

Analytic and Algebraic Geometry 2

Lódź University Press 2016, 161–173

DOI: http://dx.doi.org/10.18778/8088_099_18

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FORMAL AND CONVERGENT SOLUTIONS OF ANALYTIC EQUATIONS

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Impressed by the power of the Preparation Theorem – indeed, it prepares us so well! – I considered “Weierstrass Preparation Theorem and its immediate consequences” as a possible title for the entire book.

Sheeram S. Abhyankar, Preface to [1]

ABSTRACT. We provide the detailed proof of a sharpened version of the M. Artin Approximation Theorem.

1. INTRODUCTION

The famous Approximation Theorem of M. Artin [2] asserts that any formal solution of a system of analytic equations can be approximated by convergent solutions up to a given order. In my PhD thesis [7] I was able by analysis of the argument used in [2] to sharpen the Approximation Theorem: any formal solution can be obtained by specializing parameters in a convergent parametric solution. The theorem was announced with a sketch of proof in [8]. The aim of these notes is to present the detailed proof of this result. It is based on the Weierstrass Preparation Theorem. The other tools are: a Jacobian Lemma which is an elementary version of the Regularity Jacobian Criterion used in [2], the trick of Kronecker (introducing and specializing variables) and a generalization of the Implicit Function Theorem due to Bourbaki [4] and Tougeron [10]. All these ingredients are vital in the proofs of some other results of this type (see [3], [11]).

2010 *Mathematics Subject Classification.* 14B12.

Key words and phrases. Formal solution, parametric solution.

This article is an updated version of my notes “Les solutions formelles et convergentes des équations analytiques” prepared for the Semestre de Singularités à Lille, 1999.

For more information on approximation theorems in local analytic geometry we refer the reader to Teissier's article [9] and to Chapter 8 of the book [5].

Let \mathbb{K} be a field of characteristic zero with a non-trivial valuation. We put $\mathbb{K}[[x]] = \mathbb{K}[[x_1, \dots, x_n]]$ the ring of formal power series in variables $x = (x_1, \dots, x_n)$ with coefficients in \mathbb{K} . If $f = \sum_{k \geq p} f_k$ is a nonzero power series represented as the sum of homogeneous forms with $f_p \neq 0$ then we write $\text{ord } f = p$. Additionally we put $\text{ord } 0 = +\infty$ and use the usual conventions on the symbol $+\infty$. The constant term of any series $f \in \mathbb{K}[[x]]$ we denote by $f(0)$. A power series $u \in \mathbb{K}[[x]]$ is a unit if $uv = 1$ for a power series $v \in \mathbb{K}[[x]]$. Note that u is a unit if and only if $u(0) \neq 0$. The non-units of $\mathbb{K}[[x]]$ form the unique maximal ideal \mathbf{m}_x of the ring $\mathbb{K}[[x]]$. The ideal \mathbf{m}_x is generated by the variables x_1, \dots, x_n . One has $f \in \mathbf{m}_x^c$, where $c > 0$ is an integer, if and only if $\text{ord } f \geq c$. Recall that if $g_1, \dots, g_n \in \mathbb{K}[[y]]$, $y = (y_1, \dots, y_n)$ are without constant term then the series $f(g_1, \dots, g_n) \in \mathbb{K}[[y]]$ is well-defined. The mapping which associates with $f \in \mathbb{K}[[x]]$ the power series $f(g_1, \dots, g_n)$ is the unique homomorphism sending x_i for g_i for $i = 1, \dots, n$. Let $\mathbb{K}\{x\}$ be the subring of $\mathbb{K}[[x]]$ of all convergent power series. Then $\mathbb{K}\{x\}$ is a local ring. If $g_1, \dots, g_n \in \mathbb{K}\{y\}$ then $f(g_1, \dots, g_n) \in \mathbb{K}\{y\}$ for any $f \in \mathbb{K}\{x\}$.

In what follows we use intensively the Weierstrass Preparation and Division Theorems. The reader will find the basic facts concerning the rings of formal and convergent power series in [1], [6] and [12].

Let $f(x, y) = (f_1(x, y), \dots, f_m(x, y)) \in \mathbb{K}\{x, y\}^m$ be convergent power series in the variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_N)$ where m, n, N are arbitrary non-negative integers. The theorem quoted below is the main result of [2].

