



OPTIMAL CONTROL OF A TWO TEAMS PREY-PREDATOR INTERACTION MODEL

M. F. ELETREBY AND SHABAN ALY

This paper is dedicated to the memory of Professor Miklós Farkas .

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Abstract. In this paper, a model of two teams of predators interacting with two teams of preys is proposed. The predator teams help each other and so do the prey teams. Therefore, some stability analysis are carried out on the model to detect the stability and instability of the system's equilibrium solutions. Also, the persistence of the solutions is investigated. Finally, the optimal control is applied to the unstable equilibrium solutions. Some numerical simulations are carried out to verify the analytic results.

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1. INTRODUCTION

The work of Malthus [10] and Verhulst [13] is the base to the field of the growth and decline of a population [3, 16]. In the nature, species do not exist alone. So, there will be interactions with the other species. When species interact, the population dynamics of each species is affected. There are three main types of interaction [7, 11]:

- (1) If the growth rate of one population is increased while the growth rate of the other population decreased, then the two populations are in a predator–prey situation.
- (2) If the growth rate of each population is decreased affected by the other species, then the two populations are in a competition case.
- (3) If the growth rate of each population is enhanced affected by the other species, then it is a mutualism situation.

The work that was proposed independently by Lotka [9] and Volterra [14] opened the gate for studying the predator–prey interaction. Lotka-Volterra model is the simplest model of these interactions. Since their pioneering work, many other notable contributions were made [12, 15].

Many creatures form teams. This has, at least, two main advantages: the first is the improvement in foraging, since looking for food in a team is more efficient than doing it alone. The second advantage is that living in a team reduces predation risk due to early spotting of predators and that existing in a team gives a higher probability that the predator will attack another member of the team. In this paper, a model is given where two teams of predators interact with two teams of preys. The teams of each group (predators or preys) help each other.

A biological realization of our study is represented by herds of zebras and gazelles living side by side and attacked by two types of predators (like lions and tigers). Multi-team game theory has been studied previously in [1,2].

The model is presented in Section 2. The equilibrium points, their local stability and persistence are studied in Section 3. In Section 4, an optimal control to the model is investigated.

2. THE MODEL

Consider two teams of preys with densities x_1, x_2 interacting with two teams of predators with densities y_1, y_2 , respectively. We assume that, in the absence of the predators, each team of preys grows logistically, $a_i x_i (1 - x_i)$. Each team of the preys will contribute to the growth of the other team, $a_i x_j, i \neq j$. Also, the two teams of preys help each other against the predator. The effect of the predation is to reduce the prey growth rate by a term proportional to the prey and predator populations, $x_i (y_1 + y_2)$. Finally, in the absence of any prey for sustenance, the predators death rate results in inverse decay, $c_i y_i$. Using these assumptions, the following model is proposed [6];

$$\begin{aligned} \frac{dx_1}{dt} &= a_1 [x_1(1 - x_1) + x_2] - x_1(y_1 + y_2), \\ \frac{dx_2}{dt} &= a_2 [x_2(1 - x_2) + x_1] - x_2(y_1 + y_2), \\ \frac{dy_1}{dt} &= -c_1 y_1 + (x_1 + x_2)(y_1 + y_2), \\ \frac{dy_2}{dt} &= -c_2 y_2 + (x_1 + x_2)(y_1 + y_2), \end{aligned} \quad (2.1)$$

where the positive coefficients a_1 and a_2 represent the intrinsic growth rates of the preys densities x_1 and x_2 , respectively. Both the positive parameters c_1 and c_2 are the death rates of the predator y_1 and y_2 , respectively. It is clear that the two preys (predators) help each other e.g. in foraging and (in the case of preys) in early warning against predation.

3. THE ANALYSIS OF THE MODEL

The steady state (equilibrium) solutions are important [4]. The system (2.1) has the following equilibria. The trivial one $E_1 = (0, 0, 0, 0)$, the boundary one $E_2 = (2, 2, 0, 0)$ and the coexistence (interior) $E_3 = (x_1, x_2, y_1, y_2)$. The components of the interior solutions are given by the relations;

$$y_2 = \frac{c_1}{c_2} y_1, \quad y_1 = \frac{a_1 c}{x_1 c_1} (c - x_1^2), \quad x_2 = (c - x_1), \quad (3.1)$$

and

$$x_1^3 - c(1 + A_2)x_1^2 - c(1 - cA_2)x_1 + c^2A_1 = 0, \quad (3.2)$$

where

$$A_1 = \frac{a_1}{a_1 + a_2}, \quad A_2 = \frac{a_2}{a_1 + a_2} \quad \text{and} \quad c = \frac{c_1 c_2}{c_1 + c_2}.$$

Proposition 1. The system (2.1) has at least one and at most two interior equilibrium solutions [6].

