Robustness of Interdependent Random Geometric Networks

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***Abstract*—We propose an interdependent random geometric graph (RGG) model for interdependent networks. Based on this model, we study the robustness of two interdependent spatially embedded networks where interdependence exists between geographically nearby nodes in the two networks. We study the emergence of the giant mutual component in two interdependent RGGs as node densities increase, and define the percolation threshold as a pair of node densities above which the mutual giant component first appears. In contrast to the case for a single RGG, where the percolation threshold is a unique scalar for a given connection distance, for two interdependent RGGs, multiple pairs of percolation thresholds may exist, given that a smaller node density in one RGG may increase the minimum node density in the other RGG in order for a giant mutual component to exist. We derive analytical upper bounds on the percolation thresholds of two interdependent RGGs by discretization, and obtain 99% confidence intervals for the percolation thresholds by simulation. Based on these results, we derive conditions for the interdependent RGGs to be robust under random failures and geographical attacks.**

I. Introduction

Cyber-physical systems such as smart power grids and smart transportation networks are being deployed towards the design of smart cities. The integration of communication networks and physical networks facilitates network operation and control. In these integrated networks, one network depends on another for information, power, or other supplies in order to properly operate, leading to interdependence. For example, in smart grids, communication networks rely on the electric power from power grids, and simultaneously control power generators [1]. Failures in one network may cascade to another network, which potentially makes the interdependent networks vulnerable to failures and attacks.

Cascading failures in interdependent networks have been extensively studied in the statistical physics literature since the seminal work in [2], where each of the two interdependent networks is modeled as a random graph. Nodes in the two random graphs are interdependent, and a node is functioning if both itself and its interdependent node are in the giant components of the respective random graphs. After initial node failures in the first graph, their interdependent nodes in the second graph fail. Thus, a connected component in the second graph may become disconnected, and the failures of the disconnected nodes cascade back to (their interdependent) nodes in the first graph. As a result of the cascading failures, removing a small fraction of nodes in the first random graph destroys the giant components of both graphs.

To model spatially embedded networks, an interdependent lattice model was studied in [3]. Under this model, geographical attacks may cause significantly more severe cascading failures than random attacks. Removing nodes in a finite region (*i.e.*, a zero fraction of nodes) may destroy the infinite clusters in both lattices [4].

If every node in one network is interdependent with multiple nodes in the other network, and a node is content to have at least one supply node, failures are less likely to cascade [5], [6]. Although the one-to-multiple interdependence exists in real-world spatially embedded interdependent networks (*e.g.*, a control center can be supported by the electric power generated by more than one power generator), it has not been previously studied using spatial graph models.

We use a random geometric graph (RGG) to model each of the two interdependent networks. The two RGGs are allowed to have different connection distances and densities, which can represent two networks that have different average link lengths and scales. These differences between the two networks were not captured in the lattice model studied in the previous literature. Moreover, the interdependent RGG model is able to capture the one-to-multiple interdependence in spatially embedded networks, and provides a more general framework for studying interdependent networks.

Robustness is a key design objective for interdependent networks. We study the conditions under which a positive fraction of nodes are functioning in interdependent RGGs as the number of nodes approaches infinity. In this case, the interdependent RGGs *percolate*. Consistent with previous research [2], [3], [5], the robustness of interdependent RGGs under random failures and geographical attacks is measured by whether percolation exists after the failures and attacks. To the best of our knowledge, our paper is the first to study the percolation of interdependent spatial network models using a mathematically rigorous approach.

The main contributions of this paper are as follows.

1) We propose an interdependent RGG model for two interdependent networks, which captures the differences in the scales of the two networks as well as the one-to-multiple interdependence in spatially embedded networks.

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2) We derive the first analytical upper bounds on the percolation thresholds of the interdependent RGGs, above which a positive fraction of nodes are functioning.
3) We obtain 99% confidence intervals for the percolation thresholds, by mapping the percolation of interdependent RGGs to the percolation of a square lattice where the probability that a bond in the lattice is open is evaluated by simulation.
4) We characterize sufficient conditions for the interdependent RGGs to percolate under random failures and geographical attacks. In particular, if the node densities are above any upper bound on the percolation threshold obtained in this paper, the interdependent RGGs remain percolated after a geographical attack.

The rest of the paper is organized as follows. We state the model and preliminaries in Section II. We derive analytical upper bounds on percolation thresholds in Section III, and obtain confidence intervals for percolation thresholds in Section IV. In Section V, we study the robustness of interdependent RGGs under random failures and geographical attacks. Section VI concludes the paper.

II. Model

A. Preliminaries on RGG and percolation

An RGG in a two-dimensional square consists of nodes generated by a Poisson point process and links connecting nodes within a given connection distance [7]. Let $G(\lambda, d, a^2)$ denote an RGG with node density λ and connection distance d in an $a \times a$ square. The studies on RGG focus on the regime where the expected number of nodes $n = \lambda a^2$ is large. We first present some preliminaries useful for developing our model. The *giant component* of an RGG is a connected component that contains $\Theta(n)$ nodes. A node belongs to the giant component with a positive probability $\Theta(n)/n$ if the giant component exists. For a given connection distance, the *percolation threshold* is a node density above which a node belongs to the giant component with a positive probability (*i.e.*, a giant component exists) and below which the probability is zero (*i.e.*, no giant component exists). By scaling, if the percolation threshold is λ^* under connection distance d, then the percolation threshold is $\lambda^* c^2$ under connection distance d/c. Therefore, without loss of generality, in this paper, we study the percolation thresholds represented by node densities, for given connection distances.