The Artin Approximation Theorem. *Suppose that there exists a sequence of formal power series $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x))$ without constant term such that*

$$f(x, \bar{y}(x)) = 0.$$

Then for any integer $c > 0$ there exists a sequence of convergent power series $y(x) = (y_1(x), \dots, y_N(x))$ such that

$$f(x, y(x)) = 0 \text{ and } y(x) \equiv \bar{y}(x) \pmod{\mathbf{m}_x^c}.$$

The congruence condition means that the power series $y_\nu(x) - \bar{y}_\nu(x)$ are of order $\geq c$ i.e. the coefficients of monomials of degree $< c$ agree in $y_\nu(x)$ and $\bar{y}_\nu(x)$. We will deduce the Artin Approximation Theorem from the following result stated with a sketch of proof in [8].

Theorem. *With the notation and assumptions of the Artin theorem there exists a sequence of convergent power series $y(x, t) = (y_1(x, t), \dots, y_N(x, t)) \in \mathbb{K}\{x, t\}^N$, $y(0, 0) = 0$, where $t = (t_1, \dots, t_S)$ are new variables, $S \geq 0$, and a sequence of formal power series $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_S(x)) \in \mathbb{K}[[x]]^S$, $\bar{t}(0) = 0$ such that*

$$f(x, y(x, t)) = 0 \text{ and } \bar{y}(x) = y(x, \bar{t}(x)).$$

The construction of the parametric solution $y(x, t)$ depends on the given formal solution $\bar{y}(x)$. To get the Artin Approximation Theorem from the stated above result fix an integer $c > 0$. Let $y(x, t)$ and $\bar{t}(x)$ be series such as in the theorem and let $t(x) = (t_1(x), \dots, t_S(x)) \in \mathbb{K}\{x\}^S$ be convergent power series such that $t(x) \equiv \bar{t}(x) \pmod{\mathfrak{m}_x^c}$. Therefore $y(x, t(x)) \equiv y(x, \bar{t}(x)) \pmod{\mathfrak{m}_x^c}$ and it suffices to set $y(x) = y(x, t(x))$. \square

Before beginning the proof of the theorem let us indicate two corollaries of it.

Corollary 1. *Assume that $m = N$, $f(x, \bar{y}(x)) = 0$ and*

$$\det \frac{J(f_1, \dots, f_N)}{J(y_1, \dots, y_N)}(x, \bar{y}(x)) \neq 0 .$$

Then the power series $\bar{y}(x)$ are convergent.

Proof. Let $y(x, t)$ and $\bar{t}(x)$ be power series without constant term such that $f(x, y(x, t)) = 0$ and $\bar{y}(x) = y(x, \bar{t}(x))$. It is easy to check by differentiation of equalities $f(x, y(x, t)) = 0$ that $(\partial y_\nu / \partial t_\sigma)(x, t) = 0$ for $\nu = 1, \dots, N$ and $\sigma = 1, \dots, S$. Therefore the series $y(x, t)$ are independent of t and the series $\bar{y}(x)$ are convergent. \square

Corollary 2. *If $f(x, y) \in \mathbb{K}\{x, y\}$ is a nonzero power series of $n + 1$ variables $(x, y) = (x_1, \dots, x_n, y)$ and $\bar{y}(x)$ is a formal power series without constant term such that $f(x, \bar{y}(x)) = 0$ then $\bar{y}(x)$ is a convergent power series.*

Proof. By Corollary 1 it suffices to check that there exists a power series $g(x, y) \in \mathbb{K}\{x, y\}$ such that $g(x, \bar{y}(x)) = 0$ and $(\partial g / \partial y)(x, \bar{y}(x)) \neq 0$. Let $I = \{g(x, y) \in \mathbb{K}\{x, y\} : g(x, \bar{y}(x)) = 0\}$. Then $I \neq \mathbb{K}\{x, y\}$ is a prime ideal of $\mathbb{K}\{x, y\}$. Assume the contrary, that is, that for every $g \in I$: $(\partial g / \partial y) \in I$. Then we get by differentiating the equality $g(x, \bar{y}(x)) = 0$ that $(\partial g / \partial x_i) \in I$ for $i = 1, \dots, n$ and, by induction, all partial derivatives of g lie in I . Consequently $g = 0$ for every $g \in I$ i.e. $I = (0)$. A contradiction since $0 \neq f \in I$. \square

2. REDUCTION TO THE CASE OF SIMPLE SOLUTIONS

We keep the notation introduced in Introduction. We will call a sequence of formal power series $\bar{y}(x) \in \mathbb{K}[[x]]$, $\bar{y}(0) = 0$ a *simple solution* of the system of analytic equations $f(x, y) = 0$ if $f(x, \bar{y}(x)) = 0$ and

$$\text{rank} \frac{J(f_1, \dots, f_m)}{J(y_1, \dots, y_N)}(x, \bar{y}(x)) = m .$$

Thus, in this case, $m \leq N$.

