Miskolc Mathematical Notes
Vol. 17 (2017), No. 2, pp. 1021–1031

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2017.1186

SOME FIXED POINT THEOREMS FOR NONSELF GENERALIZED CONTRACTION

IOAN A. RUS AND MARCEL-ADRIAN ŞERBAN

Received 31 March, 2014

Abstract. In this paper we give a new proof of a result by S. Reich and A.J. Zaslavski (S. Reich and A.J. Zaslavski, A fixed point theorem for Matkowski contractions, Fixed Point Theory, 8(2007), No. 2, 303–307). Some new fixed point theorems for nonself generalized contractions are also given.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: metric space, nonself generalized contraction, fixed point, comparison function, maximal displacement functional

1. INTRODUCTION

There are many techniques in the fixed point theory of nonself operators (see [10], [4], [6], [9], [19], [20], [2], ...). An exotic result is given in [14] (see also, [13] and [15]). This result read as follows:

Theorem 1. *Let (X,d) be a complete metric space, $Y \subset X$ a nonempty closed subset and $f : Y \to X$ be a φ-contraction, where φ is a comparison function. We suppose that there exists a bounded sequence $(x_n)_{n\in\mathbb{N}^*}$ such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$. Then f has a unique fixed point x^* and $f^n(x_n) \to x^*$.*

The aim of this paper is to give a new proof of this theorem and to obtain other results of this type.

2. PRELIMINARIES

2.1. *Notations*

$\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{N}^* = \{1,2,3,\ldots\}$.
$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$
Let (X,d) be a metric space. We will use the following symbols:
$\mathcal{P}(X) = \{Y \mid Y \subset X\}$

This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

$P(X) = \{Y \subset X \mid Y \text{ is nonempty}\}$, $P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}$,
$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$, $P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$.
If $f : X \to X$ is an operator then $F_f := \{x \in X \mid x = f(x)\}$ denotes the fixed point set of the operator f. In the case when the operator f has an unique fixed point $x^* \in X$ then we write $F_f = \{x^*\}$.

The diameter functional $\delta : P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$\delta(A) := \sup\{d(a,b) \mid a,b \in A\}.$$

2.2. *Comparison functions*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. We consider the following conditions relative to φ:

(i_φ) φ is increasing;
(ii_φ) $\varphi(t) < t$, $\forall t > 0$;
(iii_φ) $\varphi(0) = 0$;
(iv_φ) $\varphi^n(t) \to 0$ as $n \to \infty$, $\forall t \in \mathbb{R}_+$;
(v_φ) $t - \varphi(t) \to \infty$ as $t \to \infty$;
(vi_φ) $\sum_{n=0}^{\infty} \varphi^n(t) < +\infty$, $\forall t \in \mathbb{R}_+$.

Definition 1 (I.A. Rus [17])**.** By definition the function φ is a comparison function if it satisfies the conditions (i_φ) and (iv_φ).

Definition 2. A comparison function is:

(a) strict comparison function if it satisfies the condition (v_φ);
(b) strong comparison function if it satisfies the condition (vi_φ).

It is clear that if φ is a comparison function then $\varphi(t) < t$, $\forall t > 0$, and $\varphi(0) = 0$. If φ is a strong comparison function then the functions φ and $\sum_{n=0}^{\infty} \varphi^n$ are continuous in $t = 0$.

For example, if $\varphi(t) := at$, $t \in \mathbb{R}_+$, $a \in [0;1[$, then φ is a strict and strong comparison function and $\varphi(t) := \frac{t}{1+t}$, $t \in \mathbb{R}_+$, is a strict comparison function which is not a strong comparison function.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function. In this case we define the function $\theta_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by,

$$\theta_\varphi(t) = \sup\{s \in \mathbb{R}_+ \mid s - \varphi(s) \leq t\}.$$

We remark that θ_φ is increasing and $\theta_\varphi(t) \to 0$ as $t \to 0$. The function θ_φ appears when we study the data dependence of the fixed points.

For more considerations on comparison functions see [17], [1], [21] and [5].