The RGG is closely related to the *Poisson boolean model* [8], where nodes are generated by a Poisson point process on an *infinite plane*. Let $G(\lambda, d)$ denote a Poisson boolean model with node density λ and connection distance d. The difference between $G(\lambda, d)$ and $G(\lambda, d, a^2)$ is that the number of nodes in $G(\lambda, d)$ is infinite while the expected number of nodes in $G(\lambda, d, a^2)$ is large but finite. The Poisson boolean model can be viewed as a limit of the RGG as the number of nodes approaches infinity. The percolation threshold of $G(\lambda, d)$ under a given d is defined as the node density above which a node belongs to the *infinite cluster* with positive probability and below which the probability is zero. It has been shown that a node belongs to the infinite cluster with positive probability if and only if an infinite cluster exists, and thus the percolation of $G(\lambda, d)$ can be equivalently defined as the existence of the infinite cluster [8]. Moreover, the percolation threshold of $G(\lambda, d)$ is identical with the percolation threshold of $G(\lambda, d, a^2)$ [7], [9].

B. Interdependent RGGs

Two interdependent networks are modeled by two RGGs $G_1(\lambda_1, d_1, a^2)$ and $G_2(\lambda_2, d_2, a^2)$ on the *same* $a \times a$ square. A node in one graph is interdependent with *all* the nodes in the other graph within the *interdependent distance* d_{dep}. See Fig. 1 for an illustration. Nodes in one graph are *supply nodes* for nodes in the other graph within d_{dep}. The physical interpretation of supply can be either electric power or information that is essential for proper operation.

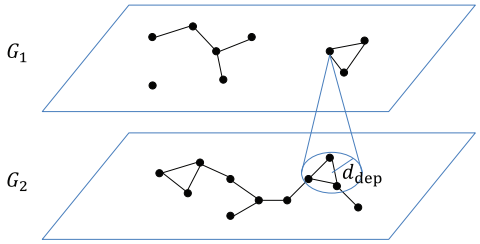

Fig. 1. Two interdependent RGGs with interdependent distance d_{dep}.

We define mutual component and giant mutual component in interdependent RGGs, in the same way as one defines the connected component and giant component in a single RGG.

Definition 1. Let V_i^0 denote nodes in a connected component in $G_i(\lambda_i, d_i, a^2)$, $\forall i \in \{1, 2\}$. If each node in $V_i \subseteq V_i^0$ has at least one supply node in $V_j \subseteq V_j^0$ within d_{dep}, $\forall i, j \in \{1, 2\}, i \neq j$, then nodes V_1 and V_2 form a *mutual component* of the interdependent RGGs.

If, in addition, V_i contains $\Theta(n_i)$ nodes, where $n_i = \lambda_i a^2$, $\forall i \in \{1, 2\}$, then V_1 and V_2 form a *giant mutual component*.

A mutual component can be viewed as an autonomous system in the sense that nodes in a mutual component have supply nodes in the same mutual component, and in each graph, nodes that belong to a mutual component are connected regardless of the existence of nodes outside the mutual component. Note that a node can receive supply from any of its supply nodes in the same mutual component. Nodes in a giant mutual component are *functioning*, since they are connected to a large number of nodes in the network. This definition of functioning is consistent with previous research on interdependent networks based on random graph models [2].

For a fixed d_{dep}, if a giant mutual component exists in interdependent RGGs $G_1(\lambda_1, d_1, a^2)$ and $G_2(\lambda_2, d_2, a^2)$, then a giant mutual component exists in interdependent RGGs $G_1'(\lambda_1', d_1, a^2)$ and $G_2(\lambda_2, d_2, a^2)$, where $\lambda_1' > \lambda_1$. This can be explained by coupling G_1' with G_1 as follows. By randomly removing each node in G_1' independently with probability

$1-\lambda_1/\lambda_1'$, the remaining nodes in G_1' has density λ_1, and a giant mutual component exists in the interdependent RGGs that consist of G_2 and the RGG formed by the remaining nodes in G_1'. Since adding nodes to a graph does not disconnect any mutual component, a giant mutual component exists in the interdependent RGGs G_1' and G_2. By the same analysis, a giant mutual component also exists in interdependent RGGs $G_1(\lambda_1, d_1, a^2)$ and $G_2'(\lambda_2', d_2, a^2)$ for a fixed d_{dep}, if $\lambda_2' > \lambda_2$.

We define a percolation threshold of two interdependent RGGs as follows.

Definition 2. A pair of node densities $(\lambda_1^*, \lambda_2^*)$ is a *percolation threshold* of two interdependent RGGs, given connection distances d_1, d_2 and the interdependent distance d_{dep}, if a giant mutual component exists in $G_1(\lambda_1, d_1, a^2)$ and $G_2(\lambda_2, d_2, a^2)$ for $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, and no giant mutual component exists otherwise.

For fixed d_1, d_2 and d_{dep}, there may exist multiple percolation thresholds. Intuitively, the larger the node density is in one graph, the smaller the required node density is in the other graph in order for the giant mutual component to exist. This is in contrast with the situation for a single RGG where there is a unique percolation threshold for a fixed d.