Proposition 2. Local stability analysis [4] shows that the equilibrium state $E_1 = (0, 0, 0, 0)$ is unstable, while $E_2 = (2, 2, 0, 0)$ is stable under the conditions $c_1 + c_2 > 8$, and $4(c_1 + c_2) < c_1 c_2$.

Proof. The Jacobian matrix of the system (2.1) is given by $J =$

$$\begin{bmatrix} a_1(1 - 2x_1) - (y_1 + y_2) & a_1 & -x_1 & -x_1 \\ a_2 & a_2(1 - 2x_2) - (y_1 + y_2) & -x_2 & -x_2 \\ (y_1 + y_2) & (y_1 + y_2) & (x_1 + x_2) - c_1 & (x_1 + x_2) \\ (y_1 + y_2) & (y_1 + y_2) & (x_1 + x_2) & (x_1 + x_2) - c_2 \end{bmatrix}.$$

At the equilibrium point $E_1 = (0, 0, 0, 0)$, the above Jacobian matrix becomes;

$$J = \begin{bmatrix} a_1 & a_1 & 0 & 0 \\ a_2 & a_2 & 0 & 0 \\ 0 & 0 & -c_1 & 0 \\ 0 & 0 & 0 & -c_2 \end{bmatrix},$$

whose eigenvalues are $\lambda = 0, -c_1, -c_1, a_1 + a_2$, which has one positive eigenvalue, $a_1 + a_2 > 0$. So, it is unstable equilibrium point.

Similarly, at the equilibrium point $E_2 = (2, 2, 0, 0)$, The Jacobian matrix becomes;

$$J = \begin{bmatrix} -3a_1 & a_1 & -2 & -2 \\ a_2 & -3a_2 & -2 & -2 \\ 0 & 0 & 4 - c_1 & 4 \\ 0 & 0 & 4 & 4 - c_2 \end{bmatrix},$$

whose eigenvalues satisfy the characteristic equations;

$$\lambda^2 + 3(a_1 + a_2)\lambda + 8a_1a_2 = 0,$$

and

$$\lambda^2 - (8 - c_1 - c_2)\lambda + c_1 c_2 - 4(c_1 + c_2) = 0.$$

The first equation has two negative eigenvalues since $3(a_1 + a_2) > 0$, $8a_1 a_2 > 0$ and the two other eigenvalues come from the second equation which are negative under the conditions

$$c_1 + c_2 > 8, \quad 4(c_1 + c_2) < c_1 c_2.$$

□

Definition 1. A system is persistent if there exists a compact region V subset of the interior of the state space such that all solutions with positive initial conditions are attracted to V [8].

Proposition 3. The system (2.1) is persistent if at least one of the above conditions is not satisfied [6].

The characteristic equation of the eigenvalue of the internal equilibrium point E_3 is given by, $\lambda^4 + D_1\lambda^3 + D_2\lambda^2 + D_3\lambda + D_4 = 0$, where

$$D_1 = (-2a_2)x_1 + (c_1 + c_2 - a_1 - a_2 + 2a_2c - 2c) + (2a_1c)x_1^{-1}$$

$$D_2 = (2a_1 - a_1^2 - 2a_1a_2)x_1^2$$

$$\begin{aligned} &+ (4a_2c + 2a_1a_2c + a_1a_2 + a_1^2 - 2a_1c - 2a_2c_1 - 2a_2c_2)x_1 \\ &- (4a_1a_2c + 4a_2c^2 + a_1c_1 + a_1c_2 + a_2c_1 + a_2c_2 - 2a_2cc_1 - 2a_2cc_2 - 2a_2c) \\ &+ (a_1c)(2c_1 + 2c_2 + 2a_2c - a_1 - a_2 - 2c)x_1^{-1} + (a_1^2c^2)x_1^{-2} \end{aligned}$$