In what follows we need

The Jacobian Lemma. *Let I be a nonzero prime ideal of the ring $\mathbb{K}\{x\}$, $x = (x_1, \dots, x_n)$. Then there exist an integer r : $1 \leq r \leq n$ and convergent power series $h_1, \dots, h_r \in I$ such that*

- (i) $\text{rank} \frac{J(h_1, \dots, h_r)}{J(x_1, \dots, x_n)} \pmod{I} = r$,
- (ii) $\forall h \in I, \exists a \notin I$ such that $ah \in (h_1, \dots, h_r)\mathbb{K}\{x\}$.

Before proving the above lemma let us note that it is invariant with respect to \mathbb{K} -linear nonsingular transformations. If Φ is an automorphism of $\mathbb{K}\{x\}$ defined by

$$\Phi(f(x_1, \dots, x_n)) = f \left(\sum_{j=1}^n c_{1j}x_j, \dots, \sum_{j=1}^n c_{nj}x_j \right)$$

with $\det(c_{ij}) \neq 0$ then the Jacobian Lemma is true for I if and only if it is true for $\Phi(I)$.

Proof of the Jacobian Lemma (by induction on the number n of variables x_i). If $n = 1$ then $I = (x_1)\mathbb{K}\{x_1\}$ and $h_1 = x_1$. Suppose that $n > 1$ and that the lemma is true for prime ideals of the ring of power series in $n - 1$ variables. Using a \mathbb{K} -linear nonsingular transformation we may assume that the ideal I contains a power series x_n -regular of order $k > 0$ i.e. such that the term x_n^k appears in the power series with a non-zero coefficients. Therefore, by the Weierstrass Preparation Theorem I contains a distinguished polynomial

$$w(x', x_n) = x_n^k + a_1(x')x_n^{k-1} + \dots + a_k(x'), \text{ where } x' = (x_1, \dots, x_{n-1}).$$

By the Weierstrass Division Theorem every power series $h = h(x)$ is of the form $h(x) = q(x)w(x', x_n) + r(x', x_n)$ where $r(x', x_n)$ is an x_n -polynomial (of degree $< k$). Therefore, the ideal I is generated by the power series which are polynomials in x_n and to prove the Jacobian Lemma it suffices to find power series h_1, \dots, h_r such that (i) holds and (ii) is satisfied for $h \in I \cap \mathbb{K}\{x'\}[x_n]$.

Let $I' = I \cap \mathbb{K}\{x'\}$ and consider the set $I \setminus I'[x_n]$. Clearly $w(x', x_n) \in I \setminus I'[x_n]$. Let

$$h_1(x', x_n) = c_0(x')x_n^l + c_1(x')x_n^{l-1} + \dots + c_l(x')$$

be a polynomial in x_n of the minimal degree $l, l \geq 0$, which belongs to $I \setminus I'[x_n]$. Since the degree $l \geq 0$ is minimal, we have

$$\begin{aligned} l &> 0, \\ c_0(x') &\notin I', \\ \frac{\partial h_1}{\partial x_n} &\in I. \end{aligned}$$

Let $h(x', x_n) \in I$ be a polynomial in x_n . Dividing $h(x', x_n)$ by $h_1(x', x_n)$ (Euklid's division) we get

$$(E) \quad c_0(x')^p h(x', x_n) = q(x', x_n) h_1(x', x_n) + r_1(x', x_n),$$

where x_n -degree of $r_1(x', x_n)$ is less than l and $p \geq 0$ is an integer. Since the x_n -degree of $r_1(x', x_n)$ is $< l$ then all coefficients of $r_1(x', x_n)$ lie in I' . If $I' = (0)$ then $r_1(x', x_n) = 0$ and (E) proves the Jacobian Lemma.

If $I' \neq (0)$ then by the induction hypothesis there exists series $h_2, \dots, h_r \in I'$ such that

- (i') $\text{rank} \frac{J(h_2, \dots, h_r)}{J(x_1, \dots, x_{n-1})} \pmod{I'} = r - 1$,
- (ii') $\forall h' \in I', \exists a' \notin I'$ such that $a'h' \in (h_2, \dots, h_r)\mathbb{K}\{x'\}$.

