Contact open books:
Classical invariants and the binding sum

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Abstract

In the light of the Giroux correspondence of open books and contact structures, it is natural to study Legendrian knots embedded in a page of a compatible open book of a contact 3-manifold. Legendrian knots, as well as their classical invariants, provide useful information on the ambient contact manifold. This thesis develops formulas for deciding whether a given knot lying on a page of a compatible open book whose monodromy is encoded as a concatenation of Dehn twists along non-isolating curves is nullhomologous, and, if so, for computing its classical invariants. Similarly, we show how to compute the Poincaré dual of the Euler class of the contact structure and how to compute the d_3-invariant of the contact structure in case the Euler class is torsion. All invariants can directly be computed from data included in the open book, namely via the intersection behaviour of the knot, an arc basis of the page and the Dehn twist curves encoding the monodromy.

We then turn to higher-dimensional manifolds, for which the relation of open books and contact structures remains partly intact. In particular, every contact structure on a manifold is supported by a compatible open book.

First purely topologically and also in even dimensions, we study fibre connected sums in the context of open book decompositions and introduce a new class of submanifolds, called nested open books, which are particularly well adapted to this setting. We show that the fibre connected sum of an open book along diffeomorphic binding components – called the binding sum – admits a natural open book decomposition provided the respective binding components admit open book structures themselves, which is no restriction in odd dimensions.

Furthermore, we prove that in case the binding sum is performed in a contact open book supporting a given contact structure on the manifold, the construction can be adapted such that the resulting natural open book decomposition is compatible with the contact structure obtained by the usual contact fibre connected sum along contactomorphic binding components.

Zusammenfassung

Nullhomologe Legendre-Knoten und ihre klassischen Invarianten liefern viele Informationen über die ambiente dreidimensionale Kontaktmannigfaltigkeit. Im Hinblick auf die Giroux-Korrespondenz von offenen Büchern und Kontaktstrukturen ist es daher natürlich, Legendreknoten zu studieren, die in eine Seite eines kompatiblen offenen Buchs einer dreidimensionalen Kontaktmannigfaltigkeit eingebettet sind. Diese Arbeit entwickelt Formeln, um zu entscheiden, ob ein gegebener, in einer Seite eines offenen Buchs enthaltener Knoten nullhomolog ist und in diesem Fall auch zur Berechnung seiner klassischen Invarianten. Dabei werden wir voraussetzen, dass die Monodromie durch eine Verkettung von Dehn-Twists entlang nicht-isolierender Kurven beschrieben wird. Außerdem wird aufgezeigt, wie das Poincaré-Duale der Euler-Klasse der Kontaktstruktur und, vorausgesetzt die Euler-Klasse ist eine Torsionsklasse, die d_3-Invariante der Kontaktstruktur bestimmt werden können. Alle Invarianten können direkt aus einer Beschreibung der Mannigfaltigkeit als abstraktes offenes Buch, genauer, durch das Schnittverhalten des Knotens, einer Bogenbasis der Seite und der Dehn-Twist-Kurven, berechnet werden.
Danach wenden wir uns höherdimensionalen Mannigfaltigkeiten zu, für welche die Beziehung zwischen Kontaktstrukturen und offenen Büchern teilweise erhalten bleibt; insbesondere wird jede Kontaktmannigfaltigkeit von einem kompatiblen offenen Buch getragen. Wir untersuchen – vorerst rein topologisch und auch in geraden Dimensionen – Fasersummen im Kontext offener Bücher und führen eine neue Klasse von Untermannigfaltigkeiten, die Daumenkinos, ein, welche gut an diese Situation angepasst sind. Wir beweisen, dass die faservebundene Summe eines offenen Buchs entlang diffeomorpher Bindungskomponenten, genannt Bindungssumme, in natürlicher Weise die Struktur eines offenen Buchs besitzt, wenn die Bindungskomponenten selbst als offenes Buch beschrieben werden können. Diese Bedingung stellt in ungeraden Dimensionen keine Einschränkung dar.
Darüberhinaus zeigen wir, dass im Falle der Kontaktbindungssumme, d.h. der Bindungssumme entlang kontaktomorpher Bindungskomponenten eines kompatiblen offenen Buchs einer Kontaktmannigfaltigkeit, die Konstruktion derart angepasst werden kann, dass das resultierende offene Buch kompatibel zur Kontaktstruktur ist, die durch die gewöhnliche kontaktgeometrische Version der faserverbundenen Summe entsteht.

Contents

Introduction

An *open book decomposition* of a manifold consists of a codimension two submanifold and a fibration of its complement over the circle, which is of a standard form in a neighbourhood of the submanifold. In 1923 Alexander [1] proved that every closed oriented 3-manifold admits an open book decomposition. In fact, combining the work of Winkelnkemper, Lawson and Quinn from the 1970s, this statement remains true for odd-dimensional manifolds in general (see [75], [54], and [69] respectively). The existence problem in even dimensions is also solved in these works but is more involved.

In 1971 Thurston and Winkelnkemper [72] used open books to construct *contact structures* on 3-manifolds. Furthermore, as was observed by Giroux [41] in 2002, contact structures in dimension three are of purely topological nature: he established a one-to-one correspondence between isotopy classes of contact structures and open book decompositions up to positive stabilisation. This correlation remains partially intact in higher dimensions. According to Giroux and Mohsen [43] any contact structure on a closed manifold of dimension at least three admits a *compatible* open book decomposition.

A contact structure on a manifold is a maximally non-integrable tangential hyperplane field. This means that contact structures are distributions which are as far from being integrable, i.e. defining a foliation, as possible. In particular, there are no surfaces tangent to the contact planes in a 3-dimensional contact manifold. Knots, however, can be everywhere tangent to the contact planes; these are called *Legendrian*. Together with the other natural class of knots, the *transverse* knots, which are everywhere transverse to the contact structure, Legendrian knots encode a lot of the geometry of a contact 3-manifold. This information is partly preserved in the classical invariants – a basic, yet useful set of invariants of *nullhomologous* Legendrian and transverse knots. For example, a contact structure is overtwisted if and only if there exists a Legendrian unknot with vanishing *Thurston–Bennequin invariant*.

The classical invariants of knots in the unique tight contact structure of the 3-sphere can easily be computed from their *front projections*. A natural extension of this to general manifolds is to consider knots in *contact surgery diagrams* and try to compute the invariants from these representations. Pioneered by Lisca, Ozsváth, Stipsicz and Szabó [57], this problem has been worked on by various people, e.g. Geiges and Onaran [38], Conway [13] and Kegel [49]. A joint article [20] of Kegel and the author on this topic can – for the sake of completeness – also be found in

Appendix A, as its results are used in the main part of this thesis.

In the light of the Giroux correspondence of open books and contact structures another natural way to present a Legendrian knot is to put it on the page of a compatible open book of the contact 3-manifold. Note that this imposes no restriction, since every Legendrian knot can be realised on a page of a compatible open book. On the other hand, a large class of simple closed curves on the page of an open book, namely the *non-isolating* ones, represent Legendrian knots.

One part of this thesis develops formulas to decide if a knot on the page of an open book is nullhomologous and if so, compute its classical invariants, as well as the *Poincaré dual to the Euler class* of the contact structure and the *d_3-invariant*, an invariant of the contact structure considered as a plane field, provided the Euler class is torsion. Previous results in this direction have been obtained by Etnyre and Özbağcı [34], who gave a formula to compute the Euler class and the d_3-invariant of a contact open book using a different approach, and Li and Wang [55], who used Etnyre and Özbağcı's result to calculate the rotation number of a Legendrian knot on the page of an open book in some cases. On the other hand, Gay and Licata [36] studied Legendrian knots in open books which in general are not contained in a page by a generalisation of the front projection, where it is possible to compute the Thurston–Bennequin invariant as well.

In Chapter 3, we will first consider knots in a more general situation: knots sitting on a *Heegaard surface* of a 3-manifold which has the additional property of being *convex* in the sense of Giroux. The presentation of the first homology of the manifold in terms of the homology of the surface provides a tool to decide whether a given Legendrian knot is nullhomologous as well as to compute its Thurston–Bennequin invariant. As a compatible open book always yields a Heegaard decomposition with the required properties, this can then be used to solve the problem in open books as well. The result is a formula to compute the Thurston–Bennequin invariant in terms of the intersection behaviour of the knot, an arc basis of the surface and the Dehn twist curves encoding the monodromy of the open book.

To compute the other classical invariant of a nullhomologous Legendrian knot, the *rotation number*, this approach is bound to fail, since – in contrast with the Thurston–Bennequin invariant – it is not purely homological. Therefore, our strategy in Chapter 4 will be to transform the open book into a suitable contact surgery diagram via an algorithm of Avdek [3] and combine this with the method of computing invariants in surgery diagrams presented in Appendix A. This will enable us to attain formulas – again in terms of the intersection behaviour of knot, Dehn twist curves and an arc basis of the page – for the classical invariants of a nullhomologous Legendrian knot and its transverse push-off and also for the Poincaré dual to the

Euler class and d_3-invariant of the contact structure.

In Chapter 5 we turn our attention to higher-dimensional open books, and we investigate how the *binding sum* construction, i.e. the fibre connected sum of two open books along diffeomorphic binding components, affects the underlying open book structures. While the mere existence of an open book decomposition on the binding sum immediately follows from the above mentioned work of Winkelnkemper, Lawson and Quinn in odd dimensions (and can easily be shown also in even dimensions (cf. Section 2.2)), these existence results give no relation of such an open book to the open book structures of the original manifolds. We will show that – provided the respective binding components admit open book decompositions themselves – the binding sum can be performed such that the resulting open book structure is natural in the sense that it can be described in terms of the original decompositions. Furthermore, we will show that in the case of the *contact* binding sum, i.e. a binding sum of two contact manifolds with contact open book decompositions along contactomorphic binding components, the construction can also be adapted to again yield a compatible open book. This generalises the work of Klukas [53] to higher dimensions. Note that the requirement of the binding components to admit open books themselves is not a restriction in odd dimensions.

Along the way, we will introduce a new class of submanifolds, namely *nested open books*, which are submanifolds inheriting an open book structure from the ambient manifold and are thus a natural generalisation of a *spinning* as discussed in contact topology by Mori [62] and Martínez Torres [60]. Nested open books turn out to be particularly useful when performing fibre connected sums. The idea of the binding sum construction is not to form the sum along the binding components themselves but along slightly isotoped copies, realising them as nested open books.

As an application, we explain how binding sums can be used to describe compatible open book decompositions of fibrations over the circle whose fibres are convex in the sense of Giroux, as well as of manifolds containing the higher-dimensional analogue of *Giroux torsion* introduced by Massot, Niederkrüger and Wendl [59].

1
Contact manifolds

This chapter contains a brief introduction to contact topology. We will focus on terminology and results needed in later chapters of this thesis and mainly follow [37] and the short overview given in [18] (which was partially published in [19]). Parts of Section 1.4 are also based on the presentation in [50]. The first section, however, introduces the required concepts from symplectic geometry. The results in this chapter are classical and well-known with the exception of Section 1.5.1 and the observations in Remarks 1.1.11 and 1.5.3, which will be needed in Chapter 5. For a more comprehensive approach to the topic we refer the reader to [11, 37, 61].

1.1 Symplectic manifolds and symplectic vector bundles

Let V be an m-dimensional real vector space and $\Omega\colon\ V \times V \to \mathbb{R}$ a bilinear map. We call Ω **skew-symmetric** if $\Omega(u,v) = -\Omega(v,u)$ for all $u, v \in V$. A skew-symmetric form Ω on a real vector space V is called **symplectic** if it is non-degenerate, i.e. if the map $\widetilde{\Omega}\colon\ V \to V^*$, defined by $\widetilde{\Omega}(u)(v) = \Omega(u,v)$, is bijective. In that case, (V, Ω) is a **symplectic vector space**. Note that the dimension of a symplectic vector space is even. For a subspace U of (V, Ω) we define the **symplectic complement** to be

$$U^\perp := \{v \in V\colon\ \Omega(v,u) = 0\ \forall u \in U\}.$$

A subspace U of (V, Ω) is said to be **symplectic** if $\Omega|_U$ is symplectic and **isotropic** if $\Omega|_U = 0$. An isotropic subspace of dimension $1/2 \cdot \dim V$ is called **Lagrangian**.

Remark 1.1.1
If $U \subset (V, \Omega)$ is an isotropic subspace, then $U^\perp \big/ U$ inherits a well-defined symplectic structure. Indeed, if $v, v' \in U^\perp$ and $u, u' \in U$, then $\Omega(v+u, v'+u') = \Omega(v, v')$ since $U \subset U^\perp$, and if we have $w_0 \in U^\perp$ such that $\Omega(w_0, w) = 0$ for all $w \in U^\perp$, then $w_0 \in \left(U^\perp\right)^\perp = U$.

Now let W be a smooth manifold and ω a closed 2-form on W such that ω_x is symplectic on T_xW for all $x \in W$. We call the pair (W, ω) a **symplectic manifold**.

Example 1.1.2 (Standard symplectic $\mathbb{R}^{2n}$)
Consider $\mathbb{R}^{2n}$ with Cartesian coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The closed and non-degenerate, i.e. symplectic, form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is the **standard symplectic form** on $\mathbb{R}^{2n}$.

The special types of subspaces of symplectic vector spaces can of course be generalised to properties of submanifolds of symplectic manifolds. A submanifold X of (W, ω) is said to be **symplectic** if $\omega_x|_{T_xX}$ is symplectic for all $x \in X$ and **isotropic** if $\omega_x|_{T_xX} = 0$ for all $x \in X$. An isotropic submanifold of (W, ω) of dimension $1/2 \cdot \dim W$ is called **Lagrangian**.

A diffeomorphism $\varphi\colon (W_1, \omega_1) \to (W_2, \omega_2)$ between symplectic manifolds is a **symplectomorphism** if $\varphi^*\omega_2 = \omega_1$.

Theorem 1.1.3 (Moser (relative version))
Let W be a manifold and X a compact submanifold. Let ω_0, ω_1 be symplectic forms on W such that $\omega_0|_p = \omega_1|_p$ for all $p \in X$. Then there are neighbourhoods U_0, U_1 of X in W and a diffeomorphism $\varphi\colon U_0 \to U_1$ such that $\varphi|_X = id_X$ and $\varphi^*\omega_1 = \omega_0$.

The proof uses a relative version of the Poincaré Lemma, see [11] for details. We will also need another type of Moser theorem in later applications. It can be proved by choosing a metric on the manifold and applying Hodge theory to ensure the smoothness of a family of 1-forms, which are the primitives of the t-derivative of the family of symplectic forms.

Theorem 1.1.4 (Moser, cf. [61, Theorem 3.17])
Let W be a closed manifold and ω_t a family of cohomologous symplectic forms on W. Then there is an isotopy ψ_t (in particular, $\psi_0 = \mathrm{id}_W$) with $\psi_t^*\omega_t = \omega_0$.

Using the above relative Moser theorem, one can prove Darboux's theorem, which says that all symplectic manifolds of a given dimension are locally symplectomorphic. In particular, there are no local invariants in symplectic geometry.

Theorem 1.1.5 (Darboux)
Let (W, ω) be a $2n$-dimensional symplectic manifold and $p \in W$. Then there is a coordinate system $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ about p such that $\omega|_U = \sum_{i=1}^n dx_i \wedge dy_i$.

Let (W, ω) be a symplectic manifold and $H\colon W \to \mathbb{R}$ a smooth function. We call H a **Hamiltonian** on W. A vector field X on (W, ω) is called **symplectic** if $\mathcal{L}_X\omega = 0$ (i.e. the flow of X preserves ω). By Cartan's formula, X is symplectic if and only if $i_X\omega$ is closed. A vector field X on (W, ω) is called **Hamiltonian** if $i_X\omega$ is exact. We define the (unique) Hamiltonian vector field X_H of a Hamiltonian function H by $i_{X_H}\omega = -dH$.[1] A vector field Y on (W, ω) is called **Liouville** if $\mathcal{L}_Y\omega = \omega$. Note that the Lie derivative of the symplectic form in the direction of a Hamiltonian vector field vanishes by Cartan's formula. As a consequence, the sum of a Liouville and a Hamiltonian vector field is Liouville.

[1] There are different sign conventions in the literature.

Example 1.1.6
The radial vector field

$$Y = \frac{1}{2}\sum_{i=1}^{n}(x_i\partial_{x_i} + y_i\partial_{y_i})$$

is a Liouville vector field on the standard symplectic $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$.

Example 1.1.7 (Cotangent bundles)
Let M be a manifold. Then the cotangent bundle T^*M of M carries a canonical symplectic structure. Indeed, there is a canonical 1-form, the so-called **Liouville form**, λ on T^*M:

$$\lambda(v) = \xi\Big(d\pi(v)\Big)$$

for $v \in T_{(x,\xi)}T^*M$. Choosing compatible local coordinate systems $(x_1, \dots, x_n)$ on M and $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on T^*M, the Liouville form can be expressed as

$$\lambda = \sum_{i=1}^{n} \xi_i dx_i.$$

Its exterior derivative $d\lambda =: \omega$ is clearly a symplectic form with local expression

$$\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i.$$

By non-degeneracy of the symplectic form, ω and λ can be used to define a Liouville vector field ν by the condition $i_\nu\omega = \lambda$, which, in our suitable local coordinates, is just the radial vector field

$$\nu = \sum_{i=1}^{n} \xi_i\partial_{\xi_i}.$$

A smooth vector bundle $E \to B$ over a manifold B equipped with a smoothly varying symplectic form ω_b on each fibre is called **symplectic vector bundle**. A **complex structure** on a vector bundle $E \to B$ is a smooth section J of $\mathrm{End}(E)$ such that $(J_b)^2 = -\mathrm{id}_{E_b}$ for all $b \in B$. A complex structure J on a symplectic vector bundle (E, ω) is called **ω-compatible** if J is fibre-wise compatible, i.e. if $\omega_b(Ju, Jv) = \omega_b(u, v)$ for all $b \in B$ and $u, v \in E_b$ and $\omega_b(u, Ju) > 0$ for all $b \in B$ and $u \in E_b \setminus \{0\}$. A complex structure on the tangent bundle TW of a smooth manifold W is called **almost complex structure** on W. Note that the space $\mathcal{J}(\omega)$ of ω-compatible complex structures on a symplectic vetor bundle E is non-empty and contractible (see [37, Proposition 2.4.5]).

Apart from Darboux's theorem, there are also neighbourhood theorems for the special types of submanifolds introduced above. We will only state the symplectic neighbourhood theorem, which says that a neighbourhood of a symplectic submanifold is determined by a symplectic vector bundle over the submanifold, the *symplectic normal bundle*. The **symplectic normal bundle** $\mathrm{SN}_W(X)$ of a symplectic submanifold X of a symplectic manifold (W, ω) is defined as the symplectic vector bundle over X with fibre the symplectic complement of the tangent space of X, i.e. $\mathrm{SN}_W(X) := (TX)^\perp$.

Theorem 1.1.8 (Symplectic neighbourhood theorem)
Let (W_j, ω_j) be symplectic manifolds with compact symplectic submanifolds $X_j \subset W_j$ ($j = 0, 1$). Suppose there exists an isomorphism $\Phi\colon SN_{W_0}(X_0) \to SN_{W_1}(X_1)$ of the symplectic normal bundles that covers a symplectomorphism

$$\phi\colon (X_0, \omega_0|_{TX_0}) \to (X_1, \omega_1|_{TX_1}).$$

Then ϕ extends to a symplectomorphism $\psi\colon \mathcal{N}(X_0) \to \mathcal{N}(X_1)$ of neighbourhoods such that $T\psi$ induces Φ on $SN_{W_0}(X_0)$.

To prove the neighbourhood theorem, one uses the exponential map to transform the bundle map into a map of neighbourhoods. Pulling back the symplectic form leads to the situation of two symplectic forms agreeing along a symplectic submanifold. The theorem then follows by using Theorem 1.1.3.

We will also need an extension theorem for symplectic isotopies.

Theorem 1.1.9 (Banyaga, cf. [61, Theorem 3.19])
Let (W, ω) be a compact symplectic manifold and $X \subset W$ compact such that X is a deformation retract of a neighbourhood of X. Assume that $H^2(W, X; \mathbb{R}) = 0$ and suppose furthermore that $\phi_t\colon U \to W$ is a symplectic isotopy of an open neighbourhood U of X in W. Then there exist a neighbourhood $\mathcal{N} \subset U$ of X and a symplectic isotopy $\psi_t\colon W \to W$ such that $\psi_t|_\mathcal{N} = \phi_t|_\mathcal{N}$.

Proof. Choose a neighbourhood $\mathcal{N} \subset U$ of X which retracts to X. Then we have $H^*(\mathcal{N}, X; \mathbb{R}) = 0$ and thus also $H^2(W, \mathcal{N}; \mathbb{R}) = 0$ by the long exact sequence of the triple $(X, \mathcal{N}, W)$. The restriction $\phi_t|_\mathcal{N}$ can be extended to W by *diffeomorphisms* ρ_t. This defines a family of symplectic forms $\omega_t := \rho_t^*\omega$ and we can consider the derivative $\tau_t := \frac{d}{dt}\omega_t$, which is closed. As ρ_t extends ϕ_t, which is symplectic on $\mathcal{N}$, the restriction of τ_t to $\mathcal{N}$ vanishes. Therefore τ_t defines a class in $H^2(W, \mathcal{N}; \mathbb{R}) = 0$. By Hodge theory, there are 1-forms σ_t satisfying $\sigma_t|_\mathcal{N} = 0$ and $d\sigma_t = \tau_t$. These determine a time-dependent vector field, whose flow pulls back ω_t to ω and is the identity on $\mathcal{N}$. Composing the flow with the diffeomorphisms ρ_t yields the desired symplectic isotopy. □

The cohomological condition is essential in the above proof. However, if we restrict to symplectic submanifolds, there is also a version of the theorem due to Auroux omitting the requirement on cohomology. It is worth noting that Auroux's theorem does not *extend* an isotopy of symplectic submanifolds or even an isotopy of open neighbourhoods, but yields an isotopy with image a given family of symplectic submanifolds. These in turn can always be assumed to *arise* by an isotopy by a Moser argument.

Theorem 1.1.10 (Auroux [2, Proposition 4])
Let $(X_t)_{t\in[0,1]}$ be a family of symplectic submanifolds in a compact symplectic manifold (W,ω). Then there exists an isotopy $\psi_t\colon W\to W$ satisfying $\psi_t(X_0)=X_t$.

Proof. As mentioned above, the submanifolds X_t can be assumed to arise as the image of an isotopy of symplectic submanifolds $\phi_t\colon X=X_0\to X_t\subset W$. Then the symplectic neighbourhood theorem 1.1.8 provides a tool for extending this isotopy to a tubular neighbourhood $\mathcal{N}$ of X. As in the proof of Banyaga's extension theorem 1.1.9, this can be extended to a family of diffeomorphisms ρ_t. Also as above, we define $\omega_t:=\rho_t^*\omega$ and observe that $\tau_t:=\frac{d}{dt}\omega_t$ defines the zero class in $H^2(W;\mathbb{R})$ since the ω_t are cohomologous. Thus, there exist 1-forms σ_t on W with $d\sigma_t=\tau_t$. However, we cannot guarantee the existence of such σ_t that also vanish on $\mathcal{N}$ unless τ_t also represents the zero class in the relative cohomology group $H^2(W,\mathcal{N};\mathbb{R})$. This means that if we use σ_t to define a vector field ξ_t via $i_{\xi_t}\omega=-\sigma_t$, the resulting flow composed with ρ_t is a symplectomorphism but will in general not map X_0 to X_t. We now want to achieve this by using σ_t to find an appropriate antiderivative α_t by hand.

For the resulting isotopy ψ_t to be symplectic, we need $d\alpha_t=\tau_t$. Furthermore, we need that the flow of the corresponding vector field ξ_t preserves X_0, i.e. ξ_t is tangent to X_0 for all t. In terms of the forms α, this translates into the condition that the symplectic complement of the tangent space to X_0 has to be contained in the kernel of α_t.

We have that $d\sigma_t|_{\mathcal{N}}=\tau_t|_{\mathcal{N}}=0$, i.e. σ_t defines a class in $H^1(\mathcal{N};\mathbb{R})$. Also, restricting σ_t to the tangent space of X_0 yields closed 1-forms on X_0. We can identify the tubular neighbourhood $\mathcal{N}$ with a neighbourhood of the zero section of the symplectic normal bundle $\mathrm{SN}_W(X)$ and denote the bundle map by π. Then $\gamma_t:=\pi^*(\sigma_t|_{TX_0})$ defines a family of closed 1-forms on $\mathcal{N}$ containing the symplectic normal spaces to X_0 in its kernel. Furthermore, the classes induced by γ_t and $\sigma_t|_{\mathcal{N}}$ agree in $H^1(\mathcal{N};\mathbb{R})$. Thus, there are functions $f_t\colon\mathcal{N}\to\mathbb{R}$ such that $\gamma_t=\sigma_t+df_t$. If we extend the functions f_t to all of W and denote the extension by g_t, then the 1-forms $\alpha_t=\sigma_t+dg_t$ are as desired. Indeed, on $\mathcal{N}$ they agree with γ_t and thus contain the symplectic normal spaces to X_0 in their kernel and we also have $d\alpha_t=d\sigma_t=\tau_t$, i.e. the flow of the vector field ξ_t induced by α_t pulls back ω_t to $\omega_0=\omega$. □

Remark 1.1.11 (A special choice of isotopy for trivial normal bundle)
If $X=X_0$ in the setting of Theorem 1.1.10 has trivial symplectic normal bundle,

the resulting isotopy can be assumed to be of a special form. By the symplectic neighbourhood theorem 1.1.8 we can identify a neighbourhood X_t with $X_t \times D^2_\varepsilon$ with symplectic form given as $\omega|_{TX_t} + dx \wedge dy$. The first step in the proof of Auroux's theorem is to extend the isotopy of symplectic submanifolds ϕ_t to an open neighbourhood and then extend this to diffeomorphisms ρ_t to the whole manifold. In our setting, these extensions can be chosen such that the restriction of ρ_t to the neighbourhood $X \times D^2_\varepsilon$ of X is $\phi_t \times \mathrm{id}_{D^2_\varepsilon}$. Furthermore, observe that the vector fields ξ_t induced by the 1-forms $\gamma_t = \pi^*(\sigma_t|_{TX})$ are not only tangent to X but in fact tangent to $X \times \{p\} \subset X \times D^2_\varepsilon$ for any $p \in D^2_\varepsilon$. Thus, the resulting isotopy is of the form $\widetilde{\phi}_t \times \mathrm{id}_{D^2_\varepsilon}$ on the neighbourhood $X \times D^2_\varepsilon$, where $\widetilde{\phi}_t$ is a family of symplectomorphisms on X.

1.2 Contact structures and elementary results

A **contact structure** on a smooth manifold M of dimension $2n+1$ is a maximally non-integrable hyperplane field $\xi \subset \mathrm{T}M$. Locally, a tangential hyperplane field can be written as the kernel of a 1-form α. Non-integrability then translates into the condition $\alpha \wedge (d\alpha)^n \neq 0$ for a defining 1-form α. There is a global expression $\xi = \ker \alpha$ if and only if the quotient bundle TM/ξ is trivial (see [37, Lemma 1.1.1]), in which case we call ξ **coorientable**. In this thesis we will only consider coorientable contact structures and refer to a defining 1-form on M as a **contact form**. The pair (M, ξ) consisting of a manifold M and a contact structure ξ on M is called **contact manifold**.

Remark 1.2.1

1. The condition $\alpha \wedge (d\alpha)^n \neq 0$ is independent of the choice of α.
2. The 2-form $d\alpha_p$ is non-degenerate on ξ_p for all $p \in M$.
3. In dimension three, the contact condition $\alpha \wedge d\alpha \neq 0$ is equivalent to the non-existence of a surface tangent of order two to the plane field in any point (see [37, Theorem 1.6.2]).

Example 1.2.2 (Standard contact structure on $\mathbb{R}^{2n+1}$)
Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ and the 1-form

$$\alpha_{st} = dz + \sum_{i=1}^{n} x_i dy_i.$$

Then α_{st} defines a contact structure on $\mathbb{R}^{2n+1}$:

$$\alpha_{st} \wedge (d\alpha_{st})^n = n! \cdot dz \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n \neq 0.$$

We call this the **standard contact structure** ξ_{st} on $\mathbb{R}^{2n+1}$.

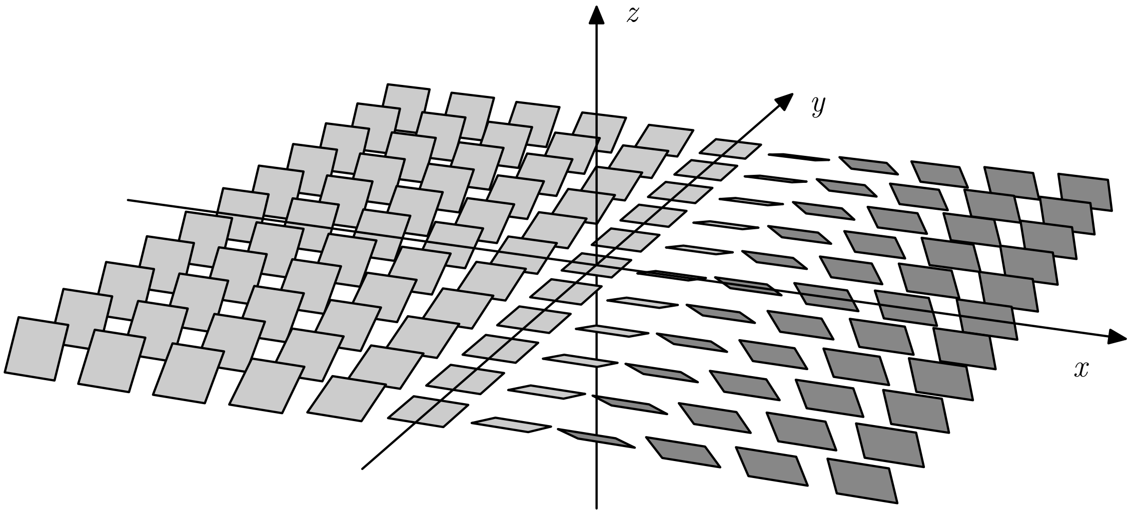


Figure 1.1: The standard contact structure on $\mathbb{R}^3$ (slightly modified version of `https://en.wikipedia.org/wiki/File:Standard_contact_structure.svg`).

The above contact structure is the prototype of a contact structure in the sense that any contact structure on any manifold of dimension $2n+1$ locally, i.e. in a neighbourhood of a point, looks like the standard contact structure on $\mathbb{R}^{2n+1}$ (cf. Theorem 1.2.9). To formulate results like these, it is reasonable to first define an appropriate notion of equivalence, i.e. maps respecting contact structures. A diffeomorphism $f\colon (M_1, \xi_1 = \ker\alpha_1) \to (M_2, \xi_2 = \ker\alpha_2)$ between two contact manifolds is called **contactomorphism** if its differential Tf maps the contact structure ξ_1 on M_1 to the contact structure ξ_2 on M_2, i.e. if there is a function $\lambda\colon M_1 \to \mathbb{R}\setminus\{0\}$ with $f^*\alpha_2 = \lambda\alpha_1$. Two contact manifolds are said to be **contactomorphic** if there exists a contactomorphism between them.

Fixing a contact form α on a manifold M yields a distinguished vector field on M, the so-called **Reeb vector field** associated with the contact form α. It is defined as the unique vector field R_α on M satisfying $d\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) \equiv 1$. Note that the Reeb vector field and its dynamics are not data that can be assigned to a contact structure but only to contact forms. Changing the contact form or applying a contactomorphism can drastically change the Reeb dynamics.

Example 1.2.3
The Reeb vector field of the standard contact form $dz + \sum_{i=1}^n x_i dy_i$ on $\mathbb{R}^{2n+1}$ as introduced in Example 1.2.2 form is ∂_z.

A vector field X on a contact manifold is said to be a **contact vector field** if it satisfies

$$\mathcal{L}_X\alpha = f\alpha$$

for some function $f\colon M \to \mathbb{R}$. Contact vector fields on a manifold M are in one-to-one correspondence with smooth functions on M (cf. [37, Theorem 2.3.1]). The assignment requires fixing a contact form α and is given by

$$\begin{aligned} X \mapsto &\ \ \alpha(X), \\ X_h \leftarrow\!\shortmid &\ \ h, \end{aligned}$$

where X_h is the unique vector field satisfying

$$\begin{aligned} \alpha(X_h) &= \quad h, \\ d\alpha(X_h, \cdot) &= \ dh(R_\alpha)\,\alpha - dh. \end{aligned} \tag{1.1}$$

Observe that X_h is indeed uniquely defined since $d\alpha$ restricts to a symplectic form on $\xi = \ker\alpha$. We also call a function h a **contact Hamiltonian** and the corresponding vector field X_h the **contact Hamiltonian vector field associated with** h.

Remark 1.2.4 (Rescaling the contact form)
Fix a contact form α on M and consider a positive function h on M. If we rescale the contact form α to $\widetilde{\alpha} := (1/h)\cdot\alpha$, the contact vector field X_h associated with h becomes the Reeb vector field of the new contact form $\widetilde{\alpha}$. Indeed, the property of being a contact vector field is independent of the choice of contact form, so by the one-to-one correspondence of contact vector fields and functions discussed above, we only have to verify $\widetilde{\alpha}(X_h) = 1$, which is clearly satisfied. This observation plays an essential role in [39].

A Liouville vector field which is transverse to a hypersurface in a symplectic manifold induces a contact structure on the hypersurface:

Lemma 1.2.5
Let (W,ω) be a $(2n+2)$-dimensional symplectic manifold and Y a Liouville field transverse to a hypersurface M in W. Then $\alpha := i_Y\omega$ induces a contact form on M.

Proof. We have

$$\alpha \wedge (d\alpha)^n = i_Y\omega \wedge \Big(d(i_Y\omega)\Big)^n = i_Y\omega\wedge\omega^n = \frac{1}{n+1}\, i_Y(\omega^{n+1}).$$

As Y is transverse to M, this expression does not vanish on M. □

If the hypersurface with transverse Liouville field is furthermore given as the level set of a function h, then the (symplectic) Hamiltonian vector field is tangent and coincides, up to reparametrisation, with the Reeb field of the induced contact form. The situation in the previous lemma is not exotic at all, as the following example shows.

Example 1.2.6 (Symplectisation)
Let $(M, \xi = \ker\alpha)$ be a contact manifold. Define $W := \mathbb{R} \times M$ and a 2-form $\omega := d(e^t\alpha) = e^t(dt \wedge \alpha + d\alpha)$, where t denotes a coordinate on the $\mathbb{R}$-factor and α is identified with its pull-back under the projection map $\pi\colon \mathbb{R} \times M \to M$. One can easily check that (W, ω) is a symplectic manifold and ∂_t a Liouville vector field. We call (W, ω) the **symplectisation** of $(M, \xi = \ker\alpha)$.

Example 1.2.7 (The standard contact structure on S^{2n-1})
As seen in Example 1.1.6 the radial vector field Y on $\mathbb{R}^{2n}$ is a Liouville vector field for the standard symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Hence, it induces a contact form α on the unit sphere by Lemma 1.2.5, the so-called **standard contact form** on S^{2n-1}. We have

$$\alpha = i_Y\omega = \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i).$$

Its Reeb vector field is equal to

$$R_\alpha = 2\sum_{i=1}^{n}(x_i\partial_{y_i} - y_i\partial_{x_i}).$$

For S^3, the Reeb orbits are exactly the fibres of the Hopf fibration

$$\begin{aligned} \mathbb{C}^2 \supset S^3 &\to \quad S^2 = \mathbb{C}P^1 \\ (z_1, z_2) &\mapsto \quad (z_1 : z_2) \end{aligned}$$

(see [37, Lemma 1.4.9]).

The following theorem states that a contact structure on a closed manifold cannot be deformed in a non-trivial way.

Theorem 1.2.8 (Gray stability theorem, cf. [37, Theorem 2.2.2])
Let M be a closed manifold and ξ_t, $t \in [0,1]$, a smooth family of contact structures on M. Then there is an isotopy ψ_t $(t \in [0,1])$ with $T\psi_t(\xi_0) = \xi_t$.

Probably the easiest way to prove it is, in analogy to the proofs of the Moser theorems stated in the previous subsection, via a so-called **Moser trick**. It is assumed that the searched isotopy is the flow of a (time-dependant) vector field. The requirements to the isotopy then translate into conditions on the vector field. If a vector field meeting these conditions exists, the isotopy can be obtained by integration (cf. [37, pp. 60–61]).

We are now going to state some basic neighbourhood theorems for certain types of submanifolds of contact manifolds – the simplest being Darboux's theorem, which describes a standard neighbourhood of a point.

Theorem 1.2.9 (Darboux's theorem, cf. [37, Theorem 2.5.1])
Let M^{2n+1} be a contact manifold, α a contact form on M and let $p \in M$ be a point. Then there is a coordinate system $(U, x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ about p such that $\alpha|_U = dz + \sum_{i=1}^n x_i \wedge dy_i$.

To formulate more sophisticated theorems, we first have to define special types of submanifolds and inspect their normal bundles. A submanifold L of a $(2n+1)$-dimensional contact manifold (M, ξ) is called **isotropic** if $T_pL \subset \xi_p$ for all p in N. An isotropic submanifold of dimension n is called **Legendrian**. The **(conformal) symplectic normal bundle** of an isotropic submanifold L of $(M, \xi = \ker \alpha)$ is

$$\mathrm{CSN}_M(L) = \Big((TL)^\perp \big/ TL,\ d\alpha\Big).$$

Notice that this is indeed a symplectic bundle by Remark 1.1.1. If we do not fix a contact form, the induced structure is only a conformal symplectic structure.

Theorem 1.2.10 (Isotropic neighbourhood theorem, [37, Theorem 2.5.8])
Let (M_i, ξ_i), $i = 0, 1$, be contact manifolds with closed isotropic submanifolds L_i. Suppose there is an isomorphism of conformal symplectic normal bundles

$$\Phi\colon\ CSN_{M_0}(L_0) \to CSN_{M_1}(L_1)$$

that covers a diffeomorphism $\phi\colon\ L_0 \to L_1$. Then this diffeomorphism ϕ extends to a contactomorphism $\psi\colon\ \mathcal{N}(L_0) \to \mathcal{N}(L_1)$ of suitable neighbourhoods $\mathcal{N}(L_i)$ of L_i such that the bundle maps $T\psi|_{CSN_{M_0}(L_0)}$ and Φ are bundle homotopic (as conformal symplectic bundle isomorphisms).

In particular, diffeomorphic *Legendrian* submanifolds possess contactomorphic neighbourhoods (cf. Example 1.4.1).

A submanifold M' of a contact manifold $(M, \xi = \ker \alpha)$ is a **contact submanifold** if $\xi' := TM' \cap \xi|_{M'}$ is a contact structure on M'. We then have

$$TM|_{M'} = TM' \oplus (\xi')^\perp.$$

The symplectic complement $(\xi')^\perp$ of ξ' in ξ can thus be identified with the normal bundle of M' in M. We define the **conformal symplectic normal bundle** $\mathrm{CSN}_M(M')$ of M' in M to be the bundle $(\xi')^\perp$ together with the conformal symplectic structure induced by $d\alpha$.

Theorem 1.2.11 (Contact neighbourhood theorem, [37, Theorem 2.5.15])
Let (M_i, ξ_i), $i = 0, 1$, be contact manifolds with closed contact submanifolds (M_i', ξ_i'). Suppose there is an isomorphism of conformal symplectic normal bundles

$$\Phi\colon\ CSN_{M_0}(M_0') \to CSN_{M_1}(M_1')$$

that covers a contactomorphism $\phi\colon\ (M_0', \xi_0') \to (M_1', \xi_1')$*. Then* ϕ *extends to a contactomorphism* $\psi\colon\ \mathcal{N}(M_0') \to \mathcal{N}(M_1')$ *of suitable neighbourhoods* $\mathcal{N}(M_i')$ *of* M_i' *such that the bundle maps* $T\psi|_{CSN_{M_0}(M_0')}$ *and* Φ *are bundle homotopic (as conformal symplectic bundle isomorphisms).*

Proof. As a first step, we want to construct contact forms α_i on M_i and a bundle map $TM_0|_{M_0'} \to TM_1|_{M_1'}$ inducing Φ that pulls back α_1 to α_0 and $d\alpha_1$ to $d\alpha_0$. Pick a contact form α_1' for ξ_1' on M_1' and set $\alpha_0' = \phi^*\alpha_1'$. We denote the respective Reeb vector fields by R_i' and choose any contact form α_i for ξ_i on M_i. We can scale α_i such that $\alpha_i(R_i') = 1$ along M_i'. This means that α_i coincides with α_i' when restricted to the tangent space of M_i'. We then also have $d\alpha_i|_{TM_i'} = d\alpha_i'$. Our aim is now to scale α_i again such that $R_i = R_i'$ on M_i', i.e. want to find smooth functions $f_i\colon\ M_i \to \mathbb{R}^+$ with $f_i|_{M_i'} \equiv 1$ and $i_{R_i'} d(f_i\alpha_i) = 0$ on $TM_i|_{M_i'}$. So in particular, we need

$$0 = i_{R_i'} d(f_i\alpha_i) = i_{R_i'}(df_i \wedge \alpha + f_i d\alpha_i) = -df_i + i_{R_i'} d\alpha_i.$$

Such functions f_i exist since $i_{R_i'} d\alpha_i|_{TM_i'} = i_{R_i'} d\alpha_i' \equiv 0$, i.e. we can choose $f_i \equiv 1$ on M_i' and integrate. With these scaled forms α_0 and α_1, we can now scale Φ such that it is a symplectic bundle isomorphism

$$\Big((\xi_0')^{\perp}, d\alpha_0\Big) \to \Big((\xi_1')^{\perp}, d\alpha_1\Big).$$

This yields a bundle map

$$T\phi \oplus \Phi\colon\ TM_0|_{M_0'} \to TM_1|_{M_1'},$$

which pulls back α_1 to α_0 and $d\alpha_1$ to $d\alpha_0$.

The second step of the proof is to use tubular maps and a stability argument to construct the desired contactomorphism. So let $\tau_i\colon\ N(M_i') \to M_i$ be tubular maps and transform the above bundle map into a diffeomorphism

$$\tau_1 \circ \Phi \circ \tau_0^{-1}\colon\ \mathcal{N}(M_0') \to \mathcal{N}(M_1')$$

of neighbourhoods $\mathcal{N}(M_i')$ of M_i' in M_i, which induces the bundle map. Thus, α_0 and $(\tau_1 \circ \Phi \circ \tau_0^{-1})^*\alpha_1$ are contact forms on $\mathcal{N}(M_0')$ that agree on $TM_0|_{M_0'}$ and so do their differentials. We define a family

$$\beta_t = (1-t)\alpha_0 + t(\tau_1 \circ \Phi \circ \tau_0^{-1})^*\alpha_1$$

for $t \in [0,1]$ and can, by the openness of the contact condition, assume that it is a family of contact forms on a possibly smaller neighbourhood $\mathcal{N}(M_0')$. Note that we have $d\beta_t \equiv d\alpha_0$ on $TM_0|_{M_0'}$. By the Gray stability (Theorem 1.2.8) there is an isotopy ψ_t of the neighbourhood $\mathcal{N}(M_0')$ which fixes M_0' and such that $\psi_t^*\beta_t = \lambda_t\alpha_0$ for a smooth family of smooth functions λ_t. Then $\tau_1 \circ \Phi \circ \tau_0^{-1} \circ \psi_1$ is a contactomorphism as desired. □

Example 1.2.12
Observe that the groups SO(2) and U(1) coincide. That means that there is exactly one conformal symplectic structure on an orientable rank two bundle. Thus, the preceding theorem says that codimension two contact manifolds possess standard neighbourhoods, which only depend on the topological bundle type of their normal bundle, which is classified by its Euler number.

Let $(M', \xi' = \ker \alpha') \subset (M, \xi)$ be a contact submanifold of codimension two with trivial normal bundle. Then a neighbourhood of M' is contactomorphic to $(M' \times D^2, \ker(\alpha' + r^2 d\theta))$, where (r, θ) are polar coordinates on D^2.

More generally, if the normal bundle N of M' in M is not necessarily trivial, let γ be a connection 1-form on the unit circle bundle of N, i.e. a normalised 1-form invariant under the circle-action (cf. [37, Definition 7.2.3]). This also defines a 1-form on the normal bundle with the zero-section removed via the pull-back under the natural retraction $\mathbb{R}^2 \setminus \{0\} \to S^1$, which will still be denoted by γ. Denoting the radial coordinate in the fibres of N by r, the form $r^2\gamma$ is a smooth 1-form on all of N. Its exterior derivative $d(r^2\gamma) = 2r dr \wedge \gamma + r^2 d\gamma$ restricts to a volume form on each fibre. Hence, $\alpha' + r^2\gamma$ is a contact form near M'. So a neighbourhood of M' in (M, ξ) is contactomorphic to a neighbourhood of the zero-section of its normal bundle with contact structure given by $\alpha' + r^2\gamma$.

