminus \{w, x\} = \emptyset, (N(w) \cap N(x)) \setminus \{u, v\} = \emptyset$, $(N[u] \cup N[v]) \setminus \{w, x\} = V - \{w, x\}$ and $(N[w] \cup N[x]) \setminus \{u, v\} = V - \{u, v\}$. Thus G satisfies (iv). This completes the proof. □

Proposition 5. Let $k \geq 2$ be an integer. If the graph G has $r \geq 1$ components $G_1, G_2, \ldots, G_r$ with $\sum_{i=1}^{r} |V(G_i)| \leq k-1$ then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - \sum_{i=1}^{r} |V(G_i)|.$$

Proof. Let $\bigcup_{i=1}^{r} V(G_i) = \{v_1, v_2, \ldots, v_s\}$, and let f be a $\gamma_{rk}(G)$-function. Clearly, $f(v_i) \neq \varnothing$ for each i. Define $g : V(G) \longrightarrow \mathcal{P}(\{1,2,\ldots,k\})$ by $g(v_s) = \{s, s+1, \ldots, k\}, g(v_i) = \{i\}$ for $i = 1,2,\ldots,s-1$ and $g(x) = f(x)$ for $x \in V(G) - \{v_1, v_2, \ldots, v_s\}$. Then obviously g is a GkRDF of G with weight $\omega(g) = \gamma_{rk}(G) + k - s$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - \sum_{i=1}^{r} |V(G_i)|$. □

Let H be the disjoint union of $r \leq k-1$ isolated vertices and a star $K_{1,s}$ with $s \geq k$. Then $\gamma_{rk}(H) = r + k$ and $\gamma_{grk}(H) = 2k$. This example demonstrates that Proposition 5 is tight.

Proposition 6. Let G be a graph of order $n \geq 4$ and $u, v \in V(G)$. If $uv \notin E(G)$, then

$$\gamma_{grk}(G) \leq n - \deg(u) - \deg(v) + 2|N(u) \cap N(v)| + 2k - 2,$$

and if $uv \in E(G)$, then

$$\gamma_{grk}(G) \leq n - \deg(u) - \deg(v) + 2|N(u) \cap N(v)| + 2k.$$

Proof. Define $f : V(G) \longrightarrow \mathcal{P}(\{1,2,\ldots,k\})$ as follows

$$f(z) = \begin{cases} \{1,2,\ldots,k\} & \text{if } z \in \{u,v\} \\ \varnothing & \text{if } z \in ((N(u) \cup N(v)) - \{u,v\}) \setminus (N(u) \cap N(v)) \\ \{1\} & \text{otherwise.} \end{cases}$$

It is easy to see that f is a GkRDF of G which attains the bound. This completes the proof. □

Corollary 4. If G is a connected triangle-free graph of order $n \geq 4$, then

$$\gamma_{grk}(G) \leq \min\{n - \Delta(G) - \delta(G) + 2k, \gamma_{rk}(G) + 2k - 1\}.$$

Proof. By considering a vertex of maximum degree and one of its neighbors, it follows from Proposition 6 that $\gamma_{grk}(G) \leq n - \Delta(G) - \delta(G) + 2k$. Hence, it is sufficient to show that $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 1$. If $n \leq \gamma_{rk}(G) + 2k - 1$, the result is immediate. Let $n > \gamma_{rk}(G) + 2k - 1$ and let f be a $\gamma_{rk}(G)$-function. Then there exists a vertex u such that $f(u) = \varnothing$. Then u has a neighbor v such that $|f(v)| \geq 1$. Define $g : V(G) \longrightarrow \mathcal{P}(\{1,2,\ldots,k\})$ by $g(u) = g(v) = \{1,2,\ldots,k\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a GkRDF of G and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 1$. This completes the proof. □

Proposition 7. Let $k \geq 2$ be an integer, and let G be a graph of diameter $\operatorname{diam}(G) \geq 5$. Then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 2.$$

