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TWIN ROMAN DOMINATION NUMBER OF A DIGRAPH

H. ABDOLLAHZADEH AHANGAR, J. AMJADI, S.M. SHEIKHOLESLAMI, V. SAMODIVKIN, AND L. VOLKMANN

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Abstract. Let D be a finite and simple digraph with vertex set $V(D)$. A *twin Roman dominating function* (TRDF) on D is a labeling $f : V(D) \to \{0,1,2\}$ such that every vertex with label 0 has an in-neighbor and out-neighbor with label 2. The *weight* of a TRDF f is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The *twin Roman domination number* of a digraph D, denoted by $\gamma_R^*(D)$, equals the minimum weight of a TRDF on D. In this paper we initiate the study of the twin Roman domination number in digraphs. In particular, we present sharp bounds for $\gamma_R^*(D)$ and determine the exact value of the twin Roman domination number for some classes of digraphs.

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1. INTRODUCTION

Let D be a finite simple digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The order $n = n(D)$ of a digraph D is the number of its vertices. We write $d_D^+(v)$ for the out-degree of a vertex v and $d_D^-(v)$ for its in-degree. The *minimum* and *maximum in-degree* and *minimum* and *maximum out-degree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. The *minimum degree* $\delta(D)$ of a digraph D is defined as the minimum of all in-degrees and all out-degrees of vertices in D and the *maximum degree* $\Delta(D)$ of a digraph D is defined as the maximum of all in-degrees and all out-degrees of vertices in D. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X,v)$ is the set of arcs from X to v. Consult [8] for the notation and terminology which are not defined here. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$.

A vertex u in a digraph D out-dominates itself and all vertices v such that uv is an arc of D, similarly, u in-dominates both itself and all vertices w such that wu is an arc of D. A set S of vertices of D is a twin dominating set of D if every vertex of D is out-dominated by a vertex of S and in-dominated by a vertex of S. The twin domination number $\gamma^*(D)$ is the cardinality of a minimum twin dominating set. A $\gamma^*(D)$-*function* is a twin dominating function of D with weight $\gamma^*(D)$. The twin domination, was introduced by Chartrand, Dankelmann, Schultz, and Swart [3] and has been studied by several authors (see [1,2,6]).

A *Roman dominating function* (RDF) on a digraph D is a function $f : V \longrightarrow \{0,1,2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ has a in-neighbor u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, equals the minimum weight of an RDF on D. A $\gamma_R(D)$-*function* is a Roman dominating function of D with weight $\gamma_R(D)$. The Roman domination for digraphs was introduced by Kamaraj and Hemalatha [5] and investigated in [7].

A *twin Roman dominating function* (TRDF) on D is a Roman dominating function of D such that every vertex with label 0 has an out-neighbor with label 2. The *twin Roman domination number* of a digraph D, denoted by $\gamma^*_R(D)$, equals the minimum weight of a TRDF on D. A $\gamma^*_R(D)$-*function* is a twin Roman dominating function of D with weight $\gamma^*_R(D)$. A twin Roman dominating function $f : V \longrightarrow \{0,1,2\}$ can be represented by the ordered partition (V_0,V_1,V_2) (or (V_0^f,V_1^f,V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1|+2|V_2|$. Since $V_1^f \cup V_2^f$ is a twin dominating set when f is a TRDF, and since placing weight 2 at the vertices of a twin dominating set yields a TRDF, we have

$$\gamma^*(D) \le \gamma^*_R(D) \le 2\gamma^*(D). \tag{1.1}$$

Obviously the function $f = (\varnothing,V(D),\varnothing)$ is a TRDF of D which implies that

$$\gamma^*_R(D) \le n. \tag{1.2}$$

Our purpose in this paper is to establish some sharp bounds for the twin Roman domination number of a digraph.

We make use of the following results in this paper.

Theorem 1 ([3]). *Let D be a digraph of order n and minimum degree $\delta(D) \ge 1$. Then,*

$$\gamma^*(D) \le \left\lfloor \frac{2n}{3} \right\rfloor.$$

The proof of the following observations are straightforward and therefore omitted.

Observation 1. *Let D be a digraph on n vertices. Then*

(i) *If $\gamma^*_R(D) = n$ then for any γ^*_R-function $f = (V_0,V_1,V_2)$ on D, $|V_0| = |V_2|$.*
(ii) *If $|V_0| = |V_2|$ for some γ^*_R-function $f = (V_0,V_1,V_2)$ on D, then $\gamma^*_R(D) = n$.*

Observation 2. *Let D be a digraph and $f = (V_0^f, V_1^f, V_2^f)$ a TRDF on D.*

(i) *If $x, y, z \in V_1$, $x \to y$, $y \to x$, $y \to z$ and $z \to y$ then $g = (V_0^f \cup \{x,z\}; V_1^f - \{x,y,z\}; V_2^f \cup \{y\})$ is a TRDF on D with $w(g) = w(f) - 1$.*

(ii) *If $x \in V_2$, $y \in V_1$, $x \to y$ and $y \to x$ then $g = (V_0^f \cup \{y\}; V_1^f - \{y\}; V_2^f)$ is a TRDF on D with $w(g) = w(f) - 1$.*

Observation 3. *Let D be a digraph and $f = (V_0, V_1, V_2)$ a $\gamma_R^*(D)$-function.*

(i) *If $v \in V(D)$ and $d^+(v)d^-(v) = 0$ then $f(v) \neq 0$.*

(ii) *If $x, y, z \in V_1$, $x \to y$ and $y \to x$ then $y \not\to z$ or $z \not\to y$ holds.*

(iii) *If $x \in V_1$ then at least one of the sets $N^+(x) \cap V_2$ and $N^-(x) \cap V_2$ is empty.*

(iv) $|V_2| \leq |V_0|$.

