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EXISTENCE OF POSITIVE SOLUTIONS TO nTH ORDER p-LAPLACIAN BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This work is devoted to the existence of positive solutions for an nth order p-Laplacian boundary value problem with integral boundary conditions. The proof of the main result is based on six functionals fixed point theorem. As an application, we give an example to illustrate the obtained result.

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1. INTRODUCTION

The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. For more information about the general theory of integral equations and theirs relation with boundary value problems, we refer to the books of Corduneanu [10] and Agarwal and O'Regan [1]. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attentions. To indentify a few, we refer the reader to [2, 3, 9, 12, 14, 15] and references therein. On the other hand, there are fewer results in the literature for higher-order differential equations with integral boundary conditions, see [4–6, 13]. In particular, we would like to mention some results of Boucherif [9] and Ahmad and Ntouyas [5].

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In [9], by means of the Krasnoselskii's fixed point theorem, Boucherif investigated the existence of positive solutions of following nonlocal second-order boundary value problems with integral boundary conditions

$$\begin{cases} y'' = f(t, y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) = \int_0^1 g_0(s) y(s) ds, \\ y(1) - by'(1) = \int_0^1 g_1(s) y(s) ds. \end{cases}$$

In [5], Ahmad and Ntouyas developed some existence results for the following nth order boundary value problem with four-point nonlocal integral boundary conditions by using Krasnoselskii's fixed point theorem and Leray-Schauder degree theory

$$\begin{cases} x^{(n)}(t) = f(t, x(t)), \ 0 < t < 1, \\ x(0) = \alpha \int_0^\xi x(s) ds, \ \ x'(0) = 0, \ x''(0) = 0, \ldots, x^{(n-2)}(0) = 0, \\ x(1) = \beta \int_\eta^1 x(s) ds, \ \ 0 < \xi < \eta < 1. \end{cases}$$

Motivated by the results above, in this study, we consider the following nth order p-Laplacian boundary value problem (BVP) with integral boundary conditions,

$$\begin{cases} \left(\phi_p\left(u^{(n-1)}(t)\right)\right)' + q(t) f(t, u(t), u'(t), u''(t), ..., u^{(n-3)}(t)) = 0, \ t \in [0,1], \\ au^{(n-3)}(0) - bu^{(n-2)}(0) = \int_0^1 g_1(s) u^{(n-3)}(s) ds, \\ cu^{(n-3)}(1) + du^{(n-2)}(1) = \int_0^1 g_2(s) u^{(n-3)}(s) ds, \\ u^{(n-1)}(0) = 0, \\ u^{(j)}(0) = 0, \quad 0 \le j \le n-4, \end{cases} \tag{1.1}$$

where n is an integer greater than 3, a, b, c, d are nonnegative real numbers, $\phi_p(s)$ is the p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

The following conditions are needed throughout the paper:

$(C1)$ $ac + ad + bc > 0$,

$(C2)$ $f \in C([0,1] \times \mathbb{R}_+^{n-2}, \mathbb{R}_+)$, where $\mathbb{R}_+ = [0, +\infty)$,
$\mathbb{R}_+^{n-2} = \overbrace{[0,+\infty) \times \cdots \times [0,+\infty)}^{n-2}$,

$(C3)$ $q, \ g_1, \ g_2 \in C([0,1], \mathbb{R}_+)$.

In this paper, utilizing the six functionals fixed point theorem [7], we get the existence of at least three positive solutions for the BVP (1.1). The method used in this study is new to the literature and so is the existence results to the nth order p-Laplacian boundary value problems with integral boundary conditions.

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. We give and prove our main result in Section 3. In Section 4, we give an example to demonstrate our main result. Lastly, some concluding remarks are given in Section 5.

2. Preliminaries

In this section, we present auxiliary lemmas which will be used later.

Definition 1. Let $\mathbb{B}$ be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset \mathbb{B}$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda x \in \mathcal{P}$;
(ii) $x \in \mathcal{P}$, $-x \in \mathcal{P}$ implies $x = 0$.

Every cone $\mathcal{P} \subset \mathbb{B}$ induces an ordering in $\mathbb{B}$ given by

$$x \leq y \text{ if and only if } y - x \in \mathcal{P}.$$

Definition 2. A map α is said to be a *nonnegative continuous concave functional* on a cone $\mathcal{P}$ of a real Banach space $\mathbb{B}$ if

$$\alpha : \mathcal{P} \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x,\ y \in \mathcal{P}$ and $t \in [0,1]$. Similarly, we say the map β is a *nonnegative continuous convex functional* on a cone $\mathcal{P}$ of a real Banach space $\mathbb{B}$ if

$$\beta : \mathcal{P} \to [0, \infty)$$

is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x,\ y \in \mathcal{P}$ and $t \in [0,1]$.

