Miskolc Mathematical Notes
Vol. 17 (2017), No. 2, pp. 1049–1059

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1197

ON HERMITE-HADAMARD TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA AND HÜSEYIN YILDIRIM

Received 07 April, 2014

Abstract. In this paper, we have established Hermite-Hadamard-type inequalities for fractional integrals and will be given an identity. With the help of this fractional-type integral identity, we give some integral inequalities connected with the left-side of Hermite–Hadamard-type inequalities for Riemann-Liouville fractional integrals.

2010 *Mathematics Subject Classification:* 26D07; 26D10; 26D15; 26A33

Keywords: Hermite-Hadamard's inequalities, Riemann-Liouville fractional integral, integral inequalities.

1. INTRODUCTION

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[20, p.137], [10]). These inequalities state that if $f: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1–4, 10–13, 15–17, 19, 20, 26, 27]) and the references cited therein.

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f: [a,b] \to \mathbb{R}$.

Definition 1. The function $f: [a,b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

In [15] in order to prove some inequalities related to Hadamard's inequality Kırmacı used the following lemma:

Lemma 1. *Let $f: I^{\circ} \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a,b \in I^{\circ}$ (I° is the interior of I) with $a<b$. If $f' \in L([a,b])$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\\&\quad=(b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt+\int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right].\end{aligned}$$

Also, in [15], Kırmacı obtained the following inequalities for differentiable mappings which are connected with Hermite-Hadamard's inequality:

Theorem 1. *Let $f: I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^{\circ}$ with $a<b$. If the mapping $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.2}$$

Theorem 2. *Let $f: I^{\circ} \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a,b \in I^{\circ}$ with $a<b$ and $p>1$. If the mapping $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le\\&\le \frac{(b-a)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\left|f'(a)\right|^{\frac{p}{p-1}}+3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right.\\&\qquad\left.+\left(3\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right\}\\&\le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned} \tag{1.3}$$

Meanwhile, Sarikaya et al.[25] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2. *Let $f: [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$$

$$= \frac{b-a}{2} \int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(ta + (1-t)b)\, dt.$$

It is remarkable that Sarikaya et al.[25] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3. *Let $f:[a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [14, 18, 21].

Definition 2. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

For some recent results connected with fractional integral inequalities see ([5–9, 22–25, 28]).

The aim of this paper is to establish Hermite-Hadamard's inequalities for Riemann-Liouville fractional integral similar to the method in [25] and we will investigate some integral inequalities connected with the left hand side of the Hermite-Hadamard type inequalities for fractional integrals.

2. Hermite-Hadamard's inequalities for fractional integrals

Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{2.1}$$

with $\alpha>0$.

Proof. Since f is a convex function on $[a,b]$, we have for $x,y\in[a,b]$ with $\lambda=\frac{1}{2}$

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2} \tag{2.2}$$

i.e., with $x=\frac{t}{2}a+\frac{2-t}{2}b$, $y=\frac{2-t}{2}a+\frac{t}{2}b$,

$$2f\left(\frac{a+b}{2}\right)\leq f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right). \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{a+b}{2}\right)\\
&\leq\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\
&=\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-u)\right)^{\alpha-1}f(u)\frac{2du}{a-b}+\int_a^{\frac{a+b}{2}}\left(\frac{2}{b-a}(v-a)\right)^{\alpha-1}f(v)\frac{2dv}{b-a}\\
&=\frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]
\end{aligned}$$

i.e.

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.2) we first note that if f is a convex function, then, for $\lambda\in[0,1]$, it yields

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\leq\frac{t}{2}f(a)+\frac{2-t}{2}f(b)$$

and

$$f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq\frac{2-t}{2}f(a)+\frac{t}{2}f(b).$$

By adding these inequalities we have

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le f(a)+f(b). \tag{2.4}$$

Then multiplying both sides of (2.4) by $t^{\alpha-1}$and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\alpha-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$
$$\le [f(a)+f(b)]\int_0^1 t^{\alpha-1}dt$$

i.e.

$$\frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\le\frac{f(a)+f(b)}{\alpha}.$$

The proof is completed. □

Remark 1. If in Theorem 4, we let $\alpha=1$, then the inequalities (2.1) become the inequalities (1.1).