There is a non-trivial phase transition in the interdependent RGGs. If λ_i is smaller than the percolation threshold of a single RGG $G_i(\lambda_i, d_i, a^2)$, clearly there does not exist a giant mutual component in the interdependent RGGs. Therefore, $\lambda_i^* > 0$, $\forall i \in \{1, 2\}$. As we will see in the next section, there exist percolation thresholds $\lambda_i^* < \infty$, $\forall i \in \{1, 2\}$, which concludes the non-trivial phase transition.

Given that the conditions for the percolation of $G_i(\lambda_i, d_i, a^2)$ and $G_i(\lambda_i, d_i)$ are the same, in most parts of the paper we study the percolation of two interdependent Poisson boolean models on the same infinite plane, $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$, by applying techniques in continuum percolation. The percolation of G_{IntDep} is defined as the existence of an *infinite mutual cluster*, which consists of an infinite number of connected nodes in both $G_1(\lambda_1, d_1)$ and $G_2(\lambda_2, d_2)$ where every node has at least one supply node in the same infinite mutual cluster. In the rest of the paper we sometimes use G_i to denote both $G_i(\lambda_i, d_i, a^2)$ and $G_i(\lambda_i, d_i)$. The model that it refers to will be clear from the context.

C. Related work

The model which is closest to ours is the interdependent lattice model, first proposed in [10] and further studied in [3], [4]. In this model, nodes in a network are represented by the open *sites* (nodes) of a square lattice, where every site is open independently with probability p. Network links are represented by the *bonds* (edges) between adjacent open sites. Every node in one lattice is interdependent with *one* randomly chosen node within distance r_d in the other lattice. The percolation threshold of the interdependent lattice model is characterized as a function of r_d, assuming the same p in both lattices [10]. Percolation of the model where some nodes do not need to have supply nodes was studied in [3]. The analysis relies on quantities estimated by simulation and extrapolation, such as the fraction of nodes in the infinite cluster of a lattice for any fixed p, which cannot be computed rigorously. In contrast, we study the percolation of the interdependent RGG model using a mathematically rigorous approach.

III. Analytical upper bounds on percolation thresholds

In interdependent RGGs, nodes in the giant mutual component are viewed as functioning while all the other nodes are not. Thus, a node is functioning only if it is in the giant component of its own graph, and it depends on at least one node in the giant component of the other graph. For any node b_1 in G_1, although the number of nodes in G_2 within distance d_{dep} from b_1 follows a Poisson distribution, the number of functioning nodes is hard to calculate, since the fraction of nodes in the giant component of G_2 is unknown. Moreover, the nodes in the giant component of G_2 are clustered, and thus the thinning of the nodes in G_1 due to a lack of supply nodes in G_2 is difficult to characterize. To overcome these difficulties, we consider the percolation of two RGGs jointly, instead of studying the percolation of one RGG with reduced node density due to a lack of functioning supply nodes.

We now give an overview of our approach. We develop mapping techniques (discretizations) to characterize the percolation of G_{IntDep} by the percolation of a discrete model. Mappings from a model whose percolation threshold is unknown to a model with known percolation threshold are commonly employed in the study of continuum percolation. For example, one can study the percolation threshold of the Poisson boolean model $G(\lambda, d)$ by mapping it to a triangle lattice and relating the state of a site in the triangle lattice to the point process of $G(\lambda, d)$. By the mapping, the percolation of the triangle lattice implies the percolation of $G(\lambda, d)$. Consequently, an upper bound on the percolation threshold of $G(\lambda, d)$ is given by λ for which the triangle lattice percolates, a known quantity [11], [8]. In general, more than one mapping can be applied, and the key is to search for a mapping that gives a good (smaller) upper bound. Following this idea, we propose different mappings that fit different conditions to obtain upper bounds on the percolation thresholds of G_{IntDep}.

A. Ratio d_2/d_1 is small

Given $G_{\text{IntDep}} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{\text{dep}})$, without loss of generality we assume that $d_1 \leq d_2$. Moreover, we assume that $d_{\text{dep}} \geq \max(d_1/2, d_2/2) = d_2/2$ (see the remark at the end of the section for comments on this assumption). Let $c = \lfloor d_2/d_1 \rfloor = \max\{c : d_2/d_1 \geq c, c \in \mathbf{N}\}$. For small c, we study the percolation of G_{IntDep} by mapping it to an independent bond percolation of a square lattice, and prove the following result.

Theorem 1. *If* (λ_1, λ_2) *satisfies*

$$(1 - e^{-\lambda_1 d_1^2/8})^c (1 - e^{-\lambda_2 c^2 d_1^2/8}) > 1/2,$$

then $G_{IntDep} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{dep})$ *percolates, where* $c = \lfloor d_2/d_1 \rfloor$, $d_1 \le d_2$, *and* $d_{dep} \ge d_2/2$.

Proof. We first construct a square lattice as follows. Partition the plane into small squares of side length $s = d_1/2\sqrt{2}$. A large square consists of $c \times c$ small squares and has side length cs. The *diagonals* of the large squares form the bonds of a square lattice L, illustrated by the thick line segments in Fig. 2.

The state of a bond in L is determined by the point process of G_{IntDep} in the large square that contains the bond. A bond (v_1, v_2) is open if the following conditions are both satisfied.