Proof. If G is disconnected, then the result follows from Propositions 1 and 5. Henceforth, we assume that G is connected. Let f be a $\gamma_{rk}(G)$-function. Let $v_1 v_2 \dots v_d$ be a diametral path in G. If $f(v_1) = f(v_d) = \varnothing$, then we have $\bigcup_{x \in N(v_1)} f(x) = \{1,2,\dots,k\}$ and $\bigcup_{x \in N(v_d)} f(x) = \{1,2,\dots,k\}$. Since $\operatorname{diam}(G) \geq 5$, we have $N(v_1) \cap N(v_d) = \varnothing$. It follows that f is a GkRDF of G and hence $\gamma_{grk}(G) = \gamma_{rk}(G)$. If $f(v_1) \neq \varnothing$ and $f(v_d) \neq \varnothing$, then the function $g : V \longrightarrow \mathcal{P}(\{1,2,\dots,k\})$ defined by $g(v_1) = g(v_d) = \{1,2,\dots,k\}$ and $g(x) = f(x)$ for $x \in V(G) - \{v_1, v_d\}$, is a GkRDF of G of weight at most $\omega(f) + 2k - 2$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k - 2$. Now let $f(v_1) = \varnothing$ and $f(v_d) \neq \varnothing$ (the case $f(v_1) \neq \varnothing$ and $f(v_d) = \varnothing$ is similar). Define $g : V \longrightarrow \mathcal{P}(\{1,2,\dots,k\})$ by $g(v_d) = \{1,2,\dots,k\}$ and $g(x) = f(x)$ for $x \in V(G) - \{v_d\}$. Obviously, g is a GkRDF of G of weight at most $\omega(f) + k - 1$ and so $\gamma_{grk}(G) \leq \gamma_{rk}(G) + k - 1$. This completes the proof. □

Proposition 8. If G is a graph of diameter 3 or 4, then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k.$$

Proof. Let f be a $\gamma_{rk}(G)$-function, and let u and v be two vertices of G such that $d(u,v) = \operatorname{diam}(G)$. Then the function $g : V \longrightarrow \mathcal{P}(\{1,2,\dots,k\})$ defined by $g(u) = g(v) = \{1,2,\dots,k\}$ and $g(x) = f(x)$ for $x \in V(G) - \{u,v\}$, is a GkRDF of G and therefore $\gamma_{grk}(G) \leq \gamma_{rk}(G) + 2k$. □

Theorem 2. If G is a graph of order $n \geq 4$ with minimum degree $\delta(G)$, then

$$\gamma_{grk}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1.$$

This bound is sharp for stars $K_{1,t}$ $(t \geq 2k-1)$ by Proposition 3.

Proof. If G is disconnected, then the result follows from Propositions 1 and 5. Therefore we assume that G is connected. Let u be a vertex of minimum degree $\delta(G)$, f be a $\gamma_{rk}(G)$-function and $B = \{x \in N(u) \mid f(x) = \varnothing\}$.

If $f(u) = \varnothing$, then $\bigcup_{v \in N(u) - B} f(v) = \{1,2,\dots,k\}$. Then obviously the function $g : V(G) \longrightarrow \mathcal{P}(\{1,2,\dots,k\})$ defined by $g(u) = \{1,2,\dots,k\}, g(x) = \{1\}$ if $x \in B$ and $g(z) = f(z)$ otherwise, is a GkRDF of G with weight at most $\gamma_{rk}(G) + \delta(G) + k - 1$ and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1$.

Let $|f(u)| \geq 1$. Define $g : V(G) \longrightarrow \mathcal{P}(\{1,2,\dots,k\})$ by $g(u) = \{1,2,\dots,k\}$, $g(v) = \{1\}$ if $v \in B$ and $g(z) = f(z)$ for each $z \in V(G) - (B \cup \{u\})$. It is clear that g is a GkRDF of G with weight at most $\gamma_{rk}(G) + \delta(G) + k - 1$ and hence $\gamma_{grk}(G) \leq \gamma_{rk}(G) + \delta(G) + k - 1$. This completes the proof. □

4. Global rainbow domination numbers of trees

According to Theorem 2, for any tree T of order $n \geq 4$ we have

$$\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 2. \tag{4.1}$$

In this section we characterize all extremal trees attaining equality in (4.1). We begin with some lemmas giving some sufficient conditions for a tree to have global 2-rainbow domination number less than $\gamma_{r2}(T) + 2$. As a special case, Corollary 1 and Proposition 3 imply the next results.

Corollary 5. For any tree T with $\mathrm{diam}(T) \geq 7$, $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

Corollary 6. If T is a star of order $n \geq 4$, then $\gamma_{gr2}(T) = \gamma_{r2}(T) + 2$.

Lemma 1. Let T be a tree. If T has two strong support vertices, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$.