$$\begin{aligned} D_3 = &(2a_1^2)x_1^3 + a_1(4a_2c + 2c_1 - 2a_2c_1 - 2a_2c_2 - a_1c_1 - a_1c_2)x_1^2 \\ &- a_1(c(c_1 + c_2) + (a_1 + a_2 + 2a_2c)(2c - c_1 - c_2))x_1 + (2a_1c) \\ &(2a_2(2c - c_1 - c_2) - c_1) + (a_1c)((2c - c_1 - c_2)(a_1 + a_2 - 2a_2c) \\ &+ c(c_1 + c_2) - 2a_1c)x_1^{-1} + (a_1^2c c_1 c_2)x_1^{-2} \end{aligned}$$

$$\begin{aligned} D_4 = &(2a_1)(a_1c_2 - a_2c_1 + a_2c_2)x_1^3 - (a_1c)(a_1(c_1 + c_2) + 2a_2(c_2 - c_1))x_1^2 \\ &+ (2a_1c)(a_1 + a_2)(c_1 - c_2)x_1 + (2a_1a_2c^2)(c_2 - c_1) - (2a_1^2c^2c_1)x_1^{-1} \\ &+ (a_1^2c^3)(c_1 + c_2)x_1^{-2} \end{aligned}$$

The Routh-Hurwitz criteria [4] shows that the interior point $E_3(x_1, x_2, y_1, y_2)$ is locally asymptotically stable if the determinants of all Hurwitz matrices are positive where the Hurwitz matrices are given by;

$$\begin{aligned} H_1 &= (D_1) & H_2 &= \begin{pmatrix} D_1 & 1 \\ D_3 & D_2 \end{pmatrix} \\ H_3 &= \begin{pmatrix} D_1 & 1 & 0 \\ D_3 & D_2 & D_1 \\ 0 & D_4 & D_3 \end{pmatrix} & H_4 &= \begin{pmatrix} D_1 & 1 & 0 & 0 \\ D_3 & D_2 & D_1 & 1 \\ 0 & D_4 & D_3 & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix} \end{aligned}$$

Using different values of the constants a_1, a_2, c_1 and c_2 to find the roots of equation (3.1), one gets that one of them is negative and one of the other two positive roots is greater than c i.e x_2 will be negative. So, we have only one interior equilibrium point.

Let the initial values of the preys and the predators are, respectively, $x_1(0) = 0.1$, $x_2(0) = 0.4$, $y_1(0) = 0.3$, $y_2(0) = 0.7$, and the values of the parameters of the model are $a_1 = 1.2$, $a_2 = 1.7$, $c_1 = 0.3$ and $c_2 = 0.5$. Substituting by these values in equations (3.2, 3.1), we get the interior equilibrium point $E_3 = (0.079, 0.1085, 1.7208, 1.0325)$, which is asymptotically stable since there are two negative real eigenvalues $\lambda = -3.019684381, -0.4022762114$ and another two conjugate complex eigenvalues with negative real parts $\lambda = -0.08409198332 \pm 1.009538530 I$. We simulate the model (2.1) for the mentioned initial values of the densities, we found that the system converges to the asymptotically stable point E_3 as shown in Fig. 1 and Fig. 2.

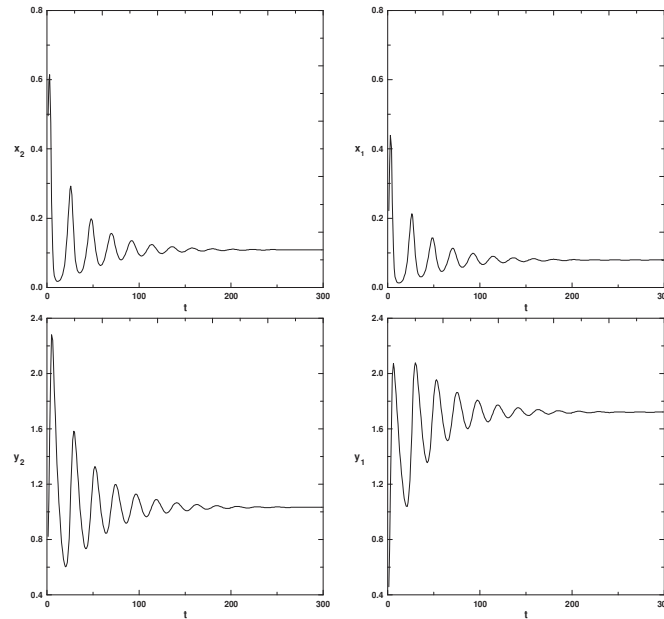


FIGURE 1. The densities of the preys and predators with time at the parameter values $a_1 = 1.2$, $a_2 = 1.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $x_1(0) = 0.1$, $x_2(0) = 0.4$, $y_1(0) = 0.3$, $y_2(0) = 0.7$.