2.3. *Maximal displacement functional*

Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous nonself operator. By the maximal displacement functional corresponding to f we understand the functional $E_f : P(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$E_f(A) := \sup\{d(x, f(x)) \mid x \in A\}.$$

We have that:

(i) $A, B \in P(Y)$, $A \subset B$ imply $E_f(A) \leq E_f(B)$;

(ii) $E_f(A) = E_f\left(\overline{A}\right)$ for all $A \in P(Y)$.

Definition 3. An operator $f : Y \to X$ is α-graphic contraction if $0 \leq \alpha < 1$ and $x \in Y$, $f(x) \in Y$ imply

$$d\left(f^2(x), f(x)\right) \leq \alpha d(x, f(x)).$$

Example 1. If $f : Y \to X$ is α-contraction then f is α-graphic contraction.

Example 2. If $f : Y \to X$ is α-Kannan operator, i.e., $0 \leq \alpha < \frac{1}{2}$, and

$$d(f(x), f(y)) \leq \alpha [d(x, f(x)) + d(y, f(y))], \ \forall x, y \in Y,$$

then f is $\frac{\alpha}{1-\alpha}$-graphic contraction.

Also, we have that:

Lemma 1. *Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be a continuous α-graphic contraction. Then:*

(a) $E_f(f(A)) \leq \alpha E_f(A)$, *for all $A \subset Y$ with $f(A) \subset Y$;*

(b) $E_f(f(A) \cap Y) \leq \alpha E_f(A)$, *for all $A \subset Y$ with $f(A) \cap Y \neq \varnothing$.*

Proof. The proof follows from the definition of E_f. Let, for example, to prove (b). We have

$$\begin{aligned} E_f(f(A) \cap Y) &= \sup\{d(x, f(x)) \mid x \in f(A) \cap Y\} = \\ &= \sup\left\{d\left(f(u), f^2(u)\right) \mid u \in A,\ f(u) \in Y\right\} \leq \\ &\leq \alpha \sup\{d(u, f(u)) \mid u \in A\} = \\ &= \alpha E_f(A) \end{aligned}$$

□

2.4. *Matrices convergent to* 0

Definition 4. A matrix $S \in \mathbb{R}_+^{m \times m}$ is called a matrix convergent to zero iff $S^k \to 0$ as $k \to +\infty$.

Theorem 2 (see [12], [16], [23], [10])**.** *Let $S \in \mathbb{R}_+^{m\times m}$. The following statements are equivalent:*

(i) *S is a matrix convergent to zero;*

(ii) $S^k x \to 0$ *as* $k \to +\infty$, $\forall x \in \mathbb{R}^m$*;*

(iii) $I_m - S$ *is non-singular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \cdots$$

(iv) $I_m - S$ *is non-singular and* $(I_m - S)^{-1}$ *has nonnegative elements;*

(v) $\lambda \in \mathbb{C}$, $\det(S - \lambda I_m) = 0$ *imply* $|\lambda| < 1$*;*

(vi) *there exists at least one subordinate matrix norm such that* $\|S\| < 1$.

The matrices convergent to zero were used by A. I. Perov [11] (see also [10] pp. 432-434) to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^m$.

3. A NEW PROOF OF THEOREM 1

Now we present a new proof of Theorem 1. Let $A \in P_{b,cl}(Y)$ be such that $x_n \in A$, for all $n \in \mathbb{N}^*$. We consider the following standard construction in the fixed point theory for the nonself operators (see for example [8] and [7]).

Let $A_1 := \overline{f(A)}$, $A_2 := \overline{f(A_1 \cap A)}$, $\ldots$, $A_{n+1} := \overline{f(A_n \cap A)}$, $n \in \mathbb{N}^*$. We remark that:

(a) $A_{n+1} \subset A_n$, $\forall n \in \mathbb{N}^*$;

(b) $f^n(x_n) \in A_n$, $\forall n \in \mathbb{N}^*$, so $A_n \neq \varnothing$, $\forall n \in \mathbb{N}^*$.