Proof. Let u and v be two strong support vertices of T and let f be a $\gamma_{r2}(T)$-function. Obviously we may assume that $f(u) = f(v) = \{1,2\}$. Since T is a tree, u and v have at most one common neighbor. If u and v have no common neighbor, then clearly f is a G2RDF of T and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. If u and v has a common neighbor, say w, then the function g defined by $g(w) = f(w) \cup \{1\}$ and $g(x) = f(x)$ otherwise, is a G2RDF of T of weight at most $\gamma_{r2}(T) + 1$ and the result follows. □

Lemma 2. Let T be a tree. If $\mathrm{diam}(T) = 6$, then $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

Proof. Let $P = v_1 v_2 \dots v_7$ be a diametral path of T and let f be a $\gamma_{r2}(T)$-function. Root T at v_1. If v_2 and v_6 are strong support vertices, then f is a $\gamma_{gr2}(T)$-function since v_2 and v_6 have no common neighbor. Hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Assume, without loss of generality, that $\deg(v_2) = 2$. By Theorem E, we may assume $f(v_1) = \{1\}$ and $2 \in f(v_3)$. If v_6 is a strong support vertex, then we can assume $f(v_6) = \{1,2\}$ and clearly f is a G2RDF of T implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Henceforth, we assume $\deg(v_6) = 2$. By Theorem E, we may assume $f(v_7) = \{1\}$ and $2 \in f(v_5)$. Define the function g by $g(v) = \{1\}$ if $v \in V(T_{v_5})$ and $f(v) = \{2\}$, $g(v) = \{2\}$ if $v \in V(T_{v_5})$ and $f(v) = \{1\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a G2RDF of T of weight $\gamma_{r2}(T)$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. This completes the proof. □

Lemma 3. Let T be a tree. If $\mathrm{diam}(T) = 5$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$.

Proof. Let $P = v_1 v_2 \dots v_6$ be a diametral path of T, and let f be a $\gamma_{r2}(T)$-function. If v_2 and v_5 are strong support vertices, then f is a $\gamma_{gr2}(T)$-function and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Assume, without loss of generality, that all support vertices adjacent to v_4 have degree 2. By Theorem E, we may assume $f(v_6) = \{1\}$ and $2 \in f(v_4)$. Then the function g defined by $g(v_3) = f(v_3) \cup \{1\}$ and $g(x) = f(x)$ otherwise, is a G2RDF of T of weight at most $\gamma_{r2}(T) + 1$ that implies $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$. □

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider*. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. The *center of a spider*, is the center of the star whose subdivision produced the spider.

Definition 1. For $1 \leq i \leq 2$, let $\mathcal{B}_i$ be the family of trees T defined as follows and let $\mathcal{B} = \bigcup_{i=1}^{2} \mathcal{B}_i$.
$\mathcal{B}_1$: T is a spider S_t for some $t \geq 2$ with exception of stars, wounded spiders S_t $(t \geq 3)$ with exactly one wounded leg or wounded spiders S_t $(t \geq 3)$ with at least four healthy legs.
$\mathcal{B}_2$: T is obtained from stars $K_{1,r_1}, K_{1,r_2}, \ldots, K_{1,r_j}$ where $r_k \geq 3$ for $1 \leq k \leq j$, with centers $y_1, y_2, \ldots, y_j$ $(j \geq 2)$ by adding a new vertex x and joining x to all vertices y_j and adding at most one pendant edge at x.

Lemma 4. Let T be a tree. If $\text{diam}(T) = 4$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$ and equality holds if and only if $T \in \mathcal{B}$.

Proof. Let $\text{diam}(T) = 4$ and let $P = v_1v_2v_3v_4v_5$ be a diametral path of T. Let f be a $\gamma_{r2}(T)$-function. Consider the following cases.
Case 1. $\deg(v_2) = 3$.
Suppose u, v_1 are the leaves adjacent to v_2. Then we can assume that $f(v_2) = \{1,2\}$. If $\deg(v_4) \geq 3$, then we may assume $f(v_4) = \{1,2\}$ and if $\deg(v_4) = 2$ then by Theorem E we can assume $f(v_5) = \{1\}$ and $2 \in f(v_3)$. Define $g : V(T) \to \mathcal{P}(\{1,2\})$ by $g(v_1) = \{1\}, g(u) = \{2\}, g(v_2) = \varnothing$ and $g(x) = f(x)$ otherwise. Obviously g is a G2RDF of T of weight $\gamma_{r2}(T)$ and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$.