Using the transformation

$$u^{(n-3)}(t) = y(t), \tag{2.1}$$

and the boundary conditions $u^{(j)}(0) = 0,\ j = 0,1,2,...,n-4$, one can obtain that

$$u^{(j)}(t) = \int_0^t \frac{(t-r)^{n-4-j}}{(n-4-j)!} y(r)dr,\ j = 0,1,2,...,n-4.$$

Thus, under the transformation (2.1), we obtain the following BVP,

$$\begin{cases} \left(\phi_p\left(y''(t)\right)\right)' + q(t) f\left(t, \int_0^t \frac{(t-r)^{n-4}}{(n-4)!} y(r)dr, \dots, y(t)\right) = 0, \ t \in [0,1], \\ ay(0) - by'(0) = \int_0^1 g_1(s) y(s) ds, \\ cy(1) + dy'(1) = \int_0^1 g_2(s) y(s) ds, \\ y''(0) = 0. \end{cases} \tag{2.2}$$

Note that, the nth order BVP (1.1) has a solution if and only if the second order BVP (2.2) has a solution.

Set

$$\triangle := \begin{vmatrix} -\int_0^1 g_1(s)(b+as)\,ds & \rho - \int_0^1 g_1(s)(d+c(1-s))\,ds \\ \rho - \int_0^1 g_2(s)(b+as)\,ds & -\int_0^1 g_2(s)(d+c(1-s))\,ds \end{vmatrix}, \tag{2.3}$$

and

$$\rho := ad + ac + bc. \tag{2.4}$$

Lemma 1. *Let* $(C1)$*-*$(C3)$ *hold. Assume that* $\triangle \neq 0$*. Then* $y(t)$ *is a solution of the BVP* (2.2) *if and only if* $y(t)$ *is a solution of the following integral equation*

$$y(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s q(\tau) f\left(\tau, \int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!} y(r)dr, \dots, y(\tau)\right) d\tau\right) ds + A(f,y)(b+at) + B(f,y)(d+c(1-t)), \tag{2.5}$$

where

$$G(t,s) = \frac{1}{\rho} \begin{cases} (b+as)(d+c(1-t)), & s \le t, \\ (b+at)(d+c(1-s)), & t \le s, \end{cases} \tag{2.6}$$

$$A(f,y) = \frac{1}{\triangle} \begin{vmatrix} \int_0^1 g_1(s) H(f,s) ds & \rho - \int_0^1 g_1(s)(d+c(1-s))ds \\ \int_0^1 g_2(s) H(f,s) ds & -\int_0^1 g_2(s)(d+c(1-s))ds \end{vmatrix}, \tag{2.7}$$

$$B(f,y) = \frac{1}{\triangle} \begin{vmatrix} -\int_0^1 g_1(s)(b+as)ds & \int_0^1 g_1(s) H(f,s) ds \\ \rho - \int_0^1 g_2(s)(b+as)ds & \int_0^1 g_2(s) H(f,s) ds \end{vmatrix}, \tag{2.8}$$

and

$$H(f,s)=\int_0^1 G(s,r)\phi_q\left(\int_0^z q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)dz. \quad (2.9)$$

Proof. Let y satisfies the integral equation (2.5), then we will show that y is a solution of the BVP (2.2). Since y satisfies equation (2.5), then we have

$$y(t)=\int_0^1 G(t,s)\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)ds \\ +A(f,y)(b+at)+B(f,y)(d+c(1-t)),$$

i.e.,

$$y(t)=\int_0^t \frac{1}{\rho}(b+as)(d+c(1-t))\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)ds \\ +\int_t^1 \frac{1}{\rho}(b+at)(d+c(1-s))\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)ds \\ +A(f,y)(b+at)+B(f,y)(d+c(1-t)),$$

$$y'(t)=-\int_0^t \frac{c}{\rho}(b+as)\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)ds \\ +\int_t^1 \frac{a}{\rho}(d+c(1-s))\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)ds \\ +A(f,y)a-B(f,y)c,$$

$$y''(t) \\ =\frac{1}{\rho}(-c(b+at)-a(d+c(1-t)))\phi_q\left(\int_0^t q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right) \\ =-\phi_q\left(\int_0^t q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right),$$

So that

$$(\phi_p(y''(t)))'+q(t)f\left(t,\int_0^t \frac{(t-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(t)\right)=0,$$

and $y''(0)=0$. Since

$$y(0)= \\ \int_0^1 \frac{b}{\rho}(d+c(1-s))\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\dots,y(\tau)\right)d\tau\right)ds \\ +bA(f,y)+(d+c)B(f,y),$$

$$y'(0) =$$
$$\int_0^1 \frac{a}{\rho}(d+c(1-s))\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\ldots,y(\tau)\right)d\tau\right)ds$$
$$+aA(f,y)-cB(f,y),$$

we have that

$$\begin{aligned} ay(0)-by'(0) &= \rho B(f,y) \quad (2.10)\\ &= \int_0^1 g_1(s)\Big[\int_0^1 G(s,z)\\ &\phi_q\left(\int_0^z q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\ldots,y(\tau)\right)d\tau\right)dz\\ &+A(f,y)(b+as)+B(f,y)(d+c(1-s))\Big]ds. \end{aligned}$$