We will say that a digraph D is a *twin Roman digraph* if $\gamma_R^*(D) = 2\gamma^*(D)$.

Observation 4. A digraph D is a twin Roman digraph if and only if it has a $\gamma_R^*(D)$-function $f = (V_0, V_1, V_2)$ with $V_1 = \varnothing$.

Proof. Let D be a twin Roman digraph, and let S be a $\gamma^*(D)$-set of D. Then $f = (V(D) - S, \varnothing, S)$ is a TRDF on D such that $\omega(f) = 2|S| = 2\gamma^*(D) = \gamma_R^*(G)$, and therefore f is a $\gamma_R^*(D)$-function with $V_1 = \varnothing$.

Conversely, let $f = (V_0, V_1, V_2)$ be a $\gamma_R^*(D)$-function with $V_1 = \varnothing$ and thus $\gamma_R^*(D) = 2|V_2|$. Then V_2 is also a twin dominating set of D implying that $2\gamma^*(D) \leq 2|V_2| = \gamma_R^*(D)$. Applying (1.1), we obtain the identity $\gamma_R^*(D) = 2\gamma^*(D)$, i.e. D is a twin Roman digraph. □

2. Basic properties and bounds on the twin Roman domination number

First we characterize the digraphs D with the properties that $\gamma_R^*(D) = 2$, $\gamma_R^*(D) = 3$, $\gamma_R^*(D) = 4$ or $\gamma_R^*(D) = 5$.

Proposition 1. (i) *For a digraph D of order $n \geq 2$, $\gamma_R^*(D) = 2$ if and only if $n = 2$ or there is a vertex v with $d^+(v) = d^-(v) = n-1$.*

(ii) *For a digraph D of order $n \geq 3$, $\gamma_R^*(D) = 3$ if and only if D has no vertex v with $d^+(v) = d^-(v) = n-1$. In addition (a) $n = 3$ or (b) D has a vertex v with $|N^+(v) \cap N^-(v)| = n-2$.*

(iii) *For a digraph D of order $n \geq 4$, $\gamma_R^*(D) = 4$ if and only if $|N^+(v) \cap N^-(v)| \leq n-3$ for any vertex $v \in V(D)$. In addition, (a) $n = 4$ or (b) there is a vertex v with $|N^+(v) \cap N^-(v)| = n-3$ or (c) there are two vertices $u, v \in V(D)$ such that $(N_D^+(u) \cup N_D^+(v)) \cap (N_D^-(u) \cap N_D^-(v)) = V(D) - \{u, v\}$.*

(iv) *For a digraph D of order $n \geq 5$, $\gamma_R^*(D) = 5$ if and only if $|N^+(v) \cap N^-(v)| \leq n-4$ for any vertex $v \in V(D)$ and $|(N_D^+(x) \cup N_D^+(y)) \cap (N_D^-(x) \cup N_D^-(y))| \leq n-3$ for all pairs of vertices $x, y \in V(D)$. In addition, (a) there are two vertices $u, v \in V(D)$ such that $|(N_D^+(u) \cup N_D^+(v)) \cap (N_D^-(u) \cup N_D^-(v))| = n-3$*

or (b) $n = 5$ or (c) D contains a vertex w with $|N^+(w) \cap N^-(w)| = n-4$ and the induced subdigraph $H = D[V(D) - (N^+[w] \cap N^-[w])]$ does not contain a vertex x with $|N_H^+(x) \cap N_H^-(x)| = 2$.

Proof. Since the proof of (i) is clear, we omit it.

(ii) Let D have no vertex v with $d^+(v) = d^-(v) = n-1$, then it follows from (i) that $\gamma_R^*(D) \geq 3$. The other two assumptions show that $\gamma_R^*(D) \leq 3$, and so we obtain $\gamma_R^*(D) = 3$.

Conversely, assume that $\gamma_R^*(D) = 3$. It follows from (i) that D has no vertex v with $d^+(v) = d^-(v) = n-1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^*(D)$-function. If $V_2 = \varnothing$, then $|V_1| = 3 = n$ and thus (a) holds. If $V_2 \neq \varnothing$, then $|V_1| = |V_2| = 1$. Suppose $V_2 = \{v\}$. Then $(u,v),(v,u) \in A(D)$ for each $u \in V_0$ and hence $|N^+(v) \cap N^-(v)| = n-2$. Thus condition (b) is proved.

(iii) Since $|N^+(v) \cap N^-(v)| \leq n-3$ for any vertex $v \in V(D)$, we deduce from (i) and (ii) that $\gamma_R^*(D) \geq 4$. The other three assumptions show that $\gamma_R^*(D) \leq 4$, and so we obtain $\gamma_R^*(D) = 4$.