3. Fractional inequalities for convex functions

We need the following lemma. With the help of this, we give some integral inequalities connected with the left-side of Hermite–Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{b-a}{4}\left\{\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\} \tag{3.1}$$

with $\alpha>0$.

Proof. Integrating by parts

$$\begin{aligned}
I_1 &= \int_0^1 t^\alpha f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\
&= t^\alpha \frac{2}{a-b} f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1} f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\frac{2}{a-b}dt \\
&= -\frac{2}{b-a} f(\frac{a+b}{2}) - \frac{2\alpha}{a-b}\int_b^{\frac{a+b}{2}} \left(\frac{2}{b-a}(b-x)\right)^{\alpha-1} \frac{2}{a-b} f(x)dx \\
&= -\frac{2}{b-a} f(\frac{a+b}{2}) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{(\frac{a+b}{2})-} f(b)
\end{aligned} \tag{3.2}$$

and similarly we get,

$$\begin{aligned}
I_2 &= \int_0^1 t^\alpha f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&= t^\alpha \frac{2}{b-a} f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\bigg|_0^1 - \frac{2\alpha}{b-a}\int_0^1 t^{\alpha-1} f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&= \frac{2}{b-a} f(\frac{a+b}{2}) - \frac{2\alpha}{b-a}\int_a^{\frac{a+b}{2}} \left(\frac{2}{b-a}(x-a)\right)^{\alpha-1} f(x)\frac{2}{b-a}dx \\
&= \frac{2}{b-a} f(\frac{a+b}{2}) - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{(\frac{a+b}{2})+} f(a).
\end{aligned} \tag{3.3}$$

By using (3.2) and (3.3), it follows that

$$I_1 - I_2 = -\frac{4}{b-a} f(\frac{a+b}{2}) + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)-} f(a)\right].$$

Thus, by multiplying the both sides by $\frac{b-a}{4}$, we have the conclusion (3.1). □

Corollary 1. *If in Lemma 3, we let $\alpha = 1$, then the equality (3.1) becomes the following equality*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\
&= \frac{b-a}{4}\left\{\int_0^1 tf'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt - \int_0^1 tf'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}.
\end{aligned} \tag{3.4}$$

Using this Lemma 3, we can obtain the following fractional integral inequality:

Theorem 5. *Let $f:[a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q \geq 1$, then the following inequality for fractional*

integrals holds:

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$
$$\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left\{\left((\alpha+1)\left|f'(a)\right|^{q}+(\alpha+3)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right. \tag{3.5}$$
$$\left.+\left((\alpha+3)\left|f'(a)\right|^{q}+(\alpha+1)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}.$$

Proof. Firstly, we suppose that $q=1$. Using Lemma 3 and the convexity of $|f'|$, we find

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$
$$\leq \frac{b-a}{4}\int_0^1 t^{\alpha}\left\{\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right\}dt$$
$$\leq \frac{b-a}{4}\left[\left|f'(a))\right|+\left|f'(b))\right|\right]\int_0^1 t^{\alpha}dt$$
$$=\frac{b-a}{4(\alpha+1)}\left[\left|f'(a))\right|+\left|f'(b))\right|\right].$$

Secondly, we suppose that $q>1$. Using Lemma 3 and the power mean inequality, and the convexity of $|f'|^q$, we find

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$
$$\leq \frac{b-a}{4}\left\{\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^{\alpha}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}$$
$$\leq \frac{b-a}{4}\frac{1}{(\alpha+1)^{\frac{1}{p}}}\left\{\left(\int_0^1\left[\frac{t^{\alpha+1}}{2}\left|f'(a)\right|^{q}+\frac{2t^{\alpha}-t^{\alpha+1}}{2}\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\frac{2t^{\alpha}-t^{\alpha+1}}{2}\left|f'(a)\right|^{q}+\frac{t^{\alpha+1}}{2}\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{b-a}{4}\frac{1}{(\alpha+1)^{\frac{1}{p}}}\left\{\left(\frac{1}{2(\alpha+2)}\left|f'(a)\right|^{q}+\frac{(\alpha+3)}{2(\alpha+1)(\alpha+2)}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\alpha+3)}{2(\alpha+1)(\alpha+2)}\left|f'(a)\right|^{q}+\frac{1}{2(\alpha+2)}\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right\}.$$