1.3 Hypersurfaces

Consider an oriented hypersurface S in a $(2n + 1)$-dimensional contact manifold $(M, \xi = \ker \alpha)$. The contact structure defines a singular 1-dimensional foliation on the hypersurface S via the distribution

$$(TS \cap \xi|_S)^{\perp}.$$

It is called the **characteristic foliation** of S. Note that if we choose a volume form Ω on S, the characteristic foliation is given by the vector field X defined by the condition $i_X \Omega = \beta \wedge (d\beta)^{n-1}$, where β is the restriction of the contact form α to S (see [37, Lemma 2.5.20]). Giroux [40] proved that the characteristic foliation of a surface in a 3-manifold determines the germ of the contact structure near the surface (cf. [37, Theorem 2.5.22]).

This can be generalised to arbitrary dimension as follows:

Proposition 1.3.1 ([15, Proposition 6.4])
Let S be a closed hypersurface in a contact manifold $(M, \xi = \ker \alpha)$. Then the germ of ξ near S is determined by the 1-form $\alpha|_{TS}$. In particular, if $\xi|_S$ is transverse to S,

then the germ of ξ near S is determined by the codimension 1 *distribution $TS \cap \xi|_S$ on S.*

A special type of surfaces in contact 3-manifolds are so-called *convex* surfaces, which have the property that the contact structure is invariant in a transverse direction in a neighbourhood of the surface. Convex surfaces were introduced by Giroux in his thesis [40] and are a powerful tool in 3-dimensional contact geometry. For instance, they play an important part in the classification of tight contact structures (cf. [37, Section 4.10]). We will only introduce some basic terminology here and point the reader to [40, 29, 37] for more information.

A surface S in a 3-manifold is called **convex** if there is a contact vector field defined near and transverse to S. It turns out that basically all contact geometric information is encoded in a 1-dimensional submanifold of the surface, its *dividing set*. The **dividing set** Γ_S of a convex surface $S \subset (M, \xi)$ with transverse contact vector field Y is the set of points in S where Y is tangent to the contact planes ξ.

Example 1.3.2
Consider the 3-Torus T^3 with the contact form $\alpha = \cos\theta dx - \sin\theta dy$. The 2-Torus $S = \{y = y_0\} \subset T^3$ is convex. Indeed, the vector field $Y = \partial_y$ is contact and transverse to S. We have $\alpha(Y) = -\sin\theta$. Thus, the dividing set is

$$\Gamma_S = \{(x, y_0, 0)\} \cup \{(x, y_0, \pi)\}.$$

The transversality of the contact vector field of a convex surface $S \subset M$ means that there is an embedding $\psi\colon\ S \times \mathbb{R} \to M$ (with $\psi|_{S\times\{0\}}$ the inclusion) inducing an $\mathbb{R}$-invariant contact structure on $S \times \mathbb{R}$, i.e. a convex surface S possesses a neighbourhood $S \times \mathbb{R}$ in which the contact structure is $\mathbb{R}$-invariant. Conversely, if such a neighbourhood exists, the surface is convex, as the $\mathbb{R}$-direction defines a transverse contact vector field. Hence, in an $\mathbb{R}$-invariant neighbourhood $S \times \mathbb{R}$, we can write the contact form as $\alpha = \beta + u dz$, where $\beta = \alpha|_S$ is a 1-form on S, $u\colon\ S \to \mathbb{R}$ a suitable function and z the coordinate of the $\mathbb{R}$-factor. In this description, the dividing set becomes $\Gamma_S = \{u = 0\}$ and can easily seen to be transverse to the characteristic foliation S_ξ (see [37, p. 230]). This motivates the general notion of a set of circles dividing a singular 1-dimensional foliation – we say that a 1-dimensional submanifold Γ **divides** a singular 1-dimensional foliation $\mathcal{F}$ on a closed surface S if

- Γ is transverse to $\mathcal{F}$,
- there is an area form Ω on S and a vector field X defining $\mathcal{F}$ with $\mathcal{L}_X\Omega|_{S\setminus\Gamma} \neq 0$,
- $S \setminus \Gamma$ splits into components $S_\pm$ of positive and negative divergence of X with respect to Ω and X points out of S_+ along Γ.

The property that the characteristic foliation of a convex surface is divided by a collection of embedded circles is in fact equivalent to convexity. Furthermore, the dividing set of a convex surface is determined up to isotopy via curves transverse to S_ξ by the characteristic foliation (cf. [37, Theorem 4.8.5]). On the other hand, any singular foliation divided by the dividing set of a convex surface can be obtained by a perturbation of the surface in an arbitrarily small neighbourhood (see [37, Theorem 4.8.11]). Combined with the above mentioned fact that the characteristic foliation of a surface determines the germ of the contact structure, this means that all information of a contact structure in a neighbourhood of a convex surface is contained in the dividing set of the surface. This is particularly useful for gluing contact structures. Note also that any closed, orientable surface is C^∞-close to a convex surface (see [37, Proposition 4.8.8]).

1.4 Knots and contact 3-manifolds

In this section we restrict ourselves to 3-dimensional contact manifolds.

There are two special types of knots in contact manifolds, which are not only interesting in their own right but also carry a lot geometric information of the ambient contact manifold itself. A knot in a contact manifold (M, ξ) is called **Legendrian** if it is a Legendrian submanifold of (M, ξ), i.e. if it is tangent to the contact structure. A knot is said to be **transverse** if it is transverse to ξ, i.e. if it is a 1-dimensional contact submanifold. Note that a transverse knot in a cooriented contact manifold comes with a preferred orientation. If it is given *some* orientation, we call it **positively** or **negatively transverse** depending on whether the orientation coincides with the one induced by the contact structure.

Legendrian knots possess a standard neighbourhood by Theorem 1.2.10 as do transverse knots by Theorem 1.2.11.

Example 1.4.1

1. Consider the contact manifold $(S^1 \times \mathbb{R}^2, \xi_n := \ker(\cos(n\theta)dx - \sin(n\theta)dy))$. Then $S^1 \times \{0\}$ is a Legendrian knot. In particular, any Legendrian knot in any contact manifold has a tubular neighbourhood which is contactomorphic to a neighbourhood of $S^1 \times \{0\}$ in this model.
2. The knot $S^1 \times \{0\} \subset (S^1 \times \mathbb{R}^2, \ker(d\theta + r^2 d\varphi))$ is transverse, and any transverse knot has a tubular neighbourhood which is contactomorphic to a neighbourhood of $S^1 \times \{0\}$ in this model (cf. Example 1.2.12).

Remark 1.4.2

The contact structures ξ_n on $S^1 \times \mathbb{R}^2$ are all contactomorphic. A contactomorphism from $(S^1 \times \mathbb{R}^2, \ker(dx + y\, d\theta))$ to $(S^1 \times \mathbb{R}^2, \xi_n)$ is given by

$$f_n(\theta, x, y) := \Big(\theta, x\cos(n\theta) + \frac{y}{n}\sin(n\theta), -x\sin(n\theta) + \frac{y}{n}\cos(n\theta)\Big).$$

Also, the map $(\theta, x, y) \mapsto (\theta, rx, ry)$ is a contactomorphism of $(S^1 \times \mathbb{R}^2, \xi_n)$ for every $r > 0$. Thus, the standard neighbourhood of a Legendrian knot described above can be chosen arbitrarily big. For transverse knots the situation is different. The maximal size of a standard neighbourhood $S^1 \times D_\varepsilon$ is an invariant of the transverse knot (cf. [6, 13, 63]).

We now want to describe distinguished push-offs of a Legendrian knot. We identify a Legendrian knot with $S^1 \times \{0\} \subset (S^1 \times \mathbb{R}^2, \ker(\cos\theta dx - \sin\theta dy))$ as above. Then the torus $S = \{x^2 + y^2 = \delta\}$, which is contained in a tubular neighbourhood for small δ, is convex since the contact vector field $X = x\partial_x + y\partial_y$ is transverse to S. The dividing set Γ_S consists of the two curves

$$\gamma_\pm(\theta) = (\theta, \pm\delta\sin\theta, \pm\delta\cos\theta).$$

We call γ_+ (γ_-) the positive (negative) **transverse push-off** of our Legendrian knot. This makes sense indeed, as these curves arise by pushing the knot into the direction of the vector field $\pm(\sin\theta\partial_x + \cos\theta\partial_y)$, which is tangent to the contact planes. Thus, different choices of δ result in push-offs which are isotopic as transverse knots.

There are also two distinguished *Legendrian* curves on S:

$$\gamma_L(\theta) = (\theta, \pm\delta\cos\theta, \mp\delta\sin\theta).$$

We call γ_L a **Legendrian push-off**. Varying the parameter δ shows that the original knot and its Legendrian push-off are isotopic as Legendrian knots.

Note that by considering a standard neighbourhood of a transverse knot one easily sees that a transverse knot possesses Legendrian push-offs. However, there is no canonical choice and two Legendrian push-offs are not necessarily isotopic as Legendrian knots.

1.4.1 The front projection

Consider a smooth curve $\gamma\colon (a, b) \to \big(\mathbb{R}^3, \xi_{st} = \ker(xdy + dz)\big)$ with parametrisation $\gamma(t) = (x(t), y(t), z(t))$. The **front projection** of γ is

$$\gamma_F(t) = (y(t), z(t)).$$

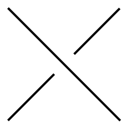
If γ is a Legendrian immersion, i.e. we have $\alpha_{st}(\dot\gamma(t)) = 0$ and $\dot\gamma(t) \neq 0$ for all $t \in (a, b)$, then the front projection γ_F does not have vertical tangencies. This is because if $\dot y$ vanishes, then so does $\dot z$ by the Legendre condition

$$0 = \alpha_{st}(\dot\gamma(t)) = \dot z(t) + x(t)\dot y(t).$$

Instead of vertical tangencies, the front projection of a Legendrian curve has *cusps*, which after a small perturbation can be assumed to be isolated and semi-cubical, i.e. around a cusp in $t = 0$ the curve is given by

$$\gamma(t) = (t + a, \lambda t^2 + b, -\lambda(2/3 \cdot t^3 + at^2) + c)$$

(see [37, Lemma 3.2.3]). The Legendrian curve γ is uniquely determined by its front projection γ_F, the x-coordinate of γ can be recovered as the negative slope $-dz/dy$ of γ_F. It follows that γ is embedded if and only if γ_F has transverse crossings only. In this case, the crossing behaviour in a front projection is always of the following form:



Furthermore, any regular curve in $\mathbb{R}^2$ with semi-cubical cusps and without vertical tangencies is the front projection of a Legendrian (cf. [37, Lemma 3.2.3]). This has the consequence that arbitrary knots in contact 3-manifolds can be C^0-approximated by Legendrian as well as transverse knots (cf. [37, Theorem 3.3.1]). To obtain a Legendrian approximation, the knot is covered by finitely many Darboux charts. In such a chart, one can consider the front projection and approximate this by a *zigzag*-curve with semi-cubical cusps and no vertical tangencies. This curve then lifts to a Legendrian curve approximating a segment of the original knot. To obtain a transverse approximation of a knot, one can approximate first by a Legendrian knot – the transverse push-off of a Legendrian approximation then yields the desired transverse approximation of the original knot.

So far, we have only discussed front projections of Legendrian knots, but they are also useful for studying transverse knots. Let $\gamma(t) = (x(t), y(t), z(t)) \in (\mathbb{R}^3, \xi_{st})$ be a positively transverse parametrised curve, i.e. we have $\dot{z} + x\dot{y} > 0$. Hence, the following holds:

- if $\dot{y} = 0$, then $\dot{z} > 0$,
- if $\dot{y} > 0$, then $x > -\dot{z}/\dot{y}$,
- if $\dot{y} < 0$, then $x < -\dot{z}/\dot{y}$.

This means that the situations depicted in Figure 1.3 cannot occur in the front projection of a positively transverse knot. In fact, these are the only restrictions that have to be imposed on a diagram so that it lifts to a positively transverse curve. However, in contrast to the front projection of a Legendrian knot, the front projection diagram of a transverse knot only determines the knot up to transverse isotopy (see [37, Section 3.2.2] for details).

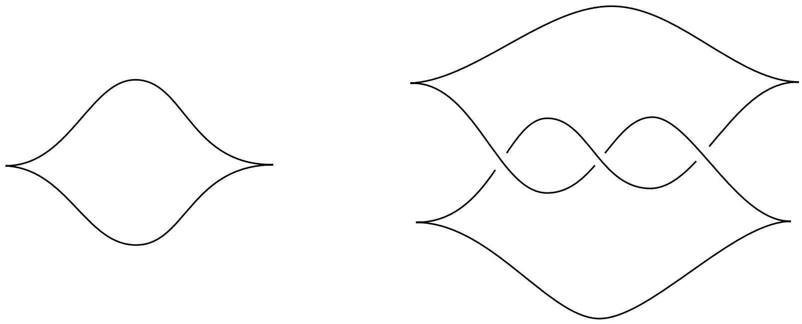


Figure 1.2: The front projections of a Legendrian unknot and trefoil.

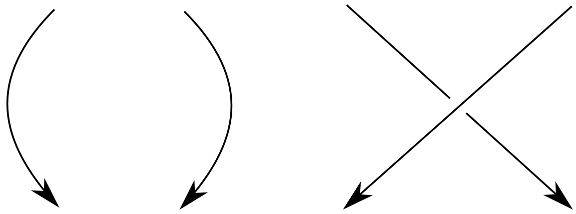


Figure 1.3: Impossible front projections of a positively transverse curve.

1.4.2 The classical invariants

Legendrian or transverse knots in contact manifolds can have the same topological knot type but may differ as Legendrian or transverse knots. This requires knot invariants adapted to the contact setting. We will briefly introduce the so-called *classical invariants*, a basic, yet useful set of invariants for nullhomologous Legendrian and transverse knots.

A nullhomologous Legendrian knot K has two distinguished longitudes. The **Seifert longitude** λ_s obtained by pushing K into a Seifert surface, i.e. an oriented connected surface bounded by K, and the **contact longitude** λ_c which is obtained by pushing K into a direction transverse to the contact structure. In the standard neighbourhood discussed above, the contact longitude is given by the Legendrian push-off K_L. We consider longitudes as curves on the boundary torus of a tubular neighbourhood νK and will usually identify them with the homology classes they represent. Two longitudes differ by a multiple of the meridian of the torus. The difference between contact and Seifert longitude is measured by the *Thurston–Bennequin invariant*.

Definition 1.4.3
Let K be a nullhomologous Legendrian knot in a contact manifold (M, ξ), let λ_s and λ_c denote its Seifert and contact longitude and μ its meridian. The **Thurston–Bennequin invariant** $\mathrm{tb}(K) \in \mathbb{Z}$ is defined by the equation

$$\lambda_c = \mathrm{tb}(K)\mu + \lambda_s \in H_1(\partial\nu K, \mathbb{Z}).$$

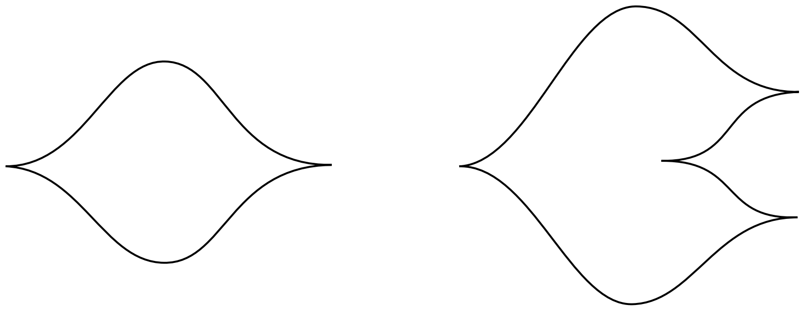


Figure 1.4: Legendrian unknots with different Thurston–Bennequin invariants.

Remark 1.4.4
A priori, the definition requires K to be oriented. However, it is independent of the choice of orientation. The longitudes are oriented as push-offs of K and the meridian will always be assumed to be the *positive meridian* of K, i.e. an oriented curve in $\partial\nu K$ bounding a disc in νK such that the pair (μ, λ), with λ a longitude of K, gives the orientation of $\partial\nu K$ induced from νK.

If a Legendrian knot K is given in a front projection, the contact longitude is given as a translation of K in the z-direction, as ∂_z is transverse to the contact planes. The Thurston–Bennequin invariant is equal to the linking number of K with this translated knot K', i.e. it can be computed by counting signed crossings where K runs over K' in the diagram. These crossings correspond exactly to self-crossings and right cusps of the front projection of K. Using the fact that the number of left and right cusps agree, one can compute the Thurston–Bennequin invariant of a knot $K \subset (\mathbb{R}^3, \xi_{st})$ via its front projection as

$$\mathrm{tb}(K) = -\frac{1}{2}c + w,$$

where c denotes the total number of cusps and w the writhe of the diagram, i.e. the signed count of self-crossings.

The pair of unknots depicted in Figure 1.4 can be distinguished by their Thurston–Bennequin invariants, the ones in Figure 1.5 cannot. They are different, however, as the second classical invariant for Legendrian knots will show.

Definition 1.4.5
Let $K \subset (M, \xi)$ be a nullhomologous oriented Legendrian knot and Σ a Seifert surface for K. The contact structure ξ can be trivialised over Σ, i.e. $\xi|_\Sigma = \Sigma \times \mathbb{R}^2$. A regular parametrisation γ of the knot K then induces a map $\gamma'\colon\ S^1 \to \mathbb{R}^2 \setminus \{0\}$. The **rotation number** $\mathrm{rot}(K, \Sigma)$ of K with respect to the Seifert surface Σ is defined to be the degree of the map $\gamma'\colon\ S^1 \to \mathbb{R}^2 \setminus \{0\}$.

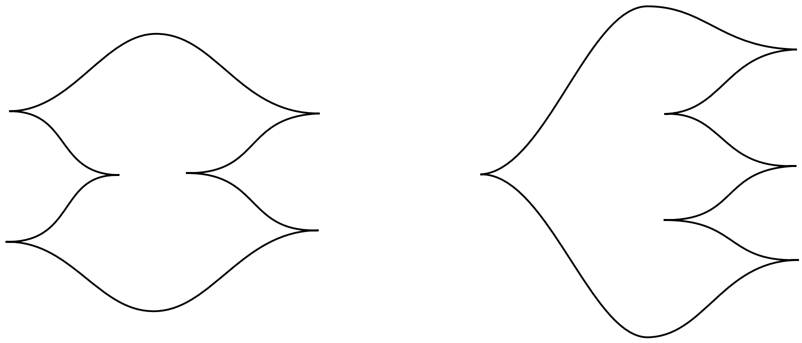


Figure 1.5: Legendrian unknots with equal Thurston–Bennequin invariants.

Remark 1.4.6
Equivalently, the rotation number of a nullhomologous Legendrian knot K with respect to a Seifert surface Σ can also be defined as

$$\mathrm{rot}(K,\Sigma) = \langle \mathrm{e}(\xi,K),[\Sigma]\rangle = \mathrm{PD}(\mathrm{e}(\xi,K)) \bullet [\Sigma],$$

where $\mathrm{e}(\xi,K)$ is the relative Euler class of the contact structure ξ relative to the trivialisation given by a positive tangent vector field along the knot K, and $[\Sigma]$ the relative homology class represented by the surface Σ. This definition of the rotation number is useful for calculations (see also [68]).

Clearly, the rotation number does only depend on the class of the chosen Seifert surface, not on the particular choice of surface itself. Furthermore, it does not depend on the particular choice of trivialisation (see [37, Lemma 3.5.14]). Note also that the rotation number is independent of the class of the Seifert surface if the Euler class $\mathrm{e}(\xi)$ of the contact structure vanishes (see [37, Proposition 3.5.15]).

As the standard contact structure on $\mathbb{R}^3$ admits a global trivialisation by ∂_x and $\partial_y - x\partial_z$, the rotation number of a Legendrian knot K in $(\mathbb{R}^3,\xi_{\mathrm{st}})$ is given by the signed count of crossings of the positive tangent vector of K over ∂_x. These crossings correspond to left cusps oriented downwards and right cusps oriented upwards in the front projection of K. One can, of course, also count how often the tangent vector of K crosses $-\partial_x$, which happens in right cusps oriented downwards and left cusps oriented upwards. Averaging the resulting formulas of both methods yields

$$\mathrm{rot}(K) = \frac{1}{2}(c_- - c_+),$$

where $c_\pm$ is the number of cusps oriented upwards or downwards, respectively (see [37, Proposition 3.5.19] for details).

The classical invariant for transverse knots is the *self-linking number*.

Definition 1.4.7
Let K be an oriented nullhomologous transverse knot in a contact manifold (M, ξ) and let Σ be a Seifert surface for K. The **self-linking number** $\mathrm{sl}(K, \Sigma)$ of K is defined as the linking number of K and K' where K' is obtained by pushing K in the direction of a non-vanishing section of $\xi|_\Sigma$.

The self-linking number of a transverse knot is independent of its orientation and does only depend on the homology class of the chosen Seifert surface (cf. [37, Section 3.5.2]).

As with Legendrian knots, the front projection of a transverse knot in standard $\mathbb{R}^3$ can be used to compute its self-linking number. If one chooses the section required in Definition 1.4.7 to be the global section ∂_x of ξ_{st}, it becomes clear that the self-linking number of a transverse knot equals the writhe of its front projection (cf. [37, Proposition 3.5.32]).

A natural question to ask is how the classical invariants of a nullhomologous Legendrian knot and the self-linking number of its transverse push-off relate, which is answered in the following proposition.

Proposition 1.4.8 ([37, Proposition 3.5.36])
Let $K \subset (M, \xi)$ be an oriented nullhomologous Legendrian knot with Seifert surface Σ and let $K_\pm$ denote its positive or negative, respectively, transverse push-off. Then

$$\mathrm{sl}(K_\pm, \Sigma) = \mathrm{tb}(K) \mp \mathrm{rot}(K, [\Sigma]),$$

where we regard Σ as a Seifert surface also for $K_\pm$, which is topologically isotopic to K.

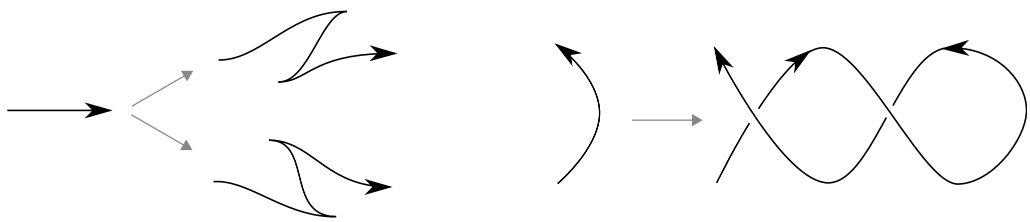


Figure 1.6: Stabilisations of Legendrian and transverse knots.

The Legendrian or transverse isotopy class of a Legendrian or transverse knot, respectively, can be changed by **stabilisations**. These are the local operations which, in the front projection, have the effect shown in Figure 1.6. The Thurston–Bennequin invariant of a Legendrian knot decreases by 1 in a stabilisation. The rotation number of the stabilised and the original Legendrian knot differ by 1. The stabilisation is said to be *positive* or *negative* accordingly. The self-linking number of a transverse knot decreases by 2 in a stabilisation. In particular, we can realise a given topological knot as a Legendrian or transverse knot with arbitrary small classical invariants. However, the invariants of a knot in standard contact $\mathbb{R}^3$ are bounded from above.

Theorem 1.4.9 (Bennequin inequality (version 1), [7])
Let K be a topological knot in $\mathbb{R}^3$. Then we have the following inequalities for every Legendrian realisation L and every transverse realisation T of K in $(\mathbb{R}^3, \xi_{st})$:

$$\operatorname{sl}(T) \leq 2g(K) - 1,$$
$$\operatorname{tb}(L) + |\operatorname{rot}(L)| \leq 2g(K) - 1.$$

Here $g(K)$ denotes the genus of the knot, i.e. the minimal genus of a Seifert surface for K.

In the next section we will state a generalisation of the above theorem due to Eliashberg [27], which can be used as a criterion for tightness (see Theorem 1.4.14).

It is reasonable to ask to what extent Legendrian and transverse knots are determined by their classical invariants. It turns out, that *unknots* in the standard contact $\mathbb{R}^3$ are completely classified by their classical invariants, as the next two theorems will show.

Theorem 1.4.10 (Classification of Legendrian unknots, [28])
Every Legendrian unknot in $(\mathbb{R}^3, \xi_{st})$ can be obtained by stabilising the Legendrian unknot in Figure 1.7. In particular, Legendrian unknots are classified by their classical invariants.

Theorem 1.4.11 (Classification of transverse unknots, [27])
Every positively transverse unknot in $(\mathbb{R}^3, \xi_{st})$ can be obtained by stabilising the transverse unknot in Figure 1.7. In particular, transverse unknots are classified by their self-linking number.

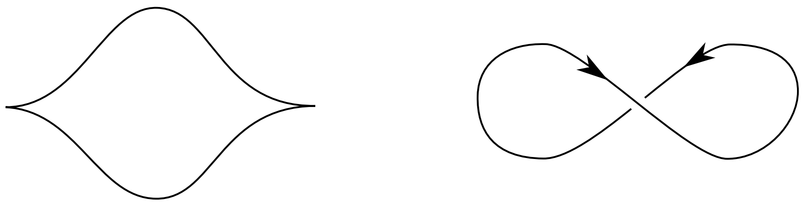


Figure 1.7: A Legendrian and a transverse unknot.

In general, the classical invariants are not sufficient to determine the Legendrian or transverse isotopy class of a knot, see e.g. [12] for examples of non-isotopic Legendrians with the same classical invariants and [32] for the transverse case.

1.4.3 Eliashberg's dichotomy and classification results

We briefly introduce Eliashberg's dichotomy of tight and overtwisted contact structures and state some classification results.

Example 1.4.12 (Standard overtwisted contact structure)
Consider the 1-form

$$\alpha_{ot} = \cos r dz + r \sin r d\theta = \cos r dz + r^2 \frac{\sin r}{r} d\theta$$

on $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z). Since the function $\sin(r)/r$ can be smoothly extended by 1 for $r = 0$, this is indeed a smooth 1-form, and one computes

$$\alpha_{ot} \wedge d\alpha_{ot} = \left(1 + \frac{\sin r}{r} \cos r\right) r dr \wedge d\theta \wedge dz,$$

i.e. α_{ot} defines a contact structure ξ_{ot} which we call the **standard overtwisted contact structure** on $\mathbb{R}^3$. Consider the disc $D := \{(r, \theta, 0) \in \mathbb{R}^3 | r \leq \pi, \theta \in S^1\}$. The vector field ∂_z is a contact vector field and transverse to D, so D is convex. Observe that the contact planes are tangent to D along its boundary $L := \partial D$. In particular, L is a Legendrian unknot whose contact and Seifert framing coincide, i.e. it has vanishing Thurston–Bennequin invariant. The characteristic foliation $D_{\xi_{ot}}$ consists of the radial lines with the centre and all boundary points being singular (see left part of Figure 1.8). By pushing the interior of D slightly into the z-direction, the boundary is turned into a closed leaf of the foliation, which then looks like depicted on the right hand side of Figure 1.8. The disc D is called the **standard overtwisted disc**.

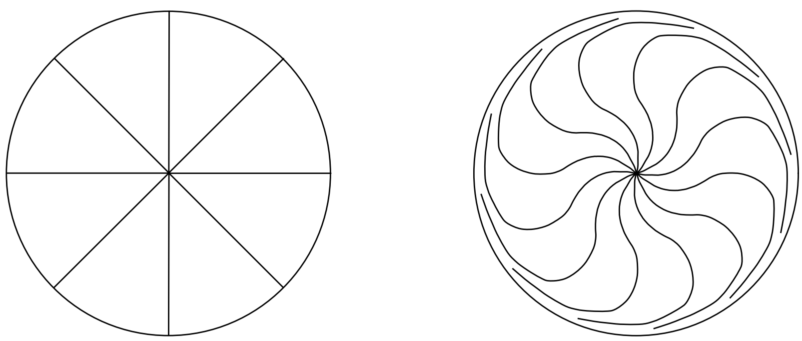


Figure 1.8: The unperturbed and the perturbed standard overtwisted disc.

Definition 1.4.13
An embedded disc $D \subset (M, \xi)$ is called **overtwisted disc** if ∂D is Legendrian with $\text{tb}(\partial D) = 0$ and the characteristic foliation D_ξ contains a unique singular point in

the interior of D. A contact structure ξ on a manifold M is called **overtwisted** if (M, ξ) contains an overtwisted disc. A contact structure is called **tight** if it is not overtwisted.

By the Bennequin inequality (Theorem 1.4.9) there are no Legendrian unknots with vanishing Thurston–Bennequin invariant in $(\mathbb{R}^3, \xi_{st})$. In particular, $(\mathbb{R}^3, \xi_{st})$ is tight. The same argument shows that the standard contact structure on the 3-sphere as introduced in Example 1.2.7 is tight because for every Legendrian knot in S^3, one can obtain a Legendrian in standard $\mathbb{R}^3$ with the same classical invariants by removing a point in its complement.

In fact, we can characterise tightness by properties of Legendrian and transverse knots.

Theorem 1.4.14 (Bennequin inequality (version 2), [27])
Let (M, ξ) be a contact 3-manifold. Then the following statements are equivalent.

1. *(M, ξ) is tight.*
2. *For any nullhomologous transverse knot $K \subset (M, \xi)$ and any Seifert surface Σ of K we have*
$$\mathrm{sl}(K, \Sigma) \leq 2g(K) - 1.$$
3. *For any nullhomologous Legendrian knot $K \subset (M, \xi)$ and any Seifert surface Σ of K we have*
$$\mathrm{tb}(K) + |\operatorname{rot}(K, [\Sigma])| \leq 2g(K) - 1.$$
4. *For any nullhomologous Legendrian knot $K \subset (M, \xi)$ we have*
$$\mathrm{tb}(K) \leq 2g(K) - 1.$$
5. *There is no unknot in (M, ξ) with vanishing Thurston–Bennequin invariant.*

Martinet was the first to prove that every closed, orientable 3-manifold can be equipped with a contact structure.

Theorem 1.4.15 (Martinet, [58])
Every closed, orientable 3-manifold admits a contact structure.

The original proof constructs a contact structure via a surgery description from S^3. A consequence of the special kind of transverse surgery operation used in the proof – the so-called *Lutz twist* – is that the resulting contact structures are always overtwisted. We will describe a different proof based on open book decompositions in the next chapter (see Theorem 2.3.1). But not only do 3-manifolds admit overtwisted contact structures, overtwisted contact structures are in fact completely classified by the topological data of the underlying plane field.

Theorem 1.4.16 (Classification of overtwisted contact structures, [24])
Let M be a closed, orientable 3-manifold. Then there is exactly one (up to isotopy) overtwisted contact structure in every homotopy class of tangential 2-plane fields.

The concept of overtwistedness has recently been generalised to higher dimensions and a corresponding existence and classification result was proved.

Theorem 1.4.17 (Existence and classification of overtwisted contact structures in all dimensions, [8])
Let M be a closed manifold. Then there is exactly one overtwisted contact structure in every homotopy class of almost contact structures on M.

The classification of tight contact structures is only known in special cases. Among those are S^3, $S^2 \times S^1$ and $\mathbb{R}^3$, which admit a unique tight contact structure – the standard contact structure (cf. [26] and [37, Section 4.10]), and the tight contact structures on the solid torus $S^1 \times D^2$ (cf. [47]), which play an essential role in Legendrian surgery constructions (cf. [37, Section 6.4]).

1.5 The contact fibre connected sum

We will briefly introduce the (topological) fibre connected sum, which is a method to construct manifolds using embedded submanifolds, and then discuss its contact version following [37, Section 7.4]. Let M' and M be closed oriented manifolds and let j_0 and j_1 be embeddings of M' into M with disjoint images. Assume that there exists a bundle isomorphism Ψ of the corresponding normal bundles N_0 and N_1 over $j_1 \circ j_0^{-1}|_{j_0(M')}$ that reverses the fibre orientation. Picking a bundle metric on N_0 and choosing the induced metric on N_1 turns Ψ into a bundle isometry. We furthermore identify open disjoint neighbourhoods of the $j_i(M')$ with the normal bundles N_i and denote the bundle projections by $\pi_i\colon\ N_i \to j_i(M')$.

The **fibre connected sum** is the quotient manifold

$$\#_\Psi M := \Big(M \setminus \big(j_0(M') \cup j_1(M')\big)\Big)/_\sim,$$

where $v \in N_0$ with $0 < \|v\| < \varepsilon$ is identified with $\frac{\sqrt{\varepsilon^2 - \|v\|^2}}{\|v\|}\Psi(v)$. A useful interpretation of the fibre sum is the following: suppose we identify the boundaries of the embedded normal bundles N_0 and N_1 with their induced sphere bundles. Then the fibre connected sum is diffeomorphic to the quotient

$$\Big(M \setminus \big(N_0 \cup N_1\big)\Big)/_\sim,$$

where we identify $p \in \partial N_0$ with $\Psi(p) \in \partial N_1$. Observe that (in both cases) the identification is orientation preserving. If M is disconnected and j_i maps M' into M_i with $M = M_0 \sqcup M_1$, we also write $M_0 \#_\Psi M_1$ for $\#_\Psi M$. In the case when M' is just a point the fibre connected sum coincides with the ordinary connected sum. Note also that there is a cobordism from M to $\#_\Psi M$, i.e. the fibre connected sum can be obtained by a sequence of surgeries (cf. Section 2.2).

Remark 1.5.1
Note that the fibre connected sum can also be defined as a quotient of M with tubular neighbourhoods of the $j_i(M')$ removed, i.e.

$$\#_\Psi M = \Big(M \setminus \Big(N_0^{[0,\varepsilon/2]} \cup N_1^{[0,\varepsilon/2]}\Big)\Big)/_\sim,$$

where $v \in N_0^{(\varepsilon/2,\sqrt{3}\varepsilon/2)}$ is identified with $\frac{\sqrt{\varepsilon^2-\|v\|^2}}{\|v\|}\Psi(v) \in N_1^{(\varepsilon/2,\sqrt{3}\varepsilon/2)}$. Here $N_i^{(a,b)}$ denotes the set of $v \in N_i$ with $a < \|v\| < b$.

The following theorem explains how the construction can be adapted to work in the contact setting if the dimensions of M and M' differ by two. It was first stated as an exercise in [46] and proved in full generality in [37]. For a symplectic analogue see Section 1.5.2.

Theorem 1.5.2 (Contact fibre connected sum, [46], [37, Theorem 7.4.3])
Let (M,ξ) and (M',ξ') be contact manifolds of dimension $\dim M' = \dim M - 2$, where the contact structures ξ, ξ' are assumed to be cooriented; these cooriented contact structures induce orientations of M and M'. Let $j_0, j_1\colon (M',\xi') \to (M,\xi)$ be disjoint contact embeddings that respect the coorientations, and such that there exists a fibre-orientation-reversing bundle isomorphism $\Phi\colon N_0 \to N_1$ of the normal bundles of $j_0(M')$ and $j_1(M')$. Then the fibre connected sum $\#_\Phi M$ admits a contact structure that coincides with ξ outside tubular neighbourhoods of the submanifolds $j_0(M')$ and $j_1(M')$.

Proof. We will use the description of the fibre connected sum given in Remark 1.5.1, and we want to construct suitable contact forms such that the identification map is a contactomorphism.

We start by choosing a connection 1-form γ_1 on the unit circle bundle of N_1. This also defines a 1-form on $N_1 \setminus j_1(M')$ via the pull-back under the natural retraction $\mathbb{R}^2 \setminus \{0\} \to S^1$, which will still be denoted by γ_1. The pull-back of γ_1 under Φ then defines a form on $N_0 \setminus j_0(M')$. However, as Φ is fibre orientation reversing, we will work with $\gamma_0 := -\Phi^*\gamma_1$. If we write r for the radial coordinate in the fibres of N_i, $r^2\gamma_i$ is a smooth 1-form on all of N_i. Its exterior derivative $d(r^2\gamma_i) = 2rdr \wedge \gamma_i + r^2 d\gamma_i$ restricts to a volume form on each fibre. For α' a contact form for $\xi' = \ker\alpha'$

on $M' \equiv j_i(M')$, we can write the contact form ξ as the kernel of $\pi_i^*\alpha' + r^2\gamma_i$ on a neighbourhood $N_i^{[0,2\varepsilon)}$ of the zero section $j_i(M')$ of N_i as explained in Example 1.2.12. We now want to adapt this contact form away from the zero section to make it compatible with the quotient map of the fibre connected sum. Choose a smooth function $f\colon (\varepsilon/2, 2\varepsilon) \to \mathbb{R}$ satisfying

- $f'(r) > 0$,
- $f(r) = r^2$ on an open interval containing $[\varepsilon, 2\varepsilon)$,
- $f(r) = r^2 - \varepsilon^2/2$ on $(\varepsilon/2, \sqrt{3}\varepsilon/2)$.

The 1-form $\pi_i^*\alpha' + f(r)\gamma_i$ coincides with the contact form $\pi_i^*\alpha' + r^2\gamma_i$ on a neighbourhood of $N_i^{[\varepsilon,2\varepsilon)}$ in $N_i^{[0,2\varepsilon)}$ and is itself a contact form on $N_i^{[\varepsilon/2,2\varepsilon)}$ by the first two properties of the function f. By the third condition we have $f(\sqrt{\varepsilon^2 - r^2}) = -f(r)$ on the interval $(\varepsilon/2, \sqrt{3}\varepsilon/2)$. Hence, since by definition we also have $\gamma_0 := -\Phi^*\gamma_1$, the identification map in the construction of the fibre connected sum as in Remark 1.5.1 is a contactomorphism. In particular, the contact form descends to a form on the quotient $\#_\Phi M$. $\square$

1.5.1 An alternative interpretation of the contact fibre connected sum

As mentioned above, one can also form the fibre connected sum by identifying the boundaries of closed tubular neighbourhoods of the submanifolds. In the following we want to explain how this interpretation can be used in the contact geometric setting provided the normal bundles are trivial.

Let M be a contact manifold and M' a codimension two contact submanifold with trivial normal bundle. Then M' has a neighbourhood N_ε in M which is contactomorphic to $M' \times D^2_\varepsilon$ with contact form $\alpha = \alpha' + r^2 d\theta$, where D^2_ε is a disc of radius ε with polar coordinates (r, θ) and α' a contact form on M' (see Example 1.2.12).

The characteristic foliation of $S_\delta := M' \times \partial D^2_\delta$ is defined by the vector field $X = -\delta^2 R_{\alpha'} + \partial_\theta$. Indeed,

$$X \in \ker\alpha' \oplus \langle X\rangle = TS_\delta \cap \ker\alpha|_{S_\delta},$$

and for $v \in \ker\alpha'$ we have $d\alpha(X, v) = 0$, i.e.

$$X \in \left(TS_\delta \cap \ker\alpha|_{S_\delta}\right)^{\perp}.$$

We now want to show that X also defines the characteristic foliation as an *oriented* foliation if we orient S_δ such that the positive r-direction is outward normal. We have

$$\alpha \wedge (d\alpha)^n = 2nrdr \wedge d\theta \wedge \alpha' \wedge (d\alpha')^{n-1},$$

i.e. a volume form giving the desired orientation on S_δ is

$$\Omega = d\theta \wedge \alpha' \wedge (d\alpha')^{n-1}.$$

Denoting the restriction of α to S_δ by β, we obtain

$$i_X\Omega = \alpha' \wedge (d\alpha')^{n-1} + \delta^2 d\theta \wedge (d\alpha')^{n-1} = (\alpha' + \delta^2 d\theta) \wedge (d\alpha')^{n-1} = \beta \wedge (d\beta)^{n-1}.$$

Hence, X also defines the characteristic foliation as an oriented foliation.

Now consider the manifold with boundary $M' \times S^1 \times [0, \delta]$ equipped with the contact form $\alpha' + f(r)d\theta$, where f is a strictly monotone function with $f(0) = 0$, $f'(0) = 1$ and f equal to r^2 near δ. Also here the characteristic foliation on

$$S_\delta := M' \times S^1 \times \{\delta\}$$

is given by $X = -\delta^2 R_{\alpha'} + \partial_\theta$ and we have

$$TS_\delta \cap \ker \alpha|_{S_\delta} = \ker \alpha' \oplus \langle X \rangle.$$

The characteristic foliation of S_δ is non-singular, i.e. the contact hyperplanes are transverse to S_δ. Hence, by Proposition 1.3.1, the germ of the contact structure is determined by the intersection $TS_\delta \cap \ker \alpha|_{S_\delta}$. Since this coincides for both copies of S_δ, we can glue $M' \times S^1 \times [0, \delta]$ to $M \setminus (M' \times D^2_\delta)$ and denote the resulting contact manifold by $\widetilde{M}$. Note that the characteristic foliation of the boundary $\partial\widetilde{M} \cong M' \times S^1 \times \{0\}$ is given by ∂_θ, which is non-vanishing, and that we have

$$T\partial\widetilde{M} \cap \ker \alpha|_{\partial\widetilde{M}} = \ker \alpha' \oplus \langle \partial_\theta \rangle.$$

We say that $\widetilde{M}$ is obtained from M by **blowing up** the submanifold M'.[2]

In the setting of the contact fibre connected sum, we have two contactomorphic codimension two submanifolds M'_0 and M'_1 of a contact submanifold M. Assume that both submanifolds have trivial normal bundle. An isomorphism of their normal bundles then corresponds to a choice of framing, i.e. a trivialisation of their normal bundles. Blowing up M'_0 and M'_1 yields a manifold $\widetilde{M}$ with two boundary components $M'_i \times S^1$. We can glue these together via the orientation-reversing map

$$\begin{aligned} M'_0 \times S^1 &\to M'_0 \times S^1 \\ (x, \theta) &\mapsto (x, -\theta), \end{aligned}$$

which respects the characteristic foliations as well as the codimension 1 distributions given by the intersection with the contact structure. The resulting manifold is a fibre connected sum of M along the framed submanifolds M'_i.

[2]The terminology is chosen in analogy to the 3-dimensional case discussed in [51].

1.5.2 The symplectic fibre connected sum

We will briefly show – following [61, Section 7.2] – how the fibre connected sum construction can be performed in the symplectic setting.

Let (W_j, ω_j) $(j = 0, 1)$ be two symplectic manifolds of the same dimension $2n$ and let (X, τ) be a compact symplectic manifold of dimension $2n-2$. Suppose that $i_j \colon X \to W_j$ are symplectic embeddings with trivial normal bundle. Then by the symplectic neighbourhood theorem 1.1.8 there are embeddings $f_j \colon X \times D^2(\varepsilon) \to W_j$ with $f_j^*\omega_j = \tau + dx \wedge dy$ and $f_j(x, 0) = i_j(x)$ for $x \in X$. Let A be the annulus $A := A(\delta, \varepsilon) = D^2(\varepsilon) \backslash \mathrm{int}\big(D^2(\delta)\big)$ and $\phi \colon A \to A$ an area- and orientation-preserving diffeomorphism interchanging the two boundary components.

We can now form the fibre connected sum

$$M_0 \# M_1 = \Big(M_0 \setminus f_0(X \times B^2(\varepsilon)\Big) \cup_\phi \Big(M_1 \setminus f_1(X \times B^2(\varepsilon)\Big),$$

where we identify $f_1(x, a)$ with $f_0(x, \phi(a))$ for $x \in X$ and $a \in A$. Since the the symplectic forms ω_j agree on the overlap $X \times A$, this carries a well-defined symplectic form induced by the ω_j.

Remark 1.5.3 (Relative symplectic fibre sum)
Note that the symplectic fibre sum can be adapted to work in a relative setting. Let W be a symplectic manifold with contact type boundary and let X be a codimension 2 symplectic submanifold with trivial normal bundle and contact type boundary which is a contact submanifold of ∂X. Let $\partial W \times (-\varepsilon, 0]$ be a collar neighbourhood of ∂W given by a Liouville field transverse to ∂W and suppose that

$$X \cap \Big(\partial W \times (-\varepsilon, 0]\Big) = \partial X \times (-\varepsilon, 0].$$

The symplectic fibre sum can then be performed in this setting with the effect on the boundary being a contact fibre connected sum.

2

Open books

In this chapter we will introduce open book decompositions and discuss some of their relations to contact structures, while focusing on the results relevant in later chapters of this thesis. Nice surveys on open books and their applications are Winkelnkemper's appendix to Ranicki's book [70] and Giroux [42]. More detailed material, in particular on the relation of open books and contact structures can be found in [30, 37, 73].