Since

$$y(1) =$$
$$\int_0^1 \frac{d}{\rho}(b+as)\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\ldots,y(\tau)\right)d\tau\right)ds$$
$$+(b+a)A(f,y)+dB(f,y),$$

$$y'(1) =$$
$$-\int_0^1 \frac{c}{\rho}(b+as)\phi_q\left(\int_0^s q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\ldots,y(\tau)\right)d\tau\right)ds$$
$$+aA(f,y)-cB(f,y),$$

we have that

$$\begin{aligned} cy(1)+dy'(1) &= \rho A(f,y)\\ &= \int_0^1 g_2(s)\Big[\int_0^1 G(s,z)\\ &\phi_q\left(\int_0^z q(\tau)f\left(\tau,\int_0^\tau \frac{(\tau-r)^{n-4}}{(n-4)!}y(r)dr,\ldots,y(\tau)\right)d\tau\right)dz \quad (2.11)\\ &+A(f,y)(b+as)+B(f,y)(d+c(1-s))\Big]ds. \quad (2.12) \end{aligned}$$

From (2.5), (2) and (2.12), we get that

$$\begin{cases} \left[-\int_0^1 g_1(s)\,(b+as)\,ds\right]A(f,y)+\left[\rho-\int_0^1 g_1(s)\,(d+c(1-s))\,ds)\right]B(f,y) \\ =\int_0^1 g_1(s)H(f,s)ds \\ \left[\rho-\int_0^1 g_2(s)\,(b+as)\,ds\right]A(f,y)+\left[-\int_0^1 g_2(s)\,(d+c(1-s))\,ds\right]B(f,y) \\ =\int_0^1 g_2(s)H(f,s)ds, \end{cases}$$

which implies that $A(f,y)$ and $B(f,y)$ satisfy (2.7) and (2.8), respectively. Then $y(t)$ satisfies all the conditions of (2.2), hence y(t) is a solution of (2.2).

Conversely, if $y(t)$ is a solution of the BVP (2.2), by integrating one can easily show that $y(t)$ is in the form (2.5). □

Lemma 2. *Let* $(C1)$-$(C3)$ *hold. Assume*

$(C4)$ $\triangle<0,\ \rho-\int_0^1 g_2(s)(b+as)ds>0,\ a-\int_0^1 g_1(s)ds>0.$

Then the solution $y(t)$ *of the problem* (2.2) *satisfies*

$$y(t)\geq 0\ for\ t\in[0,1].$$

Proof. It is an immediate consequence of the facts that $G\geq 0$ on $[0,1]\times[0,1]$ and $A(f,y)\geq 0,\ B(f,y)\geq 0.$ □

Lemma 3. *Let* $(C1)$-$(C4)$ *hold. Assume that*

$(C5)$ $c-\int_0^1 g_2(s)ds<0.$

Then the solution $y(t)$ *of the problem* (2.2) *satisfies* $y'(t)\geq 0$ *for* $t\in[0,1]$.

Proof. Assume that the inequality $y'(t)<0$ holds. Since $y'(t)$ is nonincreasing on $[0,1]$, one can verify that

$$y'(1)\leq y'(t),\ t\in[0,1].$$

From the boundary conditions of the problem (2.2), we have

$$-\frac{c}{d}y(1)+\frac{1}{d}\int_0^1 g_2(s)y(s)ds\leq y'(t)<0.$$

The last inequality yields

$$-cy(1)+\int_0^1 g_2(s)y(s)ds<0.$$

Therefore, we obtain that

$$\left(c-\int_0^1 g_2(s)ds\right)y(1)>0.$$

According to Lemma 2, we have that $y(1) \geq 0$. So, $c - \int_0^1 g_2(s)ds > 0$. However, this contradicts to condition $(C5)$. Consequently, $y'(t) \geq 0$ for $t \in [0,1]$. □

Let the Banach space $\mathbb{B} = \mathcal{C}([0,1])$ be equipped with the norm $\|y\| = \max_{t\in[0,1]} y(t)$, and we define a cone $\mathcal{P}$ in $\mathbb{B}$ by

$$\mathcal{P} = \{y \in \mathbb{B} : y(t) \text{ is nonnegative, nondecreasing and concave on } [0,1]\}.$$

Lemma 4. *Let $y \in \mathcal{P}$ and $k > 2$ is a constant. Then,*

$$\min_{t\in[1/k,1]} y(t) \geq \frac{1}{k}\|y\|.$$

Proof. Since $y \in \mathcal{P}$ we know that $y(t)$ is nondecreasing on $[0,1]$. So, $\min_{t\in[1/k,1]} y(t) = y(1/k)$ and $\|y\| = \max_{t\in[0,1]} y(t) = y(1)$. Since $y'(t)$ is nonincreasing on $[0,1]$, we have

$$\frac{y(1)-y(0)}{1} \leq \frac{y(1/k)-y(0)}{1/k},$$

i.e.,

$$y(1/k) \geq \frac{1}{k}y(1) + \left(1-\frac{1}{k}\right)y(0).$$

So, $y(1/k) \geq \frac{1}{k}y(1)$. The proof is completed. □

Note that by Lemmas 1 and 2, we know that if $(C1)$-$(C4)$ are satisfied, then the solutions of the BVPs (1.1) and (2.2) are both positive. Therefore, we only need to deal with the existence of the positive solutions of (2.2).