Since f is a φ-contraction, i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function such that

$$d(f(x), f(y)) \leq \varphi(d(x,y)),\ \forall x, y \in Y,$$

it follows that

$$\delta(f(B)) \leq \varphi(\delta(B)),\ \forall B \in P_b(Y).$$

From the properties of φ and δ we have

$$\begin{aligned} \delta(A_{n+1}) &= \delta\left(\overline{f(A_n \cap A)}\right) = \delta(f(A_n \cap A)) \leq \delta(f(A_n)) \leq \\ &\leq \varphi(\delta(A_n)) \leq \cdots \leq \varphi^{n+1}(\delta(A)) \to 0 \end{aligned}$$

as $n \to +\infty$. From Cantor intersection lemma we have

$$A_\infty := \bigcap_{n \in \mathbb{N}} A_n \neq \varnothing,\ \delta(A_\infty) = 0 \text{ and } f(A_\infty \cap A) \subset A_\infty.$$

From $A_\infty \neq \varnothing$ and $\delta(A_\infty) = 0$, we have that $A_\infty = \{x^*\}$. On the other hand $f^n(x_n) \in A_n$ and $f^{n-1}(x_n) \in A_{n-1} \cap Y$. This implies that $\{f^n(x_n)\}_{n\in\mathbb{N}}$ and $\left\{f^{n-1}(x_n)\right\}_{n\in\mathbb{N}}$ are fundamental sequences. Since A_n, $n \in \mathbb{N}^*$, are closed, it follows that

$$f^{n-1}(x_n) \to x^* \text{ and } f^n(x_n) \to x^* \text{ as } n \to +\infty.$$

Since f is continuous then $f^n(x_n) \to f(x^*)$, so $f(x^*) = x^*$.

With respect to the data dependence of the fixed point, in Theorem 1, we have the following result:

Theorem 3. *Let $f : Y \to X$ be as in Theorem 1, where φ is a strict comparison function. Then:*

(a) $d\left(f^n(x_n), x^*\right) \le \varphi\left(d\left(x_n, x^*\right)\right)$, $\forall n \in \mathbb{N}^*$;

(b) $d\left(x, x^*\right) \le \theta_\varphi\left(d\left(x, f(x)\right)\right)$, $\forall x \in Y$;

(c) *Let $g : Y \to X$ be such that:*

  (1) *there exists $\eta > 0$ such that $d\left(f(x), g(x)\right) \le \eta$, $\forall x \in Y$;*

  (2) $F_g \neq \varnothing$.

Then

$$d\left(x^*, y^*\right) \le \theta_\varphi(\eta), \forall y^* \in F_g.$$

Proof. Let us prove (b) and (c).

(b). The conclusion (b) follows from the following estimation

$$d\left(x, x^*\right) \le d\left(x, f(x)\right) + d\left(f(x), x^*\right) \le d\left(x, f(x)\right) + \varphi\left(d\left(x, x^*\right)\right), \ \forall x \in Y.$$

So,

$$d\left(x, x^*\right) - \varphi\left(d\left(x, x^*\right)\right) \le d\left(x, f(x)\right), \ \forall x \in Y.$$

(c). Let $y^* \in F_g$ then from (b) it follows

$$d\left(x^*, y^*\right) \le \theta_\varphi\left(d\left(y^*, f\left(y^*\right)\right)\right) = \theta_\varphi\left(d\left(g\left(y^*\right), f\left(y^*\right)\right)\right) \le \theta_\varphi(\eta).$$

□

For more considerations on data dependence of the fixed points for nonself φ-contractions see [3], [18] and [22].

4. A FIXED POINT THEOREM FOR NONSELF KANNAN OPERATORS

We have:

Theorem 4. *Let (X, d) be a complete metric space, $Y \subset X$ a nonempty bounded closed subset and $f : Y \to X$ a continuous operator. We suppose that:*

(i) *f is α-Kannan operator;*

(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*

(iii) $E_f(Y) < +\infty$.

Then:

(a) $F_f = \{x^*\}$;

(b) *$f^{n-1}(x_n) \to x^*$ and $f^n(x_n) \to x^*$ as $n \to +\infty$;*

(c) $d\left(x, x^*\right) \le (1+\alpha)\, d\left(x, f(x)\right)$, $\forall x \in Y$;

(d) $d\left(f^{n-1}(x_n), x^*\right) \le \alpha^{n-1}(1-\alpha)^{1-n}(1+\alpha)\, d\left(x_n, f(x_n)\right)$, $\forall n \in \mathbb{N}^*$;

(e) *Let $g : Y \to X$ be such that:*

  (1) *there exists $\eta > 0$ such that $d\left(f(x), g(x)\right) \le \eta$, $\forall x \in Y$;*

  (2) $F_g \neq \varnothing$.