The presentation of the material in this chapter does not follow any particular source – with the exception of the discussion of the Thurston–Winkelnkemper construction in Section 2.3 which roughly follows [37, Section 7.3] and [23, Section 2.2]. Section 2.1.2 is a slight generalisation of the discussion of stabilisations of contact open books in [73] to the general topological setting. Section 2.1.1 is inspired by Lawson's existence proof [54], whereas the content of Section 2.2 does not seem to appear elsewhere in the literature.

2.1 Topological open books

An **open book decomposition** of an n-dimensional manifold M is a pair (B, π), where B is a co-dimension 2 submanifold in M with trivial normal bundle, called the **binding** of the open book, and $\pi\colon\ M \setminus B \to S^1$ is a (smooth) fibration such that each fibre $\pi^{-1}(\theta)$, $\theta \in S^1$, corresponds to the interior of a compact hypersurface $\Sigma_\theta \subset M$ with $\partial\Sigma_\theta = B$. The hypersurfaces Σ_θ, $\theta \in S^1$, are called the **pages** of the open book.

Remark 2.1.1 (Alternative definitions)
We can also define an open book decomposition as a pair (B, π), where B is a co-dimension 2 submanifold in M with trivial normal bundle and $\pi\colon\ M \setminus B \to S^1$ is a smooth fibration which in a neighbourhood $B \times D^2$ with coordinates (b, r, θ) is given by the angular coordinate θ. These two definitions are clearly equivalent and are used in most of the literature. Another alternative definition, which uses a map globally defined on the manifold, can be found e.g. in [60]: the binding of the standard open book of $\mathbb{C} = \mathbb{R}^2$ is defined as the origin, the pages of the standard decomposition are the half-lines. An open book decomposition of a manifold M is then a map $f\colon\ M \to \mathbb{C}$ which is transverse to the standard open book on $\mathbb{C}$.

The binding and pages are defined as the preimages of the binding and the pages, respectively.

In some cases we are not interested in the exact position of the binding or the pages of an open book decomposition inside the ambient space. Therefore, given an open book decomposition (B, π) of an n-manifold M, we could ask for the relevant data to remodel the ambient space M and its underlying open book structure (B, π), say up to diffeomorphism. This leads us to the notion of *abstract* open books.

An **abstract open book** is a pair (Σ, ϕ), where Σ is a compact hypersurface with non-empty boundary $\partial\Sigma$, called the **page**, and $\phi\colon\ \Sigma \to \Sigma$ is a diffeomorphism equal to the identity near $\partial\Sigma$, called the **monodromy** of the open book. Let $\Sigma(\phi)$ denote the **mapping torus** of ϕ, that is, the quotient space obtained from $\Sigma\times[0, 2\pi]$ by identifying $(x, 2\pi)$ with $(\phi(x), 0)$ for each $x \in \Sigma$. Then the pair (Σ, ϕ) determines a closed manifold $M_{(\Sigma,\phi)}$ defined by

$$M_{(\Sigma,\phi)} := \Sigma(\phi) \cup_{\mathrm{id}} (\partial\Sigma \times D^2), \tag{2.1}$$

where we identify $\partial\Sigma(\phi) = \partial\Sigma \times S^1$ with $\partial(\partial\Sigma \times D^2)$ using the identity map. Let $B \subset M_{(\Sigma,\phi)}$ denote the embedded submanifold $\partial\Sigma \times \{0\}$. Then we can define a fibration $\pi\colon\ M_{(\Sigma,\phi)} \setminus B \to S^1$ by

$$\left.\begin{array}{l}[x, \theta]\\ {[x', r\mathrm{e}^{i\theta}]}\end{array}\right\} \mapsto [\theta],$$

where we understand $M_{(\Sigma,\phi)} \setminus B$ as decomposed as in (2.1) and $[x, \theta] \in \Sigma(\phi)$ or $[x', r\mathrm{e}^{i\theta}] \in \partial\Sigma \times D^2 \subset \partial\Sigma \times \mathbb{C}$. Clearly, (B, π) defines an open book decomposition of $M_{(\Sigma,\phi)}$.

On the other hand, an open book decomposition (B, π) of some n-manifold M defines an abstract open book as follows: identify a neighbourhood of B with $B\times D^2$ such that $B = B \times \{0\}$ and such that the fibration on this neighbourhood is given by the angular coordinate, θ say, on the D^2-factor. We can define a 1-form α on the complement $M \setminus (B \times D^2)$ by pulling back $d\theta$ under the fibration π, where this time we understand θ as the coordinate on the target space of π. The vector field ∂_θ on $\partial\big(M \setminus (B \times D^2)\big)$ extends to a nowhere-vanishing vector field X which we normalise by demanding it to satisfy $\alpha(X) = 1$. Let ϕ denote the time-2π map of the flow of X. Then the pair (Σ, ϕ), with $\Sigma = \overline{\pi^{-1}(0)}$, defines an abstract open book such that $M_{(\Sigma,\phi)}$ is diffeomorphic to M.

Open books with isotopic monodromies are diffeomorphic:

Proposition 2.1.2
Let (Σ, ϕ_0) and (Σ, ϕ_1) be two abstract open books and assume that the monodromies

ϕ_0 and ϕ_1 are isotopic. Then there is a diffeomorphism $M_{(\Sigma,\phi_0)} \to M_{(\Sigma,\phi_1)}$ which respects the induced open book structures.

Proof. Let ψ_t be an isotopy from id_Σ to $\phi_1^{-1} \circ \phi_0$ and let $h\colon [0,2\pi] \to [0,1]$ be a smooth monotone function which is constantly 0 near 0 and constantly 1 near 2π. The map

$$\begin{array}{rcl} \Sigma \times [0,2\pi] & \to & \Sigma \times [0,2\pi] \\ (x,\theta) & \mapsto & (\psi_{h(\theta)}(x), \theta) \end{array}$$

descends to a diffeomorphism $\Sigma(\phi_0) \to \Sigma(\phi_1)$ on the mapping tori, which can be extended via the identity to give rise to the desired diffeomorphism. □

Remark 2.1.3 (Existence of open book decompositions)
In 1923 Alexander [1] proved that every closed oriented 3-manifold admits an open book decomposition. Winkelnkemper [75] showed that a closed oriented simply-connected manifold of dimension at least six can be given the structure of an open book if and only if its signature vanishes. In particular, that is the case in odd dimensions by definition. For odd-dimensional manifolds the hypothesis of simply-connectedness can be dropped, as was shown by Lawson [54]. Quinn [69] extended this to 5-manifolds and also discussed the even-dimensional case, where the obstruction is more involved than in the simply-connected case.

Example 2.1.4 (Classification of open books on surfaces)
A 2-dimensional open book has to have closed intervals as pages and thus trivial monodromy (up to isotopy). Hence, the only closed, connected surface admitting an open book decomposition is the sphere and the decomposition is unique up to isotopy (see Figure 2.1).

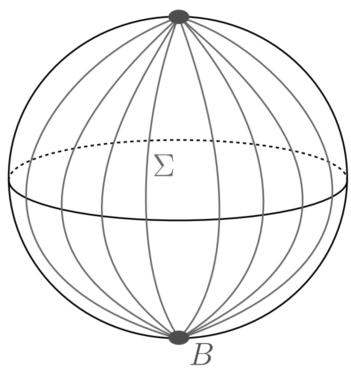


Figure 2.1: The open book decomposition of the 2-sphere.

Example 2.1.5
The 3-dimensional sphere S^3 has an open book decomposition with page a disc and trivial monodromy. This can be seen by realizing S^3 as the one-point compactification of $\mathbb{R}^3$ and extending the binding and pages in Figure 2.2 along the z-axis. To be more precise, choosing polar coordinates (r, θ) on the xy-plane we define $B := \{x = y = 0\} \cup \{\infty\}$ and $\pi\colon\ S^3 \setminus B \to S^1$ by sending (r, θ, z) to θ, where we identify $S^3 \equiv \mathbb{R}^3 \cup \{\infty\}$.

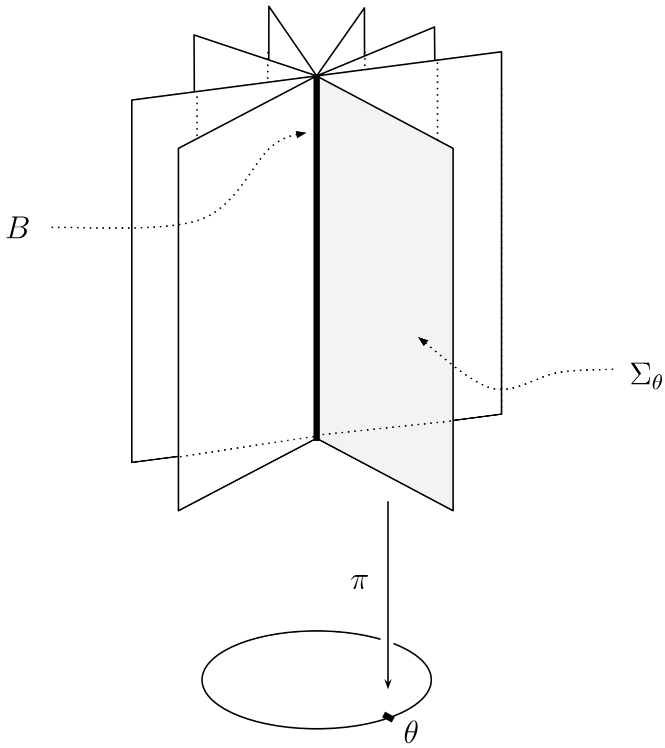


Figure 2.2: An open book for S^3.

The same open book decomposition can be realised by rotating Figure 2.3 around the vertical axis and again interpreting S^3 as $\mathbb{R}^3$ with a point at infinity.

Example 2.1.6
Consider S^3 as the unit sphere in $\mathbb{C}^2$ and denote the positive Hopf link by

$$B := \{(z_1, z_2) \in S^3 : z_1 z_2 = 0\}.$$

Then

$$\pi\colon\ S^3 \setminus B \to S^1 \subset \mathbb{C},\ (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$$

defines an open book decomposition with page an annulus and monodromy a positive Dehn-twist (see Definition 2.3.9) along the core of the annulus.

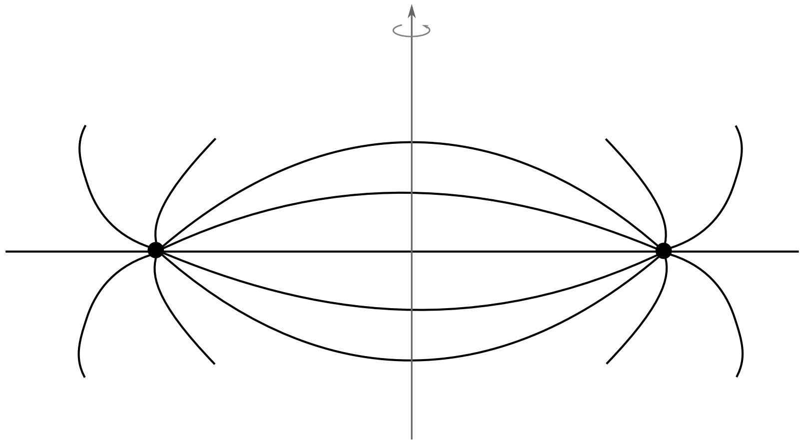


Figure 2.3: A second visualisation of the open book for S^3 with page a disc.

2.1.1 Open books and handlebodies

An open book decomposition (B, π) yields a decomposition of the underlying manifold M into two diffeomorphic solid handlebodies, where each corresponds to "one half of the open book":

$$M = \overline{\pi^{-1}([0,\pi])} \cup \overline{\pi^{-1}([\pi,2\pi])}.$$

In particular, the handlebodies in this splitting are the quotient of the product of the page Σ of the open book with the interval, where each $\{p\} \times [0,\pi]$ for $p \in \partial\Sigma$ is smashed to a point. In dimension three, the above splitting is a *Heegaard splitting* of the manifold (cf. [71, Chapter 9]). Conversely, certain (but not all) decompositions of a manifold as a double of a handlebody $M = H_1 \cup H_2$ give rise to an open book structure: Assume that ∂H_i is given the structure of a cell complex and there exists some embedded subcomplex $K \subset \partial H_i$ such that the inclusions into H_i are homotopy equivalences and such that H_i is diffeomorphic to the product of the interval I with a regular neighbourhood V of K in ∂H_i. Then M has an open book decomposition with page V.

Example 2.1.7
The open book decomposition of the 3-sphere described in Example 2.1.6 is induced by a genus 1 Heegaard splitting. Clearly, starting from the open book we get a decomposition of the sphere into two solid tori. For the converse, we need K on the torus such that the inclusions into both solid tori in the Heegaard decomposition are homotopy equivalences. This forces K to be a $(1,1)$-torus knot. The page of the induced open book is a regular neighbourhood of K on the Heegaard torus, i.e. it is an annulus, and the induced monodromy is a Dehn twist.

Remark 2.1.8
Note that it is not always possible to write a handlebody as a product of the special form described above to give rise to an open book decomposition. Consider the connected sum of two copies of $\mathbb{C}P^4$. This has a handle decomposition consisting of a single 0- and 8-handle and two handles of every even index in between. Thus, we can write $\mathbb{C}P^4\#\mathbb{C}P^4$ as the union of two diffeomorphic solid handlebodies, each consisting of a single 0- and 4-handle and two 2-handles. However, $\mathbb{C}P^4\#\mathbb{C}P^4$ has non-vanishing signature, so it does not admit an open book decomposition by the results mentioned in Remark 2.1.3.

2.1.2 Stabilisations

One way to alter an open book structure in *odd* dimensions is *stabilisation*, where the page and the monodromy change in a compatible way. In terms of the handlebody description from the previous section, the procedure will correspond to the introduction of a cancelling pair of handles. More concretely, the monodromy changes by a *Dehn–Seidel twist*, which we shortly present in the following paragraph.

Let S^n be the n-sphere embedded in some $2n$-dimensional manifold W such that the normal bundle is isomorphic to its cotangent bundle T^*S^n. We identify a neighbourhood νS^n of S^n in W with $T^*S^n \subset \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ with Cartesian coordinates $\mathbf{q}$ and $\mathbf{p}$, i.e. νS^n is given by the equations $\mathbf{q}\cdot\mathbf{q} = 1$ and $\mathbf{q}\cdot\mathbf{p} = 0$. Let σ_t be the diffeomorphism of $\nu S^n \setminus S^n$ given by

$$\sigma_t(\mathbf{q},\mathbf{p}) = \begin{pmatrix} \cos t & |\mathbf{p}|^{-1}\sin t \\ -|\mathbf{p}|\sin t & \cos t \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

and set $\tau(\mathbf{q},\mathbf{p}) = \sigma_{g(|\mathbf{p}|)}(\mathbf{q},\mathbf{p})$, where $g(r)$ is a smooth monotone function which is equal to π near $r = 0$ and vanishes for large r. We can extend τ to all of νS^n by setting $\tau(\mathbf{q},\mathbf{0}) = (-\mathbf{q},\mathbf{0})$. Then τ is a diffeomorphism of νS^n, called a **(generalised) right-handed (or positive) Dehn twist** or **Dehn–Seidel twist**. Its inverse τ^{-1} is called a **left-handed (or negative) Dehn twist**. The Dehn–Seidel twist is a natural generalisation of the usual *Dehn twist* on a surface, which we will define in Definition 2.3.9. In the following, we are applying this in the situation where the even-dimensional manifold W is the page of an open book.

We can now explain how the page of an open book changes under a stabilisation. Let $D^n \subset \Sigma^{2n}$ be an n-dimensional disc embedded into the $2n$-dimensional page of an open book (Σ,ϕ) of an odd-dimensional manifold M such that D^n meets $\partial\Sigma$ transversely and exactly in its boundary ∂D^n and such that the normal bundle of ∂D^n in $\partial\Sigma$ is trivial. Attach an n-handle H to Σ along ∂D^n in such a way that the normal bundle of the sphere $S^{n+1} = D^n \cup \operatorname{core}(H)$ is isomorphic to T^*S^n. Then the open book $(\Sigma \cup H, \phi \circ \tau)$ is called a **stabilisation** of (Σ, ϕ), where τ denotes a right-handed Dehn twist along the sphere S^{n+1}. Observe that the original open book (Σ, ϕ) and the stabilised open book $(\Sigma \cup H, \phi \circ \tau)$ give rise (up to diffeomorphism) to the same manifold M. Indeed, the sphere $\partial D^n \subset \partial\Sigma = B \subset (\Sigma, \phi)$ is a sphere with trivial normal bundle in M, since the binding B has trivial normal bundle by definition. Attaching handles to each page is equivalent to a surgery along ∂D^n. The manifold M' obtained by that surgery carries the open book structure $(\Sigma \cup H, \phi)$. Performing the Dehn twist τ along S^{n+1} is equivalent to a surgery cancelling the one corresponding to the handle attachment. For details the reader is referred to [73]. Observe that we can also define a stabilisation using left-handed Dehn twists, this corresponds to a change of orientation of the page.

Example 2.1.9
The open book decomposition of S^3 with page an annulus described in Example 2.1.6 arises as a stabilisation of the open book with page a disc described in Example 2.1.5.

A natural question to ask is how the induced handlebody decomposition changes by a stabilisation of an open book. The $(2n)$-dimensional page of an open book changes by an n-handle attachment. Abstractly, this results in attaching an n-handle to the solid handlebody. If we consider the handlebody decomposition $H_1 \cup H_2$, then adding an n-handle to H_1 cuts out an $(n+1)$-handle in its complement, which is a dualised n-handle. Combined, the two handles form a cancelling pair, i.e. stabilising an open book is stabilising the handlebody splitting in the handlebody sense. Unfortunately, this implies that stabilisation only works in odd dimensions, an analogue in even dimensions is not known.

In dimension 3, a stabilisation of an open book corresponds to a *Murasugi sum* of the open book with the open book of the sphere with annular pages from Example 2.1.6 (see [30, Section 2] for details).

2.2 Open books and fibre sums

In this section we will briefly discuss which manifolds can be obtained by fibre connected sums and how the existence of open book structures behaves under fibre connected sums. We will also introduce a natural fibre connected sum of open books, the so-called *binding sum*.

Proposition 2.2.1
Let M and N be two closed manifolds of the same dimension. Then N can be obtained from the disjoint union of M and finitely many spheres by a sequence of fibre connected sums if and only if M and N are cobordant.

Proof. Let M' and M be closed oriented manifolds and let j_0 and j_1 be embeddings of M' into M with disjoint images. Assume that there exists a bundle isomorphism Ψ of the corresponding normal bundles N_0 and N_1 over $j_1 \circ j_0^{-1}|_{j_0(M')}$ that reverses the fibre orientation. Recall from Section 1.5 that the fibre connected sum of M along the submanifolds $j_i(M')$ is the manifold obtained by identifying open neighbourhoods of $j_0(M')$ and $j_1(M')$ via Ψ.

Consider the product manifold $[0,1] \times M$ and identify *closed* tubular neighbourhoods of $\{1\} \times j_0(M')$ and $\{1\} \times j_1(M')$ via Ψ. After smoothing corners, the upper boundary is diffeomorphic to the fibre connected sum $\#_\Psi M$, and the lower boundary is M. Hence, M and $\#_\Psi M$ are cobordant and thus can also be obtained from each other by a sequence of surgeries.

On the other hand, a surgery can be interpreted as a fibre connected sum with a sphere. Let M be an n-dimensional manifold and $S \subset M$ a k-dimensional embedded sphere with trivial normal bundle. Then performing surgery means cutting out a neighbourhood $S \times D^{n-k}$ of S and gluing back in a copy of $D^{k+1} \times S^{n-k-1}$. The resulting manifold can also be obtained by performing a fibre connected sum on $M \sqcup S^n$ along S and S^k if we interpret the sphere as

$$S^n = \partial D^{n+1} = \partial(D^{k+1} \times D^{n-k}) = (S^k \times D^{n-k}) \cup (D^{k+1} \times S^{n-k-1}).$$

□

As mentioned in Remark 2.1.3, odd-dimensional closed oriented manifolds do admit open book decompositions. In general, Quinn [69] found an obstruction, which is invariant under cobordism. Hence, we have the following corollary.

Corollary 2.2.2
A fibre connected sum $\#_\Psi M$ on a manifold M admits an open book decomposition if and only if the original manifold M does.

In the following, we want to describe a natural fibre connected sum in the open book setting, called the *binding sum*.

Let M be a (not necessarily connected) smooth n-dimensional manifold with open book decomposition (Σ, ϕ) whose binding B contains two diffeomorphic components B_0 and B_1. Their normal bundles νB_0 and νB_1 admit trivialisations induced by the pages of the open book decomposition of M. Let Ψ denote the fibre orientation reversing diffeomorphism of $B \times D^2 \subset B \times \mathbb{C}$ sending (b, z) to $(b, \bar{z})$. Hence, we can perform the fibre connected sum along B_0 and B_1 with respect to the above trivialisations of the normal bundles and the map induced by Ψ and denote the result by

$$\#_{B_0,B_1} M.$$

We call $\#_{B_0,B_1}M$ the **binding sum** of M along B_0 and B_1.

Note that by the above corollary, the binding sum admits an open book structure. However, the corollary does not provide any information on how such a structure is related to the original open books. Chapter 5 of this thesis gives an explicit open book decomposition which is natural in the sense that it coincides with the original open books away from a submanifold isotopic to the binding.

2.3 Open books in contact topology

A positive contact structure ξ on an oriented manifold M is **supported** by an open book structure (B, π) if it can be written as the kernel of a contact form α inducing a positive contact structure on B and such that $d\alpha$ induces a positive symplectic structure on the fibres of π. Such a contact form α is then called **adapted** to the open book and the triple (B, π, α) is said to be a **contact open book**.

Note that a contact form is adapted to an open book (B, π), where the binding B is a contact submanifold, if and only if its Reeb vector field is positively transverse to the fibres of π and positively tangent to B (cf. [73, Lemma 2.13]).

Open books can be used to construct contact structures, as the next result shows. We will roughly follow [37, Section 7.3] and [23, Section 2.2] in the discussion and proof of the theorem.

Theorem 2.3.1 (Thurston–Winkelnkemper [72], Giroux [41])
Let M be a closed, odd-dimensional manifold with abstract open book decomposition (Σ, Φ). Suppose furthermore that the following holds:

- *the page Σ admits an exact symplectic form $\omega = d\beta$,*
- *the Liouville vector field Y defined by $i_Y\omega = \beta$ is transverse to the boundary $\partial\Sigma$ pointing outwards, and*
- *the monodromy ϕ is a symplectomorphism of the symplectic page (Σ, ω).*

Then M admits a contact structure supported by the given open book decomposition.

The original statement by Thurston and Winkelnkemper [72] was restricted to dimension three. Combined with Alexander's existence result for open book decompositions of 3-manifolds, this proves Martinet's theorem (Theorem 1.4.15), since the hypotheses on the decomposition do not impose any restriction in three dimensions. The more general version stated here is due to Giroux [41].

The idea of the proof is to construct a contact form on the mapping torus using the symplectic structure of the page and extending this over a neighbourhood of the binding in a suitable way using a contact form on the binding and a *Lutz pair*.

Definition 2.3.2
A **Lutz pair** (h_1, h_2) consists of two of smooth functions $h_1\colon [0,1] \to \mathbb{R}^+$ and $h_2\colon [0,1] \to \mathbb{R}_0^+$ such that

- $h_1(0) = 1$ and h_2 vanishes like r^2 at $r = 0$,
- $h_1'(r) < 0$ and $h_2'(r) \geq 0$ for $r > 0$,
- all derivatives of h_1 vanish at $r = 0$.

For a contact form α_B on B and a Lutz pair (h_1, h_2) the 1-form

$$\alpha = h_1(r)\alpha_B + h_2(r)d\theta$$

is a contact form on $B \times D^2$ with polar coordinates (r, θ) on the D^2-factor, and its derivative $d\alpha$ restricts to a symplectic form on the subsets with constant angle θ. Indeed, we have

$$\alpha \wedge (d\alpha)^n = h_1^{n-1}(h_1 h_2' - h_2 h_1')\alpha_B \wedge (d\alpha_B)^{n-1} \wedge dr \wedge d\theta > 0$$

and

$$(d\alpha|_{\{\theta=\text{const.}\}})^n = n h_1' h_1^{n-1} dr \wedge \alpha_B \wedge (d\alpha_B)^{n-1} > 0,$$

as α_B is contact on B and ∂_r is transverse to the sets $\{\theta = \text{const.}\}$.

The mapping torus used in the construction is in fact a generalised version of the mapping torus defined in Section 2.1. For a manifold with boundary Σ, a diffeomorphism ϕ of Σ and a positive function h on Σ which is constant near $\partial\Sigma$ we define the **generalised mapping torus** $\Sigma_h(\phi)$ as

$$\Sigma_h(\phi) = \{(x, \theta) \in \Sigma \times \mathbb{R}\colon\ 0 \leq \theta \leq h(x)\}/_{(x,h(x))\sim(\phi(x),0)}.$$

From this, we can construct a closed manifold $M^h_{(\Sigma,\phi)}$ as the quotient

$$\Big(\Sigma_h(\phi) + \partial\Sigma \times D^2\Big)/_\sim,$$

where $(x, e^{i\theta}) \in \partial(\partial\Sigma \times D^2)$ is identified with $[x, c\theta/2\pi] \in \partial\Sigma_h(\phi)$. Here, c denotes the value of h near $\partial\Sigma$. Observe that the manifold $M_{(\Sigma,\phi)}$ constructed with the usual mapping torus is diffeomorphic to the manifold $M^h_{(\Sigma,\phi)}$ obtained by the generalised mapping torus. A diffeomorphism is given by extending the map

$$[x, \theta] \mapsto \Big[x, \frac{\theta}{2\pi} h(x)\Big]$$

on the mapping tori by the identity.

Furthermore, the monodromy has to be an exact symplectomorphism, which is no real restriction, as the following lemma shows.

Lemma 2.3.3 ([37, Lemma 7.3.4])
Let ϕ be a symplectomorphism of an exact symplectic manifold $(\Sigma, \omega = d\beta)$ with boundary which is equal to the identity near the boundary. Then ϕ is isotopic to an exact symplectomorphism ϕ_1, i.e. $\phi_1^*\beta - \beta$ is exact, via symplectomorphisms equal to the identity near the boundary.

We can now give the proof of Giroux's theorem.

Sketch of proof of Theorem 2.3.1. Open books with isotopic monodromies are diffeomorphic, so we can assume that the monodromy ϕ is exact symplectic by the above lemma. Hence, we have $\phi^*\beta - \beta = dh$ for some function h on Σ. By the compactness of Σ, we can assume that h is positive and that its minimum is equal to 1. The 1-form

$$\alpha = \beta + d\theta$$

is a contact form on $\Sigma \times \mathbb{R}$. As it is invariant under the map

$$(x, \theta) \mapsto \big(\phi(x),\, \theta - h(x)\big),$$

it induces a contact form on the generalised mapping torus $\Sigma_h(\phi)$. The monodromy ϕ is equal to the identity near $\partial\Sigma$ and the function h is constant there, so we can extend α over $\partial\Sigma \times D^2$ via

$$\frac{h_1}{h_1(1)}\beta|_{T\partial\Sigma} + h_2 d\theta,$$

where (h_1, h_2) a Lutz pair with h_2 equal to 1 near 1. One can now check that this yields indeed a contact form as desired. □

Remark 2.3.4
We call the construction in the proof of Theorem 2.3.1 a **(generalised) Thurston–Winkelnkemper construction** and $(\Sigma, \phi, d\beta)$ an **abstract contact open book**.

Conversely, contact structures are always supported by open books:

Theorem 2.3.5 (Giroux–Mohsen [43], cf. [60])
Every contact structure on a closed manifold admits a supporting open book decomposition.

In dimension three, there even is a one-to-one correspondence under appropriate assumptions (see Theorem 2.3.8 below).

Dörner showed that if a contact structure is supported by an open book decomposition, it can in fact be assumed to arise as a generalised Thurston–Winkelnkemper construction.

Theorem 2.3.6 (Dörner, cf. [23, Theorem 3.1.22])
Every contact manifold supported by an open book (B, π) *is contactomorphic to a generalised Thurston–Winkelnkemper construction.*

Definition 2.3.7
Let $(M, \xi = \ker \alpha)$ be a contact manifold supported by an open book (B, π) and denote the pull-back of $d\theta$ under $\pi\colon M \setminus B \to S^1$ also by $d\theta$. A vector field X is called **monodromy vector field** if

- it is transverse to the pages and satisfies $d\theta(X) = 1$ on $M \setminus B$,
- the restriction of $\mathcal{L}_X d\alpha$ to any page vanishes,
- it equals ∂_θ on a neighbourhood $B \times D^2 \subset M$ of the binding, where (r, θ) are polar coordinates on the D^2-factor, and the open book fibration is given by the angular coordinate on D^2.

Given a monodromy field we get an associated abstract open book description of (M, ξ), and in turn an identification of (M, ξ) as a generalised Thurston–Winkelnkemper construction. In particular, such a vector field always exists if the open book and contact structure comes from a generalised Thurston–Winkelnkemper construction, i.e. we can always assume the existence of a monodromy vector field by the previous theorem.

Note that the stabilisation procedure described in Section 2.1.2 also works in the contact setting if one requires the disc to be Lagrangian in the page and intersecting the binding in a Legendrian sphere (see [73, Section 4.3] for details). To ensure that the stabilised open book yields the same *contact* manifold, one has to restrict oneself to *positive* stabilisations in this setting.

2.3.1 Open books in dimension three

As mentioned above, in dimension three the relation between open books and contact structures was shown to be even deeper:

Theorem 2.3.8 (Giroux [41])
Let M *be a closed, orientable* 3*-manifold. Then there is a one-to-one correspondence between isotopy classes of oriented contact structures on* M *and open book decompositions up to positive stabilisations.*

An accessible discussion of the theorem can be found in [30]. Note that the correspondence is between contact structures and *topological* open book decompositions. In fact, the concept of *contact* open books is not required in dimension three at all. In particular, there is a Thurston–Winkelnkemper construction for any pair (Σ, ϕ)

with Σ a surface with non-empty boundary and ϕ a diffeomorphism equal to the identity near the boundary (see [37, Section 4.4.2] for details on the 3-dimensional construction and [37, Section 7.3] for the relation of the 3-dimensional to the general case).

In addition to that, orientation-preserving diffeomorphisms of a surface can be decomposed into a sequence of maps, which are easy to understand and visualise.

Definition 2.3.9
Let γ be an embedded curve on an oriented surface S and identify an oriented neighbourhood N of γ with $S^1 \times [-1,1]$ such that γ corresponds to $S^1 \times \{0\}$. A **right-handed (or positive) Dehn twist** of S along γ is the homeomorphism of S which is equal to the identity outside N and restricts to

$$S^1 \times [-1,1] \ni (s,t) \mapsto \Big(s + \pi(1+t),\ t\Big) \in S^1 \times [-1,1]$$

on N. The inverse of this map is called a **left-handed (or negative) Dehn twist** along γ.

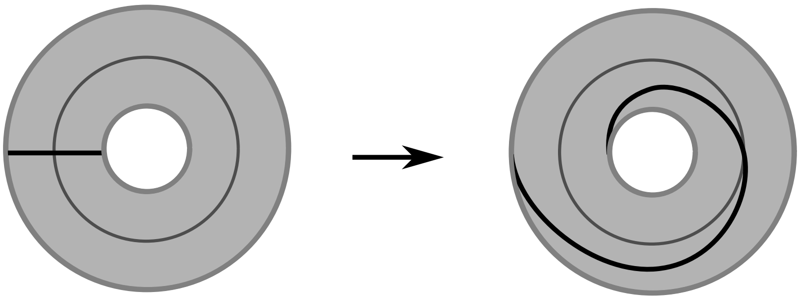


Figure 2.4: A right-handed Dehn twist along the central curve.

Note that this does not depend on the orientation of the curve and that it is possible to smoothen a Dehn twist (e.g. by constructing it as the flow of an appropriate vector field) to turn it into a diffeomorphism.

The Dehn–Seidel twists described in Section 2.1.2 are a natural generalisation of Dehn twists to higher dimension and coincide with ordinary Dehn twists in dimension two. In particular, a stabilisation of an abstract open book in dimension three is obtained by attaching a 2-dimensional 1-handle to the page and precomposing the monodromy with a Dehn twist along a curve which runs over the handle exactly once.

The class of Dehn twists provides sufficiently many building blocks to construct all diffeomorphisms of surfaces:

Theorem 2.3.10 (Lickorish, cf. [56])
Any orientation-preserving diffeomorphism of a compact oriented surface (with possibly non-empty boundary) can be written as a composition of Dehn twists and diffeomorphisms isotopic to the identity.

We have already discussed that the Reeb field of a contact form adapted to an open book is transverse to the fibres of the corresponding open book fibration. In particular, the interior of a page of an open book supporting the contact structure is a convex surface. However, more is true. As described in Section 2.1.1, an open book decomposition (B, π) of a 3-manifold yields a Heegaard splitting. The Heegaard surface S is the union of the two “opposite” pages $\overline{\pi^{-1}(0)}$ and $\overline{\pi^{-1}(\pi)}$ along their common boundary B. If the open book supports the contact structure and α is an adapted contact form, one can choose a volume form Ω on S which defines the same orientation as $d\alpha$ on $\pi^{-1}(0)$ and the opposite one on $\pi^{-1}(\pi)$. The characteristic foliation is given by the vector field X satisfying $i_X\Omega = \alpha|_{TS}$ and is divided by B (in the sense of Section 1.3). Hence, the surface S is a convex surface (for details see [37, Example 4.8.4(4)]).

A Legendrian knot sitting on a page of an open book, or more general on a convex surface S, possesses two natural framings. The framing given by the contact planes and the framing given by the surface. Using an $\mathbb{R}$-invariant neighbourhood of S in which the contact form is given by $\alpha = \beta + u dz$ as discussed in Section 1.3, one can observe that the contact framing makes a negative half-twist relative to the surface framing exactly when the knot crosses the dividing set. This yields the following theorem (see [29, Theorem 2.30] for details).

Theorem 2.3.11 (cf. [29, Theorem 2.30])
Let L be a Legendrian knot on a convex surface S with dividing set Γ. Then the framing induced by the contact structure ξ and the framing induced by the surface S differ by

$$-\frac{1}{2}|L \cap \Gamma|.$$

In particular, the contact and surface framing coincide if L sits on the page of an open book.

The situation of a Legendrian curve sitting on a convex surface is not exotic at all. A large class of curves sitting on a convex surface can be realised as Legendrians, as the next theorem shows.

Theorem 2.3.12 (Legendrian realisation principle, Honda [47, Theorem 3.7])
Let S be a convex surface with dividing set Γ and let γ be a properly embedded arc or closed curve on S such that every component of $S \setminus \gamma$ contains a component of $\Gamma \setminus \gamma$. Then S can be isotoped through convex surfaces such that γ is Legendrian.

We will call a simple closed curve L on a convex surface S with the above property **non-isolating**, i.e. L is non-isolating if every component of $S \setminus L$ has non-empty intersection with the dividing set Γ of S. In fact, the converse holds as well, i.e. Legendrian simple closed curves on convex surfaces are non-isolating.

Lemma 2.3.13
Let L be a Legendrian knot on a convex surface S. Then L is non-isolating.

Proof. Let L be not non-isolating, i.e. there is a component S_0 of the complement of L in S with $S_0 \cap \partial S = \emptyset$, and assume that L represents a Legendrian knot. Without loss of generality, we have $\mathrm{div}_\Omega(X) > 0$ on $\overline{S}_0$, where Ω is a volume form on S and X the vector field defining the characteristic foliation. Hence,

$$0 < \int_{\overline{S}_0} \mathrm{div}_\Omega(X) = \int_{\overline{S}_0} d(i_X\Omega) = \int_L i_X\Omega = \int_L \alpha = 0,$$

where α denotes the contact form and the last equality holds because L is Legendrian. □

Moreover, there is also an even stronger result in the opposite direction of Theorem 2.3.12. If we fix a Legendrian knot, we can always realise it on the page of a compatible open book decomposition.

Theorem 2.3.14 (cf. [30, Corollary 4.23])
Let L be a Legendrian link in a contact 3-manifold (M, ξ). Then there is an open book decomposition of M supporting ξ such that the knot L sits on a page.

This is a corollary to the proof of the 3-dimensional version of Theorem 2.3.5, namely, that every oriented contact structure on a closed 3-manifold admits a supporting open book decomposition. The idea of the proof is as follows. The first step is to construct a *contact cell decomposition* of the manifold, i.e. a finite cell decomposition such that the 1-skeleton is Legendrian, for every 2-cell D the contact structure makes exactly one negative twist along the boundary ∂D relative to D, and the contact structure is tight on every 3-cell. This can be achieved by covering M with finitely many Darboux balls and picking a cell decomposition such that every 3-cell is contained in a Darboux ball. The 1-skeleton can then be made Legendrian, and the condition on the 2-cells can be obtained by subdividing the 2-cells by Legendrian arcs in a suitable way. One can then explicitly construct an open book fibration with fibres retracting to the 1-skeleton (cf. [30, 3]). A given Legendrian knot can simply be included in the 1-skeleton of the cell decomposition and will thus sit on a page of the resulting open book.

3
Computing the Thurston–Bennequin invariant in open books

This chapter presents explicit formulas for computing the Thurston–Bennequin invariant of a Legendrian knot sitting on a convex Heegaard surface or on the page of an open book in terms of its intersection behaviour with the Heegaard curves or in terms of the monodromy, respectively. The chapter is based on joint work with Marc Kegel and Mirko Klukas, which was published in [21].

We first state and prove the formula for Heegaard surfaces in Section 3.1, which we then adapt to the setting of open books in Section 3.2. First we compute the homology of the knot exterior from the Heegaard diagram and then present contact and Seifert framing in this homology. Comparing these two classes then yields the Thurston–Bennequin invariant. We furthermore present some examples and applications in Section 3.3 and extend the obtained results to rationally nullhomologous Legendrian knots in Section 3.4.

3.1 The Thurston–Bennequin invariant in Heegaard diagrams

Let (M, ξ) be a closed 3-dimensional contact manifold. Fix a contact Heegaard splitting $M = V_1 \cup V_2$, i.e. a Heegaard splitting such that the Heegaard surface is convex in the sense of Section 1.3. In particular, the handlebodies V_1 and V_2 are not assumed to be standard contact handlebodies. Let $K \subset M$ be a Legendrian knot on $\partial V_1 = \partial V_2$ which is nullhomologous in M and intersects the dividing set Γ of the convex Heegaard surface ∂V_1 transversely. We denote the number of intersection points by $|K \cap \Gamma|$. Note that for a given knot in a contact manifold it is always possible to find a contact Heegaard splitting such that the knot lies on the Heegaard surface (by Theorem 2.3.14 the knot can even be realised on the page of an open book).

We give a formula to calculate the Thurston–Bennequin invariant of K in this setting. Let n denote the genus of the Heegaard surface. We may assume that the solid handlebody V_1 consists of a single 0-handle and n 1-handles and the solid handlebody V_2 consists of n 2-handles and a single 3-handle. Let g_i, g_i^*, $i = 1, \ldots, n$, be a set of generators of $H_1(\partial V_1; \mathbb{Z})$ such that the g_i^* are trivial in $H_1(V_1; \mathbb{Z})$ and

$g_i \bullet g_j^* = \delta_{ij}$, $g_i \bullet g_j = 0 = g_i^* \bullet g_j^*$, where $\bullet$ denotes the intersection product in $H_1(\partial V_1; \mathbb{Z})$ (see Figures 3.1 and 3.2). For ease of notation we will not differentiate between an oriented curve and the homology class it represents. Furthermore, the Heegaard curves on ∂V_1, i.e. the images of the attaching spheres c_i of the 2-handles, are called c_i'. We fix orientations of K and of the c_i. This is needed for the calculations, but the results are independent of the particular choice.

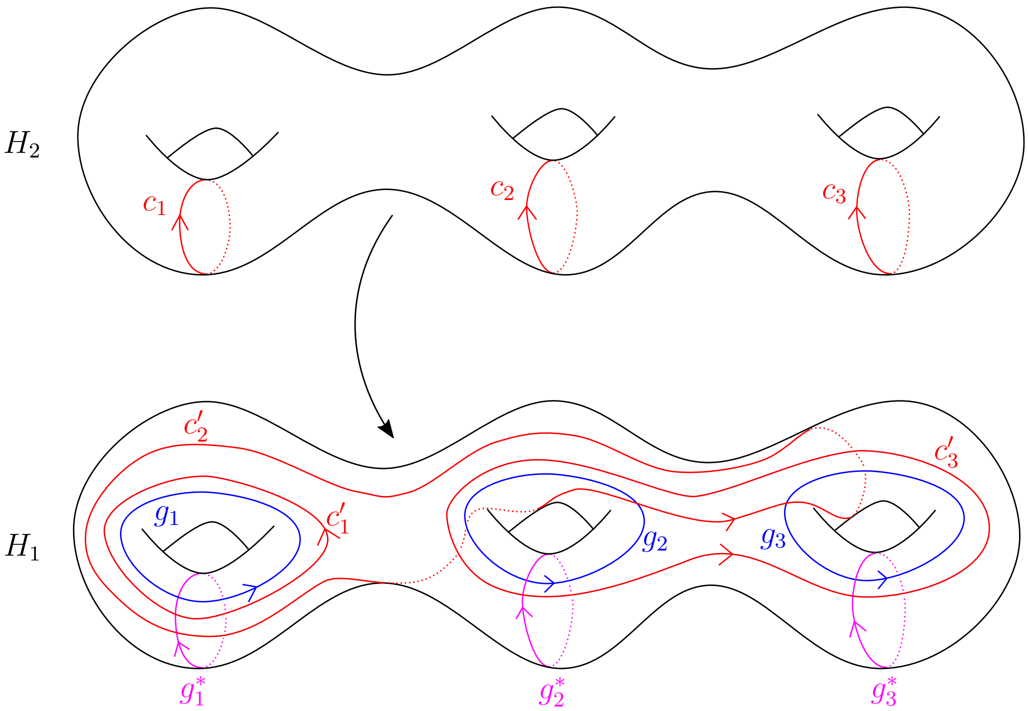


Figure 3.1: A Heegaard diagram of $S^1 \times S^2$.

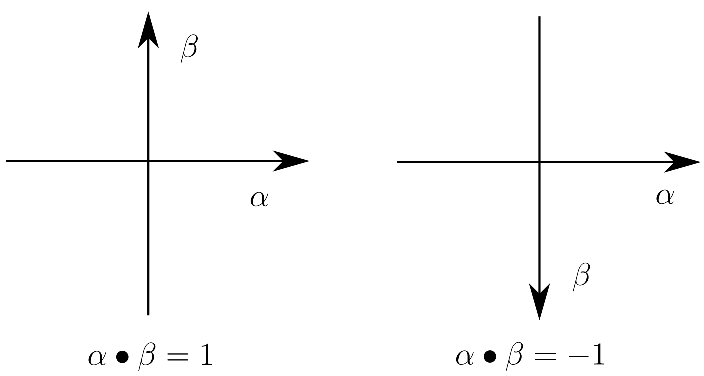


Figure 3.2: The intersection pairing in $\mathbb{R}^2$ with standard orientation.

Observe that $H_1(M; \mathbb{Z})$ is generated by the g_i and there is a relation for every Heegaard curve c_j' (whose expression in terms of the generators can be read off by counting intersections of c_j' with the g_i^*, i.e. $c_j' = \sum (c_j' \bullet g_i^*) g_i$, cf. [71, Chapter 9]).

We will omit the coefficient group $\mathbb{Z}$ and all homology groups are understood to be integral if not stated otherwise. So we have the presentation

$$H_1(M) = \langle g_1, \dots, g_n \mid c'_1, \dots, c'_n \rangle.$$

A knot is nullhomologous in M if and only if its class is a linear combination of the relations in $H_1(M)$ over the integers, i.e. as a class in $H_1(V_1)$ we can write the nullhomologous knot K as

$$K = \sum_{i=1}^{n} E_i c'_i$$

for appropriate integers E_i.