Define an operator $T : \mathcal{P} \to \mathbb{B}$ by

$$Ty(t) = \int_0^1 G(t,s)\phi_q\left(\int_0^s q(\tau)F(\tau,y(\tau))d\tau\right)ds + A(F,y)(b+at) + B(F,y)(d+c(1-t)), \tag{2.13}$$

where

$$F(t,y(t)) = f\left(t, \int_0^t \frac{(t-r)^{n-4}}{(n-4)!}y(r)dr, \dots, \int_0^t y(r)dr, y(t)\right),$$

and G, $A(F,y)$, $B(F,y)$ are respectively defined as in (2.6), (2.7), (2.8).

By Lemmas 1, 2 and the definition of T, it is well known that the BVP (2.2) has positive solution if and only the operator T has a fixed point.

Lemma 5. *Let $(C1)$-$(C5)$ hold. Then $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. For all $y \in \mathcal{P}$, Lemmas 1, 2, 3 and the definition of T, we have

$$(Ty)(t) \geq 0, \ (Ty)'(t) \geq 0, \ \text{and } (Ty)'(t) \text{ is concave on } [0,1].$$

Then $Ty \in \mathcal{P}$. So T is an operator from $\mathcal{P}$ to $\mathcal{P}$. By Arzela-Ascoli theorem, one can easily prove that operator T is completely continuous. □

3. Main results

We are now ready to apply the six functionals fixed point theorem [7] to the operator T in order to get sufficient conditions for the existence of at least three positive solutions to the problem (1.1).

Let α be a nonnegative continuous concave functional on $\mathcal{P}$, and let β be a nonnegative continuous convex functional on $\mathcal{P}$; then for positive numbers r and R we define the sets:

$$Q(\beta, R) = \{u \in \mathcal{P} : \ \beta(u) \leq R\}, \tag{3.1}$$

$$Q(\alpha, \beta, r, R) = \{u \in \mathcal{P} : \ r \leq \alpha(u) \text{ and } \beta(u) \leq R\}. \tag{3.2}$$

Lemma 6. *[7] Suppose $\mathcal{P}$ is a cone in a real Banach space $\mathbb{B}$, α, ψ and ζ are nonnegative continuous concave functionals on $\mathcal{P}$, β, θ and η are nonnegative continuous convex functionals on $\mathcal{P}$, and there exist nonnegative numbers h, h', r, r', R and R' such that*

$$T : Q(\beta, R) \to \mathcal{P}$$

is a completely continuous operator and

- (a) *$Q(\beta, R)$ is a bounded set,*
- (b) *$Q(\eta, h)$ and $Q(\alpha, \beta, r, R)$ are disjoint subsets of $Q(\beta, R)$,*
- (c) *$\{u \in \mathcal{P} : \ \theta(u) < r', \ r < \alpha(u), \ R' < \psi(u), \text{ and } \beta(u) < R\} \neq \varnothing$,*
- (d) *$\{u \in \mathcal{P} : \ h' < \zeta(u) \text{ and } \eta(u) < h\} \neq \varnothing$, and*
- (e) *$\{u \in \mathcal{P} : \ h < \eta(u) \text{ and } \alpha(u) < r\} \neq \varnothing$.*

Let the following properties be satisfied:

- (i) *$\alpha(Tu) > r$, for all $u \in \mathcal{P}$ with $\alpha(u) = r$, $\beta(u) \leq R$, and $r' < \theta(Tu)$,*
- (ii) *$\alpha(Tu) > r$, for all $u \in \mathcal{P}$ with $\alpha(u) = r$, $\beta(u) \leq R$, and $\theta(u) \leq r'$,*
- (iii) *$\beta(Tu) < R$, for all $u \in \mathcal{P}$ with $r \leq \alpha(u)$, $\beta(u) = R$, and $\psi(Tu) < R'$,*
- (iv) *$\beta(Tu) < R$, for all $u \in \mathcal{P}$ with $r \leq \alpha(u)$, $\beta(u) = R$, and $R' \leq \psi(u)$,*
- (v) *$\eta(Tu) < h$, for all $u \in \mathcal{P}$ with $\eta(u) = h$ and $\zeta(Tu) < h'$, and*
- (v) *$\eta(Tu) < h$, for all $u \in \mathcal{P}$ with $\eta(u) = h$ and $h' \leq \zeta(u)$,*

then T has at least three fixed points u_1, u_2 and u_3 in $Q(\beta, R)$ such that

$$\eta(u_1) \leq h, \ r \leq \alpha(u_2) \text{ with } \beta(u_2) \leq R, \text{ and } h < \eta(u_3) \text{ with } \alpha(u_3) < r.$$