Conversely, assume that $\gamma_R^*(D) = 4$. It follows from (i) and (ii) that $|N^+(v) \cap N^-(v)| \leq n-3$ for any vertex $v \in V(D)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^*(D)$-function. If $V_2 = \varnothing$, then $n = |V_1| = 4$ and so (a) holds. We distinguish two cases.

Case 1. Assume that $|V_2| = 1$ and $|V_1| = 2$. If $V_2 = \{v\}$, then we deduce that $|N^+(v) \cap N^-(v)| = n-3$ and the condition (b) holds.

Case 2. Assume that $|V_2| = 2$. If $V_2 = \{u,v\}$, then we conclude that $(N_D^+(u) \cup N_D^+(v)) \cap (N_D^+(u) \cup N_D^+(v)) = V(D) - \{u,v\}$, and we obtain condition (c).

(iv) By (i), (ii), (iii), the conditions $|N^+(v) \cap N^-(v)| \leq n-4$ for any vertex $v \in V(D)$ and $|(N_D^+(x) \cup N_D^+(y)) \cap (N_D^-(x) \cup N_D^-(y))| \leq n-3$ for all pairs of vertices $x, y \in V(D)$ imply that $\gamma_R^*(D) \geq 5$. The other three assumptions show that $\gamma_R^*(D) \leq 5$, and so we obtain $\gamma_R^*(D) = 5$.

Conversely, assume that $\gamma_R^*(D) = 5$. Using (i), (ii) and (iii), we can see that $|N^+(v) \cap N^-(v)| \leq n-4$ for any vertex $v \in V(D)$ and $|(N_D^+(x) \cup N_D^+(y)) \cap (N_D^-(x) \cup N_D^-(y))| \leq n-3$ for all pairs of vertices $x, y \in V(D)$. Let $f = (V_0, V_1, V_2)$ a $\gamma_R^*(D)$-function. If $V_2 = \varnothing$, then $|V_1| = 5$ and thus $n = 5$. Again, we distinguish two cases.

Case 1. Assume that $|V_2| = 1$ and $|V_1| = 3$. If $V_2 = \{w\}$, then we deduce that $|N^+(w) \cap N^-(w)| = n-4$. Let $\{a,b,c\} = V(D) - (N^+[w] \cap N^-(w))$. If $H = D[\{a,b,c\}]$ contains a vertex x with $|N_H^+(x) \cap N_H^-(x)| = 2$, then we have condition (a). If $D[\{a,b,c\}]$ does not contain a vertex x with $|N_H^+(x) \cap N_H^-(x)| = 2$, then we have condition (c).

Case 2. Assume that $|V_2| = 2$ and $|V_1| = 1$. If $V_2 = \{u,v\}$, then it follows that $|(N_D^+(u) \cup N_D^+(v)) \cap (N_D^-(u) \cup N_D^-(v))| = n-3$ and condition (a) is proved. □

Corollary 1. *For any oriented graph D of order $n \geq 4$, $\gamma_R^*(D) \geq 4$.*

Theorem 2. *Let D be a digraph of order n, maximum outdegree $\Delta^+ \geq 1$ and maximum indegree Δ^-. Then*

$$\gamma_R^*(D) \geq \max\left\{\left\lceil \frac{2n}{\Delta^+ + 1} \right\rceil, \left\lceil \frac{2n}{\Delta^- + 1} \right\rceil\right\}.$$

Proof. We only prove $\gamma_R^*(D) \geq \lceil (2n)/(\Delta^+ + 1) \rceil$, as $\gamma_R^*(D) \geq \lceil (2n)/(\Delta^- + 1) \rceil$ can be proved similarly. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^*(D)$-function. Then $\gamma_R^*(D) = |V_1| + 2|V_2|$ and $n = |V_0| + |V_1| + |V_2|$. Since each vertex of V_0 has at least one in-neighbor in V_2, we observe that $|V_0| \leq \Delta^+|V_2|$. Since $\Delta^+ \geq 1$, we deduce that

$$\begin{aligned}(\Delta^+ + 1)\gamma_R^*(D) =& (\Delta^+ + 1)(|V_1| + 2|V_2|) = (\Delta^+ + 1)|V_1| + 2|V_2| + 2\Delta^+|V_2| \\ \geq& (\Delta^+ + 1)|V_1| + 2|V_2| + 2|V_0| = 2n + (\Delta^+ - 1)|V_1| \geq 2n.\end{aligned}$$

This inequality chain leads to $\gamma_R^*(D) \geq \lceil (2n)/(\Delta^+ + 1) \rceil$. □

If K_n^* is the complete digraph of order $n \geq 2$, then Proposition 1 (i) implies that $\gamma_R^*(K_n^*) = 2$. If $K_{n,n}^*$ is the complete bipartite digraph with $n \geq 4$, then it follows from Theorem 2 that $\gamma_R^*(K_{n,n}^*) \geq 4$. Now it is easy to see that $\gamma_R^*(K_{n,n}^*) = 4$. These examples show that Theorem 2 is sharp.

If D is the empty digraph of order n, then clearly $\gamma_R^*(D) = n$. Therefore Theorem 2 yields to the next result immediately.