Theorem 3.1.1

The Thurston–Bennequin invariant of the Legendrian nullhomologous knot K lying on a convex Heegaard surface, transversely intersecting its dividing set, computes as

$$\mathrm{tb}(K) = -\frac{1}{2}|K \cap \Gamma| + \sum_{i=1}^{n} E_i \cdot (K \bullet c'_i).$$

Proof. First we consider the case in which K does not intersect the dividing set Γ of the convex Heegaard surface. Then the contact framing of K coincides with the *Heegaard framing*, i.e. the framing induced by a parallel copy of K on the Heegaard surface. We want to use the above presentation of $H_1(M)$ to construct a presentation of $H_1(M \setminus \nu K)$, where νK denotes a tubular neighbourhood of K in M. To that end, we slightly push the curves g_i and c'_i into the handlebody V_1 in a neighbourhood of the intersection points with K and denote the resulting curves by $\widetilde{g}_i$ and $\widetilde{c}'_i$ (see Figure 3.3). Let μ be a positive meridian of K in M. Then $H_1(M \setminus \nu K)$ is generated by μ together with the $\widetilde{g}_i$ and the relations are $\widetilde{c}'_i - (K \bullet c'_i)\mu$, so

$$H_1(M \setminus \nu K) = \langle \widetilde{g_1}, \dots, \widetilde{g_n}, \mu \mid \widetilde{c}'_1 - (K \bullet c'_1)\mu,\ \dots,\ \widetilde{c}'_n - (K \bullet c'_n)\mu \rangle.$$

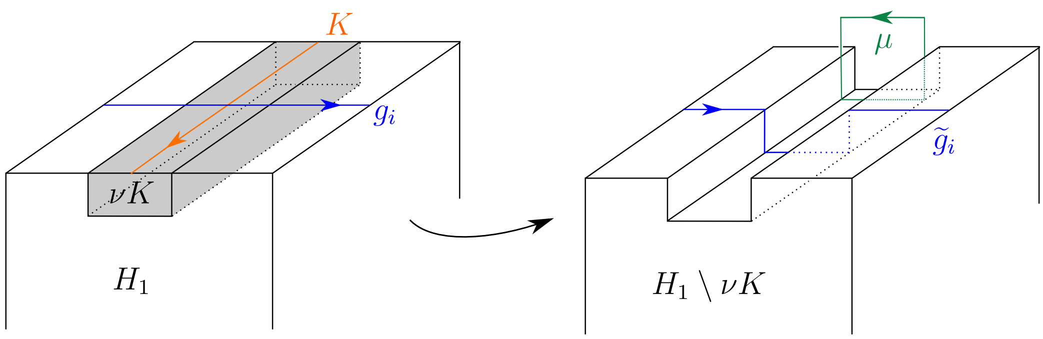


Figure 3.3: The relation of the generators in M and $M \setminus \nu K$.

Let λ_c denote the contact longitude and λ_s the Seifert longitude of K as defined in Section 1.4.2. Then the Thurston–Bennequin invariant $\operatorname{tb}(K)$ is defined by the equation

$$\lambda_c = \operatorname{tb}(K) \cdot \mu + \lambda_s$$

in $H_1(\partial\nu K)$. The Seifert longitude is defined by the condition $\lambda_s = 0$ in $H_1(M \setminus \nu K)$. This yields the equation

$$-\operatorname{tb}(K) \cdot \mu + \lambda_c = 0 \in H_1(M \setminus \nu K).$$

In our setting, the contact framing coincides with the Heegaard framing. Therefore the contact longitude λ_c is given as a parallel copy of K on the Heegaard surface, i.e. we have $\lambda_c = K$ in $H_1(\partial V_1)$ and thus $\lambda_c = \sum_{i=1}^n E_i \widetilde{c}_i'$ in $H_1(M \setminus \nu K)$. Inserting this expression for the contact longitude into the above equation for the Thurston–Bennequin invariant we get

$$-\operatorname{tb}(K) \cdot \mu + \sum_{i=1}^n E_i \widetilde{c}_i' = 0.$$

Using the relations in $H_1(M \setminus \nu K)$ this transforms to

$$\operatorname{tb}(K)\mu = \sum_{i=1}^n E_i \widetilde{c}_i' = \sum_{i=1}^n E_i \widetilde{c}_i' - \sum_{i=1}^n E_i \Big(\widetilde{c}_i' - (K \bullet c_i')\mu\Big) = \sum_{i=1}^n E_i \cdot (K \bullet c_i')\mu.$$

As the meridian of a nullhomologous knot has infinite order in the knot complement (see Appendix B), this proves the first case.

In the general case, when the intersection of K with the dividing set Γ is non-empty the result follows from the fact that the contact framing and the framing induced by the Heegaard surface differ by half the number of intersection points of K with the dividing set (see Theorem 2.3.11). □

Algorithm 3.1.2 (Computing the Thurston–Bennequin invariant)
Using the formula from Theorem 3.1.1 we can compute the Thurston–Bennequin invariant of a Legendrian nullhomologous knot lying on a convex Heegaard surface algorithmically. Define vectors

$$A := (K \bullet g_i^*)_{i=1,\dots,n}$$

and

$$I := (K \bullet c_i')_{i=1,\dots,n}$$

and a matrix

$$C := \left(c_j' \bullet g_i^*\right)_{i,j=1,\dots,n}.$$

Solve the equation

$$A = C \cdot E$$

over the integers (such a solution exists exactly if K is nullhomologous). Then the Thurston–Bennequin invariant is given by:

$$\mathrm{tb} = -\frac{1}{2}|K \cap \Gamma| + \langle E, I \rangle.$$

Remark 3.1.3

The last formula in the proof of Theorem 3.1.1 shows that the scalar product of a vector B in the kernel of C with the vector I vanishes, i.e. the particular choice of a solution E of the equation $A = CE$ does not impact the result.

Example 3.1.4

We compute the first homology group of the manifold M given by Figure 3.1 as

$$H_1(M) = \langle g_1, g_2, g_3 \mid c'_1, c'_2, c'_3 \rangle = \langle g_1, g_2, g_3 \mid g_1, g_1, g_2 + g_3 \rangle \cong \mathbb{Z},$$

where we use $c'_j = \sum (c'_j \bullet g^*_i) g_i$. In fact, one can show that M is diffeomorphic to $S^1 \times S^2$.

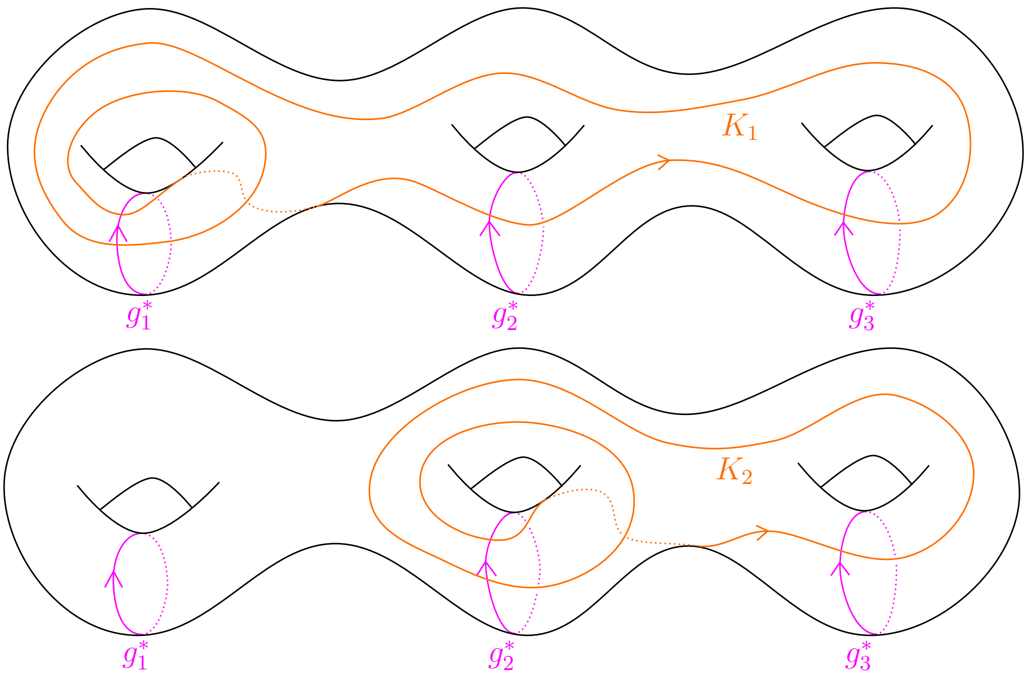


Figure 3.4: Knots on a Heegaard surface of $S^1 \times S^2$.

Now consider two knots K_1 and K_2 in M as shown in Figure 3.4. The matrix C is equal to

$$C = \left(c'_j \bullet g^*_i \right)_{i,j=1,\dots,3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the knots are encoded by

$$A_1 = (K_1 \bullet g_i^*)_{i=1,\dots,3} = (2,1,1)^\intercal$$

and

$$A_2 = (K_2 \bullet g_i^*)_{i=1,\dots,3} = (0,2,1)^\intercal.$$

The equation $A_1 = CE$ admits integral solutions, e.g. $(1,1,1)^\intercal$, which means K_1 is nullhomologous. However, $A_2 = CE$ is not solvable at all, so K_2 is not nullhomologous.

3.2 The Thurston–Bennequin invariant in open books

In this section we use the result on Heegaard surfaces to give a computable formula for the Thurston–Bennequin invariant of a nullhomologous Legendrian knot on the page of an open book and furthermore a way to check whether a knot on a page is nullhomologous. Note that by Theorem 2.3.14 it is always possible to find an open book supporting the contact structure such that a given Legendrian knot lies on a page. Let $(S, \phi = \mathrm{T}_l^{\varepsilon_l} \circ \cdots \circ \mathrm{T}_1^{\varepsilon_1})$ be a contact open book with monodromy ϕ encoded by a concatenation of Dehn twists. Here $\mathrm{T}_k^{\varepsilon_k}$ denotes a Dehn twist along the curve T_k with sign ε_k. Let (M, ξ) be the resulting contact manifold. Choose an arc basis a_i, $i = 1, \dots, n$, i.e. a system of arcs such that S becomes a disc when cutting along them, in such a way that the arcs meet the curves T_k transversely. Using the intersection product on S we define a matrix C via

$$c_{ij} := \sum_{m=1}^{l} \sum_{1\le k_1<\dots<k_m\le l} \varepsilon_{k_1} \cdots \varepsilon_{k_m} (T_{k_m} \bullet T_{k_{m-1}}) \cdots (T_{k_2} \bullet T_{k_1})(T_{k_1} \bullet a_j)(T_{k_m} \bullet a_i).$$

Theorem 3.2.1

Let $(S, \phi = \mathrm{T}_l^{\varepsilon_l} \circ \cdots \circ \mathrm{T}_1^{\varepsilon_1})$ be a contact open book with monodromy ϕ encoded by a concatenation of Dehn twists and fixed arc basis a_i, $i = 1, \dots, n$ of S as above. Let K be a Legendrian knot on S. Define a vector A by $A = (K \bullet a_i)_{i=1,\dots,n}$.

1. *K is nullhomologous if and only if there exists an integer solution E of*

$$A = C \cdot E.$$

2. *If K is nullhomologous its Thurston–Bennequin invariant is equal to*

$$\mathrm{tb}(K) = -\langle E, A \rangle.$$

Proof. By Theorem 2.3.11 the contact framing of the Legendrian knot K on S coincides with the framing induced by the page S. With the chosen arc basis a_i, $i = 1, \dots, n$, we get that

$$(\Sigma := S_1 \cup S_2,\ g_i^* := (a_i)_1 \cup (a_i)_2,\ c_i' := (a_i)_1 \cup (\phi(a_i))_2)$$

is a genus $n = (2 \cdot \text{genus}(S) + r - 1)$ Heegaard diagram for M, where r is the number of boundary components of S (this is a slight variation of the classic approach published in [48]). Here S_1 and S_2 are two copies of the page S, with the orientation of S_2 reversed, glued along their boundary, i.e. the Heegaard surface Σ is the double of S, and $(a_i)_j$ denotes a copy of a_i on S_j (see Figure 3.5). Curves and arcs on S_2 are always assumed to be oriented oppositely to their counterparts on S to give rise to oriented curves on Σ. Furthermore, the curves g_i^* and c_i' are understood to be slightly isotoped to only have transverse intersections. We identify S with S_1, so the knot K lies on S_1.

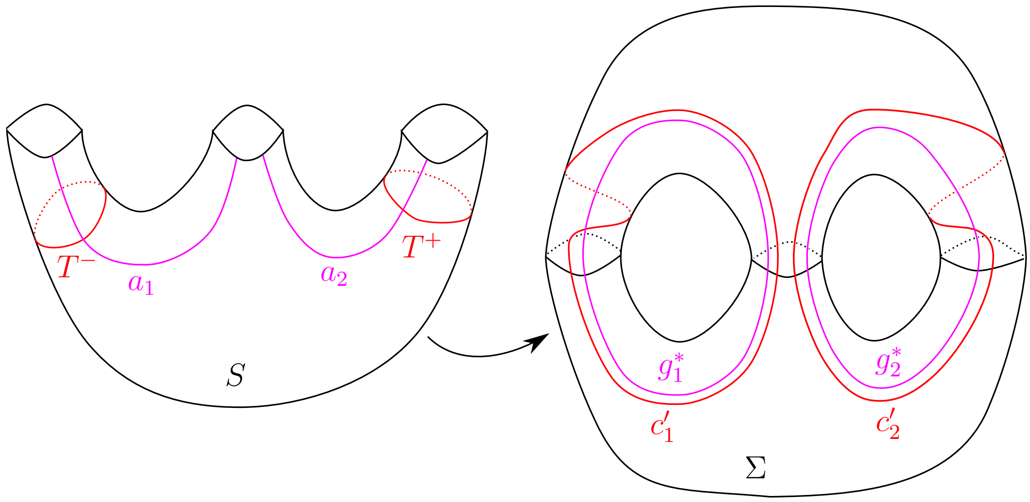


Figure 3.5: From an open book decomposition to a Heegaard diagram.

Having transformed the open book into a Heegaard diagram, Theorem 3.1.1 gives a formula for computing the Thurston–Bennequin invariant. In particular, we will use Algorithm 3.1.2 and adapt it such that it only uses input data from the open book, i.e. tb is computable without constructing a Heegaard diagram first. We have

$$A = (K \bullet g_i^*)_{i=1,\dots,n} = (K \bullet ((a_i)_1 \cup (a_i)_2))_{i=1,\dots,n} = (K \bullet a_i)_{i=1,\dots,n} ,$$

where the last equality arises from restriction to S_1 as K lies only on S_1. Analogously, the matrix C has entries

$$c_{ij} = c_j' \bullet g_i^* = ((a_j)_1 \cup (\phi(a_j))_2) \bullet ((a_i)_1 \cup (a_i)_2) = \phi(a_j) \bullet a_i,$$

where the last term is again read in S_2 and comes from restriction (we isotope the curves such that there are no intersection points on the boundary and consider the algebraic intersection number). Observe that in our current setting we have

$$I = (K \bullet c_i')_{i=1,\dots,n} = A$$

by a similar argument.

As we have shown in Section 3.1 the knot K is nullhomologous if and only if the equation

$$A = C \cdot E$$

has an integral solution E, and in that case the Thurston–Bennequin invariant computes as

$$\mathrm{tb} = \langle E, A \rangle.$$

It remains to calculate the entries of the matrix C in terms of the Dehn twists $\mathrm{T}_l^{\varepsilon_l} \circ \cdots \circ \mathrm{T}_1^{\varepsilon_1}$ encoding the monodromy. Let α be any curve on a surface and T^ε a Dehn twist. Then the homology class of the image of α under T^ε is

$$\alpha + \varepsilon (T \bullet \alpha) T,$$

where we identify curves with their classes as usual. Repeatedly applying this to the a_j yields

$$\phi(a_j) = a_j + \sum_{m=1}^{l} \sum_{1 \leq k_1 < \ldots < k_m \leq l} \varepsilon_{k_1} \cdots \varepsilon_{k_m} (T_{k_m} \bullet T_{k_{m-1}}) \cdots (T_{k_2} \bullet T_{k_1})(T_{k_1} \bullet a_j) T_{k_m}$$

and thus

$$c_{ij} = \sum_{m=1}^{l} \sum_{1 \leq k_1 < \ldots < k_m \leq l} \varepsilon_{k_1} \cdots \varepsilon_{k_m} (T_{k_m} \bullet T_{k_{m-1}}) \cdots (T_{k_2} \bullet T_{k_1})(T_{k_1} \bullet a_j)(T_{k_m} \bullet a_i),$$

where we use the intersection product on S_2. In applications, however, we want to consider intersections on the page S, which has the opposite orientation. This provides for the negative sign in the formula to compute the Thurston–Bennequin invariant in an open book, i.e. we have

$$\mathrm{tb} = -\langle E, A \rangle.$$

□

Remark 3.2.2
Note that in the case of disjoint Dehn twist curves T_k the expression of the matrix entries c_{ij} reduces to

$$c_{ij} = \sum_{k=1}^{l} \varepsilon_k (T_k \bullet a_j)(T_k \bullet a_i).$$

In particular, C is symmetric.

Remark 3.2.3
It follows from Remark 3.1.3 that the particular choice of a solution E does not impact the result. In case that the matrix C is symmetric, this is also immediate since two different solutions differ by a vector B in the kernel of C and A is in the image of C. Thus, the scalar product of A and B vanishes, see also Example 3.3.3.

3.3 Applications and Examples

Example 3.3.1 (Unknot in the standard 3-sphere)
Consider the open book decomposition of (S^3, ξ_{st}) with page S an annulus and the monodromy given by a positive Dehn twist T^+ along the central curve T and let K be a Legendrian knot parallel to T on the page S. In this example an arc basis of S consists of a single arc a only, which we choose to be a linear segment joining the boundary components of the annulus.

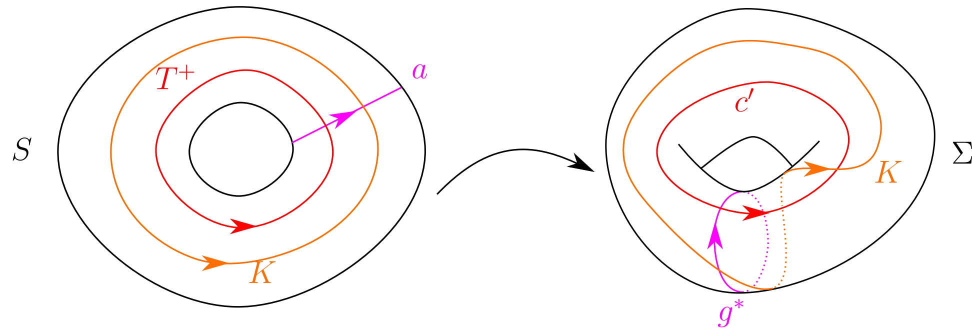


Figure 3.6: The Legendrian unknot in (S^3, ξ_{st}).

Choosing orientations as depicted in Figure 3.6 we get

$$A = K \bullet a = -1$$

and

$$C = \varepsilon(T \bullet a)^2 = 1 \cdot (-1)^2 = 1.$$

The knot K is nullhomologous since the equation $-1 = 1 \cdot E$ has the solution $E = -1$. This we knew before since any knot in S^3 is nullhomologous, but we need a particular solution E to calculate tb. We then compute the Thurston–Bennequin invariant as

$$\mathrm{tb}(K) = -\langle E, A\rangle = -1 \cdot (-1) \cdot (-1) = -1.$$

The Heegaard diagram on the right hand side of Figure 3.6 encodes the same situation. Here it becomes clear that K is the unknot. This particular Heegaard splitting arises from the open book picture on the left by performing a Dehn twist.

Example 3.3.2 (Unknot in an overtwisted 3-sphere)
We change the monodromy in the previous example to be a negative Dehn twist T^- along T. As above, we then have $A = -1$, but C becomes

$$C = \varepsilon(T \bullet a)^2 = -1 \cdot (-1)^2 = -1$$

and $E = 1$ solves $A = CE$. So we get

$$\mathrm{tb}(K) = -\langle E, A\rangle = -1 \cdot 1 \cdot (-1) = 1.$$

Stabilising K once yields an overtwisted disc, so the contact structure is indeed overtwisted.

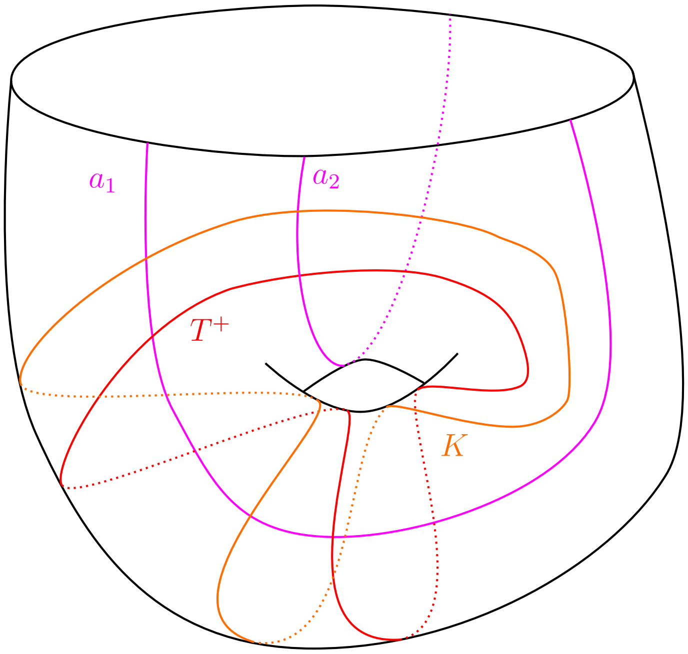


Figure 3.7: A nullhomologous knot K with non-unique E.

Example 3.3.3
Consider the open book for $(S^1 \times S^2, \xi_{st})$ depicted in Figure 3.7. We have

$$A = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

The equation $A = CE$ is solvable over the integers, so K is nullhomologous. However, the solution is non-unique. Solutions are of the form

$$E_n = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + n \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

for $n \in \mathbb{Z}$. Then

$$\mathrm{tb}(K) = -\langle E_n, A \rangle = -1 \cdot (-1) \cdot (-1) = -1,$$

i.e. the result is independent of the chosen solution E_n. This is always the case (see Remark 3.2.3).

Example 3.3.4 (Stabilisations)
Let K be a nullhomologous Legendrian knot on the page S of an open book (S, ϕ) with $\phi = T_l^{\varepsilon_l} \circ \cdots \circ T_1^{\varepsilon_1}$. We want to compute the Thurston–Bennequin invariant of the stabilised knot K_{stab} in the stabilised open book $(S_{\text{stab}}, \phi_{\text{stab}} = T_{l+1}^{\varepsilon_{l+1}} \circ T_l^{\varepsilon_l} \circ \cdots \circ T_1^{\varepsilon_1})$. Let A, C, E be the data associated to the original open book and knot. With an additional arc a and orientations chosen as in Figure 3.8, we have

$$A_{\text{stab}} = \begin{pmatrix} A \\ 1 \end{pmatrix}$$

and

$$C_{\text{stab}} = \begin{pmatrix} C & 0 \\ 0 & \varepsilon_{l+1} \end{pmatrix}$$

since T_{l+1} is disjoint from the other Dehn twists. The equation $A_{\text{stab}} = C_{\text{stab}} E_{\text{stab}}$ is then solved by the integral vector

$$E_{\text{stab}} = \begin{pmatrix} E \\ \varepsilon_{l+1} \end{pmatrix}$$

and we compute tb to be

$$\operatorname{tb}(K_{\text{stab}}) = -\langle E_{\text{stab}}, A_{\text{stab}} \rangle = -\left\langle \begin{pmatrix} E \\ \varepsilon_{l+1} \end{pmatrix}, \begin{pmatrix} A \\ 1 \end{pmatrix} \right\rangle = -\langle E, A \rangle - \varepsilon_{l+1} = \operatorname{tb}(K) - \varepsilon_{l+1}.$$

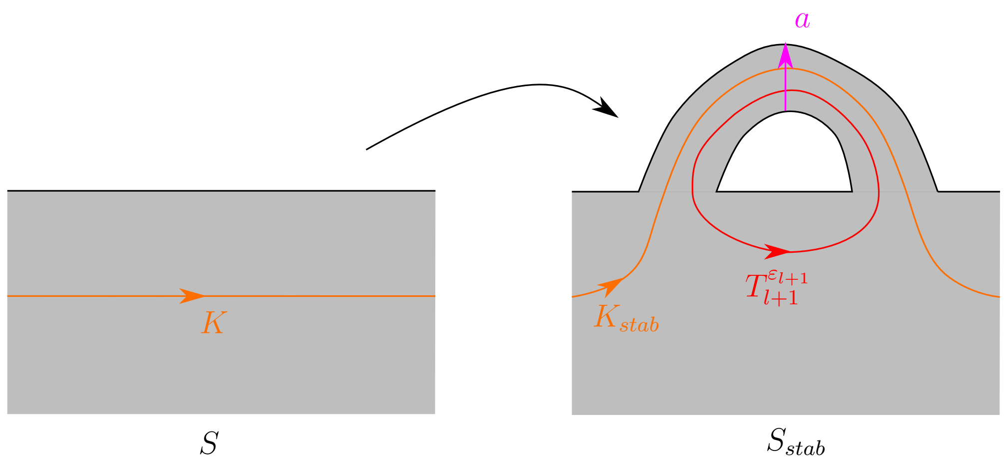


Figure 3.8: A stabilisation of K obtained by a positive stabilisation of the open book.

3.4 Rationally nullhomologous knots

In this section we study rationally nullhomologous Legendrian knots as proposed in Baker–Grigsby [5], Baker–Etnyre [4] and Geiges–Onaran [38]. In particular, we generalise Theorems 3.1.1 and 3.2.1 to rationally nullhomologous Legendrian knots. Let K be a knot in M. We call K **rationally nullhomologous** if its homology class is of finite order $d > 0$ in $H_1(M)$. Let νK be a tubular neighbourhood of K and denote the meridian by $\mu \subset \partial \nu K$.

Definition 3.4.1
A **Seifert framing** of a rationally nullhomologous knot K of order d is a class $r \in H_1(\partial \nu K)$ such that

- $\mu \bullet r = d$,
- $r = 0$ in $H_1(M \setminus \nu K)$.

It is obvious that every rationally nullhomologous knot has a Seifert framing; uniqueness however is not obvious.

Lemma 3.4.2
The Seifert framing of a rationally nullhomologous knot is unique.

Proof. Let r_1 and r_2 be Seifert framings. Let μ, λ be an oriented basis of $H_1(\partial \nu K)$, where μ is represented by a meridian of K. Then we can write

$$r_i = p_i \mu + q_i \lambda.$$

As r_i is a Seifert framing we have $q_i = d$ with d the order of K. The classes r_1 and r_2 are equal if considered in $H_1(M \setminus \nu K)$. Therefore we have $p_1 \mu = p_2 \mu$ in $H_1(M \setminus \nu K)$. But a meridian of K intersects a rational Seifert surface non-trivially, so μ cannot be a torsion element. Hence $p_1 = p_2$, i.e. the framings coincide. □

Existence and uniqueness of the Seifert framing enables us to define a rational Thurston–Bennequin invariant, which coincides with the usual definition in the nullhomologous case, and is well-defined in arbitrary contact 3-manifolds.

Definition 3.4.3
The **rational Thurston–Bennequin invariant** of a rationally nullhomologous Legendrian knot K of order d is defined as

$$\operatorname{tb}_{\mathbb{Q}}(K) = \frac{1}{d} \left(\lambda_c \bullet r \right)$$

where λ_c denotes the contact longitude and r the Seifert framing, and the intersection is taken in $H_1(\partial \nu K)$.

Observe that this means that we have the equality

$$r = d\lambda_c - d \operatorname{tb}_{\mathbb{Q}}(K) \mu$$

in $H_1(\partial\nu K)$.

Now consider a Legendrian knot K on a convex Heegaard surface not intersecting the dividing set. Using the notation from Section 3.1, such a knot is rationally nullhomologous of order d in M if and only if the equation

$$dA = C \cdot E$$

admits a solution E over the integers and d is the minimal natural number for which a solution exists. In that case, fix a solution E. Analogously to the nullhomologous case we then have

$$d\,\mathrm{tb}_{\mathbb{Q}}(K)\mu = \sum_{i=1}^{n} E_i \widetilde{c}_i' = \sum_{i=1}^{n} E_i \cdot (K \bullet c_i')\mu$$

in $H_1(M \setminus \nu K)$. Since μ has infinite order we thus proved the following theorem.

Theorem 3.4.4
The rational Thurston–Bennequin invariant of the Legendrian rationally nullhomologous knot K of order d lying on a convex Heegaard surface, transversely intersecting its dividing set Γ, computes as

$$\mathrm{tb}_{\mathbb{Q}}(K) = -\frac{1}{2}|K \cap \Gamma| + \frac{1}{d}\sum_{i=1}^{n} E_i \cdot (K \bullet c_i') = -\frac{1}{2}|K \cap \Gamma| + \frac{1}{d}\langle E, I\rangle.$$

Similarly, Theorem 3.2.1 generalises to the result stated below.

Theorem 3.4.5
Let $(S, \phi = \mathrm{T}_l^{\varepsilon_l} \circ \cdots \circ \mathrm{T}_1^{\varepsilon_1})$ be a contact open book with monodromy ϕ encoded by a concatenation of Dehn twists and fixed arc basis a_i, $i = 1, \ldots, n$, of S. Let K be a Legendrian knot on S. Define a vector A by $A = (K \bullet a_i)_{i=1,\ldots,n}$.

1. *K is rationally nullhomologous of order d if and only if there exists an integer solution E of*

$$dA = C \cdot E$$

   and d is the minimal natural number for which a solution exists.
2. *If K is rationally nullhomologous of order d its rational Thurston–Bennequin invariant is equal to*

$$\mathrm{tb}_{\mathbb{Q}}(K) = -\frac{1}{d}\langle E, A\rangle.$$

4
Computing the rotation number in open books

In this chapter we continue in the spirit of Chapter 3 and again consider a Legendrian knot sitting on the page of a contact open book. We explain how to compute the second classical invariant, the rotation number, in case the knot is nullhomologous. In particular, we prove the following theorem:

Theorem 4.0.1
Let K be a knot sitting on the page of an open book (Σ, ϕ) with monodromy ϕ given as a concatenation of Dehn twists along non-isolating curves. Then there exists an arc basis of Σ such that the intersection behaviour of K and the Dehn twist curves with the arcs give criteria and formulas to

- *(a) decide whether K is (rationally) nullhomologous,*
- *(b1) compute the (rational) Thurston–Bennequin invariant of K if K is (rationally) nullhomologous,*
- *(b2) compute the (rational) rotation number of K if K is (rationally) nullhomologous,*
- *(b3) compute the (rational) self-linking number of a transverse push-off of K if K is (rationally) nullhomologous,*
- *(c) compute the Poincaré dual of the Euler class of the contact structure,*
- *(d) decide if the Euler class of the contact structure is torsion and if so, compute its d_3-invariant*

(see Algorithm 4.4.1).

The chapter is based on joint work with Marc Kegel and is also going to be published separately.

We will first generalise an example of [57] to compute the rotation number of a Legendrian knot sitting on the page of a specific planar open book of (S^3, ξ_{st}). Afterwards we use the method of [3] to find embeddings of more general non-planar abstract open books into (S^3, ξ_{st}) and give formulas for computing the rotation number in these cases. For the general case, we first use Avdek's algorithm [3] for transforming an open book into a contact surgery diagram along a Legendrian link and then compute the invariants from the resulting surgery diagram via the methods discussed in Appendix A. Section 4.4 presents an algorithm to actually compute the invariants from Theorem 4.0.1 and also gives some examples. Finally, in Section 4.5 the results are applied to the binding number of Legendrian knots, which we propose to study in analogy to the binding number of a contact manifold as introduced in [34].

4.1 A special planar case

We begin by discussing a method to compute the rotation number in an easy planar case which is based on the idea presented in [57, Lemma 4.1].

Suppose that Σ is *planar*, i.e. Σ is a disc with holes

$$\Sigma \cong D^2 \setminus \left(\bigsqcup_{i=1}^{k} D_i^2 \right),$$

and the monodromy is given by $\phi = \beta_k^{+1} \circ \cdots \circ \beta_1^{+1}$, where β_i^{+1} denotes a positive Dehn twist along a curve β_k parallel to the inner boundary ∂D_i^2. We furthermore assume that the curves β_i are oriented consistently with the boundary orientation induced by Σ (see Figure 4.1). In particular, by destabilising the open book, we see that (Σ, ϕ) describes the standard contact 3-sphere (S^3, ξ_{st}) and from this it also follows that K is some Legendrian unknot.

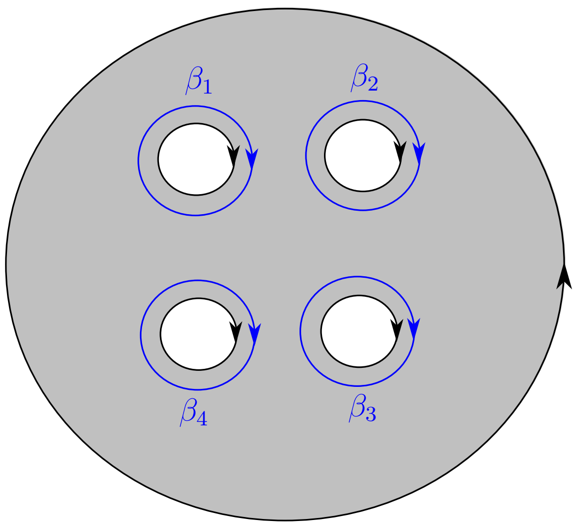


Figure 4.1: A planar open book decomposition of (S^3, ξ_{st}).

Proposition 4.1.1

Let K be a Legendrian knot sitting on the page of a planar open book (Σ, ϕ) with ϕ as described above. Then the following holds:

1. $K = \sum_{i=1}^{k} b_i \beta_i \in H_1(\Sigma)$ *such that either all* $b_i \in \{+1, 0\}$ *or all* $b_i \in \{-1, 0\}$*,*

2. the rotation number of L computes as

$$\operatorname{rot}(K) = \sum_{i=1}^{k} b_i - \operatorname{sign}\left(\sum_{i=1}^{k} b_i\right).$$

Proof. (1) First note, that a simple closed curve cannot have $|b_i| > 1$ or it would have self-intersections. With orientations chosen as above, one also observes that all non-vanishing b_i have to be equal.

(2) By the first statement, we can glue small oriented rectangular bands connecting the $b_i\beta_i$ with non-vanishing coefficients b_i inside Σ in such a way that the oriented boundary of the resulting region is isotopic to K in Σ (cf. Figure 4.2). The orientation of these rectangles coincides with the orientation of the page Σ exactly if the b_i are positive.

Note that the β_i are unknots with Thurston–Bennequin invariant -1 and vanishing rotation number. Indeed, β_i can be assumed to be parallel to a Dehn twist curve arising by a stabilisation. These curves bound a disc in the complement and by the Dehn twist, the Seifert framing differs by one from the contact framing given by the page. So β_i is a tb-(-1) unknot, i.e. the rotation number is zero. Furthermore, a Seifert surface for L is given by the union of the discs bounded by the non-vanishing $b_i\beta_i$ (in the complement of the page) and the attached bands in the page. The rotation number computes as the sum of the indices of a vector field in the contact structure extending the positive tangent of K over Σ. As $\operatorname{rot}(\beta_i) = 0$, an extension without zeros is possible over the discs bounded by β_i and we only have to study the bands. As the contact framing and the page framing coincide, this reduces the problem to extending the positive tangent vector field to the boundary of the bands over the bands in Σ. This is ± 1 for each band by Poincaré–Hopf, depending on whether the orientation of the band agrees with the orientation of the page Σ or not. Hence, the rotation number of L is a signed count of the number of bands, i.e. $\operatorname{rot}(K) = \sum_{i=1}^{k} b_i - \operatorname{sign}\left(\sum_{i=1}^{k} b_i\right)$. □

Remark 4.1.2

The formula from Proposition 4.1.1 can also be obtained by observing that a curve enclosing k-holes is the result of $(k-1)$ times stabilising a curve running around a single hole. The latter has Thurston–Bennequin invariant -1 and vanishing rotation number.

Example 4.1.3

Consider the Legendrian L on the planar open book of (S^3, ξ_{st}) as depicted in Figure 4.2. The class in the first homology group of Σ represented by L can be written as

$$L = \sum_{i=1}^{4} b_i\beta_i = \beta_2 + \beta_3 + \beta_4.$$

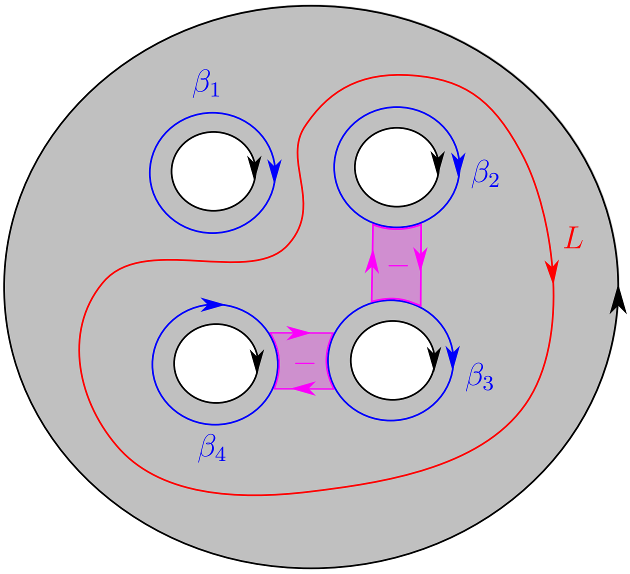


Figure 4.2: A Legendrian knot on the page of a planar open book of (S^3, ξ_{st}).

By Proposition 4.1.1, the rotation number of L is

$$\mathrm{rot}(L) = \sum_{i=1}^{4} b_i - \mathrm{sign}\left(\sum_{i=1}^{k} b_i\right) = 2.$$

This method is not known to generalise to non-planar open books. One reason is that on surfaces of higher genus, the isotopy class of a curve is not determined by its homology class.

4.2 Another special case

Next we consider knots on open books (Σ, ϕ) of the standard contact 3-sphere with an arbitrary page but a special monodromy. Denote the genus of Σ by g and the number of boundary components by $h+1$. Suppose that the monodromy is given by

$$\phi = \beta_{g+h}^{+1} \circ \cdots \circ \beta_{g+1}^{+1} \beta_g^{+1} \circ \alpha_g^{+1} \circ \cdots \circ \beta_1^{+1} \circ \alpha_1^{+1}$$

as indicated in Figure 4.3. We also choose orientations of α_i and β_i as in the picture. In particular, $\alpha_i \bullet \beta_j = \delta_{ij}$. Let r_i, $i = 1, \ldots, g+h-1$, be the depicted reducing arcs, which do not intersect the α- and β-curves, i.e. when cutting along them the page Σ decomposes into a collection of tori with a disc removed and annuli. Let a_i and b_i be arcs on the page Σ representing a basis of $H_1(\Sigma, \partial\Sigma)$ dual to $\{\alpha_i, \beta_i\}$ with respect to the intersection product (oriented such that $\alpha_i \bullet a_i = 1$, $\beta_j \bullet b_j = 1$).

The following algorithm will be applied to a word corresponding to the knot K in Proposition 4.2.2. Note that the conventions presented below for labelling vertical

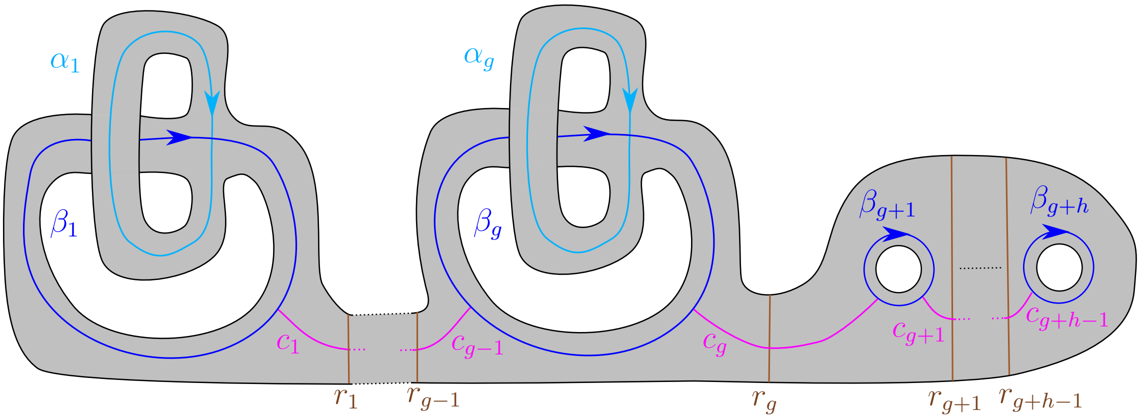


Figure 4.3: A non-planar open book of (S^3, ξ_{st}) with arbitrary many boundary components.

tangencies in this setting by ρ_+ and λ_+ do not agree with those for counting cusps of a Legendrian front projection as in [37, Proposition 3.5.19].

Algorithm 4.2.1
Let w be a word in $\langle \alpha_i, \beta_i \mid i = 1 \ldots, k\rangle$. Set λ_+ to be the number of times a β^{-1} is followed by an α^{-1} of the same index, also considering the step from the last to the first letter, and similarly, set ρ_+ equal to the number of times an α^{-1} is followed by a β^{-1} of the same index.

Denote places where the index changes by r_u (r_d) if the index increases (decreases) – including the last position if the index of the last letter is not equal to the index of the first letter. For instance, in the word

$$\alpha_1\beta_2\alpha_2\beta_4^{-1}\alpha_3^{-1}\beta_2$$

we have five positions of index changes:

$$\alpha_1 r_u \beta_2\alpha_2 r_u \beta_4^{-1} r_d \alpha_3^{-1} r_d \beta_2 r_d.$$

Now run through the index changes and increment λ_+ and ρ_+ according to the following rule:

- increment λ_+ by 1 for
  - a β^{-1} followed by r_u
  - r_d followed by an α^{-1}
- increment ρ_+ by 1 for
  - an α^{-1} followed by r_d
  - a β followed by r_d.

In the example sequence above, we have $\lambda_+ = 1$ and $\rho_+ = 2$.

Proposition 4.2.2
Let K be an oriented non-isolating knot on the abstract open book (Σ, ϕ) of (S^3, ξ_{st}) specified above. Choose a starting point on K and write K as a word in the α_i and β_i by noting intersections with a_i and b_i when traversing along K. Then the rotation number of K is

$$\mathrm{rot}(K) = \rho_+ - \lambda_+$$

with ρ_+ and λ_+ calculated from the presentation of K as described in Algorithm 4.2.1.

Proof. First note that without loss of generality, we can assume that the page Σ has only a single boundary component by stabilising the open book along arcs not intersecting the r_i connecting a hole to the outer boundary component. Then the open book (Σ, ϕ) can be embedded into (S^3, ξ_{st}) with the front projection shown in Figure 4.4 (in lightly shaded regions the orientation of Σ agrees with the blackboard orientation, in darkly shaded regions the orientations disagree) – the embedded page Σ is the ribbon of the Legendrian graph displayed in the upper half of Figure 4.4 (see [3] for details). Note that in particular, the contact vector field ∂_z is transverse to the embedded page. Furthermore, after rescaling the embedding can be assumed to be such that in $\mathbb{R}^3 \subset S^3$ we have

$$[-1, 1] \times \Sigma \to \big(\mathbb{R}^3, \xi_{\mathrm{st}} = \ker(x dy + dz)\big), \ (t, p) \mapsto p + (0, 0, t),$$

i.e. we can relate to a specific page by its shift in the z-direction.

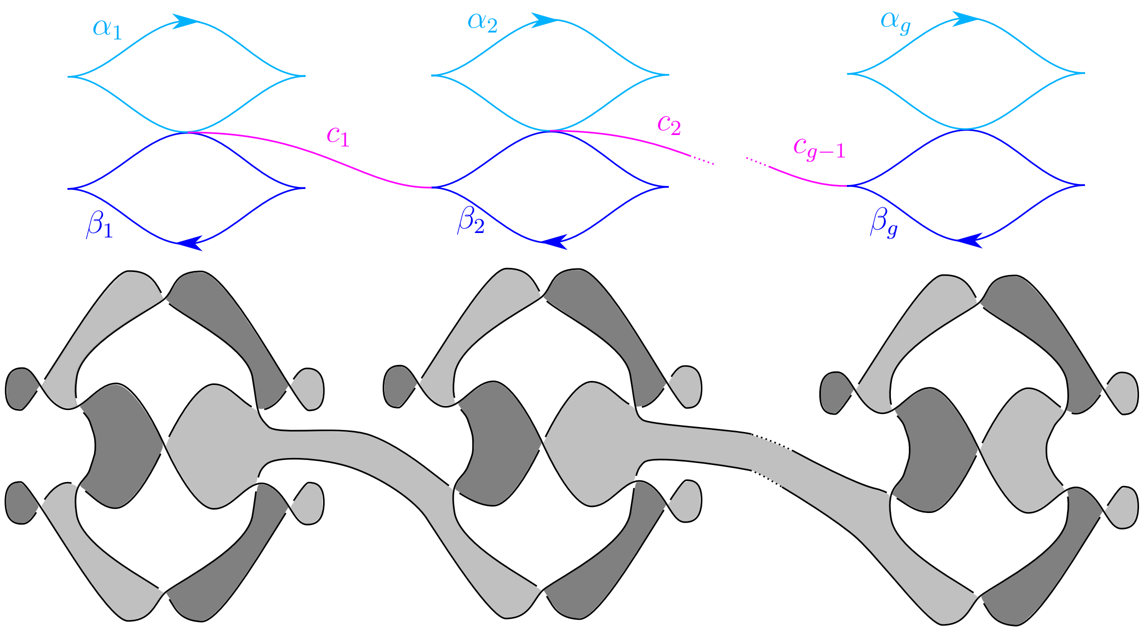


Figure 4.4: An embedding into (S^3, ξ_{st}) of the (stabilised) abstract open book from Figure 4.3 and the Legendrian graph shown in the front projection.