1) There is at least one node from G_1 in each of the two small squares that contain the ends (v_1 and v_2) of the bond, and they are connected through nodes from G_1, *all within the large square of side length* cs.
2) There is at least one node from G_2 in the large square that contains the bond.

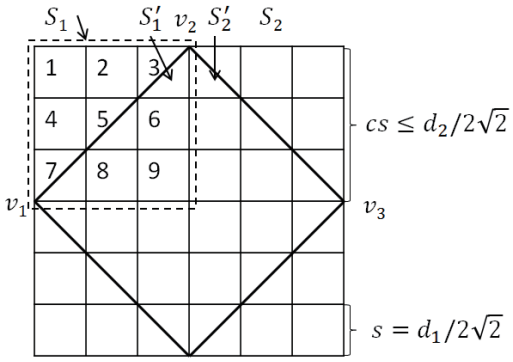

Fig. 2. Mapping to a square lattice for $c = 3$.

The first condition is satisfied if there exists a sequence of adjacent small squares, each of which contains at least one node in G_1, from the small square that contains v_1 to the small square that contains v_2. (Each small square is *adjacent* to its eight immediate neighbors.) In the example of Fig. 2, these sequences include 3-5-7, 3-2-4-7, and 3-6-8-7.

To obtain a closed-form formula, instead of computing the exact probability, we compute a lower bound on the probability that the first condition is satisfied. The probability is lower bounded by the probability that the c small squares that intersect the bond each contain at least one node from G_1, given by

$$p_1 \ge (1 - e^{-\lambda_1 d_1^2/8})^c.$$

The probability that the second condition is satisfied is

$$p_2 = 1 - e^{-\lambda_2 c^2 d_1^2/8}.$$

Given that the two Poisson point processes in G_1 and G_2 are independent, the probability that a bond is open is $p_1 p_2$.

It remains to prove that the percolation of L implies the percolation of G_{IntDep}. Consider two adjacent open bonds $(v_1, v_2), (v_2, v_3)$ in L. Let S_1 and S_2 denote the two adjacent large squares of side length cs that contain the two open bonds. Let S_1' and S_2' denote two adjacent small squares of side length s that contains v_2, within S_1 and S_2, respectively. See Fig. 2 for an illustration. Since $(v_1, v_2), (v_2, v_3)$ are open, under the second condition, nodes of G_2 exist in S_1 and S_2 and they are connected, because they are within distance $2\sqrt{2}cs \le d_2$. Under the first condition, nodes of G_1 form a connected path from the small square (within S_1, marked as 7 in Fig. 2) containing v_1 to S_1', and another path from the small square (within S_2) containing v_3 to S_2'. Moreover, the two paths are joined, because any pair of nodes in S_1' and S_2' are within distance $2\sqrt{2}s = d_1$. Given that any pair of nodes within a large square have distance at most $\sqrt{2}cs \le d_2/2 \le d_{\text{dep}}$, all the nodes have at least one supply node inside the large square that contains an open bond. To conclude, if the open bonds in L form an infinite cluster, then the nodes in G_{IntDep} form an infinite mutual cluster.

The event that a bond is open depends on the point processes in the large square that contains the bond, and is independent of whether any other bonds are open. As long as the probability that a bond is open, $p_1 p_2$, is larger than $1/2$, which is the threshold for independent bond percolation in a standard square lattice [12], G_{IntDep} percolate. □

The bound can be made tighter for any given $c = \lfloor d_2/d_1 \rfloor$, by computing more precisely the probability that the first condition is satisfied. We provide an example to illustrate the computation of an improved upper bound.

Example: Consider an example where $d_1 = 1, d_2 = 2d_{\text{dep}} = 3$. The probability that there is at least one node from G_2 in the large square of side length $3/2\sqrt{2}$ is $p_2 = 1 - e^{-9\lambda_2/8}$.

The probability that a small square contains at least one node from G_1 is $p_s = 1 - e^{-\lambda_1/8}$. The probability that the first condition is satisfied is

$$p_1 = p_s^3 + (1 - p_s)p_s^4 + (1 - p_s)p_s^4 - (1 - p_s)p_s^6, \quad (1)$$

obtained by considering all the sequences of adjacent small squares. Since p_1 computed by Eq. (1) is larger than p_s^3 for any fixed p_s, the bound on λ_2 is smaller for any fixed λ_1.

B. Ratio d_2/d_1 is large

In the mapping from G_{IntDep} to the square lattice L, the condition for a bond to be open becomes overly restrictive as d_2/d_1 increases. A path joining the two large squares that contain two adjacent bonds do not have to cross the small squares that contain the common end of the two bonds. We obtain another upper bound on the percolation threshold of G_{IntDep}, given by the following theorem. This upper bound is tighter than the bound in Theorem 1 for larger values of d_2/d_1.

Theorem 2. *If* (λ_1, λ_2) *satisfies*

$$\left[1-\frac{4}{3}(m+1)e^{m\log 3(1-p)}\right]\left[1-\frac{4}{3}(2m+1)e^{m\log 3(1-p)}\right]p' > 0.8639,$$

then $G_{IntDep} = (G_1(\lambda_1, d_1), G_2(\lambda_2, d_2), d_{dep})$ *percolates, where* $p = 1 - e^{-\lambda_1 d_1^2/8}$, $p' = 1 - e^{-2D^2\lambda_2}$, $D = \min(d_2/\sqrt{10}, d_{dep}/\sqrt{5}), m = \lfloor 2D/d_1 \rfloor$, $d_1 \le d_2$, *and* $d_{dep} \ge d_2/2$.