The proof of Theorem 5 is complete. □

Remark 2. If we take $\alpha=1$ and $q=1$ in Theorem 5, then the inequality (3.5) becomes the inequality (1.2) of Theorem 1.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^{q}$ is convex on $[a,b]$ for $q>1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$
$$\leq\frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'a)|+3|f'(b))|}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'a)|+|f'(b))|}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right] \tag{3.6}$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 3, using the well-known Hölder's inequality and the convexity of $|f'|^{q}$, we find

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$
$$\leq\frac{b-a}{4}\left(\int_{0}^{1}t^{\alpha p}dt\right)^{\frac{1}{p}}\left\{\left(\int_{0}^{1}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{0}^{1}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left(\int_{0}^{1}\left[\frac{t}{2}|f'(a))|^{q}+\frac{2-t}{2}|f'(b)|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{0}^{1}\left[\frac{2-t}{2}|f'(a))|^{q}+\frac{t}{2}|f'(b)|^{q}\right]dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a))|+3|f'(b))|}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a))|+|f'(b))|}{4}\right)^{\frac{1}{q}}\right].$$

Let $a_1 = 3|f'(a)|^q, b_1 = |f'(b)|^q, a_2 = |f'(a)|^q, b_2 = 3|f'(b)|^q$. Here, $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s$$

For $(0 \leq s < 1)$, $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)+}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f(\frac{a+b}{2})\right|$$

$$\leq \frac{b-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(a)\right|^q+3\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

$$\leq \frac{b-a}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}4\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

which completed proof. □

Remark 3. If we take $\alpha = 1$ in Theorem 6, then the inequality (3.6) becomes the inequality (1.3) of Theorem 2.

REFERENCES

[1] M. Alomari and M. Darus, "On the Hadamard's inequality for log-convex functions on the coordinates," *Journal of Inequalities and Applications*, vol. 2009, p. 13 pages, 2009, doi: 10.1155/2009/283147.

[2] A. Azpeitia, "Convex functions and the Hadamard inequality," *Rev. Colombiana Math.*, vol. 28, pp. 7–12, 1994.

[3] M. K. Bakula and J. Pečarić, "Note on some Hadamard-type inequalities," *J. Ineq. Pure and Appl. Math.*, vol. 5, no. 3, p. Art. 74, 2004.

[4] M. Bakula, M. Özdemir, and J. Pečarić, "Hadamard type inequalities for $m-convex$ and (α,m)-*convex functions*," *J. Ineq. Pure and Appl. Math.*, vol. 9, no. 4, p. 96, 2008.

[5] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *J. Ineq. Pure and Appl. Math.*, vol. 10, no. 3, p. Art. 86, 2009.

[6] Z. Dahmani, "New inequalities in fractional integrals," *International Journal of Nonlinear Scinece*, vol. 9, no. 4, pp. 493–497, 2010.

[7] Z. Dahmani, "On Minkowski and Hermite-Hadamard integral inequalities via fractional integration," *Ann. Funct. Anal.*, vol. 1, no. 1, pp. 51–58, 2010.

[8] Z. Dahmani, L. Tabharit, and S. Taf, "New generalizations of Gruss inequality usin Riemann-Liouville fractional integrals," *Bull. Math. Anal. Appl.*, vol. 2, no. 3, pp. 93–99, 2010.