Then

$$d\left(x^*,y^*\right)\le(1+\alpha)\eta,\forall y^*\in F_g.$$

Proof. $(a)+(b)$. Let $Y_1:=\overline{f(Y)}$, $Y_2:=\overline{f(Y_1\cap Y)}$, ..., $Y_{n+1}:=\overline{f(Y_n\cap Y)}$, $n\in\mathbb{N}^*$. We remark that $Y_{n+1}\subset Y_n$ and $f^n(x_n)\in Y_n$, so $Y_n\neq\varnothing$, $n\in\mathbb{N}^*$. Since f is α-Kannan operator, from Example 2 and Lemma 1, we have that:

$$\begin{aligned}\delta(Y_{n+1})&=\delta\left(\overline{f(Y_n\cap Y)}\right)=\delta(f(Y_n\cap Y))\le 2\alpha\cdot E_f(Y_n\cap Y)=\\&=2\alpha\cdot E_f\left(\overline{f(Y_{n-1}\cap Y)}\cap Y\right)=2\alpha\cdot E_f(f(Y_{n-1}\cap Y)\cap Y)\le\\&\le\frac{2\alpha^2}{1-\alpha}E_f(Y_{n-1}\cap Y)\le\cdots\le\frac{2\alpha^{n+1}}{(1-\alpha)^n}E_f(Y)\to 0\text{ as }n\to+\infty.\end{aligned}$$

Now the proof is similar with the proof of Theorem 1.

(c). Let $x\in Y$. From the definition of the Kannan operator we have:

$$d\left(x,x^*\right)\le d(x,f(x))+d\left(f(x),x^*\right)\le d(x,f(x))+\alpha d(x,f(x)),\ \forall x\in Y.$$

(d) and (e) follow from (c). □

5. Other nonself generalized contractions

5.1. *Ćirić-Reich-Rus operators*

Let (X,d) be a metric space, $Y\in P_{cl}(X)$ and $f:Y\to X$ be a nonself operator. An operator $f:Y\to X$ is a Ćirić-Reich-Rus operator (see [4], [20], [22], ...) if there exist $a,b\in\mathbb{R}_+$ with $a+2b<1$ such that

$$d(f(x),f(y))\le ad(x,y)+b[d(x,f(x))+d(y,f(y))],\ \forall x,y\in Y.$$

Lemma 2. *Let (X,d) be a metric space, $Y\in P_{cl}(X)$ and $f:Y\to X$ a nonself Ćirić-Reich-Rus operator then f is a nonself α-graphic contraction with $\alpha=\frac{a+b}{1-b}$.*

Proof. Let $x\in Y$ such that $f(x)\in Y$ then

$$d\left(f^2(x),f(x)\right)\le ad(f(x),x)+b\left[d\left(f(x),f^2(x)\right)+d(x,f(x))\right],$$

so

$$d\left(f^2(x),f(x)\right)\le\frac{a+b}{1-b}d(x,f(x)).$$

□

Lemma 3. *Let (X,d) be a metric space, $Y\in P_{cl}(X)$ and $f:Y\to X$ a nonself Ćirić-Reich-Rus operator then:*

(a) $\delta(f(A)\cap Y)\le a\delta(A)+2bE_f(A)$, *for all $A\subset Y$;*

(b) $E_f(f(A)\cap Y)\le\alpha E_f(A)$, *for all $A\subset Y$, where $\alpha=\frac{a+b}{1-b}$.*

Proof. (a). Let $A \subset Y$ then

$$\begin{aligned}\delta(f(A)\cap Y) &= \sup\{d(x,y) \mid x,y \in f(A)\cap Y\} = \\ &= \sup\{d(f(u),f(v)) \mid u,v \in A,\ f(u),f(v) \in Y\} \leq \\ &\leq a\sup\{d(u,v) \mid u,v \in A\} + 2b\sup\{d(u,f(u)) \mid u \in A\} = \\ &= a\delta(A) + 2bE_f(A)\end{aligned}$$

(b). The proof follows from Lemma 2 and Lemma 1. □

Also, for the next result we need the following lemma

Lemma 4 (Cauchy Lemma, [21])**.** *Let a_n, $b_n \in \mathbb{R}_+$, $n \in \mathbb{N}$. We suppose that:*

(i) $\sum_{k=0}^{\infty} a_k < +\infty$;
(ii) $b_n \to 0$ *as* $n \to \infty$.