For the convenience, we take the notations

$$A = \frac{1}{\triangle}\begin{vmatrix} \int_0^1 g_1(s)\left(\int_0^1 G(s,r)\phi_q\left(\int_0^r q(\tau)d\tau\right)dr\right)ds & \rho - \int_0^1 g_1(s)(d+c(1-s))ds \\ \int_0^1 g_2(s)\left(\int_0^1 G(s,r)\phi_q\left(\int_0^r q(\tau)d\tau\right)dr\right)ds & -\int_0^1 g_2(s)(d+c(1-s))ds \end{vmatrix},$$

$$B = \frac{1}{\triangle}\begin{vmatrix} -\int_0^1 g_1(s)(b+as)ds & \int_0^1 g_1(s)\left(\int_0^1 G(s,r)\phi_q\left(\int_0^r q(\tau)d\tau\right)dr\right)ds \\ \rho - \int_0^1 g_2(s)(b+as)ds & \int_0^1 g_2(s)\left(\int_0^1 G(s,r)\phi_q\left(\int_0^r q(\tau)d\tau\right)dr\right)ds \end{vmatrix},$$

$$\Lambda = \int_{1/k}^1 G(1/k,s)\phi_q\left(\int_{1/k}^s q(\tau)d\tau\right)ds,$$

$$\Omega = \int_0^1 G(1,s)\phi_q\left(\int_0^s q(\tau)d\tau\right)ds + (b+a)A + dB,$$

$$[0,2R]^{n-2} = \overbrace{[0,2R]\times\cdots\times[0,2R]}^{n-2}.$$

Define the concave functionals $\alpha,\ \psi$ and ζ by

$$\alpha(y) = \min_{t\in[1/k,1]} y(t) = y(1/k),$$

$$\psi(y) = \zeta(y) = \int_{1/k}^1 ky(s)ds,$$

and the convex functionals $\theta,\ \beta$ and η by

$$\theta(y) = \max_{t\in[0,1]} y(t) = y(1),$$

$$\beta(y) = \eta(y) = \int_0^1 y(s)ds.$$

Theorem 1. *Assume* $(C1)$*-*$(C5)$ *hold. If there exist positive real numbers* r *and* R *with* $r < \min\left\{\frac{R}{k^2+1}, \frac{\Lambda}{\Omega}R\right\} < kr$, *and suppose that* f *satisfies the following conditions*

$(C6)$ $f(x_1,x_2,x_3,\ldots,x_{n-1}) < \phi_p\left(\frac{R}{\Omega}\right)$ *for all* $(x_1,x_2,x_3,\ldots,x_{n-1}) \in [0,1]\times[0,2R]^{n-2}$,

$(C7)$ $f(x_1,x_2,x_3,\ldots,x_{n-1}) < \phi_p\left(\frac{r}{k\Omega}\right)$ *for all* $(x_1,x_2,x_3,\ldots,x_{n-1}) \in [0,1]\times\left[0,\frac{2r}{k}\right]^{n-2}$,

$(C8)$ $f(x_1,x_2,x_3,\dots,x_{n-1}) > \phi_p\left(\frac{r}{\Lambda}\right)$ *for all*

$$(x_1,x_2,x_3,\dots,x_{n-1}) \in \left[\frac{1}{k},1\right] \times [0,kr]^{n-3} \times [r,kr].$$

Then the nth order p-Laplacian BVP (1.1) has at least three positive solutions.

Proof. Let $R' = \frac{R}{k^2+1}$, $r' = kr$, $l = \frac{r}{k}$ and $l' = \frac{l}{k^2+1}$. By Lemma 5, we have that

$$T : Q(\beta,R) \to \mathcal{P}$$

is completely continuous. Applying a standard calculus argument, we have that the set $Q(\beta,R)$ is bounded, since if $y \in Q(\beta,R)$, then y' is nonincreasing, and hence

$$\frac{y(1)-y(0)}{2} \le \int_0^1 y(s)ds \le R.$$

Also, it can easily be shown that

$$\frac{R'+kr}{2} \in \left\{y \in \mathcal{P} : \theta(y) < r',\ r < \alpha(y),\ R' < \psi(y), \text{ and } \beta(y) < R\right\},$$

$$\frac{l}{k} \in \left\{y \in \mathcal{P} : l' < \zeta(y) \text{ and } \eta(y) < l\right\}, \text{ and}$$

$$\frac{r+l}{2} \in \{y \in \mathcal{P} : l < \eta(y) \text{ and } \alpha(y) < r\},$$

and hence the sets are nonempty. Moreover, if $y \in Q(\eta,l)$, then we have

$$\frac{y(1)-y(0)}{2} \le \int_0^1 y(s)ds \le l,$$

and since $y(t)$ is concave on $[0,1]$, one can easily obtain that $y(t) \ge ty(1)$, $t \in [0,1]$. Integrating the last inequality from 0 to 1, we get $\int_0^1 y(s)ds \ge \int_0^1 sy(1)ds = \frac{1}{2}y(1)$. Then we get

$$y(1) \le 2\int_0^1 y(s)ds, \tag{3.3}$$

and hence

$$\alpha(y) = y(1/k) \le y(1) \le 2l < 2\frac{r}{k} < r.$$

Therefore, $y \notin Q(\alpha,\beta,r,R)$. Thus, the set conditions (a), (b), (c), (d) and (e) of Lemma 6 are fulfilled. Now we verify the functional conditions.