Corollary 2. *Let D be a digraph of order n. If $\gamma_R^*(D) < n$, then $\Delta^+(D) \geq 2$ and $\Delta^-(D) \geq 2$.*

Let C_n^* be the digraph of order $n \geq 3$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $v_i \to v_{i+1}$, $v_{i+1} \to v_i$ for $1 \leq i \leq n-1$, $v_n \to v_1$ and $v_1 \to v_n$. Now it is straightforward to verify that $\gamma_R^*(C_n^*) = \lceil (2n)/3 \rceil < n$ for $n \geq 3$. The digraph C_n^* demonstrates that $\Delta^+(D) = 2$ and $\Delta^-(D) = 2$ in Corollary 2 is possible. In addition, this is a further example showing the sharpness of Theorem 2.

Proposition 2. *Let D be a digraph of order n, maximum out-degree Δ^+ and maximum in-degree Δ^-. If $\Delta^+ + \Delta^- \geq n + 3$, then $\gamma_R^*(D) < n$.*

Proof. Let $d^+(v) = \Delta^+$.

First we assume that $d^-(v) = \Delta^-$. In this case the condition $\Delta^+ + \Delta^- \geq n + 3$ leads to $|N^+(v) \cap N^-(v)| \geq 4$. Then the function $f = (N^+(v) \cap N^-(v), V(D) - ((N^+(v) \cap N^-(v)) \cup \{v\}, \{v\})$ is a TRDF on D of weight $\omega(f) \leq n - 3$ and thus $\gamma_R^*(D) \leq n - 3$.

Second we assume that $d^-(u) = \Delta^-$ for a vertex $u \neq v$. The condition $\Delta^+ + \Delta^- \geq n + 3$ implies that $|N^+(v) \cap N^-(u)| \geq 3$. Therefore the function $f = (N^+(v) \cap N^-(u), V(D) - ((N^+(v) \cap N^-(u)) \cup \{u, v\}), \{u, v\})$ is a TRDF on D of weight $\omega(f) \leq n - 1$ and thus $\gamma_R^*(D) \leq n - 1$. This completes the proof. □

Let H be the digraph with vertex set $\{v,u_1,u_2,\ldots,u_{n-1}\}$ with $n \geq 5$ such that $v \to u_i$ for $i = 1,2,\ldots,n-1$, $u_2 \to u_1$ and $u_3 \to u_1$. Then $\Delta^+(H)+\Delta^-(H) = n-2$ and $\gamma_R^*(H) = n$. This example demonstrates that the condition $\Delta^+ + \Delta^- \geq n+3$ in Proposition 2 is best possible in some sense.

Proposition 3. *Let D be a digraph. The following statements are equivalent.*

(i) $\gamma^*(D) = \gamma_R^*(D)$.
(ii) $\gamma^*(D) = |V(D)|$.
(iii) *There is no a directed path of length* 2 *in* D.

Proof. (i) $\Rightarrow$ (ii): Let $\gamma^*(D) = \gamma_R^*(D)$. Then for any $\gamma_R^*(D)$-function $f = (V_0,V_1,V_2)$ on D we have $\gamma^*(D) \leq |V_1|+|V_2| \leq |V_1|+2|V_2| = \gamma_R^*(D)$. Hence $V_2 = \varnothing$ implying that $V_0 = \varnothing$. Therefore $\gamma^*(D) = \gamma_R^*(D) = |V_1| = |V(D)|$.

(ii) $\Rightarrow$ (i): The result follows immediately by (1.1) and (1.2).

(ii) $\Leftrightarrow$ (iii): Obvious. □

Proposition 4. *If D is a digraph on n vertices, then* $\gamma_R^*(D) \geq \min\{n,\gamma^*(D)+1\}$.

Proof. If $\gamma_R^*(D) = n$, then the result is immediate. Assume now that $\gamma_R^*(D) < n$, and suppose to the contrary that $\gamma_R^*(D) \leq \gamma^*(D)$. By (1.1) we have $\gamma_R^*(D) = \gamma^*(D)$. Now Proposition 3 implies $\gamma_R^*(D) = n$, a contradiction. □

Proposition 5. *Let D be a digraph of order $n \neq 3$ with $\delta(D) \geq 1$. Then $\gamma_R^*(D) = \gamma^*(D)+1$ if and only if there is a vertex $v \in V(D)$ with $|N^+(v)\cap N^-(v)| = n-\gamma^*(D)$.*

Proof. Let D have a vertex v with $|N^+(v)\cap N^-(v)| = n-\gamma^*(D)$. Then clearly $f = (N^+(v)\cap N^-(v), V(D)-(N^+[v]\cap N^-[v]), \{v\})$ is a TRDF on D of weight $\gamma^*(D)+1$. Hence $\gamma_R^*(D) \leq \gamma^*(D)+1$, and the result follows by Proposition 4.

Conversely, let $\gamma_R^*(D) = \gamma^*(D)+1$ and let $f = (V_0,V_1,V_2)$ be a $\gamma_R^*(D)$-function. Then either (1) $|V_1| = \gamma^*(D)+1$ and $|V_2| = 0$ or (2) $|V_1| = \gamma^*(D)-1$ and $|V_2| = 1$.