We claim that h_1, \dots, h_r satisfy (i) and (ii) of the Jacobian Lemma. To check (i) observe that

$$\det \frac{J(h_1, \dots, h_r)}{J(x_{i_1}, \dots, x_{i_{r-1}}, x_n)} = \det \frac{J(h_2, \dots, h_r)}{J(x_{i_1}, \dots, x_{i_{r-1}})} \cdot \frac{\partial h_1}{\partial x_n},$$

where $i_1, \dots, i_{r-1} \in \{1, \dots, n-1\}$ and use (i'). Applying (ii') to the coefficients of $r_1(x', x_n)$ we find a power series $a'(x')$ such that $a'(x') r_1(x', x_n) \in (h_2, \dots, h_r)\mathbb{K}\{x\}$. By (E) we get $a(x') h(x', x_n) \in (h_1, \dots, h_r)\mathbb{K}\{x\}$ where $a(x') = a'(x') c_0(x')^p \notin I$ which proves (ii). □

Now, we can check

Proposition 2.1. *Let $f(x, y) = (f_1(x, y), \dots, f_m(x, y)) \in \mathbb{K}\{x, y\}^m$, $f(x, y) \neq 0$, $\bar{y}_1(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x)) \in \mathbb{K}[[x]]$, $\bar{y}(0) = 0$, be formal power series such that $f(x, \bar{y}(x)) = 0$. Then there exist convergent power series $h(x, y) = (h_1(x, y), \dots, h_r(x, y)) \in \mathbb{K}\{x, y\}^r$ such that*

- (i) $h(x, \bar{y}(x)) = 0$,
- (ii) $\text{rank} \frac{J(h_1, \dots, h_r)}{J(y_1, \dots, y_N)}(x, \bar{y}(x)) = r$,
- (iii) *suppose that there exist formal power series $y(x, t) = (y_1(x, t), \dots, y_N(x, t))$, $y(0, 0) = 0$ and $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_S(x))$, $\bar{t}(0) = 0$, such that $h(x, y(x, t)) = 0$ and $\bar{y}(x) = y(x, \bar{t}(x))$. Then $f(x, y(x, t)) = 0$.*

Proof. Consider the prime ideal

$$I = \{g(x, y) \in \mathbb{K}\{x, y\} : g(x, \bar{y}(x)) = 0\}.$$

Clearly $f_1(x, y), \dots, f_m(x, y) \in I$ and $I \neq (0)$. By the Jacobian Lemma there exist formal power series $h_1(x, y), \dots, h_r(x, y) \in I$ such that

- $\text{rank} \frac{J(h_1, \dots, h_r)}{J(x_1, \dots, x_n, y_1, \dots, y_N)}(x, \bar{y}(x)) = r$,
- $\forall g \in I, \exists a \notin I$ such that $a(x, y) g(x, y) \in (h_1, \dots, h_r)\mathbb{K}\{x, y\}$.

We claim that h_1, \dots, h_r satisfy the conditions (i), (ii), (iii). Condition (i) holds since $h_1, \dots, h_r \in I$. To check (ii) it suffices to observe that

$$(J) \text{rank} \frac{J(h_1, \dots, h_r)}{J(x_1, \dots, x_n, y_1, \dots, y_N)}(x, \bar{y}(x)) = \text{rank} \frac{J(h_1, \dots, h_r)}{J(y_1, \dots, y_N)}(x, \bar{y}(x)).$$

Indeed, differentiating the equations $h_i(x, \bar{y}(x)) = 0$, $i = 1, \dots, r$, we get

$$\frac{\partial h_i}{\partial x_j}(x, \bar{y}(x)) + \sum_{\nu=1}^N \frac{\partial h_i}{\partial y_\nu}(x, \bar{y}(x)) \frac{\partial \bar{y}_\nu}{\partial x_j} = 0 \text{ for } j = 1, \dots, n$$

and (J) follows. To check (iii) let us write

$$a_i(x, y)f_i(x, y) = \sum_{k=1}^r a_{i,k}(x, y)h_k(x, y) \text{ in } \mathbb{K}\{x, y\} ,$$

where $a_i(x, y) \notin I$ for $i = 1, \dots, m$. Thus $a_i(x, \bar{y}(x)) \neq 0$ and $a_i(x, y(x, t)) \neq 0$ since $\bar{y}(x) = y(x, \bar{t}(x))$ and (iii) follows. □

3. THE BOURBAKI–TOUGERON IMPLICIT FUNCTION THEOREM

Let $f(x, y) = (f_1(x, y), \dots, f_m(x, y)) \in \mathbb{K}\{x, y\}^m$ be convergent power series in variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_N)$. Suppose that $m \leq N$ and put

$$J(x, y) = \frac{J(f_1, \dots, f_m)}{J(y_{N-m+1}, \dots, y_N)} \text{ and } \delta(x, y) = \det J(x, y) .$$

