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TG-SUPPLEMENTED MODULES

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Abstract. In this work, we define tg-supplemented modules and investigate some properties of these modules. We prove that the finite t-sum of tg-supplemented modules is tg-supplemented. We also prove that the homomorphic image of a distributive tg-supplemented module is tg-supplemented. We give some examples separating tg-supplemented modules from supplemented and generalized ⊕-supplemented modules.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule K of M by $K \leq M$. Let M be an R-module and $K \leq M$. If $T = M$ for every submodule T of M such that $K + T = M$, then K is called a *small submodule* of M and denoted by $K \ll M$. Let M be an R-module and $K \leq M$. If there exists a submodule T of M such that $K + T = M$ and $K \cap T = 0$, then K is called a *direct summand* of M and it is denoted by $M = K \oplus T$. For any module M, the intersection of maximal submodules of M is called the *radical* of M and denoted by $Rad M$. If M have no maximal submodules, then we define $Rad M = M$. A module M is called *distributive* [8] if for every submodules K, L, T of M, $K \cap (L + T) = K \cap L + K \cap T$ or equivalently $(K + L) \cap (K + T) = K + L \cap T$. Let U and V be submodules of a module M. If $U + V = M$ and V is minimal with respect to this property, or equivalently, $U + V = M$ and $U \cap V \ll V$, then V is called a *supplement* [10] of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. M is called ([5],[6]) ⊕*-supplemented module* if every submodule of M has a supplement that is a direct summand of M. Let M be an R-module and U, V be submodules of M. V is called a *generalized supplement* ([1],[9],[11]) of U in M if $M = U + V$ and $U \cap V \leq Rad V$. M is called *generalized supplemented* or briefly a *GS-module* if every submodule of M has a generalized supplement in M. Clearly we see that every supplemented module is a generalized supplemented module. M is called a *generalized $\oplus$-supplemented* ([2],[4],[7],[8]) module if every submodule of M has a generalized supplement that is a direct summand of M. In this paper we generalize these modules.

Lemma 1. *Let V be a supplement of U in M and $L, K \leq V$. Then K is a supplement of L in V if and only if K is a supplement of $U + L$ in M. ([3], Exercise 20.39)*

Proof. ($\Rightarrow$) Let $U + L + T = M$, for some $T \leq K$. Since V is a supplement of U in M and $L + T \leq V$, $L + T = V$ and by K being a supplement of L in V, $T = K$. Hence K is a supplement of $U + L$ in M.

($\Leftarrow$) Let $L + T = V$, for some $T \leq K$. Then $U + L + T = M$, and by K being a supplement of $U + L$ in M, $T = K$. Hence K is a supplement of L in V. □

Lemma 2. *Let M be a π-projective module and K, L be two submodules of M. If K and L are mutual supplements in M, then $K \cap L = 0$ and $M = K \oplus L$.*

Proof. See ([10], 41.14(2)). □

2. TG-SUPPLEMENTED MODULES

Definition 1. Let M be an R-module and K, L be two submodules of M. If $K$and L are mutual supplements in M, then M is called a *t-sum* of K and L. This equivalent to $M = K + L$, $K \cap L \ll K$ and $K \cap L \ll L$. This case K and L are called *t-summands* of M.

Definition 2. Let M be an R-module and $\{A_i\}_{i \in I}$ be a family of submodules of M. M is called a *t-sum* of $\{A_i\}_{i \in I}$, if A_k and $\sum_{j \neq k} A_j$ are mutual supplements in M for every $k \in I$.

Lemma 3. *Let M be an R-module, V be a t-summand of M and $K \leq V$. Then $K \ll M$ if and only if $K \ll V$.*

Proof. Clear from ([12], Lemma 1.1). □

Lemma 4. *Let M be a t-sum of U and V. If K is a supplement of S in U and L is a supplement of T in V, then $K + L$ is a supplement of $S + T$ in M.*

Proof. Since U is a supplement of V in M and K is a supplement of S in U, by Lemma 1.1, K is a supplement of $V + S$ in M. Hence $(V + S) \cap K \ll K$. Similarly, we prove that $(U + T) \cap L \ll L$. Then $(S + T) \cap (K + L) \leq (S + T + K) \cap L + (S + T + L) \cap K = (U + T) \cap L + (V + S) \cap K \ll K + L$, and by $M = U + V = S + K + T + L = S + T + K + L$, $K + L$ is a supplement of $S + T$ in M. □

Lemma 5. *Let M be a t-sum of U and V, and $L,T \leq V$. Then V is a t-sum of L and T if and only if M is a t-sum of $U+L$ and T, and M is a t-sum of $U+T$ and L.*