In case (1), since $|V_2| = 0$, we have $|V_0| = 0$. Hence $n = \gamma^*(D)+1$. It follows from Theorem 1 that $n = \gamma^*(D)+1 \leq \frac{2n}{3}+1$, a contradiction when $n \geq 4$. If $n = 2$, then the hypothesis $\delta(D) \geq 1$ implies that D consists of two vertices x and y such that $x \to y \to x$ and thus $|N^+(x)\cap N^-(x)| = 1 = 2-1 = n-\gamma^*(D)$.

In case (2), let $V_2 = \{v\}$. Then $(v,u),(u,v) \in A(D)$ for each $u \in V_0$. Since $N^+(v)\cap N^-(v)\cap V_1 = \varnothing$, we obtain $|N^+(v)\cap N^-(v)| = |V_0| = n-|V_1|-|V_2| = n-\gamma^*(D)$. □

Proposition 6. *Let D be a digraph on $n \geq 7$ vertices with $\delta(D) \geq 1$. Then $\gamma_R^*(D) = \gamma^*(D)+2$ if and only if:*

(i) *D does not have a vertex v with with $|N^+(v)\cap N^-(v)| = n-\gamma^*(D)$.*
(ii) *either D has a vertex v with with $|N^+(v)\cap N^-(v)| = n-\gamma^*(D)-1$ or D contains two vertices v,w such that*

$$|(N^+[v]\cup N^+[w])\cap(N^-[v]\cup N^-[w])| = n-\gamma^*(D)+2.$$

Proof. Let $\gamma_R^*(D) = \gamma^*(D) + 2$. It follows from Proposition 5 that D does not have a vertex v with $|N^+(v) \cap N^-(v)| = n - \gamma^*(D)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R^*(D)$-function. Then either (1) $|V_1| = \gamma^*(D) + 2$ and $|V_2| = 0$, (2) $|V_1| = \gamma^*(D)$ and $|V_2| = 1$, or (3) $|V_1| = \gamma^*(D) - 2$ and $|V_2| = 2$.

In case (1), we have $|V_0| = 0$. Then $V(D) = V_1$. By Theorem 1, we have $n = \gamma^*(D) + 2 \le \frac{2n}{3} + 2$ which leads to a contradiction because $n \ge 7$.

In case (2), let $V_2 = \{v\}$. Obviously $(v,u) \in A(D)$ and $(u,v) \in A(D)$ for each $u \in V_0$. Since for each $x \in V_1$, $(v,u) \notin A(D)$ or $(u,v) \notin A(D)$, we obtain $|N^+(v) \cap N^-(v)| = n - \gamma^*(D) - 1$.

In case (3), let $V_2 = \{v, w\}$. Since $(v,u) \in A(D)$ or $(w,u) \in A(D)$ and $(u,v) \in A(D)$ or $(u,w) \in A(D)$ for each $u \in V_0$ and since $x \notin (N^+[v] \cup N^+[w]) \cap (N^-[v] \cup N^-[w])$ for each $x \in V_0$, we deduce that $|(N^+[v] \cup N^+[w]) \cap (N^-[v] \cup N^-[w])| = n - |V_1| = n - (\gamma^*(D) - 2) = n - \gamma^*(D) + 2$.

Conversely, let D satisfy (i) and (ii). It follows from Proposition 5 and (i) that $\gamma_R^*(D) \ge \gamma^*(D) + 2$. If D has a vertex v with $|N^+(v) \cap N^-(v)| = n - \gamma^*(D) - 1$, then obviously $f = (N^+(v) \cap N^-(v), V(D) - (N^+[v] \cap N^-[v]), \{v\})$ is a TRDF on D of weight $\gamma^*(D) + 2$ implying that $\gamma_R^*(D) \le \gamma^*(D) + 2$. If D has two vertices v, w such that $|(N^+[v] \cup N^+[w]) \cap (N^-[v] \cup N^-[w])| = n - \gamma^*(D) + 2$, then $f = ((N^+[v] \cup N^+[w]) \cap (N^-[v] \cup N^-[w]), V(D) - (N^+[v] \cup N^+[w]) \cap (N^-[v] \cup N^-[w]), \{v, w\})$ is a TRDF on D of weight $\gamma^*(D) + 2$ and the result follows again. This completes the proof. □

3. Twin Roman domination in oriented graphs

An *orientation* of a graph G is a digraph D obtained from G by choosing an orientation ($x \to y$ or $y \to x$) for every edge $xy \in E(G)$. Clearly, two distinct orientations of a graph can have distinct twin domination numbers. Motivated by this observation Chartrand et al. [3] introduced the concept of the lower orientable twin domination number $\mathrm{dom}^*(G)$ and the upper orientable twin domination number $\mathrm{DOM}^*(G)$ of a graph G, as

$$\mathrm{dom}^*(G) = \min\{\gamma^*(D) \mid \text{ D is an orientation of } G\},$$

and

$$\mathrm{DOM}^*(G) = \max\{\gamma^*(D) \mid \text{ D is an orientation of } G\}.$$

This concepts have been studied in [2].