Let $M(x, y)$ be the adjoint of the matrix $J(x, y)$. Thus we have

$$M(x, y)J(x, y) = J(x, y)M(x, y) = \delta(x, y)I_m$$

where I_m is the identity matrix of m rows and m columns. Let $g(x, y) = (g_1(x, y), \dots, g_m(x, y)) \in \mathbb{K}\{x, y\}^m$ be convergent power series defined by

$$\begin{bmatrix} g_1(x, y) \\ \vdots \\ g_m(x, y) \end{bmatrix} = M(x, y) \begin{bmatrix} f_1(x, y) \\ \vdots \\ f_m(x, y) \end{bmatrix} .$$

It is easy to see that

(a) $g_i(x, y) \in (f_1(x, y), \dots, f_m(x, y))\mathbb{K}\{x, y\}$ for $i = 1, \dots, m$

and

(b) $\delta(x, y)f_i(x, y) \in (g_1(x, y), \dots, g_m(x, y))\mathbb{K}\{x, y\}$ for $i = 1, \dots, m$.

Now, we can state

The Bourbaki-Tougeron implicit function theorem. *Suppose that there exists a sequence of formal power series $y^0(x) = (y_1^0(x), \dots, y_N^0(x))$, $y^0(0) = 0$, such that*

$$g_i(x, y^0(x)) \equiv 0 \pmod{\delta(x, y^0(x))^2 \mathbf{m}_x} \quad \text{for } i = 1, \dots, m .$$

Then

- I. Let $y_\nu(x, t) = y_\nu^0(x) + \delta(x, y^0(x))^2 t_\nu$ for $\nu = 1, \dots, N - m$ where $t = (t_1, \dots, t_{N-m})$ are new variables. Then there exists a unique sequence of formal power series $u(x, t) = (u_{N-m+1}(x, t), \dots, u_N(x, t)) \in \mathbb{K}[[x, t]]^m$, $u(0, 0) = 0$, such that if we let $y_\nu(x, t) = y_\nu^0(x) + \delta(x, y^0(x)) u_\nu(x, t)$ for $\nu = N - m + 1, \dots, N$ and $y(x, t) = (y_1(x, t), \dots, y_N(x, t))$ then

$$f(x, y(x, t)) = 0 \text{ in } \mathbb{K}[[x, t]] .$$

If the series $y^0(x)$ are convergent then $u(x, t)$ and $y(x, t)$ are convergent as well.

- II. For every sequence of formal power series $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x))$, $\bar{y}(0) = 0$, the following two conditions are equivalent
- (i) there exists a sequence of formal power series $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_{N-m}(x))$, $\bar{t}(0) = 0$, such that $\bar{y}(x) = y(x, \bar{t}(x))$,
 - (ii) $f(x, \bar{y}(x)) = 0$ and

$$\begin{aligned} \bar{y}_\nu(x) &\equiv y_\nu^0(x) \pmod{\delta(x, y^0(x))^2 \mathbf{m}_x} \text{ for } \nu = 1, \dots, N - m \\ \bar{y}_\nu(x) &\equiv y_\nu^0(x) \pmod{\delta(x, y^0(x)) \mathbf{m}_x} \text{ for } \nu = N - m + 1, \dots, N . \end{aligned}$$

Remark. In what follows we call

- $y^0(x)$ an approximate solution of the system $f(x, y) = 0$,
- $y(x, t)$ a parametric solution determined by the approximate solution $y^0(x)$
- $\bar{y}(x)$ satisfying (i) or (ii) a subordinate solution to the approximate solution $y^0(x)$

Proof. Let $v = (v_1, \dots, v_N)$ and $h = (h_1, \dots, h_n)$ be variables. Taylor's formula reads

$$\begin{aligned} \text{(T)} \quad \begin{bmatrix} f_1(x, v + h) \\ \vdots \\ f_m(x, v + h) \end{bmatrix} &= \begin{bmatrix} f_1(x, v) \\ \vdots \\ f_m(x, v) \end{bmatrix} + \frac{J(f_1, \dots, f_m)}{J(y_1, \dots, y_{N-m})}(x, v) \begin{bmatrix} h_1 \\ \vdots \\ h_{N-m} \end{bmatrix} \\ &+ J(x, v) \begin{bmatrix} h_{N-m+1} \\ \vdots \\ h_N \end{bmatrix} + \begin{bmatrix} P_1(u, v, h) \\ \vdots \\ P_m(u, v, h) \end{bmatrix} \end{aligned}$$

where $P_i(x, v, h) \in (h_1, \dots, h_N)^2 \mathbb{K}\{x, v, h\}$ for $i = 1, \dots, m$. Let $u = (u_{N-m+1}, \dots, u_N)$ be variables and put

$$\begin{aligned} F_i(x, t, u) &= f_i(x, y_1(x, t), \dots, y_{N-m}(x, t), y_{N-m+1}^0(x) + \delta(x, y^0(x)) u_{N-m+1}, \\ &\dots, y_N^0(x) + \delta(x, y^0(x)) u_N) . \end{aligned}$$

