

**TAUBERIAN CONDITIONS UNDER WHICH λ -STATISTICAL
CONVERGENCE FOLLOWS FROM STATISTICAL
SUMMABILITY (V, λ)**

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Received 18 May, 2014

Abstract. In this paper, we will show Tauberian conditions under which λ -statistical convergence follows from (V, λ) -statistical convergence. Our results generalize the ones given in [3].

2010 *Mathematics Subject Classification:* 40E05; 40G05

Keywords: statistical convergence, λ -statistical convergence, statistical summability (V, λ) , one-sided and two-sided Tauberian conditions

1. INTRODUCTION

We shall denote by \mathbb{N} the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $d(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [1, 6]), i.e. for each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely.

The idea of λ -statistical convergence was introduced in [5] as follows: Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, n - 1 < \lambda_n$. The generalized de la Vallée-Poussin mean is defined by $T_n(x) =: \frac{1}{\lambda_n} \sum_{j \in I_n} x_j$, where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_j)$ is said to be (V, λ) -summable to a number L (see [2]) if $T_n(x) \rightarrow L$ as $n \rightarrow \infty$. In this case L is called the λ -limit of x . And we say that $x = (x_n)$ is λ -statistical convergent to L , if

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

for every given $\epsilon > 0$. And will write $st_\lambda - \lim_n x_n = L$.

In paper of Mursaleen et al. [4], the definition of the statistically λ -convergent sequences was given as follows:

A sequence $x = (x_n)$ is said to be statistically λ -convergent to L if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{n} |\{k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \quad (1.1)$$

holds. In this case we write that $st - \lim_n T_n = L$.

In what follows we will define the following type of the statistical convergence. A sequence $x = (x_n)$ is said to be (V, λ) -statistically convergent to L if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \quad (1.2)$$

holds. In this case we write that $st_\lambda - \lim_n T_n = L$.

In the sequel we will show conditions under which for every bounded sequence (x_k) the implication

$$st_\lambda - \lim_k x_k = L \quad \text{implies} \quad st_\lambda - \lim_k T_k = L$$

holds.

Theorem 1. *Let us suppose that (x_k) is a bounded sequence such that exists $st_\lambda - \lim_k x_k = L$, then it follows that $st_\lambda - \lim_k T_k = L$, but not conversely.*

Proof. Let us suppose that $st_\lambda - \lim_k x_k = L$. Let $\epsilon > 0$ be any given number, and $B_\epsilon = \{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}$. Then

$$\begin{aligned} |T_k(x) - L| &= \left| \frac{1}{\lambda_k} \sum_{j \in I_k} x_j - L \right| = \left| \frac{1}{\lambda_k} \sum_{j \in I_k} (x_j - L) \right| \leq \left| \frac{1}{\lambda_k} \sum_{j \in B_\epsilon} (x_j - L) \right| \leq \\ &\frac{1}{\lambda_k} (\sup_j |x_j - L|) \cdot B_\epsilon \rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. Hence $T_k(x) \rightarrow L$, as $k \rightarrow \infty$, respectively $st_\lambda - \lim_k T_k = L$.

Example 1. Let us consider that $\lambda_n = n$ and $x = (x_n)$ defined as follows:

$$x_n = \begin{cases} 1 & \text{if } k \text{ is odd} \\ -1 & \text{if } k \text{ is even} \end{cases}$$

Of course this sequence is not st_λ -convergent. On the other hand, x is (V, λ) -summable to 0 and hence (V, λ) -statistically convergent to 0. \square

In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for the defined type of convergence. Exactly, we will prove under which conditions λ -statistical convergence follows from (V, λ) -statistically convergence. This method generalize the method given in [3], as it is shown by the following example.

Example 2. In case where $\lambda_n = n$, then (V, λ) -summability method is the Cesaro summability method $(C, 1)$ as given in [3].

2. MAIN RESULTS

Theorem 2. Let (λ_n) be a sequence of real numbers defined as above and

$$st_\lambda - \liminf \frac{\lambda_{t_n}}{n} > 1, \quad t > 1 \tag{2.1}$$

where t_n , denotes the integral parts of the $[tn]$ for every $n \in \mathbb{N}$, and let (T_k) be a sequence of real numbers such that $st_\lambda - \lim_k T_k = L$. Then (x_k) is st_λ -convergent to the same number L if and only if the following to conditions holds:

$$\inf_{t > 1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \leq -\epsilon \right\} \right| = 0 \tag{2.2}$$

and

$$\inf_{0 < t < 1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^k (x_k - x_j) \leq -\epsilon \right\} \right| = 0. \tag{2.3}$$

Remark 1. Let us suppose that $st_\lambda - \lim_k x_k = L$; $st_\lambda - \lim_k T_k = L$ and relation (2.1) satisfies, then for every $t > 1$, the following relation is valid:

$$st_\lambda - \lim_n \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} x_j = 0 \tag{2.4}$$

and in case where $0 < t < 1$,

$$st_\lambda - \lim_n \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^k x_j = 0. \tag{2.5}$$

In what follows we will show some auxiliary lemmas which are going to be used in the sequel.