4. OPTIMAL CONTROL PROBLEM

In some cases the presence of the predators is harmful. So, we should keep the situation in which the predator does not exist. Such as the case when the predators are viruses or tumors. Also, there are some animals that present in few numbers, like

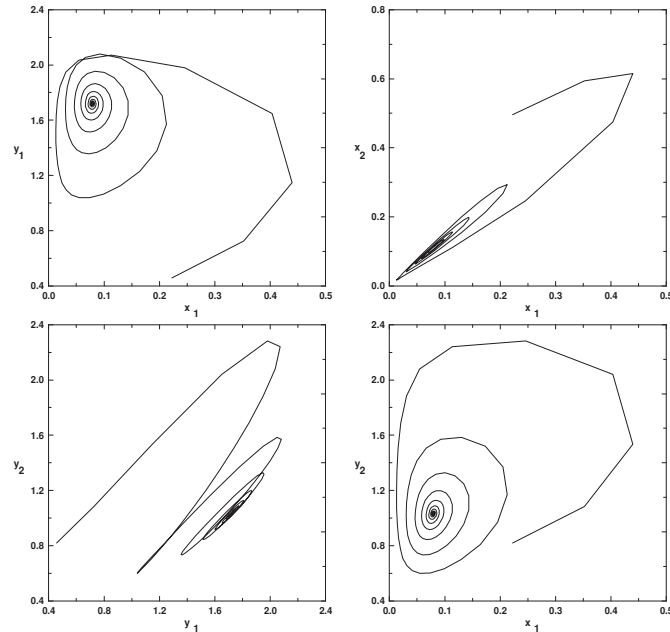


FIGURE 2. The system converges to the stable point E_3 for the parameter values $a_1 = 1.2$, $a_2 = 1.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $x_1(0) = 0.1$, $x_2(0) = 0.4$, $y_1(0) = 0.3$, $y_2(0) = 0.7$.

Panda, we have to keep them from extinction or predation. In cases like these, it is required to make these equilibrium solutions stable.

We saw that the first equilibrium solution E_1 is unstable and the second equilibrium solution E_2 is unstable if one of the conditions $c_1 + c_2 > 8$ or $4(c_1 + c_2) < c_1 c_2$ is not satisfied. Then, we will use the optimal control technique to stabilize the second solution E_2 in this section.

Here, we will use the same feedback control approach which is presented in [5] for a different model. Consider the above system in the following form;

$$\begin{aligned}
 \frac{dx_1}{dt} &= a_1 [x_1(1 - x_1) + x_2] - x_1(y_1 + y_2) + u_1, \\
 \frac{dx_2}{dt} &= a_2 [x_2(1 - x_2) + x_1] - x_2(y_1 + y_2) + u_2, \\
 \frac{dy_1}{dt} &= -c_1 y_1 + (x_1 + x_2)(y_1 + y_2) + u_3, \\
 \frac{dy_2}{dt} &= -c_2 y_2 + (x_1 + x_2)(y_1 + y_2) + u_4,
 \end{aligned} \tag{4.1}$$

where u_i , ($i = 1, 2, 3, 4$), are the control inputs. Some performance measure is used to determine these inputs from the conditions of asymptotic stability of the system (2.1) about its unstable equilibrium states.