Here, we propose similar concepts the lower orientable twin Roman domination number $\mathrm{dom}_R^*(G)$ and the upper orientable twin Roman domination number $\mathrm{DOM}_R^*(G)$ as follows.

$$\mathrm{dom}_R^*(G) = \min\{\gamma_R^*(D) \mid \text{ D is an orientation of } G\},$$

and

$$\mathrm{DOM}^*_R(G) = \max\{\gamma^*_R(D) \mid \text{ D is an orientation of } G\}.$$

Clearly $\mathrm{dom}^*_R(G) \le \mathrm{DOM}^*_R(G) \le n$ for every graph G of order n.

Proposition 7. *Let G be a graph of order n with at most one cycle. Then* $\mathrm{dom}^*_R(G) = n$.

Proof. By (1.2), it is enough to prove $\gamma^*_R(\overrightarrow{G}) \ge n$. First let G be not a cycle. We proceed by induction on n. The result can be easily verified for all graphs with at most 3 vertices. Hence, suppose that $n \ge 4$ and the result is true for all graphs of order less than n. Let G be a graph of order n. By assumption G has an end vertex, say x. Let $\overrightarrow{G}$ be an orientation of G. Then obviously for any $\gamma^*_R(\overrightarrow{G})$-function $f = (V_0, V_1, V_2)$, $f(x) \ne 0$. If $f(x) = 1$ then $g = (V_0, V_1 - \{x\}, V_2)$ is a TRDF on $\overrightarrow{G} - \{x\}$ and it follows from the induction hypothesis that $\gamma^*_R(\overrightarrow{G}) = \omega(g) + 1 \ge \gamma^*_R(\overrightarrow{G} - \{x\}) + 1 \ge n$, as desired. Now let $f(x) = 2$. Then $f(y) = 0$, where y is the support vertex of x in G. This implies that the function $h = (V_0 - \{y\}; V_1 \cup \{y\}; V_2 - \{x\})$ is a TRDF on $\overrightarrow{G} - \{x\}$ with $\omega(h) = \gamma^*_R(D) - 1$. Now the result follows by the induction hypothesis as above.

Now let $G = C_n$ and let $\overrightarrow{G}$ be an orientation of G. Assume to the contrary that $\gamma^*_R(D) < n$. Suppose $f = (V_0^f, V_1^f, V_2^f)$ is a $\gamma^*_R(\overrightarrow{G})$-function. Then both of V_0^f and V_2^f are nonempty. Hence (a) each vertex in V_0^f has exactly 2 neighbors and they both are in V_2^f, and (b) each vertex in V_2^f has at most 1 neighbor not in V_0^f. From (a) and (b) it immediately follows that $|V_0^f| \le |V_2^f|$. Hence $\gamma^*_R(\overrightarrow{G}) = |V_1^f| + 2|V_2^f| = |V_0^f| + |V_1^f| + |V_2^f| = n$ and the proof is completed. □

The next results are immediate consequences of Proposition 7.

Corollary 3. *For $n \ge 1$, $\mathrm{dom}^*_R(K_{1,n}) = n$.*

Corollary 4. $\mathrm{dom}^*_R(C_n) = \mathrm{dom}^*_R(P_n) = \mathrm{Dom}^*_R(C_n) = \mathrm{Dom}^*_R(P_n) = n$.

Proposition 8. *For any graph G of order $n \ge 4$ with clique number $c \ge 4$,* $\mathrm{dom}^*_R(G) \le n - c + 4$.

Proof. Let $S = \{v_1, v_2, \ldots, v_c\}$ be a clique in G. Let $\overrightarrow{G}$ be an orientation of G such that the edges are oriented from v_1 to $v_2, v_3, \ldots, v_c$ and from $v_3, v_4, \ldots, v_c$ to v_2 and the other edges oriented arbitrary. Then $f = (\{v_3, v_4, \ldots, v_c\}, V(G) - S, \{v_1, v_2\})$ is a twin Roman dominating function of $\overrightarrow{G}$ which yields $\mathrm{dom}^*_R(G) \le n - c + 4$. □

An independent set is a set of vertices that no two of which are adjacent. A maximum independent set is an independent set of largest possible size. This size is called the independence number of G, and denoted by $\alpha(G)$.

Proposition 9. *For any graph G of order $n \geq 4$ with $\delta(G) \geq 2$,* $\mathrm{dom}_R^*(G) \leq 2(n - \alpha(G))$.

Proof. Let $S = \{v_1, v_2, \ldots, v_{\alpha(G)}\}$ be an independent set of G. Since S is independent and $\delta(G) \geq 2$, each v_i has two neighbors u_i, w_i in $V - S$. Let $\overrightarrow{G}$ be an orientation of G such that $(v_i, u_i), (w_i, v_i) \in A(\overrightarrow{G})$. Then the function $f = (S, \emptyset, V - S)$ is a twin Roman dominating function of $\overrightarrow{G}$ that implies that $\mathrm{dom}_R^*(G) \leq 2(n - \alpha(G))$. □

The next results are immediate consequences of Corollary 1 and Propositions 8 and 9.

Corollary 5. *For $n \geq 4$,* $\mathrm{dom}_R^*(K_n) = 4$.

Corollary 6. *For $n \geq 2$,* $\mathrm{dom}_R^*(K_{2,n}) = 4$.