By Case 1, we may assume that all support vertices adjacent to v_3 have degree different from 3.
Case 2. $\deg(v_2) > 3$.
Then $f(v_2) = \{1,2\}$. If $\deg(v_4) = 2$, then by Theorem E we may assume $f(v_5) = \{1\}$ and $2 \in f(v_3)$, and clearly f is a G2RDF of T and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. So we assume that each support vertex adjacent to v_3 has degree at least 4. If v_3 is a strong support vertex, then $f(v_3) = \{1,2\}$ and clearly f is a G2RDF of T and hence $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Let v_3 be not a strong support vertex. Then $T \in \mathcal{B}_2$ and T has at most two $\gamma_{r2}(T)$-functions which none of them is G2RDF of T and hence $\gamma_{gr2}(T) \geq \gamma_{r2}(T) + 1$. On the other hand, the function g defined by $g(v_3) = \{1\}$ and $g(x) = f(x)$ otherwise is a G2RDF of T of weight $\gamma_{r2}(T) + 1$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$.

By Cases 1 and 2, we may assume that all support vertices adjacent to v_3 have degree 2. Thus T is a spider of diameter 4. If T is a wounded spiders S_t $(t \geq 3)$ with exactly one wounded leg, then the function g that assigns $\varnothing$ to all support vertices of T with exception of the center of spider, $\{1\}$ to the center of spider and the leaf adjacent to the center of spider, and $\{2\}$ to the other leaves, is a G2RDF of T of weight $\gamma_{r2}(T)$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Now T is a wounded spider S_t $(t \geq 3)$ with at least four healthy legs. Suppose x is the center of T and u_1, u_2, u_3, u_4 are leaves at distance two from x. Then the function g that assigns $\{1,2\}$ to x, $\varnothing$ to all support vertices of T, $\{1\}$ to u_1, u_2, and $\{2\}$ to the other leaves, is a G2RDF of T of weight $\gamma_{r2}(T)$ implying that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Finally let T be a spider that is not a wounded spider S_t $(t \geq 3)$ with exactly one wounded leg or a wounded spider S_t $(t \geq 3)$ with at least four healthy legs, that is $T \in \mathcal{B}_1$. It is easy to see that in this case $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$ and the proof is complete. □

For $p, q \geq 1$, a double star $DS(p,q)$ is a tree with exactly two vertices that are not leaves, with one adjacent to p leaves and the other to q leaves.

Lemma 5. Let T be a tree. If $\operatorname{diam}(T) = 3$, then $\gamma_{gr2}(T) \leq \gamma_{r2}(T) + 1$ and equality holds if and only if $T = DS(p,q)$ with $q \geq p = 1$.

Proof. Let $\operatorname{diam}(T) = 3$. Then T is a double star $DS(p,q)$ with $q \geq p \geq 1$. Let u, v be the vertices of T of degree p and q, respectively. If $p \geq 2$, then u, v are strong support vertices with no common neighbor and it follows from the proof of Lemma 1 that $\gamma_{gr2}(T) = \gamma_{r2}(T)$. Henceforth, assume $p = 1$. If $q = 1$, then $T = P_4$ and clearly $\gamma_{gr2}(T) = \gamma_{r2}(T) + 1$. Let $q \geq 2$ and u' be the leaf adjacent to u. Then T has exactly two $\gamma_{r2}(T)$-functions f_i $(i = 1,2)$ defined by $f_i(v) = \{1,2\}$, $f_i(u') = \{i\}$ and $f_i(x) = \varnothing$ otherwise. Obviously, none of f_1 or f_2 is not a G2RDF of T and also the function g defined by $g(u) = \{1\}$ and $g(x) = f_1(x)$ for $x \in V(T) - \{u\}$ is a G2RDF of T that yields $\gamma_{gr2}(T) \geq \gamma_{r2}(T) + 1$. □

The next theorem is an immediate consequence of (4.1), Corollaries 5, 6 and Lemmas 2, 3, 4, 5.

Theorem 3. Let T be a tree of order $n \geq 4$. Then $\gamma_{gr2}(T) = \gamma_{r2}(T) + 2$ if and only if T is the star $K_{1,t}$ for some $t \geq 3$.

ACKNOWLEDGEMENT

The authors would like to thank anonymous referees for their remarks and suggestions that helped improve the manuscript.

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