Then

$$\sum_{k=0}^{n} a_{n-k} b_k \to 0 \text{ as } n \to \infty.$$

Theorem 5. *Let (X,d) be a complete metric space, $Y \subset X$ a nonempty bounded closed subset and $f : Y \to X$ a continuous operator. We suppose that:*

(i) *f is Ćirić-Reich-Rus operator;*
(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$;*
(iii) $E_f(Y) < +\infty$.

Then:

(a) $F_f = \{x^*\}$;
(b) *$f^{n-1}(x_n) \to x^*$ and $f^n(x_n) \to x^*$ as $n \to +\infty$;*
(c) $d(x,x^*) \leq (1+b)(1-a)^{-1} d(x,f(x))$, $\forall x \in Y$;
(d) $d\left(f^{n-1}(x_n),x^*\right) \leq (1+b)(1-a)^{-1}\alpha^{n-1} d(x_n,f(x_n))$, $\forall n \in \mathbb{N}^*$, *where* $\alpha = \frac{a+b}{1-b}$.
(e) *Let $g : Y \to X$ be such that:*
   (1) *there exists $\eta > 0$ such that $d(f(x),g(x)) \leq \eta$, $\forall x \in Y$;*
   (2) $F_g \neq \varnothing$.
   Then

$$d\left(x^*,y^*\right) \leq (1+b)(1-a)^{-1}\eta, \forall y^* \in F_g.$$

Proof. $(a)+(b)$. Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1\cap Y)}$, $\ldots$, $Y_{n+1} := \overline{f(Y_n\cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n(x_n) \in Y_n$, so $Y_n \neq \varnothing$, $n \in \mathbb{N}^*$. Since f is Ćirić-Reich-Rus operator, from Lemma 3 (a), we have that:

$$\delta(Y_{n+1}) = \delta\left(\overline{f(Y_n\cap Y)}\right) = \delta(f(Y_n\cap Y)) \leq$$

$$\leq a\delta\left(Y_n\cap Y\right)+2bE_f\left(Y_n\cap Y\right)\leq$$
$$\leq a\delta\left(Y_n\right)+2bE_f\left(Y_n\cap Y\right)\leq\cdots\leq$$
$$\leq a^{n+1}\delta\left(Y\right)+a^n 2b\cdot E_f\left(Y\right)+$$
$$+a^{n-1}2b\cdot E_f\left(Y_1\cap Y\right)+\cdots+2b\cdot E_f\left(Y_n\cap Y\right).$$

On the other hand, from Lemma 3 (b) we get

$$E_f\left(Y_k\cap Y\right)=E_f\left(\overline{f\left(Y_{n-1}\cap Y\right)}\cap Y\right)=E_f\left(f\left(Y_{n-1}\cap Y\right)\cap Y\right)\leq$$
$$\leq\alpha E_f\left(Y_{k-1}\cap Y\right)\leq\cdots<\alpha^k E_f\left(Y\right),\ k\in N^*,$$

where $\alpha=\frac{a+b}{1-b}$. Applying Lemma 4 for $a_n=a^n$ and $b_n=2b\cdot E_f\left(Y_n\cap Y\right)$ and we get that

$$\delta\left(Y_n\right)\to 0 \text{ as } n\to+\infty$$

and the proof is similar with the proof of Theorem 1.

(c). Let $x\in Y$. From the definition of the Ćirić-Reich-Rus operator we have:

$$d\left(x,x^*\right)\leq d\left(x,f\left(x\right)\right)+d\left(f\left(x\right),x^*\right)\leq$$
$$\leq d\left(x,f\left(x\right)\right)+ad\left(x,x^*\right)+bd\left(x,f\left(x\right)\right),\ \forall x\in Y,$$

so

$$d\left(x,x^*\right)\leq\frac{1+b}{1-a}d\left(x,f\left(x\right)\right),\ \forall x\in Y.$$