This upper bound is obtained by mapping G_{IntDep} to a dependent bond percolation model L_D. The mapping from the Poisson boolean model $G(\lambda, d)$ to L_D was first proposed in [13] to study the percolation threshold of $G(\lambda, d)$, and later applied to the study of the robustness of random geometric networks [14]. We briefly describe the method in the previous literature that uses L_D to study the percolation of $G(\lambda, d)$, and then prove Theorem 2 based on a similar method.

1) 1-dependent bond percolation model L_D: In the standard bond percolation model on a square lattice L, the event that a bond is open is independent of the event that any other bond is open. If in a square lattice L_D, the event that a bond is open may depend on the event that its adjacent bond is open, but is independent of the event that any non-adjacent bond is open, then L_D is a *1-dependent bond percolation model* on a square lattice. With the additional restriction that each bond is open with an identical probability, an upper bound on the percolation threshold of L_D is 0.8639 [13].

The 1-dependent bond percolation model L_D can be used to study the percolation of G' where the points are generated by homogeneous Poisson point processes. To construct a mapping from G' to L_D, consider two adjacent $D \times D$ squares S_1 and S_2 and let R be the rectangle formed by the two squares. A bond (v_1, v_2) that connects the centers of S_1 and S_2 is associated with R. Figure 3 illustrates the square lattice formed by the bonds, represented by thick line segments.

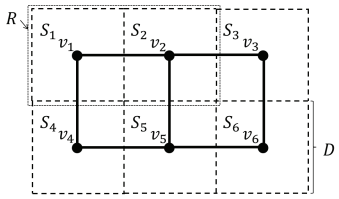

Fig. 3. Square lattice L_D formed by the bonds.

Lemma 3. *Let the state of bond (v_1, v_2) be determined by the homogeneous Poisson point processes of G' inside R, and the conditions for a bond to be open be identical for all bonds. Then the bonds form a 1-dependent bond percolation model L_D with identical bond open probabilities.*

The proof of this lemma can be found in the technical report [15]. By properly setting the conditions for a bond to be open, the percolation of L_D can imply the percolation of G'. We first look at an example in [12] that studies the percolation of $G(\lambda, d)$, and extend the technique to study G_{IntDep}.

Example [12]: Let a bond be open if a path in $G(\lambda, d)$ crosses[1] R' horizontally and another path in $G(\lambda, d)$ crosses S_1' vertically, where R' is a $(2D - 2d) \times (D - 2d)$ rectangle that has the same center as R, and S_1' is a $(D-2d)\times(D-2d)$ square that has the same center as S_1. The reason for considering R' and S_1' is that the existence of the two crossing paths over R' and S_1' is entirely determined by the point process within R, while the existence of links within distance d from the boundaries (and thus the crossings over R) may depend on nodes outside R.

If two adjacent bonds are open, the paths in $G(\lambda, d)$ in the two rectangles are joined. To see this, note that in Fig. 4, if the black and blue bonds (same direction) are both open, the crossings 1 and 2 intersect. If the black and red bonds (perpendicular) are both open, the crossings 1 and 3 intersect.

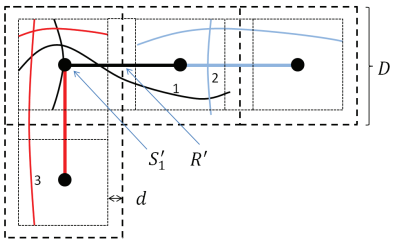

Fig. 4. Crossings over rectangles associated with two adjacent open bonds are joined.

If the square lattice L_D percolates, open bonds form an infinite cluster. Paths in $G(\lambda, d)$ across the rectangles associated with the open bonds are connected and form an infinite cluster. Therefore, a node density above which L_D percolates is an upper bound on the percolation threshold of $G(\lambda, d)$.

2) Proof of Theorem 2: We map G_{IntDep} to L_D by letting a bond in L_D be open if the following three conditions are satisfied in its associated rectangle $R = S_1 \cup S_2$. The size of the rectangle satisfies $D = \min(d_2/\sqrt{10}, d_{\text{dep}}/\sqrt{5}) \geq d_2/2\sqrt{5}$.

1) A path from G_1 crosses R' horizontally, where R' is a $(2D-2d_1)\times(D-2d_1)$ rectangle that has the same center as R.
2) A path from G_1 crosses S_1' vertically, where S_1' is a $(D-2d_1)\times(D-2d_1)$ square that has the same center as S_1.
3) There is at least one node from G_2 in R.

To bound the percolation thresholds of G_{IntDep}, in the technical report [15], we prove that the percolation of L_D implies the percolation of G_{IntDep}, and compute the probability that the three conditions are satisfied using a method similar to [16].