Substituting in Taylor's formula (T) $v_i = y_i^0(x)$ for $i = 1, \dots, N$, $h_i = \delta(x, y^0(x))^2 t_i$ for $i = 1, \dots, N - m$ and $h_i = \delta(x, y^0(x)) u_i$ for $i = N - m + 1, \dots, N$ we get

$$\begin{aligned} \begin{bmatrix} F_1(x, t, u) \\ \vdots \\ F_m(x, t, u) \end{bmatrix} &= \begin{bmatrix} f_1(x, y^0(x)) \\ \vdots \\ f_m(x, y^0(x)) \end{bmatrix} + \delta(x, y^0(x))^2 \frac{J(f_1, \dots, f_m)}{J(y_1, \dots, y_{N-m})}(x, y^0(x)) \begin{bmatrix} t_1 \\ \vdots \\ t_{N-m} \end{bmatrix} \\ &+ \delta(x, y^0(x)) J(x, y^0(x)) \begin{bmatrix} u_{N-m+1} \\ \vdots \\ u_N \end{bmatrix} + \delta(x, y^0(x))^2 \begin{bmatrix} Q_1(x, t, u) \\ \vdots \\ Q_m(x, t, u) \end{bmatrix} \end{aligned}$$

where $Q_i(x, t, u) \in (t, u)^2 \mathbb{K}\{x, t, u\}$ for $i = 1, \dots, m$. Multiplying the above identity by the matrix $M(x, y^0(x))$ and taking into account that $M(x, y^0(x)) J(x, y^0(x)) = \delta(x, y^0(x)) I_m$ and $g_i(x, y^0(x)) \equiv 0 \pmod{\delta(x, y_0(x))^2 \mathbf{m}_x}$ for $i = 1, \dots, m$, we get

$$(*) \quad M(x, y^0(x)) \begin{bmatrix} F_1(x, t, u) \\ \vdots \\ F_m(x, t, u) \end{bmatrix} = \delta(x, y^0(x))^2 \begin{bmatrix} G_1(x, t, u) \\ \vdots \\ G_m(x, t, u) \end{bmatrix}$$

where $G_i(0, 0, 0) = 0$ for $i = 1, \dots, m$. Differentiating (*) we obtain

$$M(x, y^0(x)) \frac{J(F_1, \dots, F_m)}{J(u_{N-m+1}, \dots, u_N)}(x, t, u) = \delta(x, y^0(x))^2 \frac{J(G_1, \dots, G_m)}{J(u_{N-m+1}, \dots, u_N)}(x, t, u)$$

which implies

$$(**) \quad \det \frac{J(F_1, \dots, F_m)}{J(u_{N-m+1}, \dots, u_N)}(x, t, u) = \delta(x, y^0(x))^{m+1} \det \frac{J(G_1, \dots, G_m)}{J(u_{N-m+1}, \dots, u_N)}(x, t, u)$$

since $\det M(x, y^0(x)) = \delta(x, y^0(x))^{m-1}$. On the other hand

$$\frac{J(F_1, \dots, F_m)}{J(u_{N-m+1}, \dots, u_N)}(x, 0, 0) = \delta(x, y^0(x)) J(x, y^0(x))$$

and

$$\det \frac{J(F_1, \dots, F_m)}{J(u_{N-m+1}, \dots, u_N)}(x, 0, 0) = \delta(x, y^0(x))^{m+1}.$$

Therefore we get from (**)

$$\det \frac{J(G_1, \dots, G_m)}{J(u_{N-m+1}, \dots, u_N)}(x, 0, 0) = 1,$$

in particular

$$\det \frac{J(G_1, \dots, G_m)}{J(u_{N-m+1}, \dots, u_N)}(0, 0, 0) = 1.$$

By the Implicit Function Theorem there exist formal power series

$$u(x, t) = (u_{N-m+1}(x, t), \dots, u_N(x, t))$$

such that

$$(G_1(x, t, u), \dots, G_m(x, t, u))\mathbb{K}[[x, t, u]] \\ = (u_{N-m+1} - u_{N-m+1}(x, t), \dots, u_N - u_N(x, t))\mathbb{K}[[x, t, u]].$$