Let

$$z_1 = x_1 - \bar{x}_1, \quad z_2 = x_2 - \bar{x}_2, \quad w_1 = y_1 - \bar{y}_1, \quad w_2 = y_2 - \bar{y}_2, \quad (4.2)$$

where z_1, z_2, w_1, w_2 represent perturbed states about its unstable equilibrium states and $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ denote to the coordinates of the unstable equilibrium states of system (2.1). Substituting by (4.2) into the system (4.1) taking into account the identities that satisfied by the equilibrium states, we get the following system,

$$\begin{aligned} \dot{z}_1 &= -(2a_1 \bar{x}_1 + \bar{y}_1 + \bar{y}_2)z_1 + a_1(z_1 + z_2 - z_1^2) - (\bar{x}_1 + z_1)(w_1 + w_2) + u_1, \\ \dot{z}_2 &= -(2a_2 \bar{x}_2 + \bar{y}_1 + \bar{y}_2)z_2 + a_2(z_1 + z_2 - z_2^2) - (\bar{x}_2 + z_2)(w_1 + w_2) + u_2, \\ \dot{w}_1 &= -c_1 w_1 + (z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) + (\bar{x}_1 + \bar{x}_2)(w_1 + w_2) + u_3, \\ \dot{w}_2 &= -c_2 w_2 + (z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) + (\bar{x}_1 + \bar{x}_2)(w_1 + w_2) + u_4. \end{aligned} \quad (4.3)$$

The system (4.3) admits the trivial solution,

$$z_1 = z_2 = w_1 = w_2 = 0, \quad u_1 = u_2 = u_3 = u_4 = 0. \quad (4.4)$$

This solution (from equation (4.2)) represents the equilibrium states of the system (2.1).

Theorem 1. *Using the optimal nonlinear control inputs;*

$$\begin{aligned} u_1^* &= -a_1(z_1 + z_2 - z_1^2) + (\bar{x}_1 + z_1)(w_1 + w_2), \\ u_2^* &= -a_2(z_1 + z_2 - z_2^2) + (\bar{x}_2 + z_2)(w_1 + w_2), \\ u_3^* &= -(z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) - (\bar{x}_1 + \bar{x}_2)(w_1 + w_2), \\ u_4^* &= -(z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) - (\bar{x}_1 + \bar{x}_2)(w_1 + w_2), \end{aligned} \quad (4.5)$$

the unstable equilibrium states of the system (2.1) will be optimally asymptotically stable with respect to the performance measure;

$$\begin{aligned} I &= \int_0^\infty \Omega dt \\ &= \int_0^\infty \left\{ [u_1 + a_1(z_1 + z_2 - z_1^2) - (\bar{x}_1 + z_1)(w_1 + w_2)]^2 \right. \\ &\quad + [u_2 + a_2(z_1 + z_2 - z_2^2) - (\bar{x}_2 + z_2)(w_1 + w_2)]^2 \\ &\quad + [u_3 + (z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) + (\bar{x}_1 + \bar{x}_2)(w_1 + w_2)]^2 \\ &\quad + [u_4 + (z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) + (\bar{x}_1 + \bar{x}_2)(w_1 + w_2)]^2 \\ &\quad \left. + (2a_1 \bar{x}_1 + \bar{y}_1 + \bar{y}_2)z_1^2 + (2a_2 \bar{x}_2 + \bar{y}_1 + \bar{y}_2)z_2^2 + c_1 w_1^2 + c_2 w_2^2 \right\} dt. \end{aligned} \quad (4.6)$$

Proof. We will use Bellman dynamic programming technique to obtain the optimal control inputs and Liapunov asymptotic stability theorem to prove our theorem.

Suppose that we have the nonlinear system (4.3) with the four perturbed state variables z_1, z_2, w_1, w_2 and the four control inputs $u_i, (i = 1, 2, 3, 4)$ with \dot{z}_i, \dot{w}_i , are definite functions of $z_i, w_i, i = 1, 2$, and $u_j, j = 1, 2, 3, 4$. If we consider the minimization of an integral of the form (4.6) we can write the condition for minimization of the integral of the form;

$$\begin{aligned} \min_{\vec{u}} \left\{ \frac{\partial V}{\partial z_1} \dot{z}_1 + \frac{\partial V}{\partial z_2} \dot{z}_2 + \frac{\partial V}{\partial w_1} \dot{w}_1 + \frac{\partial V}{\partial w_2} \dot{w}_2 \right. \\ + [u_1 + a_1(z_1 + z_2 - z_1^2) - (\bar{x}_1 + z_1)(w_1 + w_2)]^2 \\ + [u_2 + a_2(z_1 + z_2 - z_2^2) - (\bar{x}_2 + z_2)(w_1 + w_2)]^2 \\ + [u_3 + (z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) + (\bar{x}_1 + \bar{x}_2)(w_1 + w_2)]^2 \\ + [u_4 + (z_1 + z_2)(w_1 + w_2 + \bar{y}_1 + \bar{y}_2) + (\bar{x}_1 + \bar{x}_2)(w_1 + w_2)]^2 \\ \left. + (2a_1\bar{x}_1 + \bar{y}_1 + \bar{y}_2)z_1^2 + (2a_2\bar{x}_2 + \bar{y}_1 + \bar{y}_2)z_2^2 + c_1w_1^2 + c_2w_2^2 \right\} = 0. \end{aligned} \quad (4.7)$$