Proof. ($\Rightarrow$) Let V be a t-sum of L and T. Since T is a supplement of L in V and V is a supplement of U in M, then by Lemma 1, T is a supplement of $U+L$ in M. Then $(U+L)\cap T \ll T$. Similarly, we can prove that $(U+T)\cap L \ll L$. Then by $U\cap V \ll U$, $(U+L)\cap T \leq U\cap(L+T)+L\cap(U+T) = U\cap V+(U+T)\cap L \ll U+L$. Since $(U+L)\cap T \ll T, (U+L)\cap T \ll U+L$ and $M = U+V = U+L+T$, then by Definition 1 M is a t-sum of $U+L$ and T. Similarly, we prove that M is a t-sum of $U+T$ and L.
($\Leftarrow$)Clear from Lemma 1. □

Corollary 1. *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If K_i is a supplement of T_i in U_i $(i=1,2,\ldots,n)$, then $K_1+K_2+\cdots+K_n$ is a supplement of $T_1+T_2+\cdots+T_n$ in M.*

Proof. Clear from Lemma 5. □

Corollary 2. *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If U_i is a t-sum of K_i and T_i $(i=1,2,\ldots,n)$, then M is a t-sum of $K_1+K_2+\cdots+K_n$ and $T_1+T_2+\cdots+T_n$.*

Proof. Clear from Corollary 1. □

Corollary 3. *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If K_i is a supplement in U_i $(i=1,2,\ldots,n)$, then $K_1+K_2+\cdots+K_n$ is a supplement in M.*

Proof. Clear from Corollary 1. □

Corollary 4. *Let M be a t-sum of $U_1,U_2,\ldots,U_n$. If K_i is a t-summand of U_i $(i=1,2,\ldots,n)$, then $K_1+K_2+\cdots+K_n$ is a t-summand of M.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be a distributive R-module and $N \leq M$. Then $(K+N)/N$ is a t-summand of M/N for every t-summand K of M.*

Proof. Let K be a t-summand of M. Then there exists a submodule L of M such that $M = L+K, L\cap K \ll L$ and $L\cap K \ll K$. Since $M = L+K$, then $M/N = (L+N)/N+(K+N)/N$. Since M is distributive, then we have $(L+N)\cap(K+N) = L\cap K+N$. Since $L\cap K \ll L$ and $L\cap K \ll K$, then we have $((L+N)/N)\cap((K+N)/N) = (L\cap K+N)/N \ll (L+N)/N$ and $((L+N)/N)\cap((K+N)/N) = (L\cap K+N)/N \ll (K+N)/N$. Hence $(K+N)/N$ is a t-summand of M/N. □

Theorem 1. *Let M be a t-sum of $\{A_i\}_{i\in I}$. Then $Rad M = \sum_{i\in I} Rad A_i$.*

Proof. Let $x \in RadM$. Since $x \in M = \sum_{i \in I} A_i$, there exist $i_1, i_2, \ldots, i_n \in I$ and $x_{i_1} \in A_{i_1}, x_{i_2} \in A_{i_2}, \ldots, x_{i_n} \in A_{i_n}$ such that $x = x_{i_1} + x_{i_2} + \cdots + x_{i_n}$. Let $k \in \{1, 2, \ldots, n\}, T \leq A_{i_k}$ and $Rx_{i_k} + T = A_{i_k}$. Let $a \in M$. Since $a \in M = \sum_{i \in I, i \neq i_k} A_i + A_{i_k}$, we can write $a = b + c$ for some $b \in \sum_{i \in I, i \neq i_k} A_i$ and $c \in A_{i_k}$. Since $c \in A_{i_k} = Rx_{i_k} + T$, there exist $r \in R$ and $t \in T$ such that $c = rx_{i_k} + t$. Then $a = b + c = b + rx_{i_k} + t = b + r\left(x - \sum_{s=1, s \neq i_k}^{n} x_{i_s}\right) + t = rx + b - \sum_{s=1, s \neq i_k}^{n} rx_{i_s} + t \in Rx + \sum_{i \in I, i \neq i_k} A_i + T$. Hence $M = Rx + \sum_{i \in I, i \neq i_k} A_i + T$ and since $Rx \ll M, M = \sum_{i \in I, i \neq i_k} A_i + T$. Since $M = \sum_{i \in I, i \neq i_k} A_i + T$ and M is a t-sum of $\{A_i\}_{i \in I}, T = A_{i_k}$. Thus $Rx_{i_k} \ll A_{i_k}$ and $x_{i_k} \in RadA_{i_k}$. Consequently, $x \in \sum_{i \in I} RadA_i$ and $RadM \leq \sum_{i \in I} RadA_i$. $\sum_{i \in I} RadA_i \leq RadM$ is clear. Thus $RadM = \sum_{i \in I} RadA_i$. □

Definition 3. Let M be an R-module. M is called a *tg-supplemented* module if every submodule of M has a generalized supplement that is a t-summand of M. Clearly generalized $\oplus$-supplemented modules are tg-supplemented. But the converse is not true in general (See Example 4).