If $y^0(x)$ are convergent then $G(x, t, u)$ and $u(x, t)$ are convergent as well. In particular $G(x, t, u(x, t)) = 0$ and by (*) $F(x, t, u(x, t)) = 0$ which implies $f(x, y(x, t)) = 0$ where $y_\nu(x, t) = y_\nu^0(x) + \delta(x, y^0(x))u_\nu(x, t)$ for $\nu = N - m + 1, \dots, N$. Let $\tilde{u}(x, t) = (\tilde{u}_{N-m+1}(x, t), \dots, \tilde{u}_N(x, t))$, $\tilde{u}(0, 0) = 0$, be power series such that $f(x, \tilde{y}(x, t)) = 0$ where

$$\tilde{y}(x, t) = (y_1(x, t), \dots, y_{N-m}(x, t), y_{N-m+1}^0(x) + \delta(x, y^0(x))\tilde{u}_{N-m+1}(x, t), \\ \dots, y_N^0(x) + \delta(x, y^0(x))\tilde{u}_N(x, t)).$$

Then $F(x, t, \tilde{u}(x, t)) = 0$ and by (*) $G(x, t, \tilde{u}(x, t)) = 0$. Thus we get $\tilde{u}(x, t) = u(x, t)$. This proves the first part of the Bourbaki-Tougeron Implicit Function Theorem. To check the second part it suffices to observe that for any formal power series $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_{N-m}(x))$ and $\bar{u}(x) = (\bar{u}_{N-m+1}(x), \dots, \bar{u}_N(x))$ without constant term $G(x, \bar{t}(x), \bar{u}(x)) = 0$ if and only if $\bar{u}(x) = u(x, \bar{t}(x))$. \square

4. APPROXIMATE SOLUTIONS

We keep the notions and assumptions of Section 3. Let $x' = (x_1, \dots, x_{n-1})$.

Proposition 4.1. *Let $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x))$, $\bar{y}(0) = 0$, be a formal solution of the system of analytic equations $f(x, y) = 0$, such that the power series $\delta(x, \bar{y}(x))$ is x_n -regular of strictly positive order $p > 0$. Then there exists an approximate solution $\bar{v}(x) \in \mathbb{K}[[x']][x_n]^N$ of the system $f(x, y) = 0$ and such that $\bar{y}(x)$ is a solution of $f(x, y) = 0$ subordinate to $\bar{v}(x)$.*

Proof. By the Weierstrass Preparation Theorem $\delta(x, \bar{y}(x)) = \bar{a}(x) \cdot \text{unit}$ where

$$\bar{a}(x) = x_n^p + \sum_{j=1}^p \bar{a}_j(x')x_n^{p-j}$$

is a distinguished polynomial. Using the Weierstrass Division Theorem we get

$$\bar{y}_\nu(x) = \sum_{j=0}^{2p-1} \bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)^2(c_\nu + \bar{t}_\nu(x)) \text{ for } \nu = 1, \dots, N - m$$

and

$$\bar{y}_\nu(x) = \sum_{j=0}^{p-1} \bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)(c_\nu + \bar{u}_\nu(x)) \text{ for } \nu = N - m + 1, \dots, N$$

where $c_\nu \in \mathbb{K}$ for $\nu = 1, \dots, N$, while

$$\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_{N-m}(x)) \text{ and } \bar{u}(x) = (\bar{u}_{N-m+1}(x), \dots, \bar{u}_N(x))$$

are formal power series without constant term. Let

$$\bar{v}_\nu(x) = \sum_{j=0}^{2p-1} \bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)^2 c_\nu \text{ for } \nu = 1, \dots, N-m$$

and

$$\bar{v}_\nu(x) = \sum_{j=0}^{p-1} \bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)c_\nu \text{ for } \nu = N-m+1, \dots, N.$$

Clearly $\bar{v}(x) = (\bar{v}_1(x), \dots, \bar{v}_N(x)) \in \mathbb{K}[[x']][x_n]^N$.

Property 1. $\delta(x, \bar{v}(x)) = \bar{a}(x) \cdot \text{unit}$

Proof. From $\bar{y}(x) \equiv \bar{v}(x) \pmod{\bar{a}(x)\mathbf{m}_x}$ we get $\delta(x, \bar{y}(x)) \equiv \delta(x, \bar{v}(x)) \pmod{\bar{a}(x)\mathbf{m}_x}$ and Property 1 follows since $\delta(x, \bar{y}(x)) = \bar{a}(x) \cdot \text{unit}$.

Property 2. $g_i(x, \bar{v}(x)) \equiv 0 \pmod{\bar{a}(x)^2\mathbf{m}_x}$ for $i = 1, \dots, m$.