Now, the function V is a function of $z_i, w_i, i = 1, 2$, and it represents the minimum value of the integral performance measure (4.6) achievable by starting from time t and state (z_1, z_2, w_1, w_2) . Let us assume that V is the solution of the (4.7) takes the form,

$$V = z_1^2 + z_2^2 + w_1^2 + w_2^2, \quad (4.8)$$

which is a positive definite function of the variables (z_1, z_2, w_1, w_2) and so will be used to prove the asymptotic stability of the solution (4.4).

Substituting by (4.3) and (4.8) in (4.7) and differentiating the resulting expression with respect to u_i and setting it to zero, we obtain the optimal control inputs u_i^* as given by (4.5).

Differentiating the function V along the trajectory of the system (4.3) taking into account the control inputs (4.5) we get;

$$\begin{aligned} \dot{V} &= 2z_1\dot{z}_1 + 2z_2\dot{z}_2 + 2w_1\dot{w}_1 + 2w_2\dot{w}_2 \\ &= -2 \left\{ (2a_1\bar{x}_1 + \bar{y}_1 + \bar{y}_2)z_1^2 + (2a_2\bar{x}_2 + \bar{y}_1 + \bar{y}_2)z_2^2 + c_1w_1^2 + c_2w_2^2 \right\} \\ &= -2\Omega^* \end{aligned} \quad (4.9)$$

where Ω^* is the value of the objective function Ω at the optimal control inputs.

Obviously, the function \dot{V} is a negative definite function of (z_1, z_2, w_1, w_2) only for positive equilibrium states $\bar{x}_i, \bar{y}_i > 0, (i = 1, 2)$ and so the positive equilibrium states (4.4) are asymptotically stable in the Liapunov sense if the conditions of stability are satisfied. \square

Some numerical analysis are achieved to show the stabilization behavior of the system (2.1) at the second unstable equilibrium solution E_2 . This numerical analysis supports the above analytical analysis. The model parameters are fixed as follows: $a_1 = 0.2$, $a_2 = 0.7$, $c_1 = 0.3$, $c_2 = 0.5$. Note that the conditions of stability of E_2 are not satisfied. The initial densities of preys are $z_1 = 0.1$ and $z_2 = 0.4$ while the initial densities of predators are $w_1 = 0.3$ and $w_2 = 0.7$. A lot of simulations for different values of the parameters and the initial values of the densities are done. Fig. 3 shows that the preys and predator densities perturbations decay to zero. In Fig. 4, the components of the control inputs tend to zero. So, we conclude that the unstable equilibrium solution E_2 of system (2.1) can be stabilized by using nonlinear control parameters (4.1)