Remark: We have assumed that $d_{\text{dep}} \geq \max(d_1/2, d_2/2) = d_2/2$ in this section. To see that this is a reasonable assumption, note that nodes in G_1 that have at least one functioning supply node are restricted in the region R_{dep}, where R_{dep} is a union of disks with radius d_{dep} centered at nodes in the giant component of G_2. If R_{dep} is fragmented, it is not likely for disks of radius $d_1/2 < d_2/2$ centered at random locations within R_{dep} to overlap, and it is not likely that a functioning giant component will exist in G_1, unless the node density in G_1 is large. Therefore, the interdependent distance d_{dep} should be large enough so that R_{dep} is a connected region, to avoid a large minimum node density in G_1. The region R_{dep} can be made larger by increasing either λ_2 or d_2. Setting

[1]A path crosses a rectangle $R' = [x_1, x_2] \times [y_1, y_2]$ horizontally if the path consists of a sequence of connected nodes $v_1, v_2, \ldots, v_{n-1}, v_n$, and $v_2, \ldots, v_{n-1}$ are in R', $x(v_1) \leq x_1, x(v_n) \geq x_2$, $y_1 \leq y(v_1), y(v_n) \leq y_2$, where $x(v_i)$ is the x-coordinate of v_i and $y(v_i)$ is the y-coordinate of v_i. A path crosses a rectangle vertically is defined analogously.

$d_{\text{dep}} \geq d_2/2$ avoids increasing λ_2 high above the percolation threshold of G_2, in order for R_{dep} to be connected. In Section IV, we propose more general approaches that do not require this assumption.

IV. Confidence intervals for percolation thresholds

In the previous section, we discussed a method of mapping the percolation of G_{IntDep} to the percolation of the 1-dependent bond percolation model L_D. The previous mapping and the mapping that we consider in this section both satisfy the following: 1) the percolation of L_D implies the percolation of G_{IntDep}; 2) the event that determines the state of a bond depends only on the point process within its associated rectangle, thus preserving the 1-dependent property. The event probability can be computed or bounded analytically in the previous section. In contrast, in this section, we consider events whose probabilities are larger under the same point processes but can only be evaluated by simulation. Since the events that we consider in this section are more likely to occur under the same point processes, the mappings yield tighter bounds.

Our mappings from G_{IntDep} to L_D extend the mappings from $G(\lambda, d)$ to L_D proposed in [13]. For completeness, we first briefly summarize the mappings in [13] that determine upper and lower bounds on the percolation threshold of $G(\lambda, d)$.

Upper bound for $G(\lambda, d)$ [13]: Recalling Fig. 3, the event that a bond $(v_1, v_2) \in L_D$ is open is determined by the point process of $G(\lambda, d)$ in the rectangle $R = S_1 \cup S_2$, where S_1 and S_2 are squares. Let V_i denote the largest component formed by the points of $G(\lambda, d)$ in S_i. If V_i is the *unique* largest component in S_i ($\forall i \in \{1, 2\}$) and V_1 and V_2 are connected, then the bond is open. Otherwise, the bond is closed.

If L_D percolates, open bonds form an infinite cluster. As a result, the largest components in the squares that intersect the open bonds are connected in $G(\lambda, d)$ and they form an infinite cluster. Therefore, a node density λ, above which the probability that a bond is open is larger than 0.8639, is an upper bound on the percolation threshold of $G(\lambda, d)$.

Lower bound for $G(\lambda, d)$ [13]: Let the *connection process* of $G(\lambda, d)$ be the union of nodes and links in $G(\lambda, d)$. Let the *complement* of the connection process be the union of the empty space that do not intersect nodes or links. If the complement of the connection process form a connected infinite region, then all the connected clusters in $G(\lambda, d)$ have finite sizes and $G(\lambda, d)$ does not percolate [13], [17]. Consider the complement of the connection process in rectangle R. Let a bond (in L_D) associated with rectangle R be open if the complement process forms a horizontal crossing[2] over the rectangle R' and a vertical crossing over the square S_1'.

If L_D percolates, the complement process forms an infinite region and $G(\lambda, d)$ does not percolate. To conclude, a node density, under which the probability that the complement process forms the two crossings is above 0.8639, is a lower bound on the percolation threshold of $G(\lambda, d)$.

[2]The complement of a connection process forms a horizontal crossing over a rectangle if a curve in the rectangle touches the left and right boundaries of the rectangle and the curve does not intersect any nodes or links. The vertical crossing of the complement process is defined analogously.

A. Upper bounds for G_{IntDep}

Two mutual components $M = V_1 \cup V_2$ and $\hat{M} = \hat{V}_1 \cup \hat{V}_2$ form one mutual component if and only if V_i and $\hat{V}_i$ are connected in G_i ($\forall i \in \{1, 2\}$). The necessity of the condition is obvious. To see that this condition is sufficient, note that every node in the connected component formed by V_i and $\hat{V}_i$ has at least one supply node which belongs to the connected component formed by V_j and $\hat{V}_j$ ($\forall i, j \in \{1, 2\}, i \neq j$). The condition can be generalized naturally for more than two mutual components to form one mutual component.

Algorithm 1 An algorithm that greedily computes a mutual component $M^{\text{greedy}}(S)$ in region S.

1) Identify the largest connected component $V_i^0(S)$ in $G_i(S)$, where $G_i(S)$ consists of the nodes and links of G_i in S. If there are multiple largest connected components, apply any deterministic tie-breaking rule (*e.g.*, choose the component that contains a nodes with the smallest x-coordinate).
2) Remove nodes in $V_i^0(S)$ that do not have supply nodes in $V_j^0(S)$ ($\forall i, j \in \{1, 2\}, i \neq j$). Identify the largest connected component $V_i^1(S)$ formed by the remaining nodes in $V_i^0(S)$ ($\forall i \in \{1, 2\}$), and apply the same tie-breaking rule.
3) Repeat step 2 until $V_i^{k+1}(S) = V_i^k(S)$ ($\forall i \in \{1, 2\}$). Let $M^{\text{greedy}}(S) = V_1^k(S) \cup V_2^k(S)$.