Lemma 1. For the sequence of real numbers $\lambda = (\lambda_n)$, condition given by relation (2.1) is equivalent to this one:

$$st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} > 1, \quad 0 < t < 1. \tag{2.6}$$

Proof. Let us suppose that relation (2.1) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \leq n$$

now by the nondecreasing property of the sequence $\lambda = (\lambda_n)$, we get:

$$\frac{\lambda_n}{\lambda_{t_n}} \geq \frac{\lambda_{[m/t]}}{\lambda_m} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} \geq st_\lambda - \liminf_n \frac{\lambda_{[m/t]}}{\lambda_m} > 1.$$

Conversely, let us suppose that relation (2.6) is valid. Let $t > 1$ be given and let t_1 be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \leq \frac{tn - 1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \leq t - \frac{1}{n}$, which is a case where if n is large enough. Under this conditions we have:

$$\frac{\lambda_{t_n}}{\lambda_n} \geq \frac{\lambda_{t_n}}{\lambda_{[m/t_1]}} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} \geq st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_{[m/t_1]}} > 1.$$

□

Lemma 2. Let us suppose that relation (2.1) is satisfied and let $x = (x_k)$ be a sequence of complex numbers which is (V, λ) -statistically convergent to L . Then for every $t > 0$,

$$st_\lambda - \lim_n T_{t_n} = L.$$

Proof. Let us consider that $t > 1$, then from construction of the sequence $\lambda = (\lambda_n)$, we get:

$$\lim_n (n - \lambda_n) = \lim_n (t_n - \lambda_{t_n}), \quad (2.7)$$

and for every $\epsilon > 0$ we have:

$$\{k \in I_{t_n} : |T_{t_k} - L| \geq \epsilon\} \subset \{k \in I_n : |T_k - L| \geq \epsilon\} \cup \left\{ k \in I_n : \frac{1}{\lambda_k} \sum_{j=k-\lambda_k+1}^k x_j \neq \frac{1}{\lambda_{t_k}} \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j \right\}.$$

Now proof of the lemma in this case follows from relation (2.7) and $st_\lambda - \lim_n T_{t_n} = L$.

(II) In this case we have that $0 < t < 1$. For $t_n = [t \cdot n]$, for any natural number n , we can conclude that (T_{t_n}) does not appears more than $[1 + t^{-1}]$ times in the sequence (T_n) . In fact if there exists a integers k, l such that

$$n \leq t \cdot k < t(k+1) < \dots < t(k+l-1) < n+1 \leq t(k+l),$$

then

$$n + t(l - 1) \leq t(k + l - 1) < n + 1 \Rightarrow l < 1 + \frac{1}{t}.$$

And we have this estimation:

$$\begin{aligned} \frac{1}{\lambda_{t_n}} |\{k \in I_{t_n} : |T_{t_k} - L| \geq \epsilon\}| &\leq \left(1 + \frac{1}{t}\right) \frac{1}{\lambda_{t_n}} |\{k \in I_n : |T_k - L| \geq \epsilon\}| \leq \\ &2(1 + t) \frac{1}{\lambda_n} |\{k \in I_n : |T_k - L| \geq \epsilon\}|. \end{aligned}$$

From the last relation it follows that $st_\lambda - \lim_n T_{t_n} = L$. □

Lemma 3. *Let us suppose that relation (2.1) is satisfied and let $x = (x_k)$ be a sequence of complex numbers which is (V, λ) -statistically convergent to L . Then for every $t > 1$,*

$$st_\lambda - \lim_n (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} x_j = L; \tag{2.8}$$

and for every $0 < t < 1$,

$$st_\lambda - \lim_n (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^n x_j = L. \tag{2.9}$$

Proof. Let us suppose that $t > 1$. After some calculations we get:

$$\begin{aligned} (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} x_j &= T_n + \lambda_{t_n} (\lambda_{t_n} - \lambda_n)^{-1} (T_{t_n} - T_n) + \\ &(\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n-\lambda_n+1}^{t_n} x_j - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j, \end{aligned}$$

respectively

$$\begin{aligned} (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} x_j &= T_n + \lambda_{t_n} (\lambda_{t_n} - \lambda_n)^{-1} (T_{t_n} - T_n) + \\ &(\lambda_{t_n} - \lambda_n)^{-1} \left(\sum_{j=n-\lambda_n+1}^{t_n} x_j - \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j \right). \end{aligned} \tag{2.10}$$

From definition of the sequence $\lambda = (\lambda_n)$, we obtain

$$st_\lambda - \limsup_n \sum_{j=n-\lambda_n+1}^{t_n} x_j = st_\lambda - \limsup_n \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j. \tag{2.11}$$