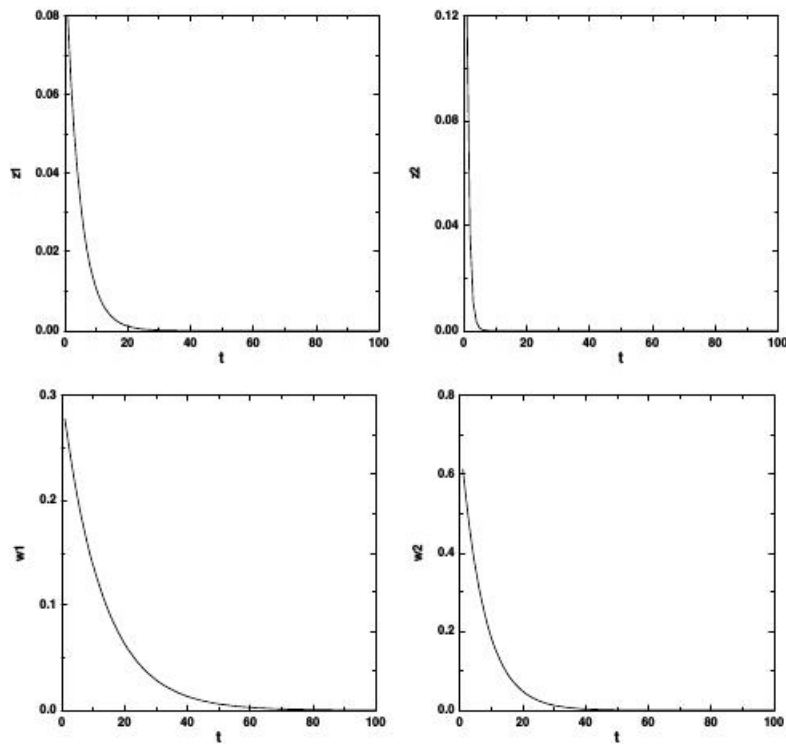


FIGURE 3. The densities perturbations of the preys and predators with time converges to zero at the parameter values $a_1 = 0.2$, $a_2 = 0.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $z_1(0) = 0.1$, $z_2(0) = 0.4$, $w_1(0) = 0.3$, $w_2(0) = 0.7$.

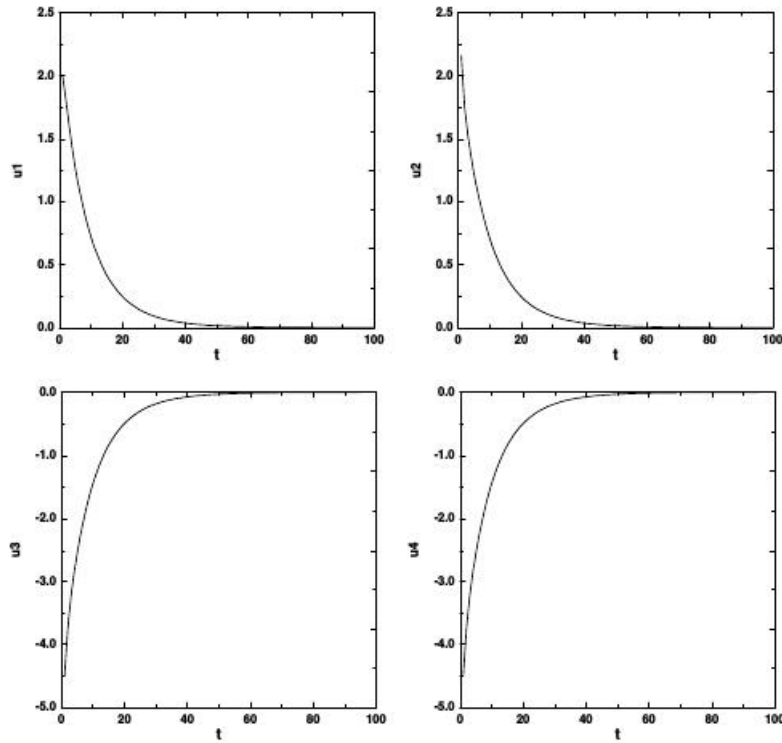


FIGURE 4. The components of the control inputs converges to the stable point state $(0, 0, 0, 0)$ at the parameter values $a_1 = 0.2$, $a_2 = 0.7$, $c_1 = 0.3$, $c_2 = 0.5$ and the initial densities $z_1(0) = 0.1$, $z_2(0) = 0.4$, $w_1(0) = 0.3$, $w_2(0) = 0.7$.

5. CONCLUSION

Many animals live in groups. Different groups share one habitat hence these groups may cooperate, compete with each other or form predator-prey system. In this work we present a model for the interaction between predator-prey teams. Equilibrium solutions are derived, their local stability and persistence. An optimal control is used to stabilize the unstable equilibrium solution. The boundary unstable equilibrium solution E_2 is stabilized using nonlinear control inputs.

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Authors' addresses

M. F. Elettrey

Current address: King Khalid University, Faculty of Science, Department of Mathematics, Applied Mathematics Group, Abha 9004, Saudi Arabia

Mansoura University, Faculty of Science, Department of Mathematics, Mansoura 35516, Egypt

E-mail address: mohfathy@mans.edu.eg

Shaban Aly

Current address: King Khalid University, Faculty of Science, Department of Mathematics, Applied Mathematics Group, Abha 9004, Saudi Arabia

Al-Azhar University, Faculty of Science, Mathematics Department, Assiut 71511, Egypt

E-mail address: shhaly70@yahoo.com