The rotation number of a nullhomologous Legendrian knot with respect to a Seifert surface S is given by the rotation of its tangent vector with respect to a fixed trivialisation of the contact planes over S. If the contact structure is globally trivialisable, one can instead fix a global trivialisation. The standard contact structure ξ_{st} on $\mathbb{R}^3 \subset (S^3, \xi_{\mathrm{st}})$ can be trivialised globally by ∂_x and $\partial_y - x\partial_z$. As the contact vector field ∂_z is transverse to the embedded page Σ of the open book, this trivialisation also induces a trivialisation of the tangent planes to Σ. Then the rotation number of the Legendrian realisation of a curve sitting on the page agrees with the rotation of its projection, i.e. the original curve, to the original page with respect to the induced trivialisation.

The projection of ∂_x to Σ along ∂_z lies in the xz-plane. Observe that the ∂_z-component changes sign when passing from a lightly shaded region to a darkly shaded region and vice-versa. To compute the rotation of a curve on the embedded page which is non-singular in the front projection diagram, we thus have to count vertical tangencies in the front projection according to the rule described in Figure 4.5. The rotation then equals $\rho_+ - \lambda_+$. Alternatively, we can also compute it as $\lambda_- - \rho_-$.

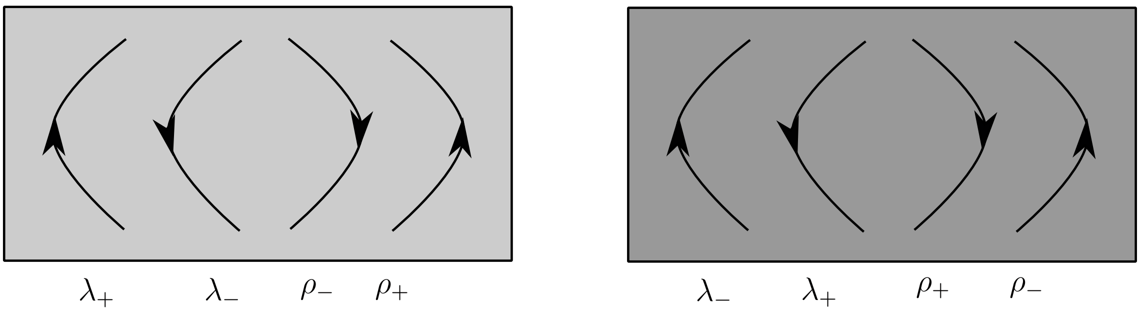


Figure 4.5: The labelling of the vertical tangencies.

In fact, we do not even have to count all vertical tangencies, but we can ignore those cancelling each other. To this end, we write K as a word in the α_i and β_i by noting intersections with a_i and b_i when traversing along K. Observe that the α- and β-curves have vanishing rotation, as they have two vertical tangencies cancelling each other. Changing from α_i to β_i accounts for a λ_-, changing from β_i to α_i for a ρ_-. Likewise, the change from α_i^{-1} to β_i^{-1} gives a ρ_+, the one from β_i^{-1} to α_i^{-1} a λ_+. It is easily verified that all other changes with fixed index do not introduce vertical tangencies. In particular, a knot not intersecting any of the reducing arcs has vanishing rotation number, since it has $\lambda_+ = \rho_+$. It thus remains to inspect those tangencies occurring before or after an intersection with a reducing arc. These intersections happen when the index of the letters change. The vertical tangencies occurring in these cases are summarised in Table 4.1.

leaving to the right from	count
α	λ_-
α^{-1}	–
β	–
β^{-1}	λ_+
coming from the left to	**count**
α	ρ_-
α^{-1}	–
β	–
β^{-1}	ρ_+
leaving to the left from	**count**
α	–
α^{-1}	ρ_+
β	ρ_+
β^{-1}	–
coming from the right to	**count**
α	–
α^{-1}	λ_+
β	λ_-
β^{-1}	–

Table 4.1: Occurrence of vertical tangencies.

Hence, the rotation number can be computed from the word according to the rule given in Algorithm 4.2.1. □

Example 4.2.3

Consider the knot on the embedded page of the open book of (S^3, ξ_{st}) given in Figure 4.6. The knot corresponds to the word $\alpha_1\beta_2\alpha_2\alpha_3^{-1}\beta_3\beta_2$. The vertical tangencies corresponding to the α- and β-curves which immediately cancel are marked in green. The remaining vertical tangencies are marked blue and labelled. We have $\rho_+ = 2$, $\lambda_+ = 0$, $\rho_- = 1$, $\lambda_- = 3$, i.e. the rotation number of the Legendrian knot represented by K is

$$\text{rot}(K) = \rho_+ - \lambda_+ = \lambda_- - \rho_- = 2.$$

We will now apply Algorithm 4.2.1 on the word $\alpha_1\beta_2\alpha_2\alpha_3^{-1}\beta_3\beta_2$. As neither a β^{-1} is followed by an α^{-1} of the same index, nor an α^{-1} by a β^{-1}, we set $\lambda_+ = 0 = \rho_+$. Next, we consider the index changes:

$$\alpha_1 r_u \beta_2 \alpha_2 r_u \alpha_3^{-1} \beta_3 r_d \beta_2 r_d.$$

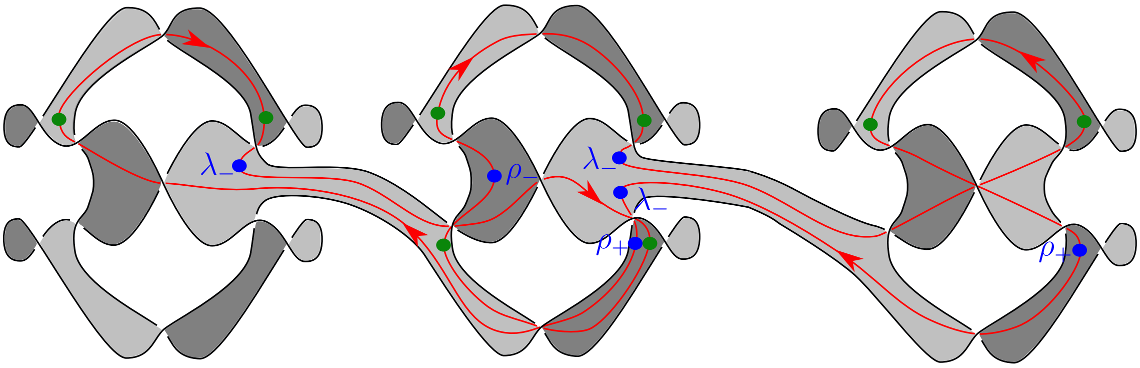


Figure 4.6: A knot on an embedded page in (S^3, ξ_{st}). Vertical tangencies cancelling each other are marked green, other vertical tangencies are marked blue and labelled.

The positions $\beta_3 r_d$ and $\beta_2 r_d$ both increase ρ_+ by one, all other positions leave the counts unchanged. Hence, also the algorithm yields

$$\operatorname{rot}(K) = \rho_+ - \lambda_+ = 2.$$

Note that we could also adapt the rules specified in the algorithm to consider ρ_- and λ_- instead using the proof of the preceding proposition.

4.3 The general case

Now we are prepared to deal with a Legendrian knot in a general open book. The idea is to change the open book to the special case discussed in the previous section by a sequence of surgeries, then compute the rotation number in (S^3, ξ_{st}) as above and finally use the results presented in Appendix A with the inverse surgeries to get the rotation number of the Legendrian in the original open book. The result will be presented in a formula that can be directly computed with the data of the original open book.

In the following remark, we will briefly recall how to compute the rotation number in contact surgery diagrams.

Remark 4.3.1 (Computing rot in a surgery diagram (see Appendix A))
For an oriented Legendrian link $L = L_1 \sqcup \ldots \sqcup L_k$ in (S^3, ξ_{st}) let (M, ξ) be the contact manifold obtained from (S^3, ξ_{st}) by contact $(1/n_i)$-surgeries $(n_i \in \mathbb{Z})$ along L_i. Denote the topological surgery coefficients by p_i/q_i, i.e.

$$\frac{p_i}{q_i} = \frac{n_i \operatorname{tb}(L_i) + 1}{n_i}.$$

Let L_0 be an oriented Legendrian knot in the complement of L and define the vector $\mathbf{l}$

with components $l_i = l_{0i}$ and the generalised linking matrix

$$Q = \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1k} \\ q_1 l_{21} & p_2 & & \\ \vdots & & \ddots & \\ q_1 l_{k1} & & & p_k \end{pmatrix},$$

where $l_{ij} := \operatorname{lk}(L_i, L_j)$. The knot L_0 is (rationally) nullhomologous in M if and only if there is an integral (rational) solution $\mathbf{a}$ of the equation $\mathbf{l} = Q\mathbf{a}$, in which case its (rational) rotation number in (M, ξ) with respect to the Seifert class $\widehat{\Sigma}$ constructed in Theorem A.2.2 is equal to

$$\operatorname{rot}_M(L_0, \widehat{\Sigma}) = \operatorname{rot}_{S^3}(L_0) - \sum_{i=1}^{k} a_i n_i \operatorname{rot}_{S^3}(L_i).$$

Proof of Theorem 4.0.1. Let $K \subset (M, \xi)$ be a Legendrian knot sitting on the page of a compatible open book

$$(\Sigma, \phi = T_l^{\pm n_l} \circ \cdots \circ T_1^{\pm n_1})$$

with monodromy encoded in a concatenation of Dehn twists, where $T^{\pm n}$ denotes n positive (resp. negative) Dehn twists along the non-isolating oriented curve T ($n \in \mathbb{N}$). We denote the genus of Σ by g and the number of boundary components by $h + 1$.

In the following, we want to choose a *special* arc basis of Σ to exactly mimic the setting from Proposition 4.2.2 (also see Remark 4.3.2). Together with a suitable monodromy yielding (S^3, ξ_{st}), this will enable us to use the proposition to compute the invariants first in (S^3, ξ_{st}) and then to apply the surgery formulas to obtain the desired result.

Choose reducing arcs $r_1, \ldots, r_{g+h-1}$ such that when cutting along r_i

- Σ decomposes into a surface Σ_i of genus i with one boundary component containing $r_1, \ldots, r_{i-1}$ and a surface of genus $g-i$ with $h+1$ boundary components for $i = 1, \ldots, g$,
- Σ decomposes into a surface Σ_i of genus g with $i + 1$ boundary components containing $r_1, \ldots, r_{i-1}$ and a disk with $h - i$ holes for $i = g + 1, \ldots, g + h - 1$.

Then choose an arc basis of $\Sigma_i \setminus \Sigma_{i-1}$, label it by a_i, b_i and orient it such that when travelling along the oriented boundary of Σ from

- r_1 to r_1
  - first a_1 is met pointing outwards, then b_1 is met pointing inwards if $g \geq 1$
  - b_1 is met and pointing outwards if $g = 0$

- r_{i-1} to r_i only b_i is met and pointing outwards ($i = 2, \dots, g+h-2$)
- r_{g+h-1} to r_{g+h-1}
  - first b_g is met pointing outwards, then a_g is met pointing outwards if $h = 0$
  - b_{g+h} is met and pointing outwards if $h > 0$.

Choose non-trivial oriented simple closed curves α_i, β_i representing a basis of $H_1(\Sigma)$ dual to the arcs with respect to the intersection product on Σ oriented such that $\alpha_i \bullet a_i = 1$, $\beta_j \bullet b_j = 1$ and $\alpha_i \bullet \beta_i = 1$ (i.e. the situation is as in Figure 4.3).

Remark 4.3.2
Note that the arc basis cannot be chosen arbitrarily, as we will use it to write the knot as a word in α_i, β_i as above and use the formula from Proposition 4.2.2 to compute the rotation from this word. For this to work with the given formula, we have to ensure that the word we get in the abstract setting is the same as the word we get in the embedded case, which coincides with the specific abstract open book depicted in Figure 4.3. In particular, the word obtained from the oriented boundary of the page is

$$\alpha_1^{-1}\beta_1\alpha_1\beta_1^{-1}\beta_2^{-1}\cdots\beta_{g+h}^{-1}\alpha_g^{-1}\beta_g\alpha_g\cdots\alpha_2^{-1}\beta_2\alpha_2.$$

A different arc basis would require a different formula to compute the rotation number from the word, see also Example 4.4.4.

This is only important for calculating the rotation number in (S^3, ξ_{st}) which is not determined by the class of the knot in the homology of the page – the linking information required to compute the rotation number via the surgery formula is purely homological and does not depend on the specific ordering. In particular, we can use an arbitrary arc basis in a planar open book if we use Proposition 4.1.1 to compute the rotation number of the involved curves in (S^3, ξ_{st}).

Observe that we can get from the open book

$$(\Sigma, \phi_{S^3} = \beta_{g+h}^{+1} \circ \cdots \circ \beta_{g+1}^{+1} \circ \beta_g^{+1} \circ \alpha_g^{+1} \circ \cdots \circ \beta_1^{+1} \circ \alpha_1^{+1})$$

to the open book (Σ, ϕ) by a sequence of contact surgeries along Legendrian knots corresponding to the Dehn twist curves.

By the algorithm presented in [3], the surgery link is as follows: every component corresponding to a Dehn twist sits on a page of the embedded open book, the shift in z-direction of the respective page relates to the position of the Dehn twists in the monodromy factorisation – the later the Dehn twist is performed, the higher the level of the page. Using Avdek's convention, we will denote a knot K sitting on the page with level t by $K(t)$.

Observe that the $\alpha_i(s), \beta_i(s)$ are unknots with rotation number zero and Thurston–Bennequin invariant -1 and that for $t \neq s$ we have

$$\operatorname{lk}\Big(\alpha_i(t), \beta_j(s)\Big) = \begin{cases} 0, & \text{if } i \neq j \text{ or } t > s, \\ -1, & \text{if } i = j \text{ and } t < s, \end{cases}$$

$$\operatorname{lk}\Big(\alpha_i(t), \alpha_j(s)\Big) = \begin{cases} 0, & \text{if } i \neq j, \\ -1, & \text{if } i = j, \end{cases}$$

$$\operatorname{lk}\Big(\beta_i(t), \beta_j(s)\Big) = \begin{cases} 0, & \text{if } i \neq j, \\ -1, & \text{if } i = j. \end{cases}$$

If $s = t$, the curves form a Legendrian graph on a single page with α and β joined by a single transverse intersection point.

The first homology class represented by a knot K on Σ can then be written as

$$K = \sum_{i=1}^{g+h} \Big((K \bullet a_i)\alpha_i + (K \bullet b_i)\beta_i)\Big)$$

and hence

$$K(t) = \sum_{i=1}^{g+h} \Big((K \bullet a_i)\alpha_i(t) + (K \bullet b_i)\beta_i(t)\Big),$$

where $(K \bullet a_i)$ is defined to be zero for $k > g$. The linking number of two knots $K_1(t)$ and $K_2(s)$ behaves linearly and distributively with respect to this decomposition, i.e. the linking number is easily computable with the linking behaviour of the α and β curves specified above.

According to Avdek's algorithm, the surgery link in S^3 to obtain (Σ, ϕ) is the link $L = L_1 \sqcup \ldots \sqcup L_{2g+h+l}$ as specified in Table 4.2. To compute the rotation number of a knot on a page of (Σ, ϕ) using the method explained in Remark 4.3.1, we need the generalised linking matrix Q – which requires us to know tb for deducing the topological surgery coefficient from the contact one as well as all linking numbers – and the rotation numbers in (S^3, ξ_{st}).

For a knot $K(t)$, we have

$$\operatorname{tb}_{S^3}\Big(K(t)\Big) = \operatorname{lk}\Big(K(t), K(t+\varepsilon)\Big)$$

and hence, for $i = 1, \ldots, l$,

$$\operatorname{tb}_{S^3}(L_{2g+h+i}) = -\sum_{k=1}^{g+h} \Big((T_i \bullet a_k)^2 + (T_i \bullet a_k)(T_i \bullet b_k) + (T_i \bullet b_k)^2\Big).$$

Therefore, the topological surgery coefficient of L_{2g+h+i} is

$$\frac{p_{2g+h+i}}{q_{2g+h+i}} = \frac{n_i \operatorname{tb}_{S^3}(T_i) \mp 1}{n_i}.$$

name	knot	contact surgery coefficient
L_1	$\beta_1(-1)$	$+1$
$\vdots$	$\vdots$	$\vdots$
L_{g+h}	$\beta_{g+h}(-1)$	$+1$
L_{g+h+1}	$\alpha_1(0)$	$+1$
$\vdots$	$\vdots$	$\vdots$
L_{2g+h}	$\alpha_g(0)$	$+1$
L_{2g+h+1}	$T_1(1/l)$	$\mp 1/n_1$
$\vdots$	$\vdots$	$\vdots$
L_{2g+h+l}	$T_l(l/l)$	$\mp 1/n_l$

Table 4.2: The surgery link for (Σ, ϕ).

Furthermore, the linking behaviour with $L_j = \beta_j$, $j = 1, \dots, g+h$ is

$$\operatorname{lk}(L_{2g+h+i}, L_j) = -(T_i \bullet b_j)$$

and similarly, for $L_{g+h+j} = \alpha_j$, $j = 1, \dots, g$

$$\operatorname{lk}(L_{2g+h+i}, L_{g+h+j}) = -\Big((T_i \bullet a_j) + (T_i \bullet b_j)\Big).$$

The linking number of two surgery knots L_{2g+h+i} and L_{2g+h+j} with $i < j$ can be computed to be

$$\operatorname{lk}(L_{2g+h+i}, L_{2g+h+j}) = -\sum_{k=1}^{g+h} \Big((T_i \bullet a_k)(T_j \bullet a_k) + (T_i \bullet a_k)(T_j \bullet b_k) + (T_i \bullet b_k)(T_j \bullet b_k)\Big).$$

Note that the knot K can be put on the page with the lowest as well as the highest level. Depending on which is chosen, the class of Seifert surface with respect to which the rotation number is given in Remark 4.3.1 might change, and hence the rotation numbers may differ. However, if the Euler class of ξ vanishes, the rotation number of a nullhomologous Legendrian knot is independent of the Seifert surface. If we choose the knot $L_0 = K(\text{low})$ to sit on a lower page than the surgery link, we get the following linking numbers

$$\begin{aligned}
\operatorname{lk}(L_0, L_j) = & -\Big((K \bullet a_j) + (K \bullet b_j)\Big),\ j = 1, \dots, g+h, \\
\operatorname{lk}(L_0, L_{g+h+j}) = & -(K \bullet a_j),\ j = 1, \dots, g, \\
\operatorname{lk}(L_0, L_{2g+h+j}) = & -\sum_{k=1}^{g+h} \Big((K \bullet a_k)(T_j \bullet a_k) + (K \bullet a_k)(T_j \bullet b_k) \\
& +(K \bullet b_k)(T_j \bullet b_k)\Big),\ j = 1, \dots, l.
\end{aligned}$$

If on the other hand $L_0 = K(\text{high})$ is assumed to sit on a page with the highest level, we get

$$\begin{aligned}
\operatorname{lk}(L_0, L_j) &= -(K \bullet b_j),\ j = 1, \dots, g+h, \\
\operatorname{lk}(L_0, L_{g+h+j}) &= -\Big((K \bullet a_j) + (K \bullet b_j)\Big),\ j = 1, \dots, g, \\
\operatorname{lk}(L_0, L_{2g+h+j}) &= -\sum_{k=1}^{g+h} \Big((K \bullet a_k)(T_j \bullet a_k) + (K \bullet b_k)(T_j \bullet a_k) \\
&\quad + (K \bullet b_k)(T_j \bullet b_k)\Big),\ j = 1, \dots, l.
\end{aligned}$$

The only data that is left to compute are the rotation numbers in S^3 of the L_i, but this can be done as in Proposition 4.2.2. Observe that using the formula from [49] also allows us to calculate the Thurston–Bennequin invariant, which is an alternative to the method presented in Chapter 3. Similarly, one can directly calculate the Poicaré-dual of the Euler class and the d_3-invariant of the contact structure (see Theorem A.5.1).

Thus, we have proved Theorem 4.0.1. □

4.4 Algorithm and examples

We summarise the process and all required formulas in the following algorithm and illustrate them by giving examples. This section is meant as a self-contained guideline to do actual computations and can be used independently.

Algorithm 4.4.1

The setting.

Given is a non-isolating curve K on the page of an open book

$$(\Sigma_{g,h+1},\ \phi = T_l^{\pm n_l} \circ \cdots \circ T_1^{\pm n_1})$$

with $n_i \in \mathbb{N}$ and $\Sigma_{g,h+1}$ a surface of genus g with $h+1$ boundary components. The monodromy is given as a sequence of Dehn twists along non-isolating oriented curves T_i.

The choices.

Choose reducing arcs $r_1, \dots, r_{g+h-1}$ such that when cutting along r_i

- Σ decomposes into a surface Σ_i of genus i with one boundary component containing $r_1, \dots, r_{i-1}$ and a surface of genus $g-i$ with $h+1$ boundary components for $i = 1, \dots, g$,
- Σ decomposes into a surface Σ_i of genus g with $i+1$ boundary components containing $r_1, \dots, r_{i-1}$ and a disk with $h-i$ holes for $i = g+1, \dots, g+h-1$.

Then choose an arc basis of $\Sigma_i \setminus \Sigma_{i-1}$ and label it by a_i, b_i and orient it such that when travelling along the oriented boundary of Σ from

- r_1 to r_1
  - first a_1 is met pointing outwards, then b_1 is met pointing inwards if $g \geq 1$
  - b_1 is met and pointing outwards if $g = 0$
- r_{i-1} to r_i only b_i is met and pointing outwards $(i = 2, \ldots, g+h-2)$
- r_{g+h-1} to r_{g+h-1}
  - first b_g is met pointing outwards, then a_g is met pointing outwards if $h = 0$
  - b_{g+h} is met and pointing outwards if $h > 0$.

Choose non-trivial oriented simple closed curves α_i, β_i representing a basis of $H_1(\Sigma)$ dual to the arcs with respect to the intersection product on Σ oriented such that $\alpha_i \bullet a_i = 1$, $\beta_j \bullet b_j = 1$ and $\alpha_i \bullet \beta_i = 1$ (i.e. the situation is as in Figure 4.3).

The definitions.

Define an integral vector $\mathbf{l} \in \mathbb{Z}^{2g+h+l}$ with entries:

$$\begin{aligned}
\mathbf{l}_j = &\; -(K \bullet b_j), \\
&\; \textit{for } j = 1, \ldots, g+h, \\
\mathbf{l}_{g+h+j} = &\; -\Big((K \bullet a_j) + (K \bullet b_j)\Big), \\
&\; \textit{for } j = 1, \ldots, g, \\
\mathbf{l}_{2g+h+j} = &\; -\sum_{k=1}^{g+h} \Big((K \bullet a_k)(T_j \bullet a_k) + (K \bullet b_k)(T_j \bullet a_k) \\
&\; +(K \bullet b_k)(T_j \bullet b_k)\Big), \\
&\; \textit{for } j = 1, \ldots, l,
\end{aligned}$$

Define an integral $(2g+h+l) \times (2g+h+l)$-matrix Q with entries:

$$\begin{aligned}
Q_{i,j} = &\; 0, \\
&\; \textit{for } i \neq (j-g-h), j \neq (i-g-h),\ i,j = 1, \ldots, 2g+h, \\
Q_{i,i+g+h} = &\; -1 = Q_{j,j-g-h}, \\
&\; \textit{for } i = 1, \ldots, g,\ j = g+h+1, \ldots, 2g+h, \\
Q_{2g+h+i,2g+h+i} = &\; \mp 1 - n_i \sum_{k=1}^{g+h} \Big((T_i \bullet a_k)^2 + (T_i \bullet a_k)(T_i \bullet b_k) + (T_i \bullet b_k)^2\Big), \\
&\; \textit{for } i = 1, \ldots, l,
\end{aligned}$$

$$
\begin{aligned}
Q_{2g+h+i,j} &= -(T_i \bullet b_j), \\
&\quad \text{for } i = 1, \ldots, l,\ j = 1, \ldots, g+h, \\
Q_{2g+h+i,j} &= -\Big((T_i \bullet a_j) + (T_i \bullet b_j)\Big), \\
&\quad \text{for } i = 1, \ldots, l,\ j = g+h+1, \ldots, 2g+h, \\
Q_{i,2g+h+j} &= -n_j(T_j \bullet b_i), \\
&\quad \text{for } i = 1, \ldots, g+h,\ j = 1, \ldots, l, \\
Q_{g+h+i,2g+h+j} &= -n_j\Big((T_j \bullet a_i) + (T_j \bullet b_i)\Big), \\
&\quad \text{for } i = 1, \ldots, g,\ j = 1, \ldots, l, \\
Q_{2g+h+i,2g+h+j} &= -n_j \sum_{k=1}^{g+h} \Big((T_i \bullet a_k)(T_j \bullet a_k) + (T_i \bullet a_k)(T_j \bullet b_k) \\
&\quad +(T_i \bullet b_k)(T_j \bullet b_k)\Big), \\
&\quad \text{for } i < j,\ i,j = 1, \ldots, l, \\
Q_{2g+h+i,2g+h+j} &= -n_i \sum_{k=1}^{g+h} \Big((T_i \bullet a_k)(T_j \bullet a_k) + (T_i \bullet b_k)(T_j \bullet a_k) \\
&\quad +(T_i \bullet b_k)(T_j \bullet b_k)\Big), \\
&\quad \text{for } i > j,\ i,j = 1, \ldots, l.
\end{aligned}
$$

For an oriented non-isolating curve L we define the quantity $r(L)$ as follows: choose a starting point on L and write L as a word in the α_i and β_i by noting intersections with a_i and b_i when traversing along L. Set λ_+ to be the number of times a β^{-1} is followed by an α^{-1} of the same index also considering the step from the last to the first letter, and similarly, set ρ_+ equal to the number of times an α^{-1} is followed by a β^{-1} of the same index. Denote places where the index changes by r_u (r_d) if the index increases (decreases) – including the last position if the index of the last letter is not equal to the index of the first letter. Now run through the index changes and increment λ_+ and ρ_+ according to the following rule:

- increment λ_+ by 1 for
  - a β^{-1} followed by r_u
  - r_d followed by an α^{-1}
- increment ρ_+ by 1 for
  - an α^{-1} followed by r_d
  - a β followed by r_d.

Then define

$$r(L) := \rho_+ - \lambda_+.$$

The results.

Then the following holds:

(a) K is nullhomologous if and only if there is an integral solution $\mathbf{a}$ of the equation $\mathbf{l} = Q\mathbf{a}$.

(a') K is rationally nullhomologous in the manifold if and only if there is a rational solution $\mathbf{a}$ of the equation $\mathbf{l} = Q\mathbf{a}$.

(b1) If K is (rationally) nullhomologous, the (rational) Thurston–Bennequin invariant of K is

$$\begin{aligned}\operatorname{tb}(K) = &-\sum_{k=1}^{g+h}\Big((K\bullet a_k)^2 + (K\bullet a_k)(K\bullet b_k) + (K\bullet b_k)^2\Big)\\ &-\sum_{j=1}^{2g+h} a_j l_j - \sum_{j=1}^{l} a_{2g+h+j} n_j l_{2g+h+j}.\end{aligned}$$

(b2) If K is (rationally) nullhomologous, the (rational) rotation number with respect to some special Seifert surface S of K is

$$\operatorname{rot}(K,S) = r(K) - \sum_{j=1}^{l} a_{2g+h+j} n_j r(T_j).$$

(b3) Denote by $K^\pm$ the positive (resp. negative) transverse push-off of a (rationally) nullhomologous Legendrian K. Then its (rational) self-linking number with respect to the Seifert surface S from (b2) is

$$\operatorname{sl}(K^\pm, S) = \operatorname{tb}(K) \mp \operatorname{rot}(K,S).$$

(c) The Poicaré-dual of the Euler class is given by

$$\operatorname{PD}\Big(\mathrm{e}(\xi)\Big) = \sum_{i=1}^{l} n_i r(T_i)\mu_{T_i} \in H_1(M).$$

The first homology group $H_1(M)$ of M is generated by the meridians μ of the α_i, β_i and T_i and the relations are given by the generalized linking matrix $Q\mu = 0$.

(d) The Euler class $\mathrm{e}(\xi)$ is torsion if and only if there exists a rational solution $\mathbf{b}$ of $Q\mathbf{b} = \mathbf{r}$ with $\mathbf{r}_i = 0$ for $i = 1, \ldots, 2g+h$ and $\mathbf{r}_{2g+h+i} = r(T_i)$ for $i = 1, \ldots, l$. In this case, the d_3-invariant of ξ computes as

$$d_3(\xi) = g + \frac{h}{2} + \frac{1}{4}\left(\sum_{i=1}^{l} n_i b_{2g+h+i} r(T_i) - (3-n_i)\operatorname{sign}_i\right) - \frac{3}{4}\sigma(Q) - \frac{1}{2},$$

where sign_i denotes the sign of the power of the Dehn twist $T_i^{\pm n_i}$.

Remark 4.4.2
In the algorithm above, we implicitly assumed that the knot K sits on the page with a higher level than the monodromy curves. As described in Section 4.3, K could also be assumed to sit on the lowest level, which would change the formulas defining the vector $\mathbf{l}$. Note that in general, if $e(\xi) \neq 0$, the resulting rotation number might differ, as it is computed with respect to a different class of Seifert surface. However, if the open book is planar or $e(\xi) = 0$, we get the same values for both cases.

Remark 4.4.3
In the planar case, the formulas simplify to

$$\begin{aligned}
\mathbf{l}_j &= -(K \bullet b_j), \ \textit{for } j = 1, \dots, h, \\
\mathbf{l}_{h+j} &= -\sum_{k=1}^{h} (K \bullet b_k)(T_j \bullet b_k), \ \textit{for } j = 1, \dots, l, \\
Q_{i,j} &= 0, \ \textit{for } i, j = 1, \dots, h, \\
Q_{h+i,h+i} &= \mp 1 - n_i \sum_{k=1}^{h} (T_i \bullet b_k)^2, \ \textit{for } i = 1, \dots, l, \\
Q_{h+i,j} &= -(T_i \bullet b_j), \ \textit{for } i = 1, \dots, l, \ j = 1, \dots, h, \\
Q_{i,h+j} &= -n_j (T_j \bullet b_i), \ \textit{for } i = 1, \dots, h, \ j = 1, \dots, l, \\
Q_{h+i,h+j} &= -n_j \sum_{k=1}^{h} (T_i \bullet b_k)(T_j \bullet b_k), \ \textit{for } i \neq j, \ i, j = 1, \dots, l.
\end{aligned}$$

If furthermore all $n_i = 1$, we have that

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$$

with $Q_1 = 0_{h \times h}$ the zero $(h \times h)$-matrix,

$$Q_2 = Q_3^{\intercal} = -\Big(T_j \bullet b_i\Big)_{i=1,\dots,h, \ j=1,\dots,l}$$

and

$$Q_4 = Q_3 Q_2 - \operatorname{diag}\Big(\operatorname{sign}(T_1), \dots, \operatorname{sign}(T_l)\Big).$$

Example 4.4.4
In this example we want to reconsider the planar open book of (S^3, ξ_{st}) discussed in Example 4.1.3, where we calculated the rotation number to be 2 using Proposition 4.1.1. If we choose the arc basis as described above, the knot is encoded by the word $\beta_2\beta_4\beta_3$. This yields $\lambda_+ = 0$ and $\rho_+ = 2$, i.e. rot $= 2$ as expected.

Note that if we choose a different arc basis, e.g. such that the word is $\beta_2\beta_3\beta_4$, then the formula does not give the desired result, as the knot would be represented by a different word. In fact, the word $\beta_2\beta_3\beta_4$ does not even encode a simple closed curve on the embedded page.

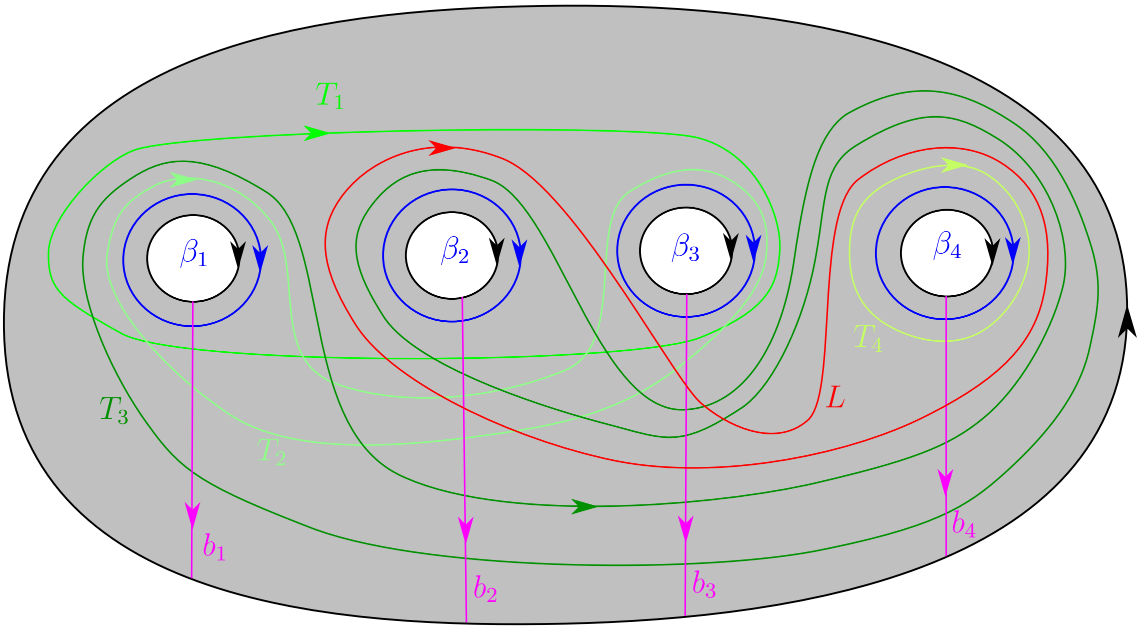


Figure 4.7: The open book $(\Sigma, \phi = T_3^{+1} \circ T_2^{+1} \circ T_1^{+1})$ of (S^3, ξ_{st}).

Example 4.4.5

Consider the open book $(\Sigma, \phi = T_3^{+1} \circ T_2^{+1} \circ T_1^{+1})$ and knot K as specified in Figure 4.7. This is an example of a non-destabilisable planar open book of (S^3, ξ_{st}) taken from [33].

By the formulas to compute rot in the special planar case, it follows directly that

$$\mathbf{r} = (0, 0, 0, 0, 2, 1, 1, 0)^\intercal$$

and $r(K) = 1$.

Using the simplified formulas for planar open books given in Remark 4.4.3, we obtain

$$\mathbf{l} = (0, -1, 0, -1, -1, 0, -1, -1)^\intercal$$

and

$$Q_2 = -\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As the manifold is S^3, it follows that Q is invertible and thus the equation $\mathbf{l} = Q\mathbf{a}$ admits a unique solution, which is easily computed to be

$$\mathbf{a} = (2, -2, -1, -1, 1, -1, 0, 1)^\intercal$$

(in particular, the calculation shows that K is nullhomologous). The Thurston–Bennequin invariant of K then computes to be

$$\mathrm{tb}(K) = -\sum_{k=1}^{4}(K \bullet b_k)^2 - \langle \mathbf{a}, \mathbf{l} \rangle = -3$$

and the rotation number is

$$\mathrm{rot}(K) = r(K) - \langle \mathbf{a}, \mathbf{r} \rangle = 0.$$

The self-linking number of both the positive and the negative transverse push-off of K is -3.

Since Q is invertible, we have $H_1 = 0$, i.e. the Poincaré dual to the Euler class of the contact structure ξ vanishes. As expected, our formula then returns

$$d_3(\xi) = -\frac{1}{2}.$$

4.5 Application to the binding number of Legendrian knots

Let K be a Legendrian knot. Then the support genus $\mathrm{sg}(K)$ is defined to be the minimal genus of the page of a contact open book decomposition in which K is contained in a single page, i.e.

$$\mathrm{sg}(K) = \min\Big\{g(\Sigma) \;\big|\; K \subset \Sigma\Big\},$$

where $g(\Sigma)$ is the genus of the surface Σ (see [64]).

In analogy to the binding number of a contact manifold as introduced in [34], we propose to define the binding number bn of K to be the minimal number of boundary components of the pages of contact open book decompositions with minimal genus containing K in a page, i.e.

$$\mathrm{bn}(K) := \min\Big\{|\partial\Sigma| \colon\; K \subset \Sigma \text{ with } g(\Sigma) = \mathrm{sg}(K)\Big\}.$$

Corollary 4.5.1
Let K be a Legendrian knot with non-vanishing rotation number and support genus $\mathrm{sg}(K) = 1$*. Then the binding number of K is at least two.*

Proof. Suppose that K has support genus and binding number both equal to one, then one can easily check using Theorem 4.0.1 or via the explicit formulas given in Algorithm 4.4.1 that the rotation number of K vanishes. □

Example 4.5.2
Every knot in (S^3, ξ_{st}) (or more generally, in a weakly fillable contact manifold) with Thurston–Bennequin invariant at least one and non-vanishing rotation number has support genus equal to one (see [64]) and thus binding number at least two.

5

Nested open books and the binding sum

In the present chapter we investigate how the binding sum construction, i.e. the fibre connected sum of two open books along diffeomorphic binding components (see Section 2.2) affects the underlying open book structures. We have already seen in Corollary 2.2.2 that the binding sum admits *some* open book decomposition. This existence result however, gives no relation of this open book to the open book structures of the original manifolds. We will show that – provided the respective binding components admit open book decompositions themselves – the binding sum can be performed such that the resulting open book structure is natural in the sense that it can be described in terms of the original decompositions (see Theorem 5.3.1). Furthermore, we will show that in the case of the contact binding sum, i.e. a binding sum of two contact manifolds with contact open book decompositions along contactomorphic binding components, the construction can also be adapted to again yield a compatible open book (see Theorem 5.4.1). This generalises the work of Klukas [53] to higher dimensions. The results in this chapter were obtained under the supervision of Mirko Klukas and are also published in a joint paper [22].

We will introduce the notion of a *nested open book* in Section 5.1, which is a submanifold inheriting an open book structure from the ambient manifold. These submanifolds turn out to be particularly useful in the context of fibre connected sums. The idea of the binding sum construction is then to not form the sum along the binding components themselves but along slightly isotoped copies, realising them as nested open books. These isotoped copies are called *push-offs* and will be discussed in detail in Section 5.2. The main result in the topological setting is stated and proved in Section 5.3. Finally, in Section 5.4, we turn our attention to the adaptation of the constructions to contact topology.

5.1 Nested open books

In this section we engage in a special class of submanifolds and introduce the notion of a *nested open book*, i.e. a submanifold carrying an open book structure compatible with the open book structure of the ambient manifold. We also discuss fibre connected sums in this context.

Let M be an n-dimensional manifold supported by an open book decomposition (B, π). Let $M' \subset M$ be a k-dimensional submanifold on its part supported by an open book decomposition (B', π') such that

$$\pi|_{M' \setminus B'} = \pi'.$$

Note that B' necessarily defines a $(k-2)$-dimensional submanifold in B. We will always assume that M' intersects the binding B transversely. We refer to M', as well as to (B', π'), as a **nested open book** of (B, π).

Remark 5.1.1
A nested open book can be compared to the images in a *flip-book*: every page of the ambient open book contains a nested page, and these nested pages "move" like the drawings in a flip-book when flipping through its pages.

Remark 5.1.2
Nested open books are a natural generalisation of *spun knots* or more general *spinnings*. A nice survey on topological spinnings is [35]. In contact topology, spinnings were used by Mori [62] and Martínez Torres [60] to construct contact immersions and embeddings of contact manifolds into higher-dimensional standard spheres.

Let (Σ, ϕ) be an abstract open book and $\Sigma' \subset \Sigma$ a properly embedded submanifold with boundary $\partial\Sigma' \subset \partial\Sigma$. We call $(\Sigma', \phi|_{\Sigma'})$ an **abstract nested open book** if Σ' is invariant under the monodromy ϕ. The equivalence of the two definitions follows analogously to the equivalence of abstract and non-abstract open books. If not indicated otherwise, we will assume the normal bundle of any nested open book to be trivial.

Example 5.1.3
Consider a k-disc $D^k \subset D^n$ inside an n-disc D^n coming from the natural inclusion $\mathbb{R}^k \subset \mathbb{R}^n$. This realises $S^{k+1} \cong (D^k, \mathrm{id})$ as a nested open book of $S^{n+1} \cong (D^n, \mathrm{id})$. The case $k = 1$ and $n = 2$ is depicted in Figure 5.1. For $k = n - 2$, the nested S^{n-1} is a *push-off*, as will be defined in Section 5.2, of the binding of (D^n, id); cf. also Example 5.3.2, where we discuss the binding sum of two copies of (D^3, id).

5.1.1 Fibre sums along nested open books

For the remainder of the section, assume the co-dimension of the nested open books to be two. The question whether the resulting manifold of a fibre connected sum operation of two open books carries an open book structure can be answered positively in the case when the sum is performed along nested open books. The page of the resulting open book is a fibre connected sum of the pages (along the page of the nested book), the binding is a fibre connected sum of the bindings (along

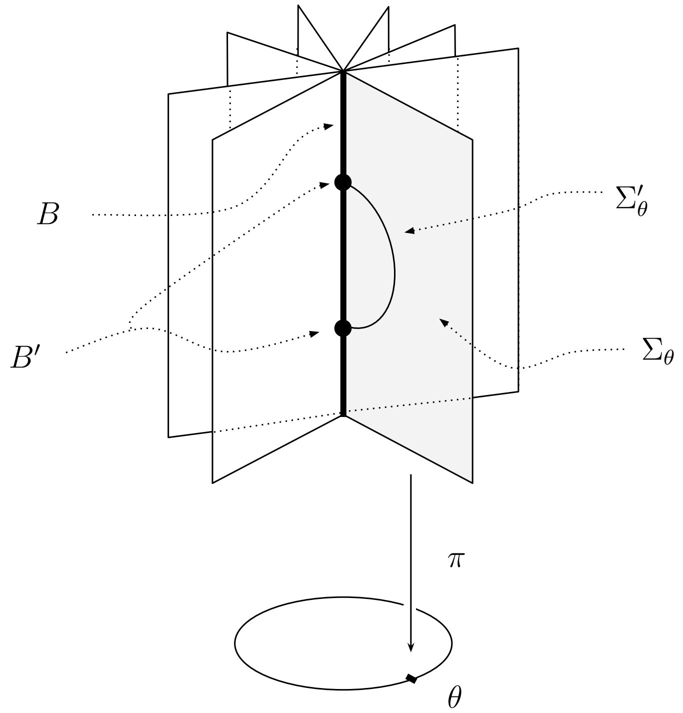


Figure 5.1: Schematic picture of an open book. A single page of a nested open book and its nested binding is indicated.

the nested binding), and the monodromies glue together. In Section 5.2 we will consider a special case in detail and discuss an explicit open book structure of the sum, namely, the case of the nested open book being a push-off of the binding. Not only is it possible to describe the monodromy in terms of the old monodromies, but we also obtain a description of the page via generalised handle attachments. This description of the new page is not always possible in the case of arbitrary nested open books, since the nested page can be entangled, i.e. cannot be isotoped to the boundary, inside the ambient page. Let us first study the situation of fibre sums along nested open books in general.

Let M' be an $(n-2)$-dimensional manifold supported by an open book (π', B'), and let $j_0, j_1\colon M' \hookrightarrow M$ be two disjoint embeddings defining nested open books of M such that their images admit isomorphic normal bundles N_i. We denote by $M_i' := j_i(M')$ the embedded copies of the nested open book M' and by $B_i' := j_i(B')$ their respective bindings. Finally let $\pi_i' := \pi' \circ j_i^{-1}$ denote the induced open book structure on $M_i' \setminus B_i'$ and let $(\Sigma_i')_\theta$ denote their pages.

Given an orientation reversing bundle isomorphism Ψ of the normal bundles $\nu M_i'$, we can perform the fibre connected sum $\#_\Psi M$. We only have to ensure that the fibres of the normal bundles of M'_0 and M'_1 lie within the pages of (π, B). In particular, we require the fibres over the nested bindings to lie within the binding of M. Moreover, we require the isomorphism Ψ of the normal bundle to respect the open book structure of M (which implies that it is compatible with the nested open book structures of M'_0 and M'_1 as well), i.e. Ψ to satisfies $\pi \circ \Psi = \pi$. Now, an open book structure of $\#_\Psi M$ is given as follows.