[9] Z. Dahmani, L. Tabharit, and S. Taf, "Some fractional integral inequalities," *Nonl. Sci. Lett. A*, vol. 1, no. 2, pp. 155–160, 2010.
[10] S. S. Dragomir and C. E. M. Pearce, "Selected topics on Hermite–Hadamard inequalities and applications," *RGMIA Monographs*, 2000.
[11] S. Dragomir, "On some new inequalities of Hermite-Hadamard type for *m-convex functions*," *Tamkang J. Math.*, vol. 3, no. 1, 2002.
[12] S. Dragomir and R. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Appl. Math. Lett.*, vol. 11, no. 5, pp. 91–95, 1998, doi: 10.1016/S0893-9659(98)00086-X.
[13] P. M. Gill, C. E. M. Pearce, and J. Pečarić, "Hadamard's inequality for *r-convex functions*," *Journal of Mathematical Analysis and Applications*, vol. 215, no. 2, pp. 461–470, 1997, doi: 10.1006/jmaa.1997.5645.
[14] R. Gorenflo and F. Mainardi, "Fractional calculus: integral and differential equations of fractional order," *Springer Verlag, Wien*, pp. 223–276, 1997.
[15] U. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula," *Appl. Math. Comput.*, vol. 147, no. 5, pp. 137–146, 2004, doi: 10.1016/S0096-3003(02)00657-4.
[16] U. Kirmaci, M. Bakula, M. Ozdemir, and J. Pecaric, "Hadamard-tpye inequalities for *s-convex functions*," *Appl. Math. Comput.*, vol. 193, pp. 26–35, 2007, doi: 10.1016/j.amc.2007.03.030.
[17] U. Kirmaci and M. Ozdemir, "On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Appl. Math. Comput.*, vol. 153, pp. 361–368, 2004, doi: 10.1016/S0096-3003(03)00637-4.
[18] S. Miller and B. Ross, "On introduction to the fractional calculus and fractional differential equations," *John Wiley & Sons, USA,*, vol. 13, p. 2, 1993.
[19] M. Ozdemir, M. Avci, and E. a. Set, "On some inequalities of Hermite-Hadamard type via *m-convexity*," *Applied Mathematics Letter*, vol. 23, no. 9, pp. 1065–1070, 2010, doi: 10.1016/j.aml.2010.04.037.
[20] J. Pecaric, F. Proschan, and Y. L. Tong, "Convex functions, partial orderings and statistical applications," *Academic Press, Boston*, 1992.
[21] I. Podlubni, "Fractional differential equations," *Academic Press, San Diego*, 1999.
[22] H. Sarikaya, M. Z.and Yaldiz, "On weighted Montogomery identities for Riemann-Liouville fractional integrals," *Konuralp Journal of Mathematics*, vol. 1, no. 1, pp. 48–53, 2013.
[23] M. Z. Sarikaya, "Ostrowski type inequalities involving the right Caputo fractional derivatives belong to L_p," *Facta Universitatis, Series Mathematics and Informatics*, vol. 27, no. 2, pp. 191–197, 2012.
[24] M. Z. Sarikaya and H. Ogunmez, "On new inequalities via Riemann-Liouville fractional integration," *Abstract and Applied Analysis*, vol. 2012, p. 10 pages, 2012, doi: 10.1155/2012/428983.
[25] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, pp. 2403–2407, 2013, doi: 10.1016/j.mcm.2011.12.048.
[26] E. Set, M. O. Ozdemir, and S. S. Dragomir, "On Hadamard-Type inequalities involving several kinds of convexity," *Journal of Inequalities and Applications*, vol. Article ID 286845, p. 12, 2010, doi: 10.1155/2010/286845.
[27] E. Set, M. O. Ozdemir, and S. S. Dragomir, "On the Hermite-Hadamard inequality and other integral inequalities involving two functions," *Journal of Inequalities and Applications*, vol. Article ID 148102, p. 9, 2010, doi: 10.1155/2010/148102.
[28] J. Wang, X. Li, M. Feckan, and Y. Zhou, "Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity," *Applicable Analysis*, pp. 1–13, 2012, doi: 10.1080/00036811.2012.727986.

Authors' addresses

Mehmet Zeki Sarıkaya
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY
E-mail address: sarikayamz@gmail.com

Hüseyin Yildirim
Department of Mathematics, Faculty of Science and Arts, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, Turkey
E-mail address: hyildir@ksu.edu.tr