Claim 1: $\alpha(Ty) > r$, for all $y \in Q(\alpha,\beta,r,R)$ with $\alpha(y) = r$ and $r' < \theta(Ty)$.

Let $y \in Q(\alpha,\beta,r,R)$ with $\alpha(y) = r$ and $r' < \theta(Ty)$. Then by Lemma 4, we have

$$\alpha(Ty) = \min_{t\in[1/k,1]} Ty(t) = Ty(1/k) \ge \frac{1}{k}Ty(1) = \frac{1}{k}\theta(Ty) > \frac{r'}{k} = r.$$

Claim 2: $\alpha(Ty) > r$, for all $y \in \{y \in Q(\alpha,\beta,r,R) : \theta(y) \le r'\}$ with $\alpha(y) = r$.

Let $y \in \{y \in Q(\alpha,\beta,r,R) : \theta(y) \le r'\}$, with $\alpha(y) = r$. Since $\theta(y) = \max_{t\in[0,1]} y(t) = \|y\| \le r' = kr$ and $\alpha(y) = \min_{t\in[1/k,1]} y(t) = y(1/k) = r$, we have $r \le y(t) \le kr$, for $t \in [1/k,1]$. It is clear that

$$\int_0^t \frac{(t-r)^{n-4}}{(n-4)!} y(r)dr \le \int_0^t \frac{(t-r)^{n-5}}{(n-5)!} y(r)dr \le \ldots \le \int_0^t y(r)dr \le y(t) \le \|y\|, t \in [0,1]. \quad (3.4)$$

So, from (3.4) we get

$$\left(t, \int_0^t \frac{(t-r)^{n-4}}{(n-4)!} y(r)dr, \int_0^t \frac{(t-r)^{n-5}}{(n-5)!} y(r)dr, \ldots, y(t)\right) \in \left[\frac{1}{k},1\right] \times [0,kr]^{n-3} \times [r,kr].$$

which implies $(C8)$ holds. Then one has

$$\begin{aligned}\alpha(Ty) = Ty(1/k) &\ge \int_0^1 G(1/k,s)\phi_q\left(\int_0^s q(\tau)F(\tau,y(\tau))d\tau\right)ds \\ &\ge \int_{1/k}^1 G(1/k,s)\phi_q\left(\int_{1/k}^s q(\tau)F(\tau,y(\tau))d\tau\right)ds \\ &> \frac{r}{\Lambda}\int_{1/k}^1 G(1/k,s)\phi_q\left(\int_{1/k}^s q(\tau)d\tau\right)ds \\ &= r.\end{aligned}$$

Claim 3: $\beta(Ty) < R$, for all $y \in Q(\alpha,\beta,r,R)$ with $\beta(y) = R$ and $\psi(Ty) < R'$. Since

$$R' > \psi(Ty) = \int_{1/k}^1 kTy(s)ds$$

and

$$\int_0^{1/k} Ty(s)ds \le \frac{1}{k-1}\int_{1/k}^1 Ty(s)ds,$$

we have

$$\frac{R'}{k} > \int_{1/k}^1 Ty(s)ds.$$

Therefore,

$$\beta(Ty) = \int_0^{1/k} Ty(s)ds + \int_{1/k}^1 Ty(s)ds < \frac{R'}{k-1} = \frac{R}{(k^2+1)(k-1)} < R.$$

Note that it can be also verified that $\eta(Ty) < l$ for all $y \in Q(\eta, l)$, with $\eta(y) = l$ and $\zeta(Ty) < l'$ respectively replacing R and R' by l and l' in Claim 3.

Claim 4: $\beta(Ty) < R$, for all $y \in \{y \in Q(\alpha, \beta, r, R) : R' \le \psi(y)\}$ with $\beta(y) = R$.