Lemma 5.1.4
The original fibration $\pi\colon\ M \setminus B \to S^1$ descends to a fibration

$$\Pi\colon\ \#_\Psi M \setminus \#_{\Psi|_{\nu B'_0}} B \to S^1.$$

In particular, the new binding is given by the fibre connected sum $\#_{\Psi|_{\nu B'_0}} B$ of the binding along the nested bindings (with respect to the isomorphism of $\nu B'_i \subset TB$ induced by Ψ), and the pages of the open book are given by the (relative) fibre sum of the original page along the nested pages (with respect to the isomorphism of $\nu \pi_i'^{-1}(\theta) \subset T\pi^{-1}(\theta)$ induced by Ψ), i.e. $\overline{\Pi^{-1}(\theta)} = \#_{\Psi|_{\nu(\Sigma'_0)_\theta}} \Sigma_\theta$. □

In the following we are going to extract the remaining information to express $\#_\Psi M$ in terms of an abstract open book, that is we describe a recipe to find the monodromy. Let us take a closer look at the nested open books within the ambient manifold. Let X be a vector field transverse to the interior of the ambient pages, vanishing on the binding, and normalised by $\pi^* d\theta(X) = 1$. Recall from Section 2.1 that the time-2π map ϕ of the flow of X yields the monodromy of the ambient open book. Furthermore, if we assume that X is tangent to the submanifolds M'_i, we obtain abstract nested open book descriptions (Σ_i, ϕ_i) of M'_i within the abstract ambient open book (Σ, ϕ). Moreover, by adapting the vector field if necessary, we can choose embeddings of the normal bundles of M'_i such that the fibres are preserved under the flow of X. The normal bundles of M'_0 and M'_1 being isomorphic translates into the condition that the normal bundles $\nu\Sigma'_i$ of the induced (abstract) nested pages in the ambient (abstract) page Σ are ϕ-equivariantly isomorphic. Note that we have not specified such an isomorphism yet. A natural choice seems to be the isomorphism induced by Ψ, to be more precise, the restriction Ψ_0 of the isomorphism Ψ to $N_0 \cap \Sigma_{\theta=0}$. However we will see below that this is not the right choice in general, i.e. the abstract description of the open book in Lemma 5.1.4 does not equal $(\#_{\Psi_0}\Sigma, \phi)$ in general. It turns out we have to adapt the monodromy and add the corresponding *twist map*, which will be described in the following.

For the remaining part of the section we identify $\nu M'_i$ with the quotient

$$\Big(\nu\Sigma'_i \times [0, 2\pi]\Big) / \sim_\phi .$$

Now let Ψ_0 be the ϕ-equivariant fibre-orientation reversing isomorphism of $\nu\Sigma'_i$ within $T\Sigma$ induced by the restriction of Ψ. Moreover, we define

$$\Psi_t := \Psi|_{\nu\Sigma'_0 \times \{t\}}.$$

Note that each Ψ_t is isotopic to Ψ_0, the whole family $\{\Psi_t\}_t$ however defines an (a priori) non-trivial loop of maps $\nu\Sigma'_0 \to \nu\Sigma'_1$ based at Ψ_0. By choosing suitable bundle metrics, this loop yields an (a priori) non-trivial loop $\{\mathcal{D}_t\}_t$ of maps $\Sigma' \to S^1$ based at the identity via

$$\mathcal{D}_t(x) \cdot \Psi_0(q) := \Psi_t(q),$$

for $x \in \Sigma'$ and $q \neq 0$ a non-trivial point in the normal-fibre over x. With this in hand we can define a monodromy-like map of $\nu\Sigma'_1$ which is the identity in a neighbourhood of the zero section and outside the unit-disc bundle by

$$\mathcal{D}(q) := \mathcal{D}_{\mathbf{r}(x)} \cdot q,$$

where $\mathbf{r}$ is a radial cut-off function in the fibre which is 1 on the zero section and vanishes away from it. We call it the **twist map** induced by ϕ and Ψ. Given this map we can now give an abstract description of the open book in Lemma 5.1.4. Recall that we already identified the page as the fibre sum of the original page along the nested pages.

Lemma 5.1.5
Let Ψ_0, ϕ and $\mathcal{D}$ be the maps described in the above paragraph (i.e. ϕ is the monodromy of $M = (B, \pi)$ adapted to the nested open books, $\mathcal{D}$ is the derived twist-map, and Ψ_0 is the restriction of Ψ to the normal bundle of the subpage at angle zero). Then the monodromy of the open book in Lemma 5.1.4 is given by $\phi \circ \mathcal{D}$, and the page is $\#_{\Psi_0}\Sigma$. □

5.2 The push-off

In this section we describe a *push-off* of the binding of an open book which realises it as a nested open book. The push-off construction will enable us to describe a natural open book structure on the fibre connected sum of two open books along their diffeomorphic bindings. The notion *push-off* may be a bit misleading, as the result of our construction is not a push-off in the ordinary sense but only close to it. In particular, the binding and its push-off will intersect. We will first describe how the binding is being pushed away from itself and then introduce a natural framing of the pushed-off copy in Subsection 5.2.1, which will be equivalent to the canonical

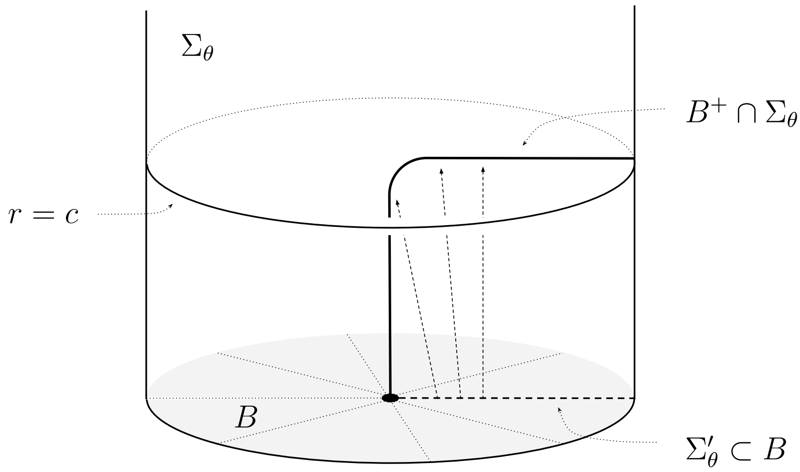


Figure 5.2: The page Σ'_θ pushed into Σ_θ.

page framing of the binding. In Subsection 5.2.2 we show that the push-off can be realised as an *abstract nested open book*.

Let M be a manifold with open book decomposition (Σ, ϕ) and binding B which also admits an open book decomposition (Σ', ϕ'). We denote the fibration maps by $\pi\colon M \setminus B \to S^1$ and $\pi'\colon B \setminus B' \to S^1$, respectively. Our aim is to define a push-off B^+ of the binding B in such a way that each page Σ'_θ of the binding open book is pushed into Σ_θ, the page corresponding to the same angle θ in the ambient open book. As all our constructions are local in a neighbourhood of the binding B, we can assume, without loss of generality, that ϕ is the identity.

Identify a neighbourhood of the binding $B' \subset B$ of the open book of the binding B with $B' \times D^2$ with coordinates (b', r', θ') such that (r', θ') are polar coordinates on the D^2-factor and θ' corresponds to the fibration π' – these are standard coordinates for a neighbourhood of a binding of an open book. We will also use Cartesian coordinates x', y' on the D^2-factor. Analogously, we have coordinates (b, r, θ) in a neighbourhood of $B \subset M$ with the corresponding properties. Combining these, we get two sets of coordinates on $(B' \times D^2) \times D^2 \subset M$:

$$(b', r', \theta', r, \theta) \quad \text{and} \quad (b', x', y', x, y).$$

First, we will describe the geometric idea of the push-off by considering just a single page Σ_θ of the open book before defining it rigorously afterwards, see Figure 5.2. The page Σ' of the binding open book is pushed into the page Σ. The push-off depends on the radial direction r' only and is invariant in the B'-component. In particular, the boundary of the page Σ' stays fixed. We divide the collar neighbourhood in Σ' into four parts by the collar parameter r'. The outermost one consisting of points in Σ' with $r' \leq \epsilon_1$ is mapped to run straight into the r-direction of the ambient page Σ. The innermost part consisting of points with $r' \geq \epsilon_3$ is translated by a constant c into the r-direction. This translation is extended over the whole of Σ'. On the rest of the collar the push-off is an interpolation between these innermost and outermost parts. This is done such that points with $\epsilon_1 \leq r' \leq \epsilon_2$ are used to interpolate in r-direction and points with $\epsilon_2 \leq r' \leq \epsilon_3$ in r'-direction.

Let $f, h\colon \mathbb{R}_0^+ \to \mathbb{R}$ be the smooth functions described in Figure 5.3. Recall that B can be decomposed as $(B' \times D^2) \cup \Sigma'(\phi')$. Let $g\colon B \to B \times D^2 \subset M$ be the embedding defined by

$$g(b) = \begin{cases} \left((b', f(r') \cdot e^{i\theta'}), h(r') \cdot e^{i\theta'}\right) & \text{for } b = (b', r' e^{i\theta'}) \in B' \times D^2 \\ \left([x', \theta'], c \cdot e^{i}\theta'\right) & \text{for } b = [x', \theta'] \in \Sigma'(\phi'). \end{cases}$$

Observe that g is well-defined and a smooth embedding.

Definition 5.2.1
We define the **push-off** B^+ **of** B as the image of the embedding g defined above, i.e. we define

$$B^+ := g(B).$$

Observe that we can easily obtain an isotopy between the binding B and the push-off B^+ by parametrising f and h.

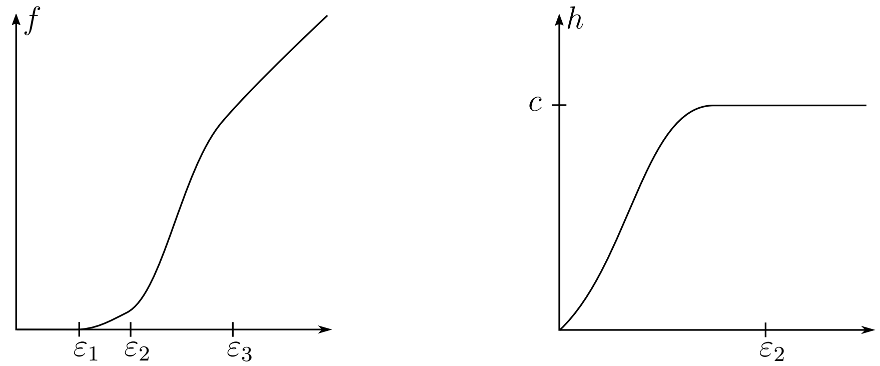


Figure 5.3: The functions f and h.

Remark 5.2.2
We call the submanifold B^+ a *push-off* of B although it is not a push-off in the usual sense since B and B^+ are not disjoint. However, it generalizes the notion of a push-off of a *transverse* knot in a contact manifold. Note that $B^+ \cap \Sigma_\theta \cap \{r = c\}$ is a copy of the interior of the page Σ'_θ of the binding and $B^+ \cap \Sigma_\theta \cap \{r = r_0 < c\} \cong B' \times \{r_0\}$.

5.2.1 Framings of the push-off

The fibre connected sum explained in Section 1.5 requires the submanifolds to have isomorphic normal bundles and explicitly uses a given bundle isomorphism. The binding of an open book has trivial normal bundle. Hence it is sufficient to specify a framing to be able to perform a fibre connected sum along the binding. In this section we will discuss a natural framing of the binding of an open book and introduce a corresponding framing for its push-off.

Let $N \subset M$ be a submanifold with trivial normal bundle. A **framing** of N is a trivialisation of its normal bundle. If the codimension of N in M equals two, we can consider the normal bundle as a complex line bundle, which can be trivialised by a nowhere-vanishing section. Thus, a framing of a codimension two submanifold with trivial normal bundle can be given by specifying a push-off, or equivalently a non-zero vector field along the submanifold that is nowhere tangent. We call two framed submanifolds **equivalent** if they are isotopic through framed submanifolds. Note that a framing of the binding B can also be specified by a homotopy class of maps from B to S^1. If B is simply-connected, such a map lifts to the universal cover $\mathbb{R}$ and thus is null-homotopic, i.e. the framing is unique.

A natural framing of the binding $B \subset M$ of an open book is the **page framing** obtained by pushing B into one fixed page of the open book. We denote the page framing given by ∂_x by F_0, i.e.

$$F_0 := \partial_x.$$

Next we are going to define a framing for the push-off B^+. Let $\widetilde{u}\colon M \to \mathbb{R}$ be a smooth function such that

- $\widetilde{u} \equiv 0$ near B and on $B' \times \{r' \leq \epsilon\} \times D^2_{c-\epsilon}$,
- $\widetilde{u} \equiv 1$ on $B' \times \{r' \geq \epsilon\} \times \{r = c\}$ and outside $\{r \leq c + \epsilon\}$,
- $\widetilde{u}$ is monotone in r'- and r-direction.

With this in hand we define a framing of the push-off B^+ by

$$F_1 := -(1 - \widetilde{u})\partial_{x'} - \widetilde{u} \cdot (\sin^2\theta\partial_{x'} - \cos\theta\partial_r).$$

One easily checks that this is indeed nowhere tangent to B^+. We now show that the push-off B^+ with the framing F_1 is equivalent to the binding B with its natural page framing F_0.

Lemma 5.2.3
The framed submanifolds (B, F_0) and (B^+, F_1) are isotopic.

Proof. The strategy of the proof is as follows. We use an *intermediate push-off* $\widetilde{B}$ which is smooth outside a singular set and consider the framings F_0 and F_1 as framings on this intermediate push-off. Here, *framing* means an honest framing on the smooth part that extends continuously. We then describe a homotopy of the framings over $\widetilde{B}$ which has the property that it still defines a framing when the intermediate push-off is smoothened (cf. Definition und Notiz (13.12) in [10]). This then yields the desired isotopy of the framed submanifolds (B, F_0) and (B^+, F_1).

Let c be the constant as in the definition of the push-off B^+. Then we define the *intermediate push-off* $\widetilde{B}$ as the image of $\widetilde{g}\colon\ B \to B \times D^2 \subset M$ defined by

$$\widetilde{g}(b) := \begin{cases} \big(b, r'(b), \pi'(b)\big) & \text{for } r' \leq c, \\ \big(b, c, \pi'(b)\big) & \text{for } r' \geq c. \end{cases}$$

Note that the intermediate push-off has singular points in $r' = c$. On the intermediate push-off the framings (which we continue to denote by F_0 and F_1 and leave out the base point for ease of notation) are as follows: The page framing is given by $F_0 = \partial_x$ or, written in polar coordinates,

$$F_0 = \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta,$$

with $\theta = \pi\big(\widetilde{g}(b)\big)$. Writing the framing F_1 of the push-off B^+ in polar coordinates gives

$$F_1 = -(1 - \widetilde{u} + \widetilde{u}\sin^2\theta)(\cos\theta'\partial_{r'} - \frac{1}{r'}\sin\theta'\partial_{\theta'}) + \widetilde{u}\cos\theta\partial_r.$$

This is isotopic to the intermediate push-off with framing

$$F_1 = -(1 - t + t\sin^2\theta)(\cos\theta'\partial_{r'} - \frac{1}{r'}\sin\theta'\partial_{\theta'}) + t\cos\theta\partial_r.$$

Here $t = t(b)$ is a function interpolating between 0 and 1 such that

- t only depends on the r'-direction,
- $t = 0$ for $r' \leq c - \epsilon$,
- $t = \frac{1}{2}$ for $r' = c$,
- $t = 1$ for $r' \geq c + \epsilon$.

We define

$$F_h := (1 - h)F_0 + hF_1$$

and show that F_h defines a framing for every $h \in [0, 1]$. The remainder of the proof is a mere calculation. The tangent space of $\widetilde{B}$ is defined outside of the set $\{r = c\}$ and is spanned by TB' and, depending on the value of t, the vectors

- $\partial_r + \partial_{r'}$ and $\partial_\theta + \partial_{\theta'}$, where $t < \frac{1}{2}$, or
- $\partial_{r'}$ and $\partial_\theta + \partial_{\theta'}$, where $t > \frac{1}{2}$.

Omitting the B'-direction and using vector notation in the ordered basis $\partial_r, \partial_{r'}, \partial_\theta$, $\partial_{\theta'}$, we have

$$F_h = \begin{pmatrix} (1-h+ht)\cos\theta \\ -(1-t+t\sin^2\theta)h\cos\theta \\ -(1-h)\frac{1}{r}\sin\theta \\ (1-t+t\sin^2\theta-t)h\frac{1}{r'}\sin\theta \end{pmatrix},$$

where we use that we have $\theta = \theta'$ on the push-off.

We consider the cases $t \geq \frac{1}{2}$ and $t \leq \frac{1}{2}$ separately.

Case 1: $t \geq \frac{1}{2}$. In the chosen basis the tangent space to $\widetilde{B}$ consists of vectors of the form

$$\Big(0,\ a,\ b,\ b\Big)^\intercal$$

with $a, b \in \mathbb{R}$.

Assume that $\cos\theta = 0$ and thus $\sin\theta = \pm 1$. Then the framing F_h becomes

$$\Big(0,\ 0,\ -(1/r)(1-h)\sin\theta,\ (1/r')h\sin\theta'\Big)^\intercal.$$

Since $(1/r)(1-h) + (1/r')h > 0$, this vector can never be of the form $(0,\ a,\ b,\ b)^\intercal$, i.e. it is not contained in the tangent space to $\widetilde{B}$.

Otherwise, i.e. if $\cos\theta \neq 0$, the tangency condition yields $(1-h+ht) = 0$, but this equation does not have a solution for $t \geq \frac{1}{2}$.

Case 2: $t \leq \frac{1}{2}$. Here the tangent space to $\widetilde{B}$ consists of vectors of the form

$$\Big(a,\ a,\ b,\ b\Big)^\intercal$$

with $a, b \in \mathbb{R}$. By the argument above we can rule out the case $\sin\theta \neq 0$. For $\sin\theta = 0$, we have $\cos\theta = \pm 1$ and F_h equal to

$$\Big(\cos\theta(1-h+ht),\ -\cos\theta h(1+t\sin^2\theta'-t),\ 0,\ 0\Big)^\intercal.$$

Thus the tangency condition leads to $1 = -h + ht + h - ht = 0$, a contradiction.

To summarise, this means that F_h is never of the form

$$\Big((1-t)a,\ a,\ b,\ b\Big)^\intercal$$

and hence does indeed define a framing on a smoothened intermediate push-off. It follows that (B, F_0), $(\widetilde{B}, F_0)$, $(\widetilde{B}, F_1)$ and (B^+, F_1) are isotopic. □

5.2.2 The push-off as an abstract nested open book

The push-off B^+ is clearly an embedded nested open book of $M = (B, \pi)$. The aim of this section is to obtain a description of the push-off as an *abstract* nested open book, and ultimately as a *framed* abstract nested open book by altering the monodromy of the abstract open book (Σ, ϕ). To this end, the monodromy of the binding is used to define a vector field X' on $B \setminus B'$, which can then be extended to a neighbourhood of B in M. By cutting off this vector field appropriately, we alter the vector field X on $M \setminus B$ corresponding to the monodromy ϕ to get a new monodromy of the form $\phi \circ \psi$. This will realise the push-off as an abstract nested open book. Finally, the monodromy of the ambient open book is changed again in such a way that the framing of a page of the push-off is independent of the ambient page. This means that the framed push-off can then be regarded as a framed abstract nested open book.

Identify a neighbourhood of $B' \subset B$ with $B' \times D^2$ as above, i.e. the pages are defined by the angular coordinate θ'. Also denoting the coordinate on S^1 by θ', we can define a non-vanishing 1-form on $B \setminus B'$ by the pull–back of $d\theta'$ under the fibration map $\pi' \colon B \setminus B' \to S^1$. With the help of this 1-form we can extend the vector field $\partial_{\theta'}$ to a vector field X' on B by prescribing the condition $(\pi')^* d\theta'(X') = 1$. The vector field X' can furthermore be extended trivially to a neighbourhood $B \times D^2$ of B in M. We obtain an abstract open book description of M by regarding the time-2π map of a certain vector field on $M \setminus B$ which satisfies conditions analogue to the ones used in the construction of X' above. Let X denote the vector field that recovers the abstract open book (Σ, ϕ). Let $u \colon \mathbb{R}_0^+ \to \mathbb{R}$ be the smooth function depicted in Figure 5.4 with c as in the definition of the push-off B^+. Then $\widetilde{X} := X + u(r)X'$ defines a vector field on $M \setminus B$.

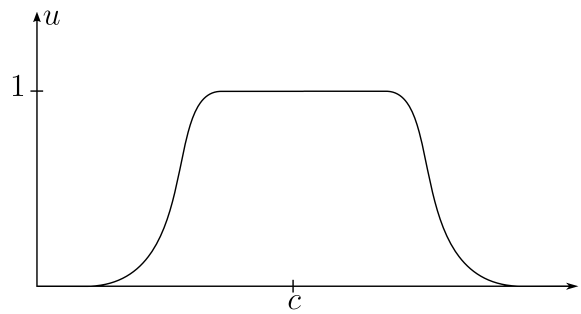


Figure 5.4: The function u.

We claim that $\widetilde{X}$ realises the push-off B^+ as an abstract nested open book of an abstract open book description of M. Observe that, as X' is tangent to the pages of (B, π), the condition $\pi^* d\theta(\widetilde{X}) = 1$ is satisfied and that $\widetilde{X}$ and X coincide near the binding B. Thus, the vector field $\widetilde{X}$ does indeed yield an abstract open book description of M. Furthermore, the vector field $\widetilde{X}$ is tangent to the push-off B^+, which means that it realises B^+ as an abstract nested open book of the abstract ambient open book. The monodromy is given by the time-2π flow of $\widetilde{X}$. However, we want to give a description that better encodes the change of the monodromy ϕ of the ambient open book we started with in terms of the monodromy of the binding.

We denote the flow of X' on B by ϕ'_t and use it to define diffeomorphisms ψ_t of a neighbourhood $B \times D^2$ of B in M:

$$\psi_t(b, r, \theta) = (\phi'_t(b), r, \theta).$$

Definition 5.2.4
Define a diffeomorphism $\psi\colon M \to M$ via $\psi := \psi_{2\pi u(r)}$ and refer to it as **Chinese burn**[1] along B. By abuse of notation, its restriction to a single page Σ_θ is also denoted by ψ.

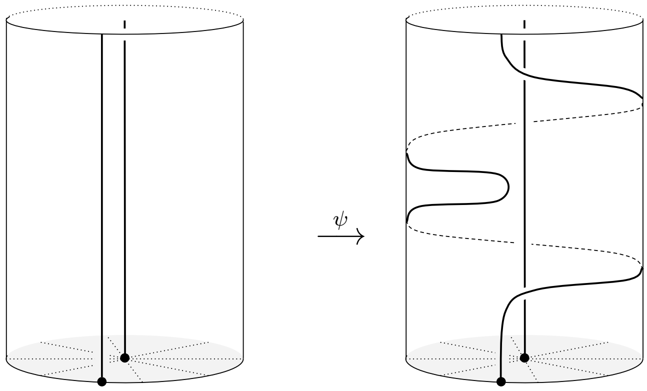


Figure 5.5: A Chinese burn along a boundary component.

Observe that the monodromy of the abstract open book obtained from the vector field $\widetilde{X}$ is $\phi \circ \psi$, i.e. the push-off B^+ yields an an abstract nested open book of $(\Sigma, \phi \circ \psi)$. We thus proved the following statement.

Lemma 5.2.5
The push-off B^+ induces an abstract nested open book of $(\Sigma, \phi \circ \psi)$ with page diffeomorphic to Σ' (more concretely, the page is $g|_{\Sigma'_0}(\Sigma'_0) \cong \Sigma'$), where ψ is a Chinese burn along B.

We constructed the push-off B^+ inside the manifold $M(\Sigma, \phi)$ and equipped it with a natural framing F_1 corresponding to the page framing. In particular, the push-off is a *framed* nested open book, i.e. a nested open book with a specified framing. The previous lemma shows that the push-off also defines an *abstract* nested open book of $(\Sigma, \phi \circ \psi)$. However, the framing F_1 does a priori not give a framing

[1]In common speech, a *Chinese burn* is the "act of placing both hands on a person's arm and then twisting it to produce a burning sensation" (cf. *www.oed.com*).

in the abstract setting since it is not invariant under the monodromy. We call an abstract nested open book with a framing which is invariant under the monodromy a **framed abstract nested open book**.

Remark 5.2.6
Given two diffeomorphic framed nested open books with pages $\Sigma_0', \Sigma_1' \subset (\Sigma, \phi)$ and isomorphic normal bundles, it is easy to obtain an open book structure of their fibre connected sum. The new page is

$$\widetilde{\Sigma} := \Big(\Sigma \setminus (\nu\Sigma_0' \cup \nu\Sigma_1')\Big)/\sim$$

with the identification induced by the given framings, and the old monodromy ϕ restricts to the new monodromy.

A natural framing of the push-off in the abstract setting is the following: We define the **constant framing** F_2 as

$$F_2\colon\ \widetilde{u}\partial_r - (1-\widetilde{u})\partial_{x'},$$

where $\widetilde{u}$ is the restriction of the function $\widetilde{u}\colon\ M \to \mathbb{R}$ defined in Section 5.2.1 to a page Σ, i.e. it is smooth with the following properties:

- $\widetilde{u} \equiv 0$ near B and on $B' \times \{r' \le \epsilon\} \times D^2_{c-\epsilon}$,
- $\widetilde{u} \equiv 1$ on $B' \times \{r' \ge \epsilon\} \times \{r = c\}$ and outside $\{r \le c + \epsilon\}$,
- $\widetilde{u}$ is monotone in r'- and r-direction.

To realise the push-off as a framed abstract nested open book with framing F_2, we have to alter the monodromy of the ambient abstract open book. We will change the monodromy by a certain diffeomorphism of the page fixing the push-off, the so-called *twist map*. Let $\sigma\colon\ \mathbb{R} \to \mathbb{R}$ be a cut-off function with $\sigma(0) = 0$ and $\sigma(\epsilon) = 1$ and

$$\tau_{\pm}^{\sigma(r)}\colon\ D^2 \to D^2$$

a smoothened Dehn twist of the disc. That is, a diffeomorphism with the qualitative behaviour of

$$(s, \theta) \mapsto (s, \theta \pm 2\pi\sigma(r)(1-s)),$$

smoothened near the boundary and the origin, such that the origin is an isolated fixed point and a neighbourhood of the boundary is fixed. This can be achieved by constructing it as the flow of an appropriate vector field. Recall that, by construction, the intersection of the push-off of the binding B with a single page Σ of the ambient open book is a copy of a page Σ' of the open book of the binding. In particular, we can identify a tubular neighbourhood of $B^+ \cap \Sigma_0$ in Σ_0 with $\Sigma' \times D^2 \subset \Sigma$.

Observe that the r-coordinate can be regarded as a collar parameter on Σ'. We define a diffeomorphism of a neighbourhood of the collar by

$$\begin{aligned} \partial\Sigma' \times [0,\epsilon] \times D^2 &\to \partial\Sigma' \times [0,\epsilon] \times D^2 \\ (b', r, p) &\mapsto (b', r, \tau_-^{\sigma(r)}(p)). \end{aligned}$$

We can extend this to a diffeomorphism $\Sigma' \times D^2 \to \Sigma' \times D^2$ of the whole tubular neighbourhood of Σ' via $\mathrm{id}_{\Sigma'} \times \tau_-^1$. Furthermore, this map can be extended to a self-diffeomorphism $\mathcal{D}\colon \Sigma \to \Sigma$ of the page Σ via the identity.

Definition 5.2.7
The diffeomorphism $\mathcal{D}\colon \Sigma \to \Sigma$ is called **twist map**.

The twist map $\mathcal{D}$ is isotopic to the identity, so we have $M_{(\Sigma,\phi\circ\psi)} \cong M_{(\Sigma,\phi\circ\psi\circ\mathcal{D})}$.

Lemma 5.2.8
The push-off B^+ with its induced framing F_1 corresponds to the framed abstract nested open book of $(\Sigma, \phi \circ \psi \circ \mathcal{D})$ with page $g|_{\Sigma_0'}(\Sigma_0') \cong \Sigma'$ framed by the natural framing F_2.

Proof. The push-off B^+ was constructed in $M_{(\Sigma,\phi)}$. To prove the lemma, we will mostly work in $M_{(\Sigma,\phi\circ\psi)}$. We begin by describing the push-off B^+ seen in this manifold. A diffeomorphism between $M_{(\Sigma,\phi)}$ and $M_{(\Sigma,\phi\circ\psi)}$ is given by the map $\Theta\colon M_{(\Sigma,\phi)} \to M_{(\Sigma,\phi\circ\psi)}$ defined by

$$\Theta(p) = \psi_{-\pi(p)u(\mathbf{r}(p))}(p),$$

where $\mathbf{r}\colon \Sigma \to \mathbb{R}$ is a function that coincides with the collar parameter r near B and is constantly extended to all of Σ. In particular, it preserves the pages Σ_θ and the r-direction, and is induced by the inverse of the time-θ map of the binding monodromy on the $\{r = c\}$-slice in Σ_θ. We denote the image of B^+ under the diffeomorphism Θ by B_1. The diffeomorphism Θ also transports the framing F_1 of B^+ to a framing of B_1, that we also denote by F_1.

On the other hand, the embedded nested open book B_1 induces the abstract nested open book from Lemma 5.2.5. Hence, as the twist map $\mathcal{D}$ fixes its pages, the framing F_2 also induces a framing of B_1 by considering the diffeomorphism from $M_{(\Sigma,\phi\circ\psi\circ\mathcal{D})}$ to $M_{(\Sigma,\phi\circ\psi)}$ coming from the twist map $\mathcal{D}$ (cf. Figure 5.6).

It remains to show the equivalence of the two framings of B_1. Note that

$$\Theta\colon M_{(\Sigma,\phi)} \to M_{(\Sigma,\phi\circ\psi)}$$

leaves the ∂_r-direction of the framing unchanged and is a rotation in the B-directions. More concretely, we have

$$\partial_{x'} \mapsto \cos(-\theta' u(h_s(r')))\partial_{x'} + \sin(-\theta' u(h_s(r')))\partial_{y'}.$$

Thus, the framing F_1 of B_1 obtained through the diffeomorphism is

$$b \mapsto \left(\widetilde{b},\ \widetilde{u}\left(-\sin^2\theta'\left(\lambda_1\partial_{x'} + \lambda_2\partial_{y'}\right) + \cos\theta'\partial_r\right) + (1-\widetilde{u})\left(-\lambda_1\partial_{x'} + \lambda_2\partial_{y'}\right)\right)$$

with

$$\widetilde{b} = \psi_{-\theta\cdot u(\mathbf{r}(g_s(b)))}\left(g_s(b)\right)$$

and

$$\lambda_1 = \cos(-\theta' u(h_s(r'))),\ \ \lambda_2 = \sin(-\theta' u(h_s(r'))).$$

The framing F_2 of B_1 in $M_{(\Sigma,\phi\circ\psi\circ\mathcal{D})}$ is mapped to

$$\widetilde{u}(\cos\theta'\partial_r + \sin\theta'\partial_{y'}) + (1-\widetilde{u})\cdot(-1)\cdot(\cos(-\theta' d(r))\partial_{x'} + \sin(-\theta' d(r))\partial_{y'})$$

under the diffeomorphism induced by the twist map $\mathcal{D}$ from $M_{(\Sigma,\phi\circ\psi\circ\mathcal{D})}$ to $M_{(\Sigma,\phi\circ\psi)}$. We can choose $d(r) = u(c_s)$ in the definition of $\mathcal{D}$ and simply interpolate between $\mathcal{D}(F_2)$ and F_1, where a little care is needed only for $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. □

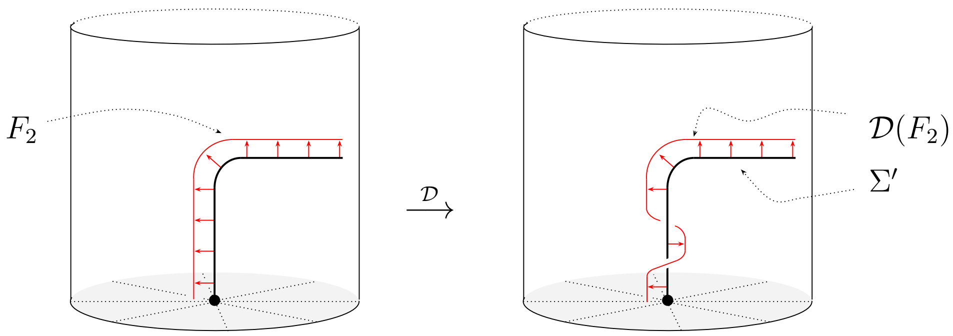


Figure 5.6: The framing F_2 and the twist $\mathcal{D}$.

5.3 An open book of the binding sum

We are now able to give an explicit open book decomposition for the binding sum operation and thus prove our main result, or, more concretely, the following theorem.

Theorem 5.3.1

Let M be a (not necessarily connected) smooth manifold with open book decomposition (Σ, ϕ) whose binding B contains two diffeomorphic components B_1, B_2 with diffeomorphic open book decompositions (Σ', ϕ'). Then the fibre connected sum of M performed along B_1 and B_2 with respect to the page framings admits an open book decomposition naturally adapted to the construction. The new page can be can be obtained from Σ by two consecutive generalised 1-handle attachments whose type depends on the pages Σ' of the binding components. The new binding is given by a fibre connected sum of B_1 and B_2 along their respective bindings $B' = \partial\Sigma$. Away from the handle attachments the monodromy remains unchanged, and over the remaining part it restricts to $\psi \circ \mathcal{D}$, where ψ is a Chinese burn *along B_i (see Definition 5.2.4) and $\mathcal{D}$ the* twist map *(see Definition 5.2.7).*

The proof is divided into two parts. First, we show the existence of a natural open book decomposition of the binding sum and specify its monodromy in the abstract setting. In the second part we prove that the resulting page can be obtained from the original pages by two consecutive generalised 1-handle attachments (for details regarding generalised 1-handles see Appendix C).

Proof of Theorem 5.3.1 I: Existence and identifying the monodromy.
Let M be a manifold with open book fibration (B, π) two of whose binding components, denoted by B_0 and B_1, admit diffeomorphic open books. By Lemma 5.2.3 B_0 and B_1 are isotopic to their respective push-offs B_i^+. Hence, the binding sum along the B_i (with respect to the page framing) can be replaced by the fibre connect sum along the push-offs B_i^+ with the framing F_1. Since the push-offs are nested open books $(B_i^+, \pi_i := \pi|_{B_i^+})$ of (B, π), the fibre connected sum carries a natural open book structure coinciding with the original open book fibration away from the push-offs by Lemma 5.1.4. A fibre of this new open book fibration is the fibre connected sum of the corresponding original fibres along the fibres of the nested open books.

In the abstract setting we have $M = M_{(\Sigma,\phi)}$ and the binding components B_0 and B_1 admit an abstract open book decomposition (Σ', ϕ'). By Lemma 5.2.8, the push-offs B_i^+ induce framed nested open books of $(\Sigma, \phi \circ \psi \circ \mathcal{D})$, and we have $M = M_{(\Sigma,\phi)} \cong M_{(\Sigma,\phi\circ\psi\circ\mathcal{D})}$. Here, ψ denotes the Chinese burn along the relevant binding components B_i (see Definition 5.2.4) and $\mathcal{D}$ the corresponding twist map (see Definition 5.2.7). As described in Remark 5.2.6, the abstract open book description of the fibre connected sum follows easily because the framing of a framed nested open book is compatible with the ambient monodromy. The new page is

$$\widetilde{\Sigma} := \Big(\Sigma \setminus (\nu\Sigma'_0 \cup \nu\Sigma'_1)\Big)/\sim$$

with the identification induced by the given framings, i.e. it is the fibre connected sum of the original page Σ along the framed nested pages Σ_i' of the framed nested open books of $(\Sigma, \phi \circ \psi \circ \mathcal{D})$ corresponding to the B_i, and the monodromy is induced by $\phi \circ \psi \circ \mathcal{D}$. □

In the following we explain how the binding connected sum of two open books with diffeomorphic bindings can be regarded as two consecutive generalised 1-handle attachments.

Proof of Theorem 5.3.1 II: Handle interpretation of the page.
Let Σ be the ambient page, i.e. $\Sigma \cong \overline{\pi^{-1}(0)}$ with (B, π) the open book, and let $B_0, B_1 \subset B = \partial\Sigma$ be two diffeomorphic binding components with common open book. We already discussed what the open book for the binding sum looks like. For ease of notation, let Σ_i' denote the subpage of the push-off B_i^+ of B_i as well as the page of B_i corresponding to a fixed angle, zero say, for $i = 0, 1$. Let $\nu\Sigma_i' \subset \Sigma$ be their embedded normal bundles. Then the new page is given by

$$\widetilde{\Sigma} := \overline{\Pi^{-1}(0)} = \#_{\Psi|_{\nu\Sigma_0'}} \Sigma.$$

Equivalently, in the abstract setting, we can identify $\nu\Sigma_i'$ with $\Sigma_i' \times \mathrm{int} D^2 \subset \Sigma_i' \times \mathbb{C}$ via the constant framing of the constant push-offs and obtain

$$\widetilde{\Sigma} := \overline{\Pi^{-1}(0)} = \Big(\Sigma \setminus \big(\nu\Sigma_0' \cup \nu\Sigma_1'\big)\Big)/(x, z) \sim (x, \bar{z}).$$

Note that $\Sigma_i' \times \{1\} \subset \nu\Sigma_0'$ is isotopic in $\Sigma \setminus \nu\Sigma_0'$ to the corresponding page $\Sigma_i' \subset B_i \subset \partial\Sigma$ of the respective binding component B_i. Let $\Sigma' \times [-1, 0]$ and $\Sigma' \times [0, 1]$ denote the traces of the obvious isotopies fixing $\partial\Sigma' \times \{1\}$. Observe furthermore that these two isotopies glue together in $\overline{\Pi^{-1}(0)}$ to an isotopy whose image we denote by $\Sigma' \times [-1, 1]$ (cf. Figure 5.7), which clearly is the co-core of a generalised 1-handle $H_{\Sigma'}^{(2)}$.

If we cut open the new page $\widetilde{\Sigma}$ along the co-core of this handle and consider the effect in the original page Σ, the areas $\Sigma_i' \times \partial D^2$ of identification become diffeomorphic to $\Sigma_i' \times [0, 2\pi]$ since $\Sigma_i' \times \{1\}$ is removed. Hence, we end up with a single embedded copy of $\Sigma' \times [0, 2\pi]$ in the quotient $\widetilde{\Sigma}$. This is the co-core of another generalised 1-handle $H_{\Sigma'}^{(1)}$. Cutting along $H_{\Sigma'}^{(1)}$ then gives back the original page Σ (cf. Figure 5.7). Thus, we have

$$\overline{\Pi^{-1}(0)} = \Big(\Sigma \sqcup H_{\Sigma'}^{(1)} \sqcup H_{\Sigma'}^{(2)}\Big)/\sim,$$

where we attach the first handle $H_{\Sigma'}^{(1)}$ along two copies of Σ', i.e. along one page of each of the two binding components B_0 and B_1. The second handle $H_{\Sigma'}^{(2)}$ is then attached to $D^1 \times \Sigma \times S^0 \subset \partial H_{\Sigma'}^{(1)}$ (see Figure 5.8). □

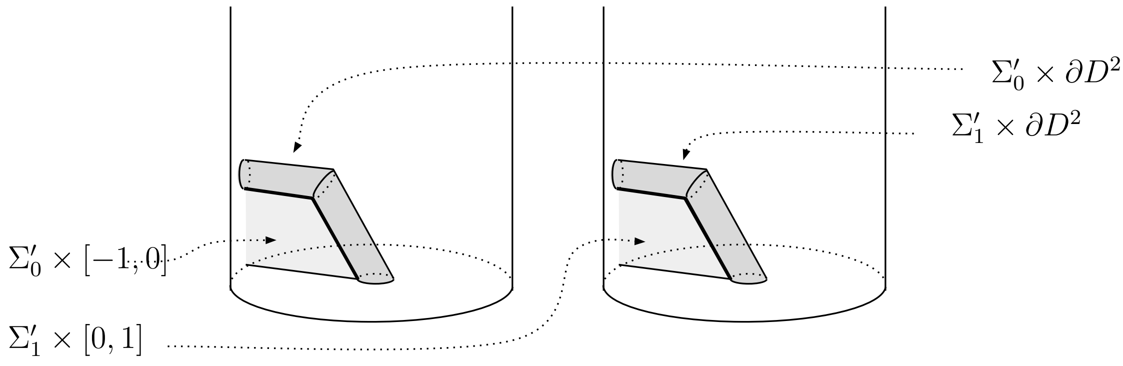


Figure 5.7: The dark grey region on each side is the neighbourhood of a single page of the push-off of binding components. The light grey region defines the co-core of a generalised 1-handle $H^{(2)}_{\Sigma'}$. After cutting along the co-core of $H^{(2)}_{\Sigma'}$, the dark grey region descends to the co-core of another generalized 1-handle $H^{(1)}_{\Sigma'}$.

We conclude the section with an example.

Example 5.3.2
Let $M = M_0 \sqcup M_1$ with M_i the four-dimensional sphere S^4 with open book decomposition $(\Sigma_i = D^3, \mathrm{id})$. Then the binding has two components B_i, both a 2-sphere with the unique open book decomposition $(\Sigma_i' = D^1, \mathrm{id})$. We have $M_i = D^3 \times S^1 \cup S^2 \times D^2$, so performing the binding sum on M along the B_i yields $S^3 \times S^1$. By Theorem 5.3.1, this has a natural open book decomposition obtained by forming the sum along the push-off of the binding.

Pushing a page $\Sigma_i' = D^1$ of the binding open book into the page $\Sigma_i = D^3$ and then removing a neighbourhood of Σ_i', turns Σ_i into a solid tours $S^1 \times D^2$. Hence, the page in the resulting open book decomposition of the binding sum consists of two solid tori glued together along a neighbourhood of $S^1 \times \{*\} \subset S^1 \times \partial D^2$, i.e. it is a solid torus. Note that we do not have to specify a framing since S^2 is simply-connected and thus possesses a unique framing.

Recall that the Chinese burn ψ is non-trivial only in a collar neighbourhood of the binding. To identify the new monodromy it is helpful to imagine the collar as split into two parts. The Chinese burn twists the outer part of the collar and then twists back on the inner part. Now observe that the new monodromy is isotopic to the identity. Indeed, the twist map $\mathcal{D}$ on the new page cancels with the Chinese burn on the outer part and the Chinese burn restricted to the inner part is isotopic to the identity also on the new page (see also the 3-dimensional case sketched at the end of this example).

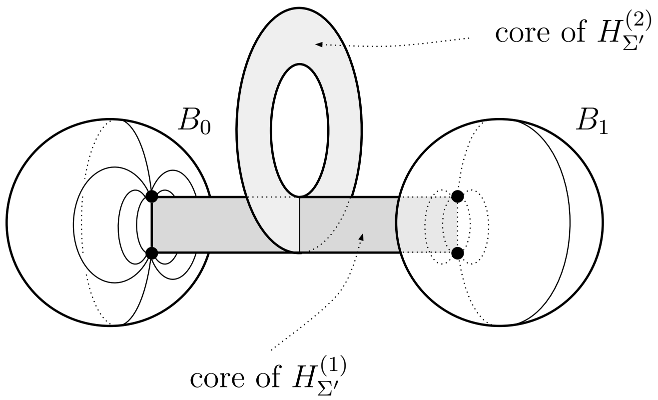


Figure 5.8: The dark and light grey region define cores of generalized 1-handles $H^{(1)}_{\Sigma'}$ and $H^{(2)}_{\Sigma'}$ respectively. Note that, in the particular case we pictured here, the 2-fold handle attachment yields a solid torus $S^1 \times D^2$.

Alternatively, we can also construct the new page by two consecutive generalised 1-handle attachments. In the present situation a generalised 1-handle $H_{\Sigma'}$ is just $D^1 \times D^1 \times D^1$. The first handle attachment described above turns $\Sigma = D^3 \sqcup D^3$ into a single ball, the second handle attachment results in a solid torus (see Figure 5.8).

It is also worth comparing this with the situation in one dimension lower as discussed in [53], i.e. the binding sum of two copies of (D^2, id) with page framing (here, a framing has to be specified). Then the new page is an annulus and the monodromy is isotopic to the identity: the Chinese burn part of the monodromy are two trivial negative Dehn twists and two non-trivial positive Dehn twists which cancel with the two negative Dehn twists forming the twist map (see [53, Theorem 3]).