Theorem 3. *([2]) For $r \geq s \geq 3$,*

$$\mathrm{dom}^*(K_{r,s}) = \begin{cases} 3 & \text{if } s = 3 \\ 4 & \text{if } s \geq 4. \end{cases}$$

Proposition 10. *For every two integers $r \geq s \geq 3$,*

$$\mathrm{dom}_R^*(K_{r,s}) = \begin{cases} 5 & \text{if } s = 3 \\ 6 & \text{if } s = 4 \\ 7 & \text{if } s = 5 \\ 8 & \text{if } s \geq 6. \end{cases}$$

Proof. Let $G = K_{r,s}$ and let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$ be the partite sets of G. Consider the following cases.

Case 1. $s = 3$.

It follows from Propositions 4, 5 and Theorem 3 that $\gamma_R^*(G) \geq \gamma^*(G) + 2 = 5$. Let $\overrightarrow{G}$ be an orientation of G such that $(x_1, y_i), (y_i, x_2) \in A(\overrightarrow{G})$ for each i. Clearly, $g = (Y, \{x_3\}, \{x_1, x_2\})$ is a TRDF of $\overrightarrow{G}$ that implies $\gamma_R^*(G) \leq 5$. Hence $\gamma_R^*(G) = 5$.

Case 2. $s = 4$.

Using an argument similar to that described in Case 1, we obtain $\gamma_R^*(G) = 6$.

Case 3. $s = 5$.

Suppose $\overrightarrow{G}$ is an orientation of G such that $(x_1, y_i), (y_i, x_2) \in A(\overrightarrow{G})$ for each i. Obviously, $g = (Y, \{x_3, x_4, x_5\}, \{x_1, x_2\})$ is a TRDF of $\overrightarrow{G}$ implying that $\gamma_R^*(G) \leq 7$. Let D be an arbitrary orientation of G. Since G has no cycle of length 2 and for any two vertices $u, v \in V(G)$, $|(N^+[v] \cup N^+[u]) \cap (N^-[v]| \cup N^-[u])| \leq n - 3 = |V(G)| - \gamma^*(G) + 1$, we deduce from Propositions 4, 5, 6 and Theorem 3 that $\gamma_R^*(G) \geq \gamma^*(G) + 3 = 7$. Thus $\gamma_R^*(G) = 7$.

Case 4. $s \geq 6$.

It follows from Theorem 3 and (1.1) that $\gamma_R^*(G) \leq 8$. Let D be an arbitrary orientation of G and $f = (V_0, V_1, V_2)$ a $\gamma_R^*(D)$-function. Since $\gamma_R^*(G) \leq 8$, we deduce that $V_0 \neq \varnothing$. If $V_0 \cap X \neq \varnothing$ and $V_0 \cap Y \neq \varnothing$, then we must have $|V_2 \cap Y| \geq 2$ and $|V_2 \cap X| \geq 2$ that implies $\gamma_R^*(G) \geq 8$ as desired. Now let, without loss of generality, $V_0 \cap X = \varnothing$. Then $V_0 \cap Y \neq \varnothing$ that implies $|V_2 \cap X| \geq 2$ and hence $\gamma_R^*(D) \geq 4 + |X| - 2 = s + 2 \geq 8$. Thus $\gamma_R^*(G) = 8$ and the proof is completed. □

Proposition 11. *Let $G = K_{m_1,m_2,\ldots,m_r}$ $(r \geq 3)$ be the complete r-partite graph with $1 \leq m_1 \leq m_2 \leq \ldots \leq m_r$. Then*

$$\mathrm{dom}_R^*(K_{m_1,m_2,\ldots,m_r}) = \begin{cases} 4 & \text{if} \quad m_1 = \cdots = m_r = 1, \\ 4 & \text{if} \quad m_1 = m_2 = 1 \text{ or } m_i = 2 \text{ for some } i, \\ 5 & \text{if} \quad m_1 = 3 \text{ or } m_1 = 1 \text{ and } m_2 = 3, \\ 6 & \text{if} \quad m_1 \geq 4. \end{cases}$$

Proof. Let $G = K_{m_1,m_2,\ldots,m_r}$ and let $X_1 = \{x_1, x_2, \ldots, x_{m_1}\}, X_2 = \{y_1, y_2, \ldots, y_{m_2}\}, X_3 = \{z_1, z_2, \ldots, z_{m_3}\}$, $X_4, X_5, \ldots, X_r$ be the partite sets of G.

If $m_1 = \cdots = m_r = 1$ then $G = K_n$ and by Corollary 5, we have $\gamma_R^*(G) = 4$. If $m_1 = m_2 = 1$, then let $\overrightarrow{G}$ be an orientation of G such that $(x_1, x), (x, y_1) \in A(\overrightarrow{G})$ for each $x \in V(G) - \{x_1, y_1\}$. Obviously, $g = (V(D) - \{x_1, y_1\}, \varnothing, \{x_1, y_1\})$ is a TRDF of $\overrightarrow{G}$ implying that $\gamma_R^*(G) = 4$ by Corollary 1. If $m_i = 2$ for some i, say $i = 2$, then assume $\overrightarrow{G}$ is an orientation of G such that $(y_1, x), (x, y_2) \in A(\overrightarrow{G})$ for each $x \in V(G) - \{y_1, y_2\}$. Clearly, $g = (V(D) - \{y_1, y_2\}, \varnothing, \{y_1, y_2\})$ is a TRDF of $\overrightarrow{G}$ that implies $\gamma_R^*(G) = 4$ again. If $m_1 = 3$ or $m_1 = 1$ and $m_2 = 3$ then as Case 1. in Proposition 10, we deduce that $\gamma_R^*(G) = 5$.