Really, let us suppose that $st_\lambda - \lim_n \sup \sum_{j=n-\lambda_n+1}^{t_n} x_j = L$, and for every $\epsilon > 0$, we get:

$$\begin{aligned} & \left| \frac{\left\{ k \in I_{t_n} : \left| \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j - L \right| \geq \epsilon \right\}}{\lambda_{t_n}} \right| \\ & \leq \left| \frac{\left\{ k \in I_n : \left| \sum_{j=k-\lambda_k+1}^{t_k} x_j - L \right| \geq \epsilon \right\}}{\lambda_n} \right| \\ & + \left| \frac{\left\{ k \in I_n : \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} x_j \neq \sum_{j=k-\lambda_k+1}^{t_k} x_j \right\}}{\lambda_n} \right|. \end{aligned}$$

The first summand in the right side of the inequality tends statistically to zero as $n \rightarrow \infty$ and second summands tends to zero, too (from relations (2.7)). And this means that $st_\lambda - \lim_n \sup \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} x_j = L$.

Since by (2.1)

$$st_\lambda - \lim_n \sup \lambda_{t_n} (\lambda_{t_n} - \lambda_n)^{-1} < \infty, st_\lambda - \lim_n \sup (\lambda_{t_n} - \lambda_n)^{-1} < \infty, \quad (2.12)$$

now relation (2.8) follows from (2.10), (2.11), (2.12), Lemma 2 and statistical convergence of T_n .

Case where $0 < t < 1$. In this case we have:

$$\begin{aligned} (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^n x_j &= T_n + \lambda_{t_n} (\lambda_n - \lambda_{t_n})^{-1} (T_n - T_{t_n}) + \\ & (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=n-\lambda_n+1}^n x_j - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=t_n-\lambda_{t_n}+1}^n x_j. \end{aligned}$$

Following Lemma 2 and the conclusions like in the previous case, we get that relation (2.9) is valid. \square

In what follows we will prove Theorem 1.

Proof of Theorem 2. Necessity. Let us suppose that $st_\lambda - \lim_k x_k = L$, and $st_\lambda - \lim_k T_k = L$. For every $t > 1$ following Lemma 2, we get relation (2.2) and in case where $0 < t < 1$, again applying Lemma 2 we obtain relation (2.3).

Sufficient: Assume that $st_\lambda - \lim_n T_n = L$, and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st_\lambda - \lim_n x_n = L$. Or equivalently, $st_\lambda - \lim_n (T_n - x_n) = 0$.

First we consider the case where $t > 1$. We will start from this estimation:

$$x_n - T_n = \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (x_j - x_n).$$

For any $\epsilon > 0$, we obtain:

$$\{k \in I_n : x_n - T_n \geq \epsilon\} \subset \left\{k \in I_n : \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) \geq \frac{\epsilon}{2}\right\} \cup \left\{k \in I_n : (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (x_j - x_n) \leq -\frac{\epsilon}{2}\right\}.$$

From relation (2.2), it follows that for every $\gamma > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{\lambda_n} \left| \left\{k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \leq -\epsilon \right\} \right| \leq \gamma.$$

By Lemma 2 and relation (2.12) we get:

$$\limsup_n \frac{1}{\lambda_n} \left| \left\{k \in I_n : |\lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n)| \geq \frac{\epsilon}{2} \right\} \right| = 0.$$

Combining last three relations we have:

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \geq \epsilon\}| \leq \gamma,$$

and γ is arbitrary, we conclude that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \geq \epsilon\}| = 0. \tag{2.13}$$

Now we consider case where $0 < t < 1$. From above we get that:

$$x_n - T_n = \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) + (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^n (x_n - x_j).$$

For any $\epsilon > 0$,

$$\{k \in I_n : x_n - T_n \leq -\epsilon\} \subset \left\{k \in I_n : \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) \leq -\frac{\epsilon}{2}\right\} \cup \left\{k \in I_n : (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^n (x_k - x_j) \leq -\frac{\epsilon}{2}\right\}.$$

For same reasons as in the case where $t > 1$, by Lemma 2, we have that for every $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : x_n - T_n \leq -\epsilon\}| = 0. \tag{2.14}$$

Finally from relations (2.13) and (2.14) we get:

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_n - T_n| \geq \epsilon\}| = 0.$$

□

In the next result we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

Theorem 3. *Let (λ_n) be a sequence of complex numbers defined above, which satisfied relation (2.1) and let us consider that $st_\lambda - \lim T_k = L$. Then (x_k) is $st_\lambda -$ statistically convergent to the same number L if and only if the following conditions holds: for every $\epsilon > 0$,*

$$\inf_{t > 1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (x_j - x_k) \geq \epsilon \right\} \right| = 0 \quad (2.15)$$

and

$$\inf_{0 < t < 1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^k (x_k - x_j) \geq \epsilon \right\} \right| = 0. \quad (2.16)$$

Remark 2. Let us suppose that $st_\lambda - \lim_k x_k = L$, $st_\lambda - \lim_k T_k = L$ and relation (2.1) satisfies. Then for every $t > 1$, relation (2.15) holds, and in case where $0 < t < 1$, relation (2.16) is valid.

Proof of Theorem 3. We omit it, because it is similar to that of Theorem 1. □

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