5.4 The contact binding sum

In this section we want to show that the binding sum of contact open books yields an open book and a contact structure that again fit nicely together.

Theorem 5.4.1
The binding sum construction can be made compatible with the underlying contact structures, i.e. the contact structure obtained by the contact fibre connected sum along contactomorphic binding components is supported by the natural open book structure resulting from the sum along the push-offs of the respective binding components.

We will now define nested open books in the contact world.

Definition 5.4.2
Let $(\Sigma, d\lambda, \phi)$ be a contact open book and $\Sigma' \subset \Sigma$ a symplectic submanifold with boundary $\partial\Sigma' \subset \partial\Sigma$ such that in a collar neighbourhood $\partial\Sigma \times (-\varepsilon, 0]$ of $\partial\Sigma$ given by an outward-pointing Liouville vector field we have $\Sigma' \cap \Big(\partial\Sigma \times (-\varepsilon, 0]\Big) = \partial\Sigma' \times (-\varepsilon, 0]$. Suppose furthermore that $\phi(\Sigma') = \Sigma'$, i.e. the monodromy leaves Σ' invariant (not necessarily pointwise). Then Σ' is a **contact abstract nested open book** of the contact open book $(\Sigma, d\lambda, \phi)$.

Recall that the fibre connected sum construction can be adapted to the contact setting (see Theorem 1.5.2).

The binding of a contact open book decomposition is a contact submanifold and hence admits an open book structure itself. Furthermore, it has trivial normal bundle. Given two contact open books with contactomorphic bindings, we can thus perform the contact fibre connected sum along their bindings, and, topologically, also along the push-offs B_i^+ of the bindings. We want to show that this topological construction can be adapted to the contact scenario.

In fact, the push-off is a contact submanifold contact isotopic to the binding. Thus, we can form the contact fibre connected sum along the push-off rather than along the binding itself when performing the contact binding sum.

Proposition 5.4.3
The push-off B^+ of the binding B of a contact open book (B, π) is a contact submanifold contact isotopic to the binding.

Proof. We first show that the push-off B^+ of the binding B is a contact submanifold of (M, ξ). The binding B is a contact submanifold, so in a neighbourhood $B \times D^2$ (as described in Section 5.2) we can assume our contact form α to be

$$\alpha = h_1\alpha_B + h_2 d\theta,$$

where α_B is a contact form on B and $(h_1(r), h_2(r))$ a Lutz pair (see Definition 2.3.2). To prove that the push-off B^+, which arises as the image of the embedding

$$g\colon\ B \to M$$

(see Definition 5.2.1), is a contact submanifold of M, we have to show that $g^*\alpha$ is a contact form on B. We will first verify this condition away from the binding B' of B. Here, g sends an element $[x', \theta']$ in the mapping torus part of the open book of B to $([x', \theta'], c, \theta') \in B \times D^2$ for constant $c > 0$. So we have

$$g^*\alpha = h_1(c)\alpha_B + h_2(c)d\theta'$$

and $dg^*\alpha = h_1(c)d\alpha_B$. Hence,

$$g^*\alpha \wedge (dg^*\alpha)^{n-1} = \big(h_1(c)\big)^n \alpha_B \wedge (d\alpha_B)^{n-1} + \big(h_1(c)\big)^{n-1} h_2(c) d\theta' \wedge (d\alpha_B)^{n-1}.$$

Observe that $(h_1(c))^{n-1}h_2(c) > 0$. Furthermore, since $d\theta'(R_{\alpha_B}) > 0$, the forms $\alpha_B \wedge (d\alpha_B)^{n-1}$ and $\big(h_1(c)\big)^{n-1} h_2(c)d\theta' \wedge (d\alpha_B)^{n-1}$ induce the same orientation, i.e. $g^*\alpha$ is indeed a contact form away from the binding.

Now we inspect the situation near the binding B' of B, i.e. we can work in a neighbourhood $B' \times D^2 \times D^2$ (as described in Section 5.2). The contact form α_B of the binding can be assumed to be of the form

$$\alpha_B = g_1\alpha_{B'} + g_2 d\theta',$$

where $\alpha_{B'}$ is a contact form on B' and $\big(g_1(r'), g_2(r')\big)$ a Lutz pair. Thus, we have

$$\alpha = h_1 g_1 \alpha_{B'} + h_1 g_2 d\theta' + h_2 d\theta.$$

Recall that the defining embedding g for the push-off is given by

$$g(b', r', \theta') = \big(b', f(r'), \theta', h(r'), \theta'\big)$$

in this neighbourhood.

We compute

$$g^*\alpha = \lambda\alpha_{B'} + \mu d\theta'$$

with

$$\lambda(r') = (h_1 \circ h)(g_1 \circ f)(r')$$

and

$$\mu(r') = \big((h_1 \circ h)(g_2 \circ f) + h_2 \circ h\big)(r').$$

So we have

$$g^*\alpha = \lambda'\alpha_{B'} + \lambda d\alpha_{B'} + \mu' d\theta',$$

$$(dg^*\alpha)^{n-1} = (n-1)\lambda^{n-2}(d\alpha_{B'})^{n-2} \wedge (\lambda' dr' \wedge \alpha_{B'} + \mu' dr' \wedge d\theta')$$

and

$$(g^*\alpha) \wedge (dg^*\alpha)^{n-1} = \frac{1}{r'}(n-1)\lambda^{n-2}(\lambda\mu' - \lambda'\mu)\Big(\alpha_{B'} \wedge (d\alpha_{B'})^{n-2} \wedge r'dr' \wedge d\theta'\Big).$$

It remains to show that the term $\lambda\mu' - \lambda'\mu$ is positive. A calculation shows

$$\begin{aligned}\lambda\mu' - \lambda'\mu \;=\;& \underbrace{(h_1 \circ h)^2\Big((g_1 \circ f)(g_2 \circ f)' - (g_1 \circ f)'(g_2 \circ f)\Big)}_{=:A} \\ &+ \underbrace{(h_1 \circ h)\Big((h_2 \circ h)'(g_1 \circ f) - (g_1 \circ f)'(h_2 \circ h)\Big)}_{=:B} \\ &+ \underbrace{\Big(-(h_1 \circ h)'(g_1 \circ f)(h_2 \circ h)\Big)}_{=:C}.\end{aligned}$$

Observe that all three summands A, B, and C are non-negative. It thus suffices to show that at least one of them is positive. Assume $C = 0$. Then either $r' = 0$ or $h' = 0$. We will first deal with the case $h' = 0$. This happens exactly where $h \equiv c$. But on this set, both f and its derivative f' are positive and therefore

$$A = \Big(h_1 \circ h(c)\Big)^2 f'\Big((g_1 g_2' - g_1' g_2) \circ f\Big) > 0,$$

because (g_1, g_2) is a Lutz pair, i.e. in particular $(g_1 g_2' - g_1' g_2) > 0$. Now consider $r' = 0$. For small r' we have $f \equiv 0$ and thus $g_1 \circ f \equiv 1$ and $g_2 \circ f \equiv 0$. Then $\lambda\mu' - \lambda'\mu$ reduces to

$$(h_1 \circ h)(h_2 \circ h)' - (h_1 \circ h)'(h_2 \circ h) = h'\Big((h_1 h_2' - h_1' h_2) \circ h\Big).$$

As h' is positive for small r' and (h_1, h_2) is a Lutz pair, this is positive. Hence, $(g^*\alpha) \wedge (dg^*\alpha)^{n-1}$ is a volume form and so $(g^*\alpha)$ a contact form, i.e. the push-off B^+ is indeed a contact submanifold.

A similar calculation using the parametrised versions of the functions f and h used in the definition of the push-off shows that the push-off B^+ is furthermore contact isotopic to the binding B. □

A first step of showing that the topological binding sum construction along the push-off can be made compatible with the underlying contact structures is to show that the push-off can be realised as a suitable abstract nested open book. To this end, we first show that there exists a monodromy vector field tangent to the push-off.

Proposition 5.4.4
There exists a monodromy vector field tangent to the push-off.

Proof. Without loss of generality, we can assume that the ambient contact manifold $(M, \ker\alpha)$ with contact open book (B, π) arises from an abstract open book $(\Sigma, d\lambda, \phi)$ by a generalised Thurston–Winkelnkemper construction. The push-off B^+ of the binding B is contained in a trivial neighbourhood $B \times D^2$, on which the monodromy ϕ restricts to the identity and the contact form α is given by $\alpha = h_1\alpha_B + h_2 d\theta$, where (h_1, h_2) is a Lutz pair and α_B a contact form on B with the induced contact structure. Hence, we can furthermore assume that $\phi = \mathrm{id}_\Sigma$, which implies that ∂_θ is a monodromy vector field.

By construction, the intersection of the push-off B^+ with the fibres of π is a cylinder over the binding B' of the compatible open book decomposition with page Σ' used in the construction of the push-off, i.e. for some small constant $k > 0$, we have

$$B^+ \cap \pi^{-1}(\theta) \cap \{r \leq \varepsilon\} = B' \times (0, k] \subset B \times D^2.$$

We will now work in $M \setminus (B \times D^2_\varepsilon) \cong \Sigma \times S^1$ with $\varepsilon < k$ (i.e. we will work in a trivial mapping torus).

Observe that $B^+ \cap \Big(\Sigma \times \{\theta\}\Big)$ is a codimension 2 symplectic submanifold with trivial normal bundle of $(\Sigma \times \{\theta\}, d\alpha|_{\Sigma\times\{\theta\}})$. Also, the symplectic structure on $\Sigma \times \{\theta\}$ induced by $d\alpha$ is independent of θ. Thus, we get a fibre-wise symplectic projection

$$p\colon\ \Sigma \times S^1 \to \Sigma.$$

This defines a family of symplectic submanifolds

$$\Sigma'_t := p\Big(B^+ \cap (\Sigma \times \{t\})\Big)$$

of Σ which all coincide near the boundary.

Auroux's version of Banyaga's isotopy extension theorem (see Theorem 1.1.10) for symplectic submanifolds then yields a symplectic isotopy

$$\phi_t\colon\ \Sigma \to \Sigma$$

with $\phi_t(\Sigma'_0) = \Sigma'_t$ in such a way that it is equal to the identity in a neighbourhood U_1 of $\partial\Sigma$ and outside a bigger neighbourhood U_2 of the boundary.

Differentiating ϕ_t yields a time-dependent vector field X_t on Σ, which can be assumed to coincide for $t = 0$ and $t = 2\pi$ (it extends a vector field along the submanifold Σ' with that property). Thus, X_t lifts to a vector field X on $\Sigma \times S^1$ with $d\theta(X) = 1$, whose projection to each fibre $\Sigma \times \{\theta\}$ is symplectic. Furthermore, the vector field X is equal to ∂_θ inside $U_1 \times S^1$ and outside $U_2 \times S^1$. To simplify notation, we set $V := (U_2 \setminus U_1) \times S^1$.

Now given any monodromy vector field Y which is equal to ∂_θ on $U_2 \times S^1$, we can replace Y by X over V to get a vector field $\widetilde{Y}$, which is tangent to the push-off by construction. We claim that $\widetilde{Y}$ is a monodromy vector field.

Indeed, we have $d\theta(\widetilde{Y}) = 1$ and near the binding $\widetilde{Y}$ equals ∂_θ. Furthermore, the Lie derivative of $d\alpha$ with respect to $\widetilde{Y}$ coincides with $\mathcal{L}_Y d\alpha$ outside of V and with $\mathcal{L}_X d\alpha$ on V. Hence, as Y is a monodromy vector field and X is symplectic on pages, the restriction of $\mathcal{L}_{\widetilde{Y}} d\alpha$ to any page vanishes, which means that $\widetilde{Y}$ is a monodromy vector field tangent to the push-off B^+. $\square$

Remark 5.4.5

By the symplectic neighbourhood theorem 1.1.8 a neighbourhood of Σ' in Σ can be written as $\Sigma' \times D^2$ with split symplectic form. A smooth family of symplectic submanifolds can be assumed to arise as the image of an isotopy. The first step in proving Auroux's theorem 1.1.10 is to extend this isotopy to an open neighbourhood. In our case, the symplectic neighbourhood theorem allows us to do this in a trivial way. Applying Auroux's construction to this trivial extension yields an isotopy which is invariant in fibre direction, i.e. the time-2π map of the resulting isotopy restricts to a map of the form $\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$ on the neighbourhood of Σ'. Hence, we have the following corollary.

Corollary 5.4.6
The push-off is an abstract nested open book of an abstract open book description with page Σ and monodromy ϕ, where ϕ restricts to

$$\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$$

for a symplectomorphism $\phi' \colon \Sigma' \to \Sigma'$ in a neighbourhood $\Sigma' \times D^2$ of Σ' given by the symplectic normal bundle. □

Remark 5.4.7
Observe that by requiring the monodromy to be trivial in fibre direction, the resulting abstract nested open book is also a framed nested abstract open book in the sense of Section 5.2.2. The monodromy does not correspond to the Chinese burn Ψ but to the concatenation $\mathcal{D} \circ \Psi$ with the twist map $\mathcal{D}$, which was used to turn a nested open book into a framed nested open book in the topological setting and exactly ensured triviality in fibre direction.

Having described the push-off as an abstract nested open book, it is natural to perform a fibre sum construction of the ambient abstract open books. This will then yield a contact structure adapted to the resulting natural open book decomposition. However, it is a priori unclear whether this contact structure is indeed the contact structure resulting from the contact fibre connected sum.

We will first show that the symplectic fibre connected sum (see Section 1.5.2) of exact symplectic manifolds is exact under suitable conditions. The following technical lemma will be useful to interpolate between Liouville forms which agree on a symplectic submanifold.

Lemma 5.4.8
Let M be a manifold and let λ_0 and λ_1 be two 1-forms on $M \times D^2$ such that $d\lambda_0 = d\lambda_1$ and $\lambda_0|_{T(M\times\{0\})} = \lambda_1|_{T(M\times\{0\})}$. Then $\lambda_1 - \lambda_0$ is exact.

Proof. The proof is just an application of Poincaré's lemma to this particular setting. Without loss of generality, we can assume that M is connected. Define $\eta := \lambda_1 - \lambda_0$. Then η is a closed 1-form on $M \times D^2$ with $\eta|_{T(M\times\{0\})} = 0$. Let p_0 be in M. For $(p, q) \in M \times D^2$ let $\gamma_{(p,q)}$ be a path of the form $\gamma_1 * \gamma_2$, where γ_1 is a path from $(p_0, 0)$ to $(p, 0)$ with trace in $M \times \{0\}$ and γ_2 the linear path connecting $(p, 0)$ and

(p,q). We then define a function $h\colon\ M \times D^2 \to \mathbb{R}$ by

$$h(p,q) := \int_{\gamma_{(p,q)}} \eta.$$

This is well-defined because η vanishes on $M \times \{0\}$ and smooth because η is a smooth 1-form and γ a piecewise smooth path. As η vanishes on $M \times \{0\}$, we can use closedness to show $dh = \eta$ exactly as in the proof of the Poincaré lemma. □

Proposition 5.4.9

Let $(W_i, \omega_i = d\lambda_i)$, $i = 0, 1$, be exact symplectic manifolds with symplectomorphic submanifolds $X_i \subset W_i$ of codimension 2 and trivial normal bundle. Assume furthermore that the restriction $\lambda|_{TX_i}$ of the Liouville forms to these submanifolds coincide. Then the symplectic fibre sum of the W_i along the X_i is again exact symplectic.

Proof. We will drop the indices in the first part of the proof and work in W_0 and W_1 separately. The submanifold $X \subset W$ is symplectic with symplectic form

$$\omega' := \omega|_{TX} = (d\lambda)|_{TX} = d(\lambda|_{TX}),$$

i.e. X is exact symplectic and a Liouville form is given by $\lambda' := \lambda|_{TX}$. As X is of codimension 2 and has trivial normal bundle, the symplectic neighbourhood theorem 1.1.8 allows us to write a neighbourhood of X in W as $X \times D^2$ with symplectic form given as $\omega = \omega' + s ds \wedge d\vartheta$, where s and ϑ are polar coordinates on the D^2-factor. In these coordinates one (local) primitive of ω is given by $\lambda' + 1/2 s^2 d\vartheta$. Hence, by Lemma 5.4.8, we have

$$\lambda|_{X\times D^2} = \lambda' + \frac{1}{2}s^2 d\vartheta + dh$$

for an appropriate function h.

Now by assumption, the restriction of the Liouville forms to the symplectic submanifolds X_i agree, so in the coordinates adapted to the symplectic normal bundle as above they are $\lambda_i = \lambda' + 1/2 s^2 d\vartheta + dh_i$.

It follows that the symplectic fibre sum along the symplectic submanifolds X_i is again exact symplectic. The Liouville form can be chosen to coincide with the original ones outside the area of identification and is given by

$$\lambda' + \frac{1}{2}s^2 d\vartheta + d\Big((1-g)h_0 + g h_1\Big)$$

on the annulus of identification. Here g is a function on the annulus equal to 1 near one boundary and equal to 0 near the other boundary component. □

We can now apply the proposition to the setting of abstract open books, which gives a contact version of the topological fibre sum of nested open books described in Remark 5.2.6.

Corollary 5.4.10
Let $\Sigma_i' \subset (\Sigma_i, d\lambda_i, \phi_i)$, $i = 0, 1$, be two contact abstract nested open books with trivial normal bundle. Let $\psi\colon \nu(\Sigma_0') \to \nu(\Sigma_1')$ be a symplectomorphism of neighbourhoods with $\psi(\Sigma_0') = \Sigma_1'$ satisfying $\psi \circ \phi_0 = \phi_1 \circ \psi$ and $(\psi^*\lambda_1)|_{T\Sigma_0'} = \lambda_0|_{T\Sigma_0'}$. Then the fibre connected sum of the Σ_i along the Σ_i' with respect to ψ yields again an abstract open book.

In particular, the symplectic fibre sum of two abstract open books along the push-offs of their contactomorphic bindings yields again an abstract open book. Hence, the topological binding sum along two contactomorphic binding components carries a contact structure which is adapted to the natural open book structure and coincides with the original structures outside a neighbourhood of the push-offs of the respective binding components.

Proof. We have to show that the symplectic fibre sum is again an exact symplectic manifold with a Liouville vector field pointing outwards at the boundary and that the original monodromies give rise to a monodromy on the fibre sum. The latter is ensured by the condition $\psi \circ \phi_0 = \phi_1 \circ \psi$. Exactness follows almost immediately from the preceding proposition. Observe that Σ_i' being contact abstract nested open books (cf. Definition 5.4.2) allows us to perform a relative version of the symplectic fibre connected sum as described in Remark 1.5.3. So by the proposition the fibre sum yields an exact symplectic manifold with boundary with Liouville field still pointing outwards.

Note that the description of the push-off as an abstract open book as in Corollary 5.4.6 fulfils the hypothesis of the first part of this corollary (the Liouville forms can be assumed to agree as the push-offs live in trivial neighbourhoods of contactomorphic binding components). Performing a generalised Thurston–Winkelnkemper construction on the resulting abstract open book then yields the second part of this corollary. □

5.4.1 Naturality of the contact structure

The preceding corollary ensures the existence of *a* contact structure adapted to the resulting open book structure on the fibre sum. It does not tell us however, that the contact structure from the *contact* fibre connected sum is adapted to this open book. This is what we want to show in the following. The problem is that the fibres of the symplectic normal bundle to the push-off B^+ are not tangent to the pages, i.e. the operation of the contact fibre connected sum, which uses these fibres, does not fit nicely to the open book structure. We are going to manipulate the abstract open book (without changing the underlying contact manifold) in such a way that the symplectic normal fibres of the push-off are tangent to the pages of the open book and thus guaranteeing compatibility of open book structure and fibre sum.

Lemma 5.4.11
Let X be a codimension 2 symplectic submanifold of an exact symplectic manifold $(W, \omega = d\lambda)$ and suppose that the normal bundle of X is trivial. Then there is a Liouville form $\widetilde{\lambda}$ such that the corresponding Liouville vector field is tangent to X.

Proof. The submanifold $X \subset W$ is symplectic with symplectic form

$$\omega' := \omega|_{TX} = (d\lambda)|_{TX} = d(\lambda|_{TX}),$$

i.e. X is exact symplectic and a Liouville form is given by $\lambda' := \lambda|_{TX}$. As X is of codimension 2 and has trivial normal bundle, the symplectic neighbourhood theorem 1.1.8 allows us to write a neighbourhood of X in W as $X \times D^2$ with symplectic form given as $\omega = \omega' + s ds \wedge d\vartheta$, where s and ϑ are polar coordinates on the D^2-factor. In these coordinates one (local) primitive of ω is given by

$$\widetilde{\lambda} := \lambda' + 1/2 s^2 d\vartheta.$$

Observe that the restriction of both λ and $\widetilde{\lambda}$ to X equals λ'. Hence, by Lemma 5.4.8, we have

$$\lambda|_{X \times D^2} = \widetilde{\lambda} + dh$$

for an appropriate function h. Consider a function $\widetilde{h}$ on W which is equal to h near X and vanishes outside a neighbourhood of X, and denote its Hamilton vector field by X_h. If Y is the Liouville vector field corresponding to the Liouville form λ, then the sum $Y + X_h$ is again Liouville. The associated Liouville form restricts to $\widetilde{\lambda}$ near X. In particular, the Liouville vector field is tangent to X. □

Remark 5.4.12
Also note that the contact structures on the open book obtained by the generalised Thurston–Winkelnkemper construction performed with two Liouville forms λ and $\lambda + dh$ coinciding near the boundary are isotopic. Indeed, a family of contact structures is given by using $\lambda + d(sh)$ for $s \in [0, 1]$ and Gray stability can be applied.

Proposition 5.4.13
Let (W, ω) be an exact symplectic manifold and $X \subset W$ a symplectic submanifold of codimension 2 with trivial symplectic normal bundle. Let λ_t, $t \in \mathbb{R}$, be a smooth family of Liouville forms such that the corresponding Liouville vector fields Y_t are tangent to X and such that the 1-form $\frac{d}{dt}\lambda_t$ vanishes on the fibres of the symplectic normal bundle of X.

Then the fibres of the conformal symplectic normal bundle of the contact submanifold $X \times \mathbb{R} \subset (W \times \mathbb{R}, \alpha = \lambda_t + dt)$ *are tangent to the fibres* $W \times \{t\}$*.*

Proof. Note that it is sufficient to work over a symplectic neighbourhood $X \times D^2$ of X and denote the restriction of ω to the tangent bundle of X by ω'. As the submanifold X of W is symplectic, we can (e.g. by choosing a suitable family of compatible almost complex structures) fix vector fields X_t tangent to X such that $\omega(X_t, Y_t) = 1$. For $p \in X$ we then have

$$T_pX = \langle X_t, Y_t\rangle \oplus \Big(\langle X_t, Y_t\rangle\Big)^{\omega'}$$

and

$$T_pW = \langle X_t, Y_t\rangle \oplus \Big(\langle X_t, Y_t\rangle\Big)^{\omega'} \oplus SN_p(X),$$

where $SN(X)$ denotes the symplectic normal bundle to X. The symplectic complement to Y_t in such points is given by

$$\Big(\langle Y_t\rangle\Big)^{\omega} = \langle Y_t\rangle \oplus \Big(\langle X_t, Y_t\rangle\Big)^{\omega'} \oplus SN_p(X).$$

Now on $W \times \mathbb{R}$ with contact form $\alpha = \lambda_t + dt = i_{Y_t}\omega + dt$ we have

$$\ker\alpha = \Big(\langle Y_t\rangle\Big)^{\omega} \oplus \langle \partial_t + X_t\rangle$$

and

$$d\alpha = d\lambda_t + \left(\frac{d}{dt}\lambda_t\right) \wedge dt = \omega + \left(\frac{d}{dt}\lambda_t\right) \wedge dt.$$

The intersection of the tangent space to $X \times \mathbb{R}$ with the kernel of α in a point (p, t) computes as

$$T_{(p,t)}(X \times \mathbb{R}) \cap \ker\alpha = \langle Y_t\rangle \oplus \Big(\langle X_t, Y_t\rangle\Big)^{\omega'} \oplus \langle \partial_t + X_t\rangle.$$

The symplectic normal bundle to $X \times \{t\}$ in $W \times \{t\}$ is contained in the kernel of α and the 1-form $\frac{d}{dt}\lambda_t$ vanishes on its fibres. Hence, the fibres of the conformal symplectic normal bundle of the contact submanifold $X \times \mathbb{R}$ (which are calculated with respect to $d\alpha$) coincide with the fibres of the symplectic normal bundle of $X \times \{t\}$ in $W \times \{t\}$ (calculated with respect to ω). In particular, they are tangent to the slices $W \times \{t\}$. □

We will only need the proposition in a special case during a generalised Thurston–Winkelnkemper construction. However, note that it also holds for a family

$$\lambda_t = \Big(1 - \mu(t)\Big)\lambda + \mu(t)\widetilde{\lambda}$$

interpolating two Liouville forms with tangent Liouville vector field provided that their difference is exact with a primitive function only depending on Σ'-directions.

This can then be used when working with Giroux domains instead of the Thurston–Winkelnkemper construction.

We can now show that the binding sum construction can be made compatible with the underlying contact structures and thus prove Theorem 5.4.1.

Proof of Theorem 5.4.1. By Corollary 5.4.6 the push-off is an abstract nested open book of an abstract open book description with page Σ and monodromy ϕ, where ϕ restricts to

$$\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$$

for a symplectomorphism $\phi'\colon \Sigma' \to \Sigma'$ in a neighbourhood $\Sigma' \times D^2$ of Σ' given by the symplectic normal bundle. Furthermore, we can assume that the Liouville form restricts to $\lambda = \lambda' + 1/2s^2 d\vartheta$ in this neighbourhood by Lemma 5.4.11 and its proof and Remark 5.4.12.

Note that the monodromy ϕ will not necessarily be exact symplectic but according to Lemma 2.3.3 it is isotopic through symplectomorphisms equal to the identity near the boundary to an exact symplectomorphism. The idea of the proof is to define a vector field X on Σ by the condition $i_X\omega = \lambda - \phi^*\lambda$ and checking that precomposing ϕ with the time-1 flow of X is an exact symplectomorphism with the desired properties. Now in our situation, observe that the vector field X is tangent to Σ', and moreover, projects to zero under the natural projection $\Sigma' \times D^2 \to D^2$ also in a neighbourhood $\Sigma' \times D^2$ as above. As a consequence, the restriction of the resulting monodromy (still denoted by ϕ by abuse of notation) will still be of the form

$$\phi|_{\Sigma' \times D^2} = \phi' \times \mathrm{id}_{D^2}$$

but now for an *exact* symplectomorphism $\phi'\colon \Sigma' \to \Sigma'$. In particular, we have $\phi^*\lambda - \lambda = dh$ for a function h which only depends on the Σ'-directions on $\Sigma' \times D^2$.

If we form the generalised mapping torus $\Sigma_h(\phi)$ and equip it with the contact form $\lambda + dt$, Proposition 5.4.13 tells us that the fibres of the conformal symplectic normal bundle of the push-off are tangent to the slices $\Sigma \times \{t\}$ and are given by the D^2-direction of $\Sigma' \times D^2 \subset \Sigma$. We now want to show that the fibres of the conformal symplectic normal bundle are then also tangent to the pages in the genuine mapping torus. For this, it is enough to observe that the diffeomorphism between the generalised and the genuine mapping tori maps a point (p,t) to a point $(p,\widetilde{t})$, where $\widetilde{t}$ only depends on t and the value of the function h at p. Thus, as in a symplectic neighbourhood $\Sigma' \times D^2$ the function h only depends on Σ', we have that for a point $p' \in \Sigma'$ and fixed t the point (p', s, ϑ, t) is mapped to $(p', s, \vartheta, \widetilde{t})$ independently of (s, ϑ). Hence, the fibres of the conformal symplectic normal bundle are tangent to the pages. Also, the resulting Reeb vector field is tangent to the push-off, as it is a multiple of the monodromy vector field outside a neighbourhood of the binding. Furthermore, by construction it is adapted to the open book structure, i.e. it is transverse to the pages. Thus, denoting the restriction of the contact form α to the push-off B^+ by α_{B^+}, the 1-form α restricts to $\alpha_{B^+} + s^2 d\vartheta$ on a tubular neighbourhood $B^+ \times D^2$ (with polar coordinates (s, ϑ)) given by the conformal symplectic normal bundle of B^+. The fibres of the symplectic normal bundle being tangent to the pages means that the contact fibre connected sum along B^+ has a natural open book structure. The contact form for the resulting contact structure used in the contact fibre connected sum construction is of the form $\tilde{\alpha} = \alpha_{B^+} + f(s)d\vartheta$ for an appropriate function f and coincides with α near $B^+ \times \partial D^2$ (see proof of Theorem 1.5.2). In particular, the Reeb vector field of $\tilde{\alpha}$ is still transverse to the fibres of the open book fibration. Hence, the resulting contact structure in the contact fibre connected sum is adapted to the resulting open book structure. □

5.4.2 Examples in the contact setting

We conclude the chapter with some applications and examples of the binding sum construction in the contact setting.

An open book of $S^4 \times S^1$

Let $M = M_0 \sqcup M_1$ with M_i the five-dimensional sphere S^5 with standard contact structure and compatible open book decomposition $(\Sigma_i = D^4, \mathrm{id})$. Then the binding has two components B_i, both a standard 3-sphere that we can equip with the compatible open book decomposition $(\Sigma_i' = D^2, \mathrm{id})$. We have $M_i = D^4 \times S^1 \cup S^3 \times D^2$, so performing the binding sum on M along the B_i yields $S^4 \times S^1$. Note that the framing of the binding is unique up to homotopy because B is simply-connected. By Theorem 5.3.1, the binding sum $S^4 \times S^1$ has a natural open book decomposition obtained by forming the sum along the push-off of the binding.

Pushing a page $\Sigma_i' = D^2$ of the binding open book into the page $\Sigma_i = D^4$ and then removing a neighbourhood of Σ_i' topologically turns Σ_i into a copy of $D^3 \times S^1$. The resulting page is then obtained by identifying two copies of $D^3 \times S^1$ along a neighbourhood of $\{*\} \times S^1 \subset \partial D^3 \times S^1$, i.e. it is $D^3 \times S^1$ as well. The new binding is the contact binding sum of the B_i with respect to the specified open book decomposition. Hence, the new binding has an open book decomposition with page an annulus and – applying the formula from [53] for the page framing – monodromy isotopic to the identity. This means that the resulting binding is $S^2 \times S^1$ with standard contact structure. This has a unique symplectic filling up to blow-up and symplectic equivalence (cf. [66, Theorem 4.2]), so the resulting page is indeed symplectomorphic to $D^3 \times S^1$ arising as the 4-ball with a 1-handle attached.

An explicit description of the monodromy is hard to obtain, as the symplectic isotopy extension theorem is used in the construction. In the following, we show that the resulting monodromy is isotopic to the identity. To this end, we equip $\Sigma_i = D^4$ with a pair of polar coordinates by considering it as the unit ball in $\mathbb{C}^2$ with standard Liouville structure, i.e.

$$\Sigma_i = \Big\{(r_1, \theta_1, r_2, \theta_2) \mid r_1^2 + r_2^2 \leq 1\Big\}.$$

The binding component $B_i = \partial \Sigma_i$ is thus given by

$$B_i = \Big\{(r_1, \theta_1, r_2, \theta_2) \mid r_1^2 + r_2^2 = 1\Big\}$$

with contact form $\alpha_{B_i} = r_1^2 d\theta_1 + r_2^2 d\theta_2$. The compatible open book structure on B' with disc pages is then given by

$$\Sigma'_\theta = \Big\{(r_1, \theta_1, r_2, \theta) \mid r_1^2 + r_2^2 = 1\Big\}$$

with $B'_i = \Big\{(1, \theta_1, 0, \theta_2)\Big\}$.

The intersection of the push-off B_i^+ with the page Σ_θ coincides with

$$\Big\{(r_1, \theta_1, 0, \theta) \mid 1 - c \leq r_1^2 \leq 1\Big\}$$

near the boundary $\partial \Sigma_\theta$ and with

$$\Big\{(r_1, \theta_1, r_2, \theta) \mid r_1^2 + r_2^2 = 1 - c\Big\}$$

away from it. The monodromy vector field of the binding is given by ∂_{θ_2}, so an isotopy of the page Σ' inside a fixed page Σ_i is a rotation in θ_2-direction near

$$r = \sqrt{r_1^2 + r_2^2} = 1 - c$$

cut-off in radial direction, i.e. it is of the form

$$(r_1, \theta_1, r_2, \theta_2) \mapsto \Big(r_1, \theta_1, r_2, \theta_2 + tf(r_1, r_2)\Big)$$

for a suitable cut-off function f. Note that in general this isotopy is not symplectic, which is why the monodromy in Corollary 5.4.6 was not constructed directly but by using the symplectic isotopy extension theorem.

Open books and contact structures on $M \times T^2$

Suppose that M is a contact manifold such that $M \times I$ admits a Stein structure. Then we get an open book decomposition and a contact structure on $M \times T^2$ by performing the binding sum of two copies of the open book $(M \times I, \mathrm{id})$. Note that the requirement for $M \times I$ to admit a Stein structure is a huge restriction (cf. [59]) and that a T^2-invariant contact structure on $M \times T^2$ for general M was constructed by Bourgeois [9].

Open books with Giroux torsion

Let M be a closed oriented manifold admitting a *Liouville pair* (α_+, α_-), i.e. a pair consisting of a positive contact form α_+ and a negative contact form α_- such that $e^{-s}\alpha_- + e^s\alpha_+$ is a positively oriented Liouville form on $\mathbb{R} \times M$ (with s denoting the coordinate on the $\mathbb{R}$-factor). Then the 1-form

$$\lambda_{GT} = \frac{1+\cos s}{2}\alpha_+ + \frac{1-\cos s}{2}\alpha_- + \sin s dt$$

defines a positive contact structure on $\mathbb{R} \times S^1 \times M$ (s and t denote the respective coordinates on the first two factors) (see [59, Proposition 8.1]). With this model, Massot, Niederkrüger and Wendl [59] define a **Giroux $2k\pi$-torsion domain** as $([0, 2k\pi] \times S^1 \times M, \lambda_{GT})$. Just as in the 3-dimensional setting this higher-dimensional version of Giroux torsion is a filling obstruction in the sense that a contact manifold admitting a contact embedding of a Giroux 2π-torsion domain is not strongly fillable (see [59, Corollary 8.2]). Observe that a Giroux 2π-torsion domain with boundary blown down (cf. [59, Section 4]) is the binding sum of two copies of the open book with page $([0, \pi] \times M, \beta)$ and trivial monodromy, where

$$\beta = \frac{1}{2}(e^{-s}\alpha_- + e^s\alpha_+)$$

along $\{\pi\} \times M$. Given any contact open book (Σ, ϕ) with Σ having two boundary components contactomorphic to M, Theorem 5.4.1 yields an open book decomposition of the binding sum $(\Sigma, \phi) \# ([0, \pi] \times M, \mathrm{id}) \# ([0, \pi] \times M, \mathrm{id})$, which is a manifold admitting an embedding of a Giroux 2π-torsion domain modelled on M.

Fibrations over the circle

Theorems 5.3.1 and 5.4.1 yield a contact open book decomposition of certain bundles over the circle with fibres being convex hypersurfaces.

Recall that an oriented hypersurface S in a contact manifold is called **convex** (in the sense of Giroux [40]) if there is a contact vector field transverse to S. A neighbourhood of the hypersurface can then be identified with $S \times \mathbb{R}$ such that the contact structure is $\mathbb{R}$-invariant, i.e. there is a contact form of type $\beta + u dt$ with β a 1-form on S and $u\colon S \to \mathbb{R}$ a function such that $(d\beta)^{n-1} \wedge (u d\beta + n\beta \wedge du)$ is a volume form on S (here $2n$ is the dimension of S). Conversely, given a triple (S, β, u) consisting of a $(2n)$-dimensional closed manifold S, a 1-form β on S and a function $u\colon S \to \mathbb{R}$ satisfying the above conditions, the 1-form $\beta + u dt$ defines an $\mathbb{R}$- or S^1-invariant contact form on $S \times \mathbb{R}$ or $S \times S^1$, respectively. Observe that S with the zero set of u removed is an exact symplectic manifold with Liouville form β/u.

Now given a triple (S, β, u) as above and a diffeomorphism ϕ of S restricting to the identity near $\Gamma := \{u = 0\}$ and to symplectomorphisms $\phi_\pm$ on the interior of $S_\pm := \{\pm u \geq 0\}$, the S-bundle M over S^1 with monodromy ϕ carries a natural contact structure, such that each fibre defines a convex surface modelled by (S, β, u). In particular, the fibration admits a contact vector field transverse to the fibres, which is tangent to the contact structure exactly over Γ. Observe that M is equal to the binding sum of the open books (S_+, ϕ_+) and (S_-, ϕ_-). Thus, Theorem 5.3.1 yields an open book description of M, which is adapted to the contact structure by Theorem 5.4.1.

A

Computing rotation and self-linking numbers in contact surgery diagrams

This first chapter of the appendix contains the article *Computing rotation and self-linking numbers in contact surgery diagrams*, which is joint work with Marc Kegel and was published in [20]. Some references have been changed to refer to the corresponding parts of this thesis.

A.1 Introduction

A lot of the geometry of a 3-dimensional contact manifold is encoded in its *Legendrian knots*, i.e. smooth knots tangent to the contact structure, and in its *transverse knots*, i.e. smooth knots transverse to the contact structure. Therefore a main topic in 3-dimensional contact geometry is the study of these knots. In particular, it is a challenge to distinguish knots within these classes. For nullhomologous knots this is mostly done by the so-called *classical invariants*, the *Thurston–Bennequin invariant* and the *rotation number* for Legendrian knots and the *self-linking number* for transverse knots. In the unique tight contact structure of the 3-sphere there are easy formulas to compute the classical invariants from a *front projection* of the knot. For this and other basic notions in contact geometry we refer the reader to [37].

A natural extension is to consider Legendrian or transverse knots in *contact surgery diagrams* along Legendrian links and to compute their classical invariants in the surgered manifold. Starting with the work of Lisca, Ozsváth, Stipsicz and Szabó [57, Lemma 6.6] several results were obtained in that setting by Geiges and Onaran [38, Lemma 2], Conway [13, Lemma 6.4] and Kegel [49, Section 8].

In Theorem A.2.2 we combine the results mentioned above to obtain a condition when a Legendrian knot is nullhomologous in the surgered manifold and give a formula computing its rotation number. In Section A.3 we explain how to represent a transverse knot in a contact surgery diagram along Legendrian knots and then compute its self-linking number in the surgered contact manifold. Finally, in Section A.4 we extend these results to rationally nullhomologous knots. On the way we present plenty of examples on how to use these formulas.

A further closely related topic is the computation of the d_3-invariant of the resulting contact structure in a surgery diagram. By translating contact $(1/n)$-

surgeries into (± 1)-surgeries, we generalise the formula by Ding–Geiges–Stipsicz [17] in Section A.5.

A.2 The rotation number in surgery diagrams

Let $L = L_1 \sqcup \ldots \sqcup L_k \subset S^3$ be an oriented link in S^3 and let $M = S^3_L(r)$ be the manifold obtained by surgery along L with coefficients p_i/q_i (for basic notions of Dehn surgery see [71]). We denote the corresponding surgery slopes by

$$r_i = p_i\mu_i + q_i\lambda_i \in H_1(\partial\nu L_i),$$

where μ_i is represented by a positive meridian of L_i and λ_i is the Seifert longitude of L_i. If no coefficient group is specified, homology groups are understood to be over the integers. Let $L_0 \subset S^3 \setminus L$ be an oriented knot in the complement of L.

Define $l_{ij} := \operatorname{lk}(L_i, L_j)$ and let $\mathbf{l}$ be the vector with components $l_i = l_{0i}$ and Q the generalised linking matrix:

$$Q = \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1k} \\ q_1 l_{21} & p_2 & & \\ \vdots & & \ddots & \\ q_1 l_{k1} & & & p_k \end{pmatrix}.$$

The knot L_0 is nullhomologous in M if and only if there is an integral solution $\mathbf{a}$ of the equation $\mathbf{l} = Q\mathbf{a}$ (see [49]).

Definition A.2.1
Let $K \subset (M, \xi)$ be a nullhomologous oriented Legendrian knot and Σ a Seifert surface for K. The **rotation number** of K with respect to the Seifert surface Σ is equal to

$$\operatorname{rot}(K, \Sigma) = \langle \mathrm{e}(\xi, K), [\Sigma] \rangle = \mathrm{PD}(\mathrm{e}(\xi, K)) \bullet [\Sigma],$$

where $\mathrm{e}(\xi, K)$ is the relative Euler class of the contact structure ξ relative to the trivialisation given by a positive tangent vector field along the knot K, and $[\Sigma]$ the relative homology class represented by the surface Σ.

This definition of the rotation number is useful for calculations (see also [68]). For an alternative equivalent definition see [37]. Clearly, the rotation number does only depend on the class of the chosen Seifert surface, not on the particular choice of surface itself. Note also that the rotation number is independent of the class of the Seifert surface if the Euler class $\mathrm{e}(\xi)$ of the contact structure vanishes (see Proposition 3.5.15 in [37]).

Theorem A.2.2

Let $L = L_1 \sqcup \ldots \sqcup L_k$ be an oriented Legendrian link in (S^3, ξ_{st}) and L_0 an oriented Legendrian knot in its complement. Let (M, ξ) be the contact manifold obtained from S^3 by contact $(1/n_i)$-surgeries ($n_i \in \mathbb{Z}$) along L. Then L_0 is nullhomologous in M if and only if there is an integral vector $\mathbf{a}$ solving $\mathbf{l} = Q\mathbf{a}$ as above, in which case its rotation number in (M, ξ) with respect to a special Seifert class $\widehat{\Sigma}$ constructed in the proof is equal to

$$\operatorname{rot}_M(L_0, \widehat{\Sigma}) = \operatorname{rot}_{S^3}(L_0) - \sum_{i=1}^{k} a_i n_i \operatorname{rot}_{S^3}(L_i).$$

The proof proceeds in two steps. First, following [13], we construct the class of a Seifert surface for L_0 in M. We then use the description of the rotation number in terms of relative Euler classes to compute rot.

Remark A.2.3

1. Notice that the matrix Q is formed using the topological surgery coefficients p_i/q_i, not the contact surgery coefficients. The topological surgery coefficient equals the sum of the contact surgery coefficient and the Thurston–Bennequin invariant of the surgery knot. Therefore, we always have $q_i = n_i$.
2. Observe that for any contact surgery coefficient $r \neq 0$ there exists a tight contact structure on the glued in solid torus compatible with the surgery. This tight contact structure on the solid torus is unique if and only if the surgery coefficient is of the form $1/n$ for $n \in \mathbb{Z}$. Therefore, contact $(1/n)$-surgery is well-defined (see [16, Proposition 7]).
For a general contact r-surgery, there is an algorithm transforming the surgery into contact $(1/n)$-surgeries. The procedure is not unique, however, the algorithm provides all possible choices of contact structures that are tight on the surgery torus (cf. [14, 17]). In contrast to the Thurston–Bennequin invariant, the rotation number in the surgered manifold does indeed depend on the choice of contact structure on the surgery tori, cf. Example A.2.6.
3. In [14] it is shown that one can get any contact 3-manifold by a sequence of contact $(1/n)$-surgeries starting from the standard tight 3-sphere. Moreover, it is easy to show that any Legendrian knot in the resulting contact manifold can be represented by a Legendrian knot in the complement of the surgery link.

Proof. Assume that L_0 is nullhomologous in M and fix Seifert surfaces $\Sigma_0, \ldots, \Sigma_k$ for $L_0, \ldots, L_k$ in S^3, such that intersections of surfaces and link components are transverse. Our aim is to use these surfaces to construct the class of a Seifert surface for L_0 in the surgered manifold M. By abuse of notation, we will identify Σ_i with its class in $H_2(S^3\setminus \nu L_i, \partial\nu L_i)$ and will denote the class in $H_2(S^3\setminus(L_0\sqcup\nu L), \partial L_0\sqcup\partial\nu L)$ induced by restriction again by Σ_i.