Let a bond (v_1, v_2) in L_D be open if the two components $M^{\text{greedy}}(S_1)$ and $M^{\text{greedy}}(S_2)$ form one mutual component, where $M^{\text{greedy}}(S_i)$ is computed by Algorithm 1. See the technical report [15] for the rationale behind this algorithm. Since $M^{\text{greedy}}(S_i)$ is unique in any square S_i, a connected cluster in L_D implies that $\{M^{\text{greedy}}(S_i)\}$ form one mutual component in G_{IntDep}, where S_i are the squares that intersect the open bonds in the connected cluster. If the probability that a bond is open is larger than 0.8639, L_D percolates and G_{IntDep} also percolate.

An alternative condition for a bond to be open is that nodes in $M^{\text{greedy}}(R)$ form a horizontal crossing over rectangle R' and a vertical crossing over square S_1' in both graphs (recall Fig. 4 and the condition for two mutual components to form one mutual component). In order for the existence of the two crossings to only depend on the point processes in R, in the definition of R' and S_1', $d = \max(d_1, d_2, d_{\text{dep}})$.

An upper bound on the percolation threshold can be obtained by either approach. The smaller bound obtained by the two approaches is a better upper bound on the percolation threshold of G_{IntDep}.

B. Lower bounds for G_{IntDep}

In G_{IntDep}, the connection process consists of nodes and links in mutual components. To avoid the heavy computation of mutual components, we study another model in which the

connection process $\tilde{P}_i$ of G_i in the new model *dominates*[3] the connection process P_i of G_i in G_{IntDep} ($\forall i \in \{1,2\}$). As a consequence, the complement of the connection process $\tilde{P}_i^c$ of G_i in the new model is dominated by P_i^c ($\forall i \in \{1,2\}$). If $\tilde{P}_i^c$ percolates, then P_i^c percolates and P_i does not percolate. If either P_1 or P_2 does not percolate, then G_{IntDep} do not percolate. Thus, node densities under which at least one of $\tilde{P}_1^c$ and $\tilde{P}_2^c$ percolates are lower bounds on the percolation thresholds of G_{IntDep}.

The new model can be viewed to have a *relaxed* supply requirement. In this model, every node (as opposed to nodes in the same mutual component) is viewed as a valid supply node for nodes in the other graph. A node b_i in G_i is removed if and only if there is no node in G_j within the interdependent distance d_{dep} from b_i ($\forall i,j \in \{1,2\}, i \neq j$). After all such nodes are removed, the remaining nodes in G_i are connected if their distances are within the connection distance d_i. The computation of the connection process $\tilde{P}_i$ is efficient and avoids the computation of mutual components in G_{IntDep} through multiple iterations.

The connection process $\tilde{P}_i$ in the new model dominates P_i in the original model G_{IntDep}. On the one hand, for any realization, all the links in P_i are present in $\tilde{P}_i$, because all the nodes in a mutual component have supply nodes, and links between these nodes are present in the new model as well. On the other hand, in the new model, nodes in a connected component $\tilde{V}_i$ in G_i may depend on nodes in several connected components in G_j. In contrast, in G_{IntDep}, $\tilde{V}_i$ may be divided into several mutual components and links do not exist between two disjoint mutual components.

An algorithm that computes a lower bound on the percolation threshold of G_{IntDep} is as follows. First obtain the connection process $\tilde{P}_i$ in the new model. Next in the $2D \times D$ rectangle R, consider the complement of the connection process $\tilde{P}_i^c$. Let p_i denote the probability that there is a horizontal crossing over R' and a vertical crossing over S_1' in the process $\tilde{P}_i^c$, where R' and S_1' are the same as before. A lower bound on the percolation threshold of G_{IntDep} is given by node densities under which $\max(p_1, p_2) \geq 0.8639$.

C. Confidence intervals

The probability that a bond is open can be represented by an integral that depends on the point processes in the rectangle R. However, direct calculation of the integral is intractable, so instead the integral is evaluated by simulation. In every trial of the simulation, nodes in G_1 and G_2 are randomly generated by the Poisson point processes with densities λ_1 and λ_2, respectively. The events that a bond is open are independent in different trials. Let the probability that a bond is open be p given (λ_1, λ_2). The probability that a bond is closed in k out of N trials follows a binomial distribution. The interval $[0.8639, 1]$ is a 99.5% confidence interval [18] for p, given that $N = 100$ and $k = 5$. If $k < 5$, $p \in [0.8639, 1]$ with a higher confidence. This suggests that if $k \leq 5$, with 99.5% confidence $p \geq 0.8639$ and the 1-dependent bond percolation model L_D percolates given (λ_1, λ_2).

Based on this method, with 99.5% confidence an upper bound on the percolation threshold of G_{IntDep} can be obtained by declaring a bond to be open using the method in Section IV-A, and with 99.5% confidence a lower bound can be obtained by declaring a bond to be open using the method in Section IV-B. For a fixed λ_2^*, a 99% confidence interval for λ_1^* is obtained, given by the interval between the upper and lower bounds. Confidence intervals for different percolation thresholds can be obtained by changing the value of λ_2^* and repeating the computation. We make a similar remark as in [13]. The confidence intervals are rigorous, and the only uncertainty is caused by the stochastic point process in the $2D \times D$ rectangle. This is in contrast with the confidence intervals obtained by estimating whether G_{IntDep} percolate based on extrapolating the observations of simulations in a finite region (which is usually not very large because of limited computational power).