We can also clearly see that every supplemented module is tg-supplemented. But the converse of this statement is not always true (See Example 1, 2, 3). Since hollow and local modules are supplemented, they are tg-supplemented modules. Clearly, every tg-supplemented module is generalized supplemented.

Lemma 7. *Let M be an R-module. If $RadM = M$, then M is tg-supplemented.*

Proof. Let N be any submodule of M. Since $N + M = M$ and $N \cap M \leq M = RadM$, we get that M is a generalized supplement of N in M. On the other hand M and 0 are mutual supplements in M. Hence M is tg-supplemented. □

Lemma 8. *Let M be a tg-supplemented R-module and $N \ll M$. Then M/N is tg-supplemented.*

Proof. Let $U/N \leq M/N$. Since M is tg-supplemented, U has a generalized supplement V that is a t-summand in M. Then by ([9], the proof of Proposition 2.6), $(V + N)/N$ is a generalized supplement of U/N in M/N. Since V is a t-summand of M, there exists a submodule L of M such that L and V are mutual supplements in M. Since L is a supplement of V in M and $N \ll M$, by ([10], 41.1(4)) L is a supplement of $V + N$ in M. Then by ([10], 41.1(7)) $(L + N)/N$ is a supplement of $(V + N)/N$ in M/N. Similarly, we can prove that $(V + N)/N$ is a supplement of $(L + N)/N$ in M/N. Hence $(L + N)/N$ and $(V + N)/N$ are mutual supplements in M/N. Thus M/N is tg-supplemented. □

Corollary 5. *Any small homomorphic image of a tg-supplemented module is tg-supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *Let M be a tg-supplemented module and $N \leq M$. If $(K+N)/N$ is a t-summand of M/N for every t-summant K of M, then M/N is tg-supplemented.*

Proof. Let $U/N \leq M/N$. Since M is tg-supplemented, U has a generalized supplement K in M such that K is a t-summand of M. Since K is a generalized supplement of U in M and $N \leq U$, we can see that $(K+N)/N$ is a generalized supplement in M/N. Since K is a t-summand of M, then by hypothesis $(K+N)/N$ is a t-summand of M/N. Hence every submodule of M/N has a generalized supplement that is a t-summand of M/N, and M/N is tg-supplemented. □

Lemma 10. *Let M be a distributive tg-supplemented R-module. Then every factor module of M is tg-supplemented.*

Proof. Clear from Lemma 6 and Lemma 9. □

Corollary 6. *Let M be a distributive tg-supplemented R-module. Then every homomorphic image of M is tg-supplemented.*

Proof. Clear from Lemma 10. □

Lemma 11. *Let M be an R-module and $RadM \ll M$. The following assertions are equivalent.*
(*i*) *M is supplemented.*
(*ii*) *M is tg-supplemented.*

Proof. (*i*)⇒(*ii*) Clear from definitions.
(*ii*)⇒(*i*) Let $U \leq M$. Since M is tg-supplemented, there exists a generalized supplement V of U that is a t-summand of M. Since V is supplement in M, then $V \cap RadM = RadV$. Since $RadM \ll M$, $RadV \ll M$ and, by Lemma 3, $U \cap V \leq RadV \ll V$. Thus V is a supplement of U in M and M is supplemented. □

Corollary 7. *Let M be a finitely generated R-module. The following assertions are equivalent.*
(*i*) *M is supplemented.*
(*ii*) *M is tg-supplemented.*

Proof. Since M is finitely generated, $RadM \ll M$. Then clearly this assertions is derived from Lemma 11. □

Lemma 12. *Let M be a t-sum of $M_1$and M_2. If $M_1$and M_2 are tg-supplemented, then M is tg-supplemented.*

Proof. Let $U \leq M$. Since M_1 is tg-supplemented, $(M_2+U)\cap M_1$ has a generalized supplement X that is a t-summand in M_1. Since M_2 is tg-supplemented, $(U+X)\cap M_2$ has a generalized supplement Y that is a t-summand in M_2. Then we get $M = M_1 + M_2 = M_2 + U + X = U + X + Y$ and $U \cap (X+Y) \leq (U+Y)\cap X + (U+X)\cap Y \leq RadX + RadY \leq Rad(X+Y)$. Hence $X+Y$ is a generalized supplement of U in M. Since M is a t-sum of M_1 and M_2, and X is a t-summand of M_1, and Y is a t-summand of M_2, then by Corollary 3, $X+Y$ is a t-summand of M. Thus M is tg-supplemented. □

Corollary 8. *Let M be a t-sum of $M_1, M_2, \ldots, M_n$. If M_i is tg-supplemented $(i = 1, 2, \ldots, n)$, then M is tg-supplemented.*

Proof. Clear from Lemma 12. □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathbb{Q}$ has no maximal submodule, we have $Rad\mathbb{Q} = \mathbb{Q}$. By Lemma 2.13, $\mathbb{Q}$ is a tg-supplemented module. But it is well known that $\mathbb{Q}$ is not supplemented (See [3], Example 20.12).