Let $y \in \{y \in Q(\alpha, \beta, r, R) : R' \le \psi(y)\}$ with $\beta(y) = R$. Thus, $\psi(y) \ge R' = \dfrac{R}{k^2+1}$, and by (3.3) and $\beta(y) = R$ we have

$$0 \le y(t) \le 2R,\ t \in [0,1].$$

Hence by (3.4), we get

$$\left(t, \int_0^t \frac{(t-r)^{n-4}}{(n-4)!} y(r)dr, \ldots, y(t)\right) \in [0,1] \times [0,2R]^{n-2}.$$

Then, by assumption $(C6)$, we obtain that

$$\begin{aligned}
\beta(Ty) &= \int_0^1 Ty(s)ds \\
&\le Ty(1) = \int_0^1 G(1,s)\phi_q\left(\int_0^s q(\tau)F(\tau, y(\tau))d\tau\right)ds \\
&\quad + A(F,y)(b+a) + B(F,y)d \\
&< \frac{R}{\Omega}\left[\int_0^1 G(1,s)\phi_q\left(\int_0^s q(\tau)d\tau\right)ds + A(b+a) + Bd\right] \\
&= R.
\end{aligned}$$

Note that Claim 4 can be used to verify that $\eta(Ty) < l$, for all $y \in \{y \in Q(\eta, l) : l' \le \zeta(y)\}$, with $\eta(y) = l$ respectively replacing R and R' by l and l'. Thus, all conditions of Lemma 6 are satisfied.

Benefiting from Claims 1-4 together with Lemma 6, we get that the operator T has at least three fixed points which are positive solutions $y_1,\ y_2$ and y_3 belonging to $Q(\beta, R)$ of (2.2) such that

$$\eta(y_1) \le l,\ r \le \alpha(y_2) \text{ with } \beta(y_2) \le R, \text{ and } l < \eta(y_3) \text{ with } \alpha(y_3) < r,$$

i.e.,

$$\int_0^1 y_1(s)ds \le l,\ r \le \min_{t\in[1/k,1]} y_2(t) \text{ with } \int_0^1 y_2(s)ds \le R,$$
$$l < \int_0^1 y_3(s)ds \text{ with } \min_{t\in[1/k,1]} y_3(t) < r.$$

Then the nth order p-Laplacian BVP (1.1) has at least three positive solutions

$$u_i(t) = \int_0^t \frac{(t-r)^{n-4}}{(n-4)!} y_i(r)dr\ (i = 1,2,3),$$

such that

$$u_1(t) \leq \frac{l}{(n-4)!},\ u_2(t) \leq \frac{R}{(n-4)!} \text{ for } t \in [0,1],$$

$$u_2(t) \geq \frac{r(t-1/k)^{(n-3)}}{(n-3)!},\ t \geq 1/k. \quad (3.5)$$

The proof is completed. □

Remark 1. In the main result, we used the six functionals fixed point theorem [7], which is a generalization of Leggett-Williams [11] and the five functionals fixed point theorems [8]. The conditions of the six functionals fixed point theorem are more strict than the conditions of the others, and at the same time the result of the former exhibits better estimations for the fixed points. For this reason, Theorem 1 provides the estimations (3.5), which can not be obtained by the applications of Leggett-Williams and the five functionals fixed point theorems. This is the main difference of our result compared to the aforementioned theorems.

4. AN EXAMPLE

Example 1. Let us consider the following BVP

$$\begin{cases} \left(\phi_{\frac{3}{2}}(u^{(4)}(t))\right)' + f(t,u(t),u'(t),u''(t)) = 0,\ t \in [0,1], \\ 3u''(0) - u^{(3)}(0) = \displaystyle\int_0^1 u''(s)ds, \\ u''(1) + u^{(3)}(1) = \displaystyle\int_0^1 2u''(s)ds, \\ u(0) = u'(0) = u''(0) = u^{(4)}(0) = 0, \end{cases} \quad (4.1)$$

where

$$f(x_1,x_2,x_3,x_4) = \begin{cases} \dfrac{t^2}{100} + 0.4, & (x_1,x_2,x_3,x_4) \in [0,1] \times \left[0, \dfrac{8}{3}\right]^3, \\ \dfrac{t^2}{100} + 8.7x_4 - 22.8, & (x_1,x_2,x_3,x_4) \in [0,1] \times [0,4]^2 \times \left(\dfrac{8}{3}, 4\right], \\ \dfrac{t^2}{100} + 12, & (x_1,x_2,x_3,x_4) \in [0,1] \times \mathbb{R}_+^2 \times (4,\infty). \end{cases}$$

By simple calculation, we get $\rho = 7,\ \triangle = -\frac{7}{2},\ A = \frac{31}{21},\ B = \frac{31}{42}$ and

$$G(t,s) = \begin{cases} (1+3s)(2-t)/7, & s \leq t, \\ (1+3t)(2-s)/7, & t \leq s. \end{cases}$$

For $k=3$, one can obtain that $\Lambda = \frac{8}{243}$ and $\Omega = \frac{571}{84}$. It is clear that $(C1)$-$(C5)$ are satisfied. Moreover, taking $r=4$ and $R=1000$, it is easy to check that

$$4 = r < \min\left\{\frac{R}{k^2+1}, \frac{\Lambda}{\Omega}R\right\} = \min\left\{100, \frac{224000}{46251}\right\} < kr = 12.$$