(d) and (e) follow from (c). □

5.2. *Perov operators*

Let (X,d) be a generalized metric space with $d:X\times X\to\mathbb{R}^m_+$, $Y\in P_{cl}\left(X\right)$ and $f:Y\to X$ be a nonself operator. By definition (see [17], [20]) $f:Y\to X$ is a nonself Perov operator if there exists a matrix convergent to zero $S\in\mathbb{R}^{m\times m}_+$ such that

$$d\left(f\left(x\right),f\left(y\right)\right)\leq S\cdot d\left(x,y\right),\ x,y\in Y.$$

We have the following fixed point results in the case of nonself Perov operators:

Theorem 6. *Let (X,d) be a complete generalized metric space with $d:X\times X\to\mathbb{R}^m_+$, $Y\subset X$ a nonempty bounded closed subset and $f:Y\to X$ an operator. We suppose that:*

(i) *f is a Perov operator;*
(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ in Y such that $f^n\left(x_n\right)$ is defined for all $n\in\mathbb{N}^*$.*

Then:

(a) $F_f=\{x^*\}$;
(b) $f^{n-1}\left(x_n\right)\to x^*$ *and* $f^n\left(x_n\right)\to x^*$ *as* $n\to+\infty$.
(c) $d\left(x,x^*\right)\leq\left(I_m-S\right)^{-1}d\left(x,f\left(x\right)\right)$, $\forall x\in Y$;

(d) $d(f^n(x_n), x^*) \le S^n d(x_n, x^*)$, $\forall n \in \mathbb{N}^*$;
(e) *Let* $g : Y \to X$ *be such that:*
(1) *there exists* $\eta \in (\mathbb{R}_+^*)^m$ *such that* $d(f(x), g(x)) \le \eta$, $\forall x \in Y$;
(2) $F_g \neq \varnothing$.
Then

$$d(x^*, y^*) \le (I_m - S)^{-1}\eta, \forall y^* \in F_g.$$

Proof. $(a)+(b)$. Let $Y_1 := \overline{f(Y)}$, $Y_2 := \overline{f(Y_1 \cap Y)}$, $\ldots$, $Y_{n+1} := \overline{f(Y_n \cap Y)}$, $n \in \mathbb{N}^*$. We remark that $Y_{n+1} \subset Y_n$ and $f^n(x_n) \in Y_n$, so $Y_n \neq \varnothing$, $n \in \mathbb{N}^*$. Since f is a Perov we have:

$$\delta(Y_{n+1}) = \delta\left(\overline{f(Y_n \cap Y)}\right) = \delta(f(Y_n \cap Y)) \le S \cdot \delta(Y_n \cap Y) = \le$$
$$\le S \cdot \delta(Y_n) \le \cdots \le S^{n+1} \cdot \delta(Y) \to 0 \text{ as } n \to +\infty.$$

Now the proof is similar with the proof of Theorem 1.

(c). Let $x \in Y$ then we have:

$$d(x, x^*) \le d(x, f(x)) + d(f(x), x^*) \le d(x, f(x)) + Sd(x, x^*), \ \forall x \in Y,$$

so

$$d(x, x^*) \le (I_m - S)^{-1} d(x, f(x)), \ \forall x \in Y.$$

(d) follows from the definition of the Perov operator and (e) is obtained from (c) for $x := y^* \in F_g$. □

6. An open problem

The above considerations give rise to the following problem:

Problem 1. *Let (X, d) be a complete metric space, Y a nonempty bounded and closed subset of X and $f : Y \to X$ a nonself operator. We suppose that there exists a sequence $(x_n)_{n \in \mathbb{N}^*}$ such that $f^n(x_n)$ is defined for all $n \in \mathbb{N}^*$. In which additional conditions on f we have that:*

(a) $F_f \neq \varnothing$?
(b) $F_f = \{x^*\}$?

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Authors' addresses

Ioan A. Rus

Babeş-Bolyai University of Cluj-Napoca, Department of Mathematics, 1 M. Kogălniceanu, 400084 Cluj-Napoca, Romania

E-mail address: iarus@math.ubbcluj.ro

Marcel-Adrian Şerban

Babeş-Bolyai University of Cluj-Napoca, Department of Mathematics, 1 M. Kogălniceanu, 400084 Cluj-Napoca, Romania

E-mail address: mserban@math.ubbcluj.ro