Example 2. Let M be a non-torsion $\mathbb{Z}$-module with $RadM = M$. Since $RadM = M$, then by Lemma 2.13, M is tg-supplemented. But M is not supplemented ([12]).

Example 3. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$, for any prime p. In this case $RadM \neq M$. Since $\mathbb{Q}$ and $p\mathbb{Z}$ are tg-supplemented, then by Lemma 12, M is tg-supplemented. But M is not supplemented.

Example 4. Let R be a commutative local ring which is not a valuation ring. Let a and b be elements of R, where neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a)\cap(b) = 0$ and $am = bm = 0$ where m is the maximal ideal of R. Let F be a free R-module with generators x_1, x_2 and x_3, K be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus, $M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R(ax_1 - bx_2)} = (R\overline{x_1} + R\overline{x_2}) \oplus R\overline{x_3}$. Here M is not $\oplus$-supplemented. But $F = Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented ([5]).

Since F is completely $\oplus$-supplemented, F is supplemented. Since a factor module of a supplemented module is supplemented, we have M is supplemented. So M is tg- supplemented. But since M is finitely generated and not $\oplus$-supplemented, M is not generalized $\oplus$-supplemented.

Lemma 13. *Let M be a t-sum of M_1 and M_2. Then M_2 is tg-supplemented if and only if for every submodule N of M such that $M_1 \leq N \leq M$, there exists a t-summand K of M_2 such that $M = K + N$ and $N \cap K \leq RadM$.*

Proof. $(\Rightarrow)$ Let $M_1 \leq N \leq M$. Since M_2 is tg-supplemented, there exists a generalized supplement K of $N \cap M_2$ in M_2 such that K is a t-summand of M_2. Then $M = M_1 + M_2 = N + N \cap M_2 + K = K + N$ and $N \cap K = N \cap M_2 \cap K \leq RadK \leq RadM$.

$(\Leftarrow)$ Let $L \leq M_2$ and $N = M_1 + L$. By hypothesis, there exists a t-summand K of M_2 such that $M = K + N$ and $N \cap K \leq RadM$. Since $K, L \leq M_2$, by Modular law, $M_2 = M_2 \cap M = M_2 \cap (K + N) = K + M_2 \cap N = K + M_2 \cap (M_1 + L) = L + K + M_2 \cap M_1$, and then by $M_2 \cap M_1 \ll M_2$, $M_2 = L + K$. Since K is a t-summand of M_2, then by Corollary 3, K is a t-summand of M. Then $RadK = K \cap RadM$ and by $N \cap K \leq RadM, L \cap K \leq N \cap K = K \cap (N \cap K) \leq K \cap RadM = RadK$. Hence K is a generalized supplement of L in M_2. Thus, M_2 is tg-supplemented. □

Theorem 2. *Let M be a tg-supplemented module. Assume that M is a t-sum of M_1 and M_2. If $K \cap M_2$ is a t-summand of M_2 for every t-summand K of M such that $M = K + M_2$, then M_2 is tg-supplemented.*

Proof. Let $M_1 \leq N \leq M$. Since M is tg-supplemented, $N \cap M_2$ has a generalized supplement K in M such that K is a t-summand of M. From this we have $M = N \cap M_2 + K$ and $N \cap M_2 \cap K \leq RadK \leq RadM$. Since $M = N \cap M_2 + K$, then by Modular law $M_2 = N \cap M_2 + M_2 \cap K$. Since $M_1 \leq N$, $M = M_1 + M_2 = M_1 + N \cap M_2 + M_2 \cap K = N + M_2 \cap K$. Since $M = K + M_2$ and K is a t-summand of M, then by hypothesis $M_2 \cap K$ is a t-summant of M_2. Hence by Lemma 13, M_2 is tg-supplemented. □

Lemma 14. *Let M be a π-projective module. Then M is tg-supplemented if and only if M is generalized $\oplus$-supplemented.*

Proof. Clear from Lemma 2. □

Theorem 3. *Let M be a projective module. The following assertions are equivalent.*

(i) *M is semiperfect.*
(ii) *M is generalized $\oplus$-supplemented.*
(iii) *M is tg-supplemented.*

Proof. $(i) \Leftrightarrow (ii)$ Clear from ([10], 42.1).
$(ii) \Leftrightarrow (iii)$ Clear from Lemma 14. □

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