Now, let us show that conditions $(C6)$-$(C8)$ are satisfied:

$$f(x_1,x_2,x_3,x_4) \le 12.01 < \phi_p\left(\frac{R}{\Omega}\right) \approx 12.1289$$
$$\text{for } (x_1,x_2,x_3,x_4) \in [0,1]\times[0,2000]^3,$$

$$f(x_1,x_2,x_3,x_4) \le 0.41 < \phi_p\left(\frac{r}{k\Omega}\right) \approx 0.4428$$
$$\text{for } (x_1,x_2,x_3,x_4) \in [0,1]\times\left[0,\frac{8}{3}\right]^3,$$

$$f(x_1,x_2,x_3,x_4) \ge 12.0011 > \phi_p\left(\frac{r}{\Lambda}\right) \approx 11.0227$$
$$\text{for } (x_1,x_2,x_3,x_4) \in \left[\frac{1}{3},1\right]\times[0,12]^2\times[4,12].$$

So, all conditions of Theorem 1 hold. Thus, according to Theorem 1, the BVP (4.1) has at least three positive solutions that belong to $Q(\beta,1000)$.

5. Conclusion

In this paper, by applying the six functionals fixed point theorem [7], which is a generalization of the five functionals fixed point theorem [8] and Leggett-Williams fixed point theorem [11], we investigate the existence of at least three positive solutions for the nth order p-Laplacian BVP with integral boundary conditions. We provide an example to support the theoretical result. In the future, we plan to study the existence of positive solutions for nth order multipoint BVPs with p-Laplacian as well as the one with impulsive effects.

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REFERENCES

[1] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*. Dordrecht: Kluwer Academic Publishers, 2001. doi: 10.1007/978-94-010-0718-4.
[2] B. Ahmad and A. Alsaedi, "Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions," *Nonlinear Anal. Real World Appl.*, vol. 10, pp. 358–367, 2009, doi: 10.1016/j.nonrwa.2007.09.004.
[3] B. Ahmad, A. Alsaedi, and B. S. Alghamdi, "Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions," *Nonlinear Anal. Real World Appl.*, vol. 9, pp. 1727–1740, 2008, doi: 10.1016/j.nonrwa.2007.05.005.
[4] B. Ahmad and S. K. Ntouyas, "Boundary value problems for n-th order differential inclusions with four-point integral boundary conditions," *Opuscula Math.*, vol. 32, pp. 205–226, 2012, doi: 10.7494/OpMath.2012.32.2.205.
[5] B. Ahmad and S. K. Ntouyas, "A study of higher-order nonlinear ordinary differential equations with four-point nonlocal integral boundary conditions," *J. Appl. Math. Comput*, vol. 39, pp. 97–108, 2012, doi: 10.1007/s12190-011-0513-0.
[6] B. Ahmad, S. K. Ntouyas, and H. H. Alsulami, "Existence results for n-th order multipoint integral boundary-value problems of differential inclusions," *Electron. J. Differential Equations*, vol. 203, p. 13pp, 2013.
[7] R. Avery, J. Henderson, and D. O'Regan, "Six functionals fixed point theorem," *Commun. Appl. Anal.*, vol. 12, pp. 69–81, 2008.
[8] R. I. Avery, "A generalization of the Leggett-Williams fixed point theorem," *Math. Sci. Res. Hot-Line*, vol. 3, no. 7, pp. 9–14, 1999.
[9] A. Boucherif, "Second-order boundary value problems with integral boundary conditions," *Nonlinear Anal.*, vol. 70, pp. 364–371, 2009, doi: 10.1016/j.na.2007.12.007.
[10] C. Corduneanu, *Integral Equations and Applications*. Cambridge: Cambridge University Press, 1991. doi: 10.1017/CBO9780511569395.
[11] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of nonlinear operators on ordered Banach spaces," *Indiana Univ. Math. J.*, vol. 28, no. 4, pp. 673–688, 1979, doi: 10.1512/iumj.1979.28.28046.
[12] Y. Li and F. Li, "Sign-changing solutions to second-order integral boundary value problems," *Nonlinear Anal.*, vol. 69, pp. 1179–1187, 2008, doi: 10.1016/j.na.2007.06.024.
[13] Y. H. Li, H. Y. Zhang, and Z. L. Zhang, "Existence of three positive solutions to a singular integral boundary value problem for systems of nonlinear nth-order ordinary differential equations," *Appl. Math. J. Chinese Univ. Ser. A.*, vol. 27, pp. 168–174, 2012.
[14] Z. Yang, "Existence of nontrivial solutions for a nonlinear Sturm-Lioville problem with integral boundary conditions," *Nonlinear Anal.*, vol. 68, pp. 216–225, 2008, doi: 10.1016/j.na.2006.10.044.
[15] X. Zhang, M. Feng, and W. Ge, "Existence results for nonlinear boundary-value problems with integral conditions in Banach spaces," *Nonlinear Anal.*, vol. 69, pp. 3310–3321, 2008, doi: 10.1016/j.na.2007.09.020.

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