The idea is to construct a class of the form

$$\Sigma = \Sigma_0 + \sum_{i=1}^{k} k_i\Sigma_i$$

such that its image under the boundary homomorphism ∂ in the long exact sequence of the pair $(S^3\setminus(L_0\sqcup\nu L), \partial L_0\sqcup\partial\nu L)$ is a linear combination of the surgery slopes r_i and a longitude of L_0, i.e. we want

$$\partial\Sigma = t\mu_0 + \lambda_0 + \sum_{i=1}^{k} m_i r_i = t\mu_0 + \lambda_0 + \sum_{i=1}^{k} m_i(p_i\mu_i + q_i\lambda_i).$$

So our aim is to solve this equation for k and m_i. We will first describe the boundary homomorphism ∂ in more detail and then compare coefficients. The surgery slopes r_i bound discs in the surgered manifold M, so Σ can be extended to give rise to a class in $H_2(M\setminus\nu L_0, \partial\nu L_0)$, which we denote by $\widehat{\Sigma}$. Geometrically, the boundary homomorphism sends Σ to its intersection with the boundary of the link complement. So we have:

$$\partial\colon\ \Sigma_j \longmapsto \lambda_j - \sum_{i\neq j} l_{ij}\mu_i,$$

and thus

$$\begin{aligned}\partial\colon\ \Sigma \longmapsto &\sum_{i=0}^{k} k_i\lambda_i - \sum_{j=0}^{k}\sum_{i\neq j} k_j l_{ij}\mu_i \\ &= -\sum_{j=1}^{k} k_j l_{0j}\mu_0 + \lambda_0 + \sum_{i=1}^{k} k_i\lambda_i - \sum_{i=1}^{k} l_{0i}\mu_i - \sum_{i=1}^{k}\sum_{j\neq i} k_j l_{ij}\mu_i,\end{aligned}$$

where we set $k_0 = 1$. Note that the minus sign stems from the induced boundary orientation of Σ (see Figure A.1). Setting $k_i = -a_iq_i$ and using that L is nullhomologous, we obtain

$$\begin{aligned}\partial\colon\ \Sigma_j \longmapsto &\sum_{j=1}^{k} a_jq_jl_{0j}\mu_0 + \lambda_0 - \sum_{i=1}^{k} a_iq_i\lambda_i - \sum_{i=1}^{k} l_{0i}\mu_i + \sum_{i=1}^{k}\sum_{j\neq i} a_jq_jl_{ij}\mu_i \\ =&\sum_{j=1}^{k} a_jq_jl_j\mu_0 + \lambda_0 - \sum_{i=1}^{k} a_iq_i\lambda_i - \sum_{i=1}^{k} l_i\mu_i + \sum_{i=1}^{k}\sum_{j\neq i} a_jQ_{ij}\mu_i \\ =&\sum_{j=1}^{k} a_jq_jl_j\mu_0 + \lambda_0 - \sum_{i=1}^{k} a_iq_i\lambda_i - \sum_{i=1}^{k} l_i\mu_i + \sum_{i=1}^{k} (l_i - a_ip_i)\mu_i \\ =&\sum_{j=1}^{k} a_jq_jl_j\mu_0 + \lambda_0 - \sum_{i=1}^{k} a_iq_i\lambda_i - \sum_{i=1}^{k} (a_ip_i)\mu_i,\end{aligned}$$

which is of the desired form with $m_i = -a_i$ and $t = \sum_{j=1}^{k} a_jq_jl_j$.

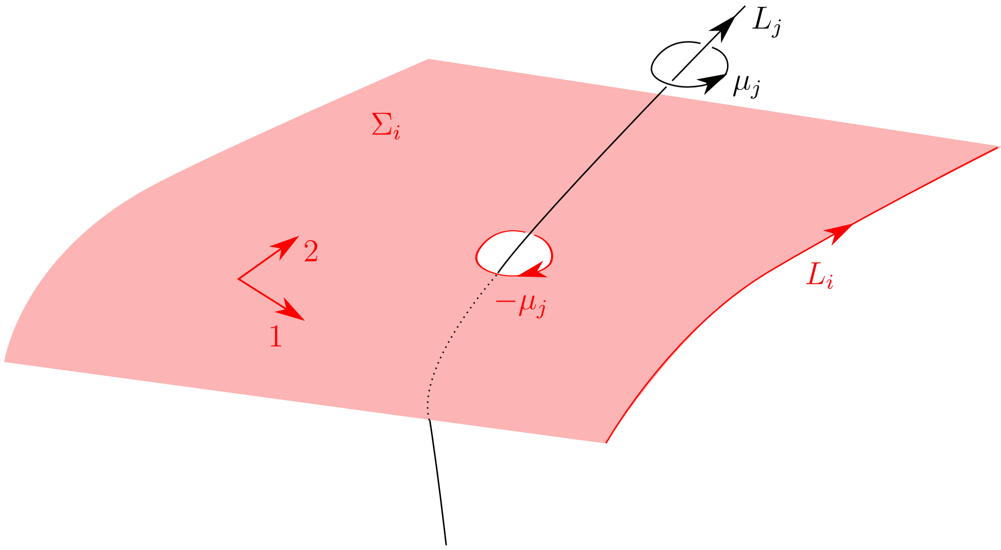


Figure A.1: Orientation of the meridian induced by the intersection.

Remark A.2.4
Observe that we can also directly obtain an embedded surface representing the capped-off class $\widehat{\Sigma}$ by resolving self-intersections in Σ. In particular, t is the negative change of the Thurston–Bennequin number of L_0 in the surgery (cf. [13], [49]), i.e. we get

$$\operatorname{tb}_M(L_0) = \operatorname{tb}_{S^3}(L_0) - \sum_{j=1}^{k} a_j n_j l_j.$$

Now consider L and L_0 to be Legendrian in (S^3, ξ_{st}) and the surgeries to be contact $(\frac{1}{n})$-surgeries. We claim that the rotation number of L_0 in the surgered contact manifold (M, ξ) with respect to $\widehat{\Sigma}$ is equal to

$$\operatorname{rot}_M(L_0, \widehat{\Sigma}) = \operatorname{rot}_{S^3}(L_0) - \sum_{i=1}^{k} a_i n_i \operatorname{rot}_{S^3}(L_i).$$

In complete analogy to [57], [38] and [13] we have the following lemma.

Lemma A.2.5
The homomorphism $H_1(S^3 \setminus (L_0 \sqcup L)) \to H_1(M \setminus L_0)$ induced by inclusion maps $\operatorname{PD}(\operatorname{e}(\xi_{st}, L_0 \sqcup L))$ *to* $\operatorname{PD}(\operatorname{e}(\xi, L_0))$.

The proof is completely analogous to the ones in [57, 38], where one uses the Legendrian rulings of the surgery tori induced by $(\frac{1}{n})$-surgery instead of (± 1)-surgery.

We thus have (cf. [13])

$$\begin{aligned}
\operatorname{rot}_M(L_0,\widehat{\Sigma}) &= \operatorname{PD}\Big(\mathrm{e}(\xi, L_0)\Big) \bullet \widehat{\Sigma} \\
&= \operatorname{PD}\Big(\mathrm{e}(\xi_{\mathrm{st}}, L_0 \sqcup L)\Big) \bullet \Sigma \\
&= \left(\sum_{i=0}^{k} \operatorname{rot}_{S^3}(L_i)\mu_i\right) \bullet \Sigma \\
&= \left(\sum_{i=0}^{k} \operatorname{rot}_{S^3}(L_i)\mu_i\right) \bullet \left(\Sigma_0 + \sum_{j=1}^{k} (-a_j n_j)\Sigma_j\right) \\
&= \operatorname{rot}_{S^3}(L_0) - \sum_{i=1}^{k} a_i n_i \operatorname{rot}_{S^3}(L_i),
\end{aligned}$$

which proves the theorem. □

If the contact surgeries are not unique, i.e. for contact surgery coefficients not of the form $1/n$ (see Remark A.2.3), the rotation number is – in contrast to the Thurston–Bennequin invariant – not independent of the chosen contact structures on the solid tori, as the following example illustrates.

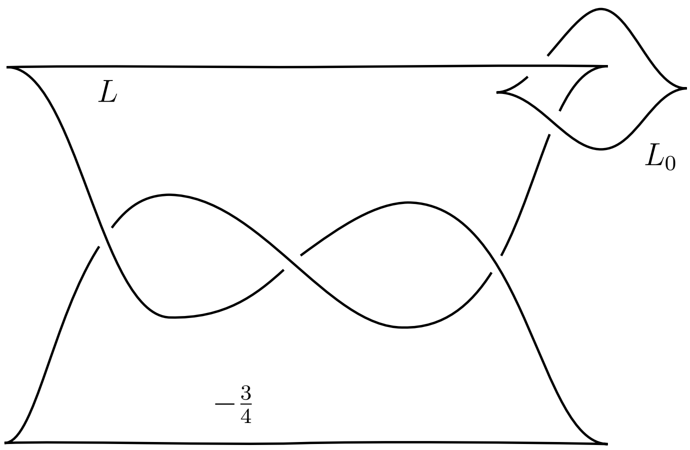


Figure A.2: Non-unique contact surgery yielding a homology sphere.

Example A.2.6

Consider the diagram depicted in Figure A.2, where L is a Legendrian trefoil with contact surgery coefficient $3/4$ and L_0 a Legendrian unknot in its complement. We have $\operatorname{tb}(L) = 1$, so the topological surgery coefficient is $\frac{1}{4}$. Thus, the surgered manifold M is a homology sphere and the rotation number of L_0 independent of the choice of Seifert surface. The contact surgery coefficient $-\frac{3}{4}$ has a continued fraction expansion $1-2-\frac{1}{-4}$, which means that there are three distinct tight contact structures on the solid torus compatible with the surgery resulting in the contact manifolds which are shown in Figure A.3 (see [17]). Topologically, these are the same, i.e. for all three diagrams we have

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \text{ and } \mathbf{l} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and hence $\mathbf{a} = (3, 1)$. Furthermore, we have $\mathrm{rot}_{S^3}(L_0) = 0$, $\mathrm{rot}_{S^3}(L_1) = 0$ and $\mathrm{rot}_{S^3}(L_2) \in \{-2, 0, 2\}$. This yields

$$\mathrm{rot}_M(L_0) = \mathrm{rot}_{S^3}(L_0) - 3\,\mathrm{rot}_{S^3}(L_1) - \mathrm{rot}_{S^3}(L_2) \in \{-2, 0, 2\},$$

depending on the chosen contact structure and orientations.

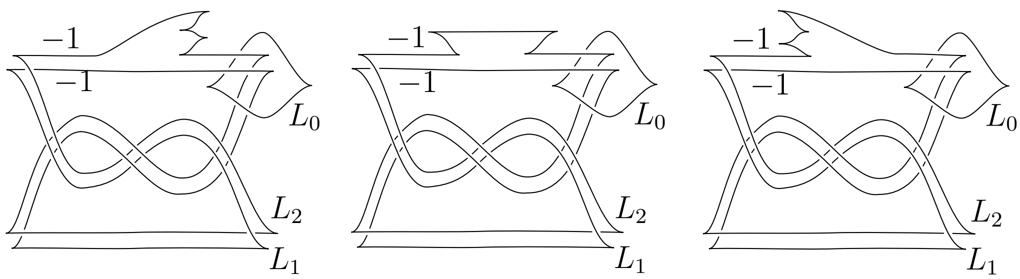


Figure A.3: Three unique contact surgeries corresponding to Figure A.2.

Example A.2.7

We consider the case of L a one component link with contact surgery coefficient $\pm\frac{1}{n}$, so the topological surgery coefficient is $\frac{(n\,\mathrm{tb}(L)\pm 1)}{n}$. We then have $Q = p = n\,\mathrm{tb}(L) \pm 1$ and L_0 is nullhomologous in the surgered manifold if and only the linking number of L_0 and L is divisible by $n\,\mathrm{tb}(L) \pm 1$, in which case $\mathbf{a}$ is the quotient $\frac{\mathrm{lk}(L_0, L)}{n\,\mathrm{tb}(L)\pm 1}$. Then the rotation number of L_0 in the surgered manifold is

$$\mathrm{rot}_M(L_0, \widehat{\Sigma}) = \mathrm{rot}_{S^3}(L_0) - \frac{n\,\mathrm{lk}(L_0, L)}{n\,\mathrm{tb}(L) \pm 1}\,\mathrm{rot}_{S^3}(L),$$

and its Thurston–Bennequin invariant is

$$\mathrm{tb}_M(L_0) = \mathrm{tb}_{S^3}(L_0) - \frac{n\,\mathrm{lk}^2(L_0, L)}{n\,\mathrm{tb}(L) \pm 1}.$$

Observe that if $n\,\mathrm{tb}(L) \pm 1$ is non-zero, the knot L_0 is rationally nullhomologous. Then the computed numbers represent the rational invariants (cf. Section A.4).

Example A.2.8

Figure A.4 shows a stabilisation of a knot L_0. Topologically, the surgery along the meridian L of L_0 again yields S^3, and there are two choices of tight contact structures on the solid torus compatible with the surgery. Here, in both cases, the resulting S^3 is tight. The topological knot type L_0 stays unchanged in M, but L_0 is either stabilised positively or negatively, depending on the particular choice of contact structure in the surgery torus.

The topological data in the diagrams with a unique choice is

$$Q = \begin{pmatrix} 0 & -1 \\ -1 & 3 \end{pmatrix} \text{ and } \mathbf{l} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and hence $\mathbf{a} = (2, -1)$. So the Thurston–Bennequin invariant of L_0 in M is

$$\operatorname{tb}_M(L_0) = \operatorname{tb}_{S^3}(L_0) - \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = \operatorname{tb}_{S^3}(L_0) - 1$$

in both cases. The rotation number of L_1 vanishes in both cases, the rotation number of L_2 is either $+1$ or -1. We thus have

$$\operatorname{rot}_M(L_0) = \operatorname{rot}_{S^3}(L_0) - \left\langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \right\rangle = \operatorname{rot}_{S^3}(L_0) \mp 1.$$

In fact, one can show that it is a stabilised copy of L_0 (see [49, Section 10]).

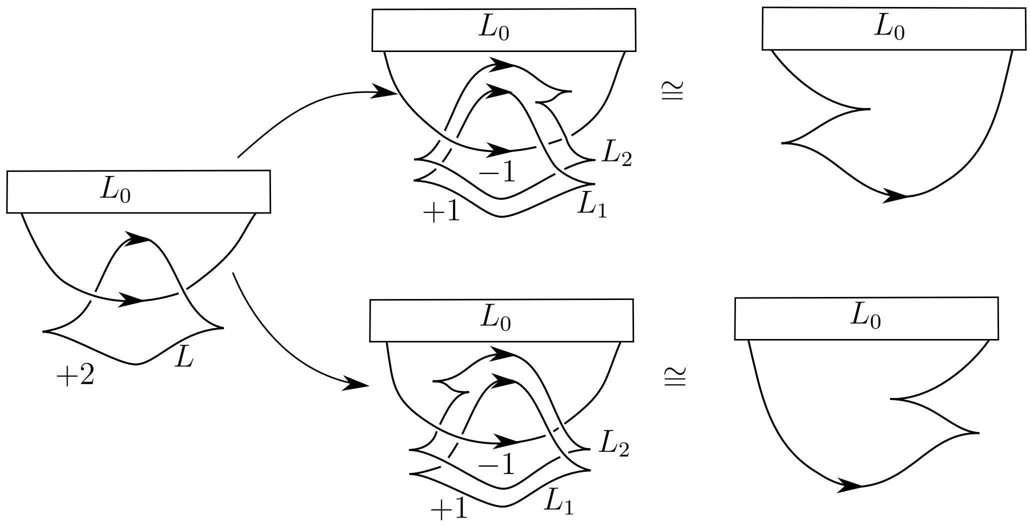


Figure A.4: Stabilisation via surgery.

A.3 The self-linking number of transverse knots

Let T be an oriented nullhomologous transverse knot in a contact manifold (M, ξ) and let Σ be a Seifert surface for T. The **self-linking number** $\operatorname{sl}(T, \Sigma)$ of T is defined as the linking number of T and T' where T' is obtained by pushing T in the direction of a non-vanishing section of $\xi|_\Sigma$.

Remark A.3.1
We consider transverse knots with arbitrary orientations. If the given orientation coincides with the orientation induced by the contact planes, we call the knot **positively transverse**, and else **negatively transverse**. The self-linking number of a transverse knot is independent of its orientation and does only depend on the homology class of the chosen Seifert surface (cf. Section 3.5.2 in [37]).

Corollary A.3.2
Let $L = L_1 \sqcup \ldots \sqcup L_k$ be an oriented Legendrian link in (S^3, ξ_{st}) and T_0 an oriented transverse knot in its complement. Let (M, ξ) be the contact manifold obtained from S^3 by contact $(1/n_i)$-surgeries ($n_i \in \mathbb{Z}$) along L. Then T_0 is nullhomologous in M if and only if there is an integral vector $\mathbf{a}$ solving $\mathbf{l} = Q\mathbf{a}$ as above, in which case its self-linking number in (M, ξ) (with respect to the special Seifert class $\widehat{\Sigma}$ as before) is equal to

$$\mathrm{sl}_M(T_0, \widehat{\Sigma}) = \mathrm{sl}_{S^3}(T_0) - \sum_{i=1}^{k} a_i n_i \Big(l_i \mp \mathrm{rot}_{S^3}(L_i) \Big),$$

where the sign is $-$ when T_0 is positively transverse and $+$ when T_0 is negatively transverse.

Remark A.3.3
An oriented transverse knot T is either positively or negatively transverse. If we pick a Legendrian knot L such that T is a transverse push-off, we orient L accordingly. Then the class of an oriented Seifert surface of T is also the class of an oriented Seifert surface of L and vice-versa. With these orientations, T is a positive (negative) push-off of L if T is positively (negatively) transverse. In particular, the topological data used in the formula in Corollary A.3.2 coincides for the two knots.

Proof. Any transverse knot is a transverse push-off of a Legendrian knot (cf. the paragraph before Theorem 2.23 in [31]), so it is enough to consider those. Now for $L_0^{\pm}$ the positive or negative push-off of the Legendrian knot L_0 and Σ a Seifert surface we have

$$\mathrm{sl}(L_0^{\pm}, [\Sigma]) = \mathrm{tb}(L_0) \mp \mathrm{rot}(L_0, [\Sigma])$$

in any contact manifold (see Proposition 1.4.8). Hence,

$$\begin{aligned}
\mathrm{sl}_M(L_0^{\pm}, \widehat{\Sigma}) &= \mathrm{tb}_M(L_0) \mp \mathrm{rot}_M(L_0, \widehat{\Sigma}) \\
&= \Big(\mathrm{tb}_{S^3}(L_0) - \sum_{j=1}^{k} a_j n_j l_j \Big) \mp \Big(\mathrm{rot}_{S^3}(L_0) - \sum_{i=1}^{k} a_i n_i \, \mathrm{rot}_{S^3}(L_i) \Big) \\
&= \mathrm{sl}_{S^3}(L_0) - \sum_{i=1}^{k} a_i n_i \Big(l_i \mp \mathrm{rot}_{S^3}(L_i) \Big).
\end{aligned}$$

□

Remark A.3.4
A front-projection that contains Legendrian as well as transverse knots has four possible types of crossings between a Legendrian and a transverse knot (see Figure A.5). Depending on whether the transverse knot is positively or negatively transverse, two of the four types of crossings have a unique crossing behaviour determined by the contact condition, in the other cases both possibilities can occur.

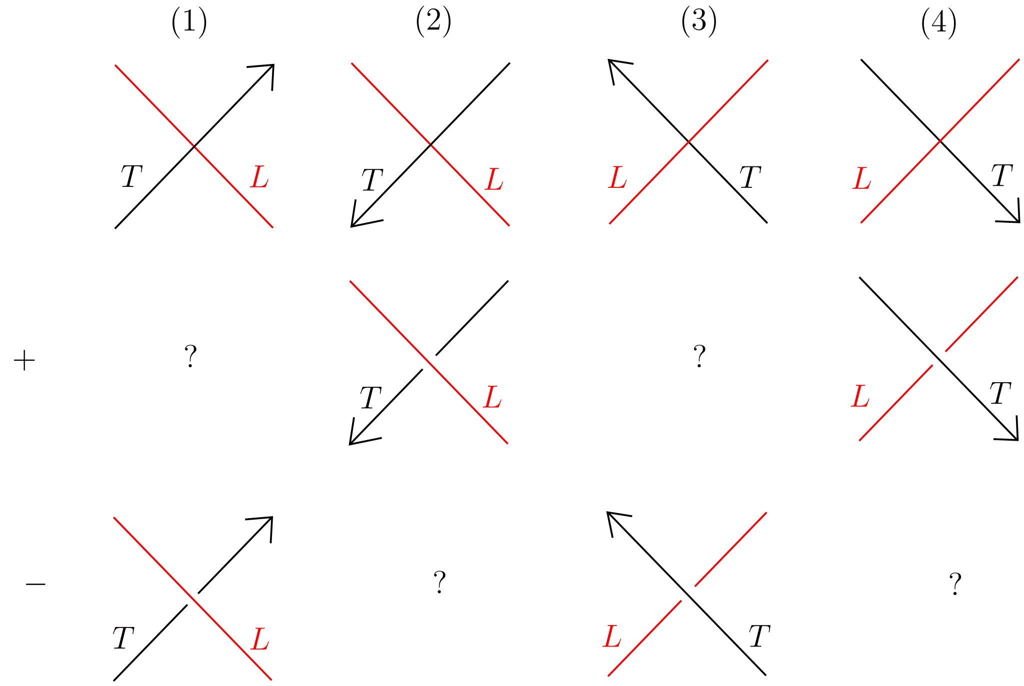


Figure A.5: Crossings between Legendrian and transverse knots. The transverse knots in the middle row are positive, the ones in the bottom row negative.

Example A.3.5

1. The left diagram in Figure A.6 shows a positive transverse knot T_0 in an overtwisted 3-sphere M. We have $\mathbf{l} = -1$, $Q = p = -1$ and thus $\mathbf{a} = 1$. The rotation number of L is 1, so we have

$$\mathrm{sl}_M(T_0) = \mathrm{sl}_{S^3}(T_0) - a_1 q_1 (l_1 - \mathrm{rot}_{S^3}(L)) = -1 - (-1 - 1) = 1.$$

   Therefore, T_0 violates the Bennequin-inequality in M, i.e. the contact structure is indeed overtwisted.
   Alternatively, we can consider a Legendrian unknot L_0 such that T_0 is its positive push-off, as shown on the right in Figure A.6. Its Thurston–Bennequin invariant in M is equal to $-1+1=0$ and its rotation number is $0-1=-1$, i.e. it bounds an overtwisted disc.

2. We can also consider T_0 as a negative transverse knot by reversing its orientation. Then $\mathbf{l} = 1$, $Q = p = -1$ and $\mathbf{a} = -1$, so

$$\mathrm{sl}_M(T_0) = \mathrm{sl}_{S^3}(T_0) - a_1 q_1 \Big(l_1 - \mathrm{rot}_{S^3}(L)\Big) = -1 - (1+1) = 1,$$

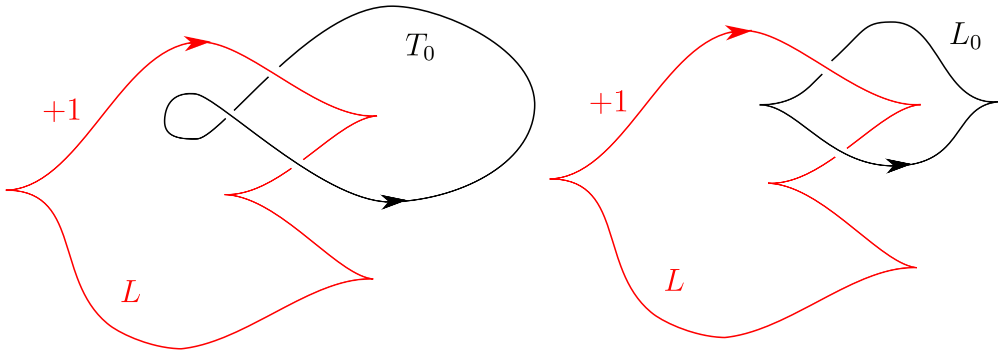


Figure A.6: Computing the self-linking number.

as expected, since the self-linking number is independent of the chosen orientation. We can again consider the corresponding Legendrian knot, which then has vanishing Thurston–Bennequin invariant and rotation number 1. As T_0 is now its negative push-off, we also get

$$\mathrm{sl}_M(T_0) = \mathrm{tb}_M + \mathrm{rot}_M = 1.$$

A.4 Rationally nullhomologous knots

The study of rationally nullhomologous knots in contact 3-manifolds has been proposed in Baker-Grigsby [5], Baker-Etnyre [4] and Geiges-Onaran [38]. In this section we generalise Theorem A.2.2 to rationally nullhomologous Legendrian knots and Corollary A.3.2 to rationally nullhomologous transverse knots. Let K be a knot in M. We call K **rationally nullhomologous** if its homology class is of finite order $d > 0$ in $H_1(M)$, i.e. it vanishes in $H_1(M;\mathbb{Q})$. Let νK be a tubular neighbourhood of K and denote the meridian by $\mu \subset \partial\nu K$.

Definition A.4.1
A *Seifert framing* of an oriented rationally nullhomologous knot K of order d is a class $r \in H_1(\partial\nu K)$ such that

- $\mu \bullet r = d$,
- $r = 0$ in $H_1(M \setminus \nu K)$.

A **rational Seifert surface** for an oriented rationally nullhomologous knot K is a surface with boundary in the complement of K whose boundary represents a Seifert framing of K.

It is obvious that every rationally nullhomologous knot has a Seifert framing. Moreover, the Seifert framing is unique (see Lemma 3.4.2).

Definition A.4.2
The **rational rotation number** of an oriented rationally nullhomologous Legendrian knot K of order d with respect to the rational Seifert surface Σ is equal to

$$\operatorname{rot}_{\mathbb{Q}}(K,\Sigma)=\frac{1}{d}\langle \mathrm{e}(\xi,K),[\Sigma]\rangle=\frac{1}{d}\operatorname{PD}(\mathrm{e}(\xi,K))\bullet[\Sigma],$$

where $\mathrm{e}(\xi,K)$ is the relative Euler class of the contact structure ξ relative to the knot K and $[\Sigma]$ the relative homology class represented by the surface Σ and the intersection is taken in $H_1(\partial\nu K)$.

Let $L_0\subset S^3\setminus L$ be an oriented knot in the complement of an oriented surgery link L. Using the notation from Section A.2, we see that L_0 is rationally nullhomologous of order d in $M=S^3_L(r)$ if and only if there is an integral solution $\mathbf{a}$ of the equation $d\mathbf{l}=Q\mathbf{a}$ and d is the minimal natural number for which a solution exists (see [49]).

Now assume that L and L_0 are Legendrian and L_0 is rationally nullhomologous of order d in M and fix Seifert surfaces $\Sigma_0,\dots,\Sigma_k$ for $L_0,\dots,L_k$ in S^3 such that intersections of surfaces and link components are transverse, as in the nullhomologous case. Again following [13], we want to construct a class of the form

$$\Sigma=d\Sigma_0+\sum_{i=1}^k k_i\Sigma_i$$

such that its image under the boundary homomorphism ∂ in the long exact sequence of the pair $(S^3\setminus(L_0\sqcup\nu L),\partial L_0\sqcup\partial\nu L)$ is a linear combination of the surgery slopes r_i and a rational longitude of L_0. Setting $k_i=-a_iq_i$, we obtain

$$\begin{aligned}\partial\colon\ \Sigma_j\longmapsto&\sum_{j=1}^k da_jq_jl_{0j}\mu_0+d\lambda_0-\sum_{i=1}^k a_iq_i\lambda_i-\sum_{i=1}^k dl_{0i}\mu_i+\sum_{i=1}^k\sum_{j\neq i}a_jq_jl_{ij}\mu_i\\=&\sum_{j=1}^k da_jq_jl_j\mu_0+d\lambda_0-\sum_{i=1}^k a_iq_i\lambda_i-\sum_{i=1}^k dl_i\mu_i+\sum_{i=1}^k(dl_i-a_ip_i)\mu_i\\=&\sum_{j=1}^k da_jq_jl_j\mu_0+d\lambda_0-\sum_{i=1}^k a_iq_i\lambda_i-\sum_{i=1}^k(a_ip_i)\mu_i.\end{aligned}$$

In complete analogy to the the nullhomologous case we then have

$$\begin{aligned}\operatorname{rot}_{\mathbb{Q},M}(L_0,\widehat{\Sigma})=&\frac{1}{d}\operatorname{PD}\Big(\mathrm{e}(\xi,L_0)\Big)\bullet\widehat{\Sigma}\\=&\operatorname{rot}_{S^3}(L_0)-\frac{1}{d}\sum_{i=1}^k a_in_i\operatorname{rot}_{S^3}(L_i).\end{aligned}$$

Thus, Theorem A.2.2 generalises as follows.

Theorem A.4.3
In the situation of Theorem A.2.2 the knot L_0 is rationally nullhomologous of order d in M if and only if there is an integral vector $\mathbf{a}$ *solving* $d\mathbf{l} = Q\mathbf{a}$ *as above with d the minimal natural number for which a solution exists, in which case its rational rotation number in* (M, ξ) *with respect to a special (rational) Seifert class* $\widehat{\Sigma}$ *is equal to*

$$\operatorname{rot}_{\mathbb{Q},M}(L_0, \widehat{\Sigma}) = \operatorname{rot}_{S^3}(L_0) - \frac{1}{d}\sum_{i=1}^{k} a_i n_i \operatorname{rot}_{S^3}(L_i).$$

The definition of the self-linking number of a transverse knot generalises to the setting of rationally nullhomologous knots by choosing a rational Seifert surface. Furthermore, the rational invariants of a Legendrian and the rational self-linking of a transverse push-off are, as in the nullhomologous case, related by

$$\operatorname{sl}_{\mathbb{Q}}(L_0^{\pm}, [\Sigma]) = \operatorname{tb}_{\mathbb{Q}}(L_0) \mp \operatorname{rot}_{\mathbb{Q}}(L_0, [\Sigma])$$

(see Lemma 1.2 in [4]). Hence, we have the following corollary.

Corollary A.4.4
In the situation of Corollary A.3.2 the knot T_0 is rationally nullhomologous of order d in M if and only if there is an integral vector $\mathbf{a}$ *solving* $d\mathbf{l} = Q\mathbf{a}$ *as above, in which case its rational self-linking number in* (M, ξ) *with respect to a special (rational) Seifert class* $\widehat{\Sigma}$ *is equal to*

$$\operatorname{sl}_{\mathbb{Q},M}(T_0, \widehat{\Sigma}) = \operatorname{sl}_{S^3}(T_0) - \frac{1}{d}\sum_{i=1}^{k} a_i n_i (l_i \mp \operatorname{rot}_{S^3}(L_i)).$$

Remark A.4.5
Observe that the formulas for rationally nullhomologous knots coincide with the ones for nullhomologous knots presented in previous sections if one allows rational coefficients.

A.5 The d_3-invariant in surgery diagrams

The so-called d_3-invariant is a homotopical invariant of a tangential 2-plane field on a 3-manifold, which is defined if the Euler class (or first Chern class) of the 2-plane field is torsion, see [45, Definition 11.3.3]. Many contact structures can be distinguished by computing the d_3-invariants of the underlying topological 2-plane fields. In [17, Corollary 3.6] Ding, Geiges and Stipsicz present a formula to compute first the Euler class and then the d_3-invariant of a contact structure given by a (± 1)-contact surgery diagram building up on the work of Gompf [44]. Both invariants are closely related to the rotation number of the surgery links.

By expressing an arbitrary $(1/n)$-contact surgery diagram as a (± 1)-contact surgery diagram and then using the result of Ding–Geiges–Stipsicz we obtain a similar result for arbitrary $(1/n)$-contact surgery diagrams, which often simplifies computations a lot.

First we recall some results from [17]: For $L = L_1 \sqcup \ldots \sqcup L_k$ an oriented Legendrian link in (S^3, ξ_{st}) and (M, ξ) the contact manifold obtained from S^3 by contact (± 1)-surgeries along L, the Poicaré-dual of the Euler class is given by

$$\mathrm{PD}\left(\mathrm{e}(\xi)\right) = \sum_{i=1}^{k} \mathrm{rot}_i \, \mu_i \in H_1(M).$$

The meridians μ_i generate the first homology $H_1(M)$ and the relations are given by $Q\mu = 0$. Observe that he generalized linking matrix Q coincides with the ordinary linking matrix, since we only have integer surgeries here. Then $\mathrm{e}(\xi)$ is torsion if and only if there exists a rational solution $\mathbf{b} \in \mathbb{Q}^k$ of $Q\mathbf{b} = \mathbf{rot}$. If this is the case, then the d_3-invariant computes as

$$d_3 = \frac{1}{4}\Big(\langle \mathbf{b}, \mathrm{rot}\rangle - 3\sigma(Q) - 2k\Big) - \frac{1}{2} + q,$$

where $\sigma(Q)$ denotes the signature of Q (i.e. the number of positive eigenvalues minus the number of negative ones) and q is the number of Legendrian knots in L with $(+1)$-contact surgery coefficient.

With the help of these results we can now state and prove a corresponding theorem for arbitrary $(1/n)$-contact surgeries.

Theorem A.5.1
Let $L = L_1 \sqcup \ldots \sqcup L_k$ be an oriented Legendrian link in (S^3, ξ_{st}) and denote by (M, ξ) the contact manifold obtained from S^3 by contact $(\pm 1/n_i)$-surgeries along L $(n_i \in \mathbb{N})$.

1. *The Poicaré-dual of the Euler class is given by*

$$\mathrm{PD}\left(\mathrm{e}(\xi)\right) = \sum_{i=1}^{k} n_i \, \mathrm{rot}_i \, \mu_i \in H_1(M).$$

   The first homology group $H_1(M)$ of M is generated by the meridians μ_i and the relations are given by the generalized linking matrix $Q\mu = 0$.

2. *The Euler class $\mathrm{e}(\xi)$ is torsion if and only if there exists a rational solution $\mathbf{b} \in \mathbb{Q}^k$ of $Q\mathbf{b} = \mathbf{rot}$. In this case, the d_3-invariant computes as*

$$d_3 = \frac{1}{4}\left(\sum_{i=1}^{k} n_i b_i \, \mathrm{rot}_i + (3 - n_i)\, \mathrm{sign}_i\right) - \frac{3}{4}\sigma(Q) - \frac{1}{2},$$

   where sign_i denotes the sign of the contact surgery coefficient of L_i.

Remark A.5.2
In the proof we will show that all eigenvalues of Q are real. Therefore, it makes sense to speak of the signature, even if Q is not symmetric.

Proof. The replacement lemma of Ding and Geiges [16, Proposition 8] states that a contact $(\pm 1/n)$-surgery along a Legendrian knot L is equivalent to n contact (± 1)-surgeries along Legendrian push-offs of L. Using this, we translate the contact $(\pm 1/n_i)$-surgeries along L to contact (± 1)-surgeries along a new Legendrian link L' and compute the invariants there.

Denote by L_i^j $(j = 1, \dots, n_i)$ the Legendrian push-offs of L_i in the new Legendrian link L'. Write μ_i for the meridian of L_i $(i = 1, \dots, k)$ and μ_i^j for the meridian of L_i^j $(i = 1, \dots, k,\ j = 1, \dots, n_i)$. We now have two surgery descriptions of the manifold M – one in terms of L and one in terms of L' – and hence two presentations of its first homology:

$$\begin{aligned} H_1(M) =& \langle \mu_i | Q\mu = 0 \rangle \text{ for the surgery presentation along } L, \\ H_1(M) =& \langle \mu_i^j | Q'\mu' = 0 \rangle \text{ for the surgery presentation along } L'. \end{aligned}$$

An isomorphism between these two presentations is given by $\mu_i^j \mapsto \mu_i$ for all i, j, and hence, as $\mathrm{rot}_i^j = \mathrm{rot}_i$,

$$\mathrm{PD}\left(\mathrm{e}(\xi)\right) = \sum_{i=1}^{k} \sum_{j=1}^{n_j} \mathrm{rot}_i^j\, \mu_i^j \longmapsto \sum_{i=1}^{k} n_i \,\mathrm{rot}_i\, \mu_i.$$

The numbers k and q compute easily as

$$k = \sum_{i=1}^{k} n_i, \qquad\qquad q = \sum_{i=1}^{k} \frac{1}{2}\left(1 + \mathrm{sign}_i\right) n_i.$$

For reasons of readability we will assume $k = 3$ in the following. The general case works exactly the same. Write $\mathbf{1}_n$ for the vector $(1, \dots, 1)^T \in \mathbb{Q}^n$.

Let $\mathbf{b} \in \mathbb{Q}^3$ a solution of $Q\mathbf{b} = \mathbf{rot}$, i.e

$$Q\mathbf{b} = \begin{pmatrix} \pm 1 + n_1 \,\mathrm{tb}_1 & n_2 l_{12} & n_3 l_{13} \\ n_1 l_{12} & \pm 1 + n_2 \,\mathrm{tb}_2 & n_3 l_{23} \\ n_1 l_{13} & n_2 l_{23} & \pm 1 + n_3 \,\mathrm{tb}_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \mathrm{rot}_1 \\ \mathrm{rot}_2 \\ \mathrm{rot}_3 \end{pmatrix} = \mathbf{rot}$$

Then for $\mathbf{b}' := (b_1, \dots, b_1, b_2, \dots, b_2, b_3, \dots, b_3)^T \in \mathbb{Q}^{n_1+n_2+n_3}$ we have

$$\begin{aligned} Q'\mathbf{b}' &= \begin{pmatrix} \pm E_{n_1} + \mathrm{tb}_1 \,\mathbf{1}_{n_1} \mathbf{1}_{n_1}^T & l_{12} \mathbf{1}_{n_1} \mathbf{1}_{n_2}^T & l_{13} \mathbf{1}_{n_2} \mathbf{1}_{n_3}^T \\ l_{12} \mathbf{1}_{n_2} \mathbf{1}_{n_1}^T & \pm E_{n_1} + \mathrm{tb}_2 \,\mathbf{1}_{n_2} \mathbf{1}_{n_2}^T & l_{23} \mathbf{1}_{n_2} \mathbf{1}_{n_3}^T \\ l_{13} \mathbf{1}_{n_3} \mathbf{1}_{n_1}^T & l_{23} \mathbf{1}_{n_2} \mathbf{1}_{n_3}^T & \pm E_{n_3} + \mathrm{tb}_3 \,\mathbf{1}_{n_3} \mathbf{1}_{n_3}^T \end{pmatrix} \mathbf{b}' \\ &= \begin{pmatrix} \mathrm{rot}_1 \,\mathbf{1}_{n_1} \\ \mathrm{rot}_2 \,\mathbf{1}_{n_2} \\ \mathrm{rot}_3 \,\mathbf{1}_{n_3} \end{pmatrix} = \mathbf{rot}' \end{aligned}$$

(conversely, every solution of $Q'\mathbf{b}' = \mathbf{rot}'$ is of this form and thus yields a solution of $Q\mathbf{b} = \mathbf{rot}$). And therefore,

$$\langle \mathbf{b}', \mathbf{rot}' \rangle = \sum_{i=1}^{3} n_i b_i \operatorname{rot}_i .$$

It remains to compute the signature $\sigma(Q')$ out of $\sigma(Q)$. Let λ be an eigenvalue of Q with eigenvector $\mathbf{v}$. Similar as above, one computes

$$Q'\mathbf{v}' = \lambda \mathbf{v}'$$

for $\mathbf{v}' := (v_1, \dots, v_1, v_2, \dots, v_2, v_3, \dots, v_3)^T \in \mathbb{Q}^{n_1+n_2+n_3}$. Thus, every eigenvalue of Q is also an eigenvalue of Q'. In particular, all eigenvalues of Q are real. Now we only have to find the other $\sum_{i=1}^{3}(n_i - 1)$ eigenvectors of Q'. To that end, consider the vector $\mathbf{v}_1 \in \mathbf{1}_{n_1}^{\perp}$ and write $\mathbf{v}_1' := (\mathbf{v}_1, 0, \dots, 0, 0, \dots, 0)^T \in \mathbb{Q}^{n_1+n_2+n_3}$. Then, as before, one computes

$$Q'\mathbf{v}_1' = \operatorname{sign}_1 \mathbf{v}_1'.$$

An analogue equation holds for all i if we define $\mathbf{v}_i$ and $\mathbf{v}_i'$ accordingly. Therefore we have

$$\sigma(Q') = \sigma(Q) + \sum_{i=1}^{k} (n_i - 1) \operatorname{sign}_i .$$

□

Example A.5.3

Consider a contact $(1/n)$-surgery ($n \in \mathbb{Z}$) along a Legendrian unknot with $\operatorname{tb} = -1$ and $\operatorname{rot} = 0$. Then the Euler class is zero because the rotation number vanishes. Hence, the d_3-invariant is defined. For $n = 1$, the signature of Q vanishes. If $n \neq 1$, the signature of Q equals -1. Thus we have

$$d_3 = \begin{cases} \frac{n}{4} - \frac{1}{2}, & n < 1, \\ 0, & n = 1, \\ 1 - \frac{n}{4}, & n > 1. \end{cases}$$

B

Homology of a knot complement

For the sake of completeness, we will compute the first homology of the complement of a nullhomologous knot in a 3-manifold.

Lemma B.0.1
Let K be a nullhomologous knot in a 3-manifold M. Then

$$H_1(M \setminus K) \cong H_1(M) \oplus \mathbb{Z},$$

where the $\mathbb{Z}$-summand is generated by a meridian of K.

Proof. Consider the Mayer-Vietoris sequence of the decomposition of the manifold $M = (M \setminus \nu K) \cup \nu K$. Observe that

$$H_1(\partial \nu K) \cong H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z},$$

generated by a meridian μ and any longitude λ of K. Furthermore, we have

$$H_1(\nu K) \cong H_1(K) \cong \mathbb{Z}.$$

Since we can assume M to be connected we get a long exact sequence

$$\cdots \to H_2(M) \xrightarrow{\Delta} \mathbb{Z}_\mu \oplus \mathbb{Z}_\lambda \xrightarrow{\alpha} H_1(M \setminus K) \oplus \mathbb{Z} \to H_1(M) \to 0.$$

We can represent any generator f of $H_2(M)$ by an embedded surface F transverse to K. The boundary homomorphism Δ thus takes f to a multiple of the meridian:

$$\Delta(f) = [F \cap \partial \nu K] = k_f \mu.$$

Hence, the image of Δ is of the form $k\mu \oplus 0$ for some $k \in \mathbb{Z}$. By exactness $(k\mu, 0)$ is in the kernel of α. But K is nullhomologous, so it bounds a surface Σ. The meridian μ intersects Σ in exactly one point, i.e. it cannot be a torsion element in $H_1(M \setminus K)$. Therefore $k = 0$ and we get a short exact sequence

$$0 \to \mathbb{Z}_\mu \oplus \mathbb{Z}_\lambda \xrightarrow{\alpha} H_1(M \setminus K) \oplus \mathbb{Z} \to H_1(M) \to 0.$$

Furthermore, we can choose the longitude λ to be the Seifert longitude. We then have $\alpha(\lambda) = (0, 1)$ and as μ is trivial in $H_1(\nu K)$, we can reduce the sequence to

$$0 \to \mathbb{Z}_\mu \to H_1(M \setminus K) \to H_1(M) \to 0.$$

The claim follows because the sequence splits. Indeed, the intersection pairing with the class of the Seifert surface yields the desired homomorphism from $H_1(M \setminus K)$ to $\mathbb{Z}_\mu$. □

C

Generalised 1-handles

We will briefly introduce generalised 1-handles in the sense of [52] and their attachment. Recall that an ordinary n-dimensional 1-handle is of the form $D^1 \times D^{n-1}$, and note that we may understand the $(n-1)$-dimensional disc as a thickened disc of dimension $n-2$. We are now simply going to replace this disc by an appropriate manifold with boundary.

Let Σ be an $(n-2)$-dimensional manifold with boundary. An **n-dimensional generalised 1-handle** is given by

$$H_\Sigma := D^1 \times (\Sigma \times D^1).$$

Its boundary decomposes into two parts, namely

$$\partial_- H_\Sigma = S^0 \times (\Sigma \times D^1) \quad \text{and} \quad \partial_+ H_\Sigma = D^1 \times \partial(\Sigma \times D^1).$$

In analogy to a regular 1-handle we define the **core** of H_Σ to be $D^1 \times (\Sigma \times \{0\})$ and its **co-core** to be $\{0\} \times (\Sigma \times D^1)$. We **attach** a generalised 1-handle H_Σ to an n-dimensional manifold M with boundary ∂M via an embedding $f\colon \partial_- H_k \to \partial M$ to obtain a manifold $M' = M \cup_f H_\Sigma$. Corners are understood to be smoothened (see Definition und Notiz (13.12) in [10]).

Attaching generalised handles with isotopic maps f_0 and f_1 results in diffeomorphic manifolds $M_0 = M \cup_{f_0} H_\Sigma$ and $M_1 = M \cup_{f_1} H_\Sigma$. An isotopy of the attaching maps defines a time-dependent vector field on its image that can be extended to the whole manifold. The corresponding time-1 map is the desired diffeomorphism. Observe that to define a generalised handle attachment it is sufficient to specify the image of the boundary $S^0 \times \Sigma \times \{0\}$ of the core of the handle under the attaching map.

It is worth noting that, in case Σ can be endowed with an appropriate exact symplectic form, the above construction can be adapted to the contact setting, and naturally extends the symplectic handle constructions due to Eliashberg [25] and Weinstein [74].

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