D. Numerical results

The simulation-based confidence intervals are much tighter than the analytical bounds. Given that $d_1 = d_2 = 2d_{\text{dep}} = 1$, and $\lambda_2^* = 2$, the upper and lower bounds on λ_1^* are 2.25 and 1.80, respectively, both with 99.5% confidence. In contrast, even if $\lambda_2^* \to \infty$, the analytical upper bound on λ_1^* is no less than 3.372, which is the best available analytical upper bound for a single G_1 [11]. Confidence intervals for the percolation thresholds are plotted in Fig. 5.

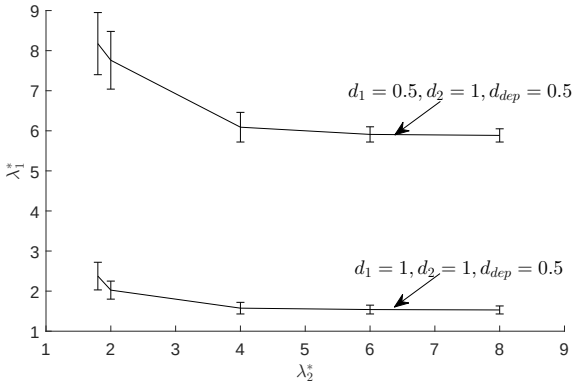

Fig. 5. The intervals between bars are 99% confidence intervals for percolation thresholds. The confidence intervals of two different G_{IntDep} are plotted.

V. Robustness of interdependent RGGs under random failures and geographical attacks

Removing nodes independently at random with the same probability in an RGG amounts to reducing the node density of the Poisson point process. To study the robustness of two interdependent RGGs G_1 and G_2 under random failures, the first step is to obtain the upper and lower bounds on percolation thresholds. With the bounds, we can determine

[3]A connection process dominates another if the nodes and links in the first process form a superset of the nodes and links in the second process, for any realization of G_i.

which graph is able to resist more random node removals, by comparing the gap between the node density λ_i and the percolation threshold λ_i^* given λ_j ($i, j \in \{1, 2\}, i \neq j$). The graph that can resist a smaller fraction of node removals is the bottleneck for the robustness of the interdependent RGGs. Besides, we can compute the maximum fraction of nodes that can be randomly removed from two graphs while guaranteeing the interdependent RGGs to be percolated.

We next show that the interdependent RGGs still percolate after a geographical attack that removes nodes in a finite connected region, if the node densities of the two graphs before the attack are above any *upper bound* on the percolation thresholds obtained in this paper (either analytical or simulation-based). Recall that we obtained upper bounds on the percolation thresholds of G_{IntDep} by mapping the percolation of G_{IntDep} to the bond percolation of either a standard square lattice L or the 1-dependent square lattice L_D. Moreover, whether a bond e is open is entirely determined by the point processes in a finite region R_e that contains the bond. After removing nodes of G_{IntDep} in a connected finite geographical region, the state of a bond e may change from open to closed only if R_e intersects the attack region. Let R_f be the union of R_e that intersects the attack region. The region R_f is also a connected finite region. As long as L or L_D still percolates after setting bonds in R_f to be closed, G_{IntDep} percolate.

Results from the percolation theory indeed indicate that setting all the bonds in a finite region R_f to be closed does not affect the percolation of L or L_D. For any percolated L, the probability that there exists a horizontal crossing of open bonds over a $kl \times l$ rectangle approaches 1 for any integer $k > 1$, as $l \to \infty$ (Lemma 8 on Page 64 of [12]). The percolation of L (after setting all bonds in R_f to be closed) is justified by the fact that the connected open bonds across rectangles form a square annulus that does not intersect R_f (shown in Fig. 6), which is a standard approach to prove the percolation of L [12]. Moreover, the percolation of L_D after all bonds in R_f are closed can be proved in the same approach, by noting that the probability that open bonds of L_D form a horizontal crossing over a rectangle approaches 1 as the rectangle size increases to infinity [13].

If the $kl \times l$ rectangle is large but finite, the probability that a horizontal crossing formed by open bonds exists is close to 1 if L or L_D percolates. Therefore, the same analysis demonstrates the robustness of finite interdependent RGGs under geographical attacks that remove a positive fraction of nodes in a connected region. Simulations for the robustness of interdependent RGGs under geographical attacks can be found in the technical report [15].

VI. Conclusion

We developed an interdependent RGG model for interdependent spatially embedded networks. We obtained analytical upper bounds and confidence intervals on the percolation thresholds. The percolation thresholds of two interdependent RGGs form a curve, which shows the tradeoff between the two node densities in order for the interdependent RGGs to percolate. The curve can be used to study the robustness of interdependent RGGs to random failures. Moreover, if the node densities are above any upper bound on the percolation thresholds obtained in this paper, then the interdependent RGGs remain percolated after a geographical attack.

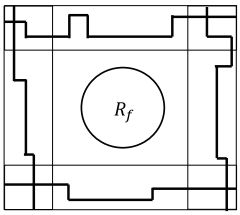

Fig. 6. Open bonds form a connected path across rectangles around R_f.

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