Finally, let $m_1 \geq 4$. It follows from Proposition 1 that $\gamma_R^*(G) \geq 6$. Let $\overrightarrow{G}$ be an orientation of G such that $(x_1, x), (x, y_1) \in A(\overrightarrow{G})$ for each $x \in V(G) - \{x_1, y_1\}$, $(z_1, x_i) \in A(\overrightarrow{G})$ for $2 \leq i \leq m_1$ and $(y_i, z_1) \in A(\overrightarrow{G})$ for $2 \leq i \leq m_2$. It is easy to see that $g = (V(D) - \{x_1, y_1, z_1\}, \varnothing, \{x_1, y_1, z_1\})$ is a TRDF of $\overrightarrow{G}$ which implies $\gamma_R^*(G) \leq 6$. Thus $\gamma_R^*(G) = 6$ and the proof is completed. □

Theorem 4. *For $n \geq 9$, $\mathrm{dom}_R^*(W_{n+1}) = \lceil \frac{2n}{3} \rceil + 2$.*

Proof. Let $W_{n+1} = x + C_n$ and $C_n = (v_1, v_2, \ldots, v_n)$. Let $\overrightarrow{W_{n+1}}$ be an orientation of W_{n+1} such that $(v_i, x) \in A(\overrightarrow{W_{n+1}})$ for each i and $(v_i, v_{i-1}), (v_i, v_{i+1}) \in A(\overrightarrow{W_{n+1}})$ for each $i \equiv 1 \pmod 3$. It is easy to see that the function f that assigns 2 to x and v_i for $i \equiv 1 \pmod 3$, $\varnothing$ to v_{i-1} and v_{i+1} for $i \equiv 1 \pmod 3$ and 1 to the other vertices, is an TRDF of $\overrightarrow{W_{n+1}}$ that yields $\mathrm{dom}_R^*(W_{n+1}) \leq \gamma_R^*(\overrightarrow{W_{n+1}}) \leq \lceil \frac{2n}{3} \rceil + 2$.

Now let D be any orientation of W_{n+1} and let f be an $\gamma_R^*(D)$-function. If $f(v) \leq 1$, then f is a TRDF of C_n and hence $\gamma_R^*(D) = \omega(f) \geq n \geq \lceil \frac{2n}{3} \rceil + 2$ by Corollary 4. Assume $f(v) = 2$. Then the function f, restricted to C_n is an RDF of

C_n and we deduce from Proposition 7 in [4] that $\gamma_R^*(D) = \omega(f) \geq \lceil \frac{2n}{3} \rceil + 2$. Thus $\mathrm{dom}_R^*(W_{n+1}) \geq \lceil \frac{2n}{3} \rceil + 2$ and the proof is completed. □

Theorem 5 ([2]). *For $n \geq 3$, $\mathrm{DOM}^*(W_{n+1}) \geq n-1$.*

Theorem 6. *For $n \geq 4$, $\mathrm{DOM}_R^*(W_{n+1}) = n+1$.*

Proof. Let D be an orientation of W_{n+1} for which $\mathrm{DOM}^*(W_{n+1}) = \gamma^*(W_{n+1}) \geq n-1$. Assume that $f = (V_0, V_1, V_2)$ is a $\gamma_R^*(D)$-function. If $V_0 = \emptyset$, then $V_2 = \emptyset$ and we have $\gamma_R^*(D) = |V_1| = n+1$. Let $V_0 \neq \emptyset$. To in-dominate and out-dominate of each vertex $u \in V_0$, we must have $|V_2| \geq 2$. Then $\gamma_R^*(D) = |V_1| + 2|V_2| \geq 2 + |V_1| + |V_2| \geq 2 + \gamma^*(D) \geq n+1$. It follows that $\mathrm{DOM}_R^*(W_{n+1}) = n+1$. □

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Authors' addresses

H. Abdollahzadeh Ahangar
Department of Basic Science, Babol University of Technology, Babol, I.R. Iran
E-mail address: ha.ahangar@nit.ac.ir

J. Amjadi
Azarbaijan Shahid Madani University, Department of Mathematics, Tabriz, I.R. Iran
E-mail address: j-amjadi@azaruniv.edu

S.M. Sheikholeslami
Azarbaijan Shahid Madani University, Department of Mathematics, Tabriz, I.R. Iran
E-mail address: s.m.sheikholeslami@azaruniv.edu

V. Samodivkin
Department of Mathematics, University of Architecture Civil Engineering and Geodesy, Hristo Smirnenski 1 Blv., 1046 Sofia, Bulgaria
E-mail address: vlsam_fte@uacg.bg

L. Volkmann
RWTH Aachen University, Lehrstuhl II für Mathematik, 52056, Aachen, Germany
E-mail address: volkm@math2.rwth-aachen.de