Proof. Substituting in Taylor's formula (T) $v = \bar{v}(x)$, $h_\nu = \bar{a}(x)^2 \bar{t}_\nu(x)$ for $\nu = 1, \dots, N-m$ and $h_\nu = \bar{a}(x)\bar{u}_\nu(x)$ for $\nu = N-m+1, \dots, N$ we get

$$\begin{aligned} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} &= \begin{bmatrix} f_1(x, \bar{v}(x)) \\ \vdots \\ f_m(x, \bar{v}(x)) \end{bmatrix} + \bar{a}(x)^2 \frac{J(f_1, \dots, f_m)}{J(y_1, \dots, y_{N-m})}(x, \bar{v}(x)) \begin{bmatrix} \bar{t}_1(x) \\ \vdots \\ \bar{t}_{N-m}(x) \end{bmatrix} \\ &+ \bar{a}(x)J(x, \bar{v}(x)) \begin{bmatrix} \bar{u}_{N-m+1}(x) \\ \vdots \\ \bar{u}_N(x) \end{bmatrix} + \bar{a}(x)^2 \begin{bmatrix} \bar{Q}_1(x) \\ \vdots \\ \bar{Q}_m(x) \end{bmatrix}. \end{aligned}$$

Multiplying the above identity by $M(x, \bar{v}(x))$ and taking into account the formula

$$M(x, \bar{v}(x))J(x, \bar{v}(x)) = \delta(x, \bar{v}(x))I_m$$

we get Property 2.

From Properties 1 and 2 it follows that

$$g_i(x, \bar{v}(x)) \equiv 0 \pmod{\delta(x, \bar{v}(x))^2\mathbf{m}_x} \quad \text{for } i = 1, \dots, m$$

i.e. $\bar{v}(x) \in \mathbb{K}[[x']][x_n]$ is an approximate solution of the system $f(x, y) = 0$. Since

$$\bar{y}_\nu(x) \equiv \bar{v}_\nu(x) \pmod{\delta(x, \bar{v}(x))^2\mathbf{m}_x} \quad \text{for } i = 1, \dots, N-m$$

and

$$\bar{y}_\nu(x) \equiv \bar{v}_\nu(x) \pmod{\delta(x, \bar{v}(x))\mathbf{m}_x} \quad \text{for } i = N-m+1, \dots, N$$

$\bar{y}(x)$ is a subordinate solution to the approximate solution $\bar{v}(x)$. □

Proposition 4.2. *Let $(c_{\nu,j}^0)$, $\nu = 1, \dots, N$, $j = 0, 1, \dots, D$, be a family of constants such that $c_{\nu,0}^0 = 0$ for $\nu = 1, \dots, N$. Suppose that*

$$\left(\sum_{j=0}^D c_{1,j}^0 x_n^j, \dots, \sum_{j=0}^D c_{N,j}^0 x_n^j \right)$$

is an approximate solution of the system of equations $f(0, x_n, y) = 0$ such that

$$\text{ord } \delta \left(0, x_n, \sum_{j=0}^D c_{1,j}^0 x_n^j, \dots, \sum_{j=0}^D c_{N,j}^0 x_n^j \right) = p, \quad 0 < p < +\infty.$$

Let $V^0 = (V_{\nu,j}^0)$, $\nu = 1, \dots, N$, $j = 0, 1, \dots, D$ be variables. Then there exists a sequence

$$F(x', V^0) = (F_1(x', V^0), \dots, F_M(x', V^0)) \in \mathbb{K}\{x', V^0\}^M$$

such that for any family $(\bar{v}_{\nu,j}^0(x'))$ of formal power series without constant term the following two conditions are equivalent

- (i) $\left(\sum_{j=0}^D (c_{1,j}^0 + \bar{v}_{1,j}^0(x')) x_n^j, \dots, \sum_{j=0}^D (c_{N,j}^0 + \bar{v}_{N,j}^0(x')) x_n^j \right)$
is an approximate solution of the system $f(x, y) = 0$,
- (ii) $F(x', (\bar{v}_{\nu,j}^0(x'))) = 0$ *in $\mathbb{K}[[x']]$.*

Proof. Let

$$v_{\bar{\nu}}(x_n) = \sum_{j=0}^D (c_{\nu,j}^0 + V_{\nu,j}^0) x_n^j, \quad v(x_n) = (v_1(x_n), \dots, v_N(x_n)).$$

It is easy to check that $\delta(x, v(x_n))$ is x_n -regular of order p . By the Weierstrass Division Theorem

$$g_i(x, v(x_n)) = Q_i(x, V^0) \delta(x, v(x_n))^2 + \sum_{j=0}^{2p-1} R_{i,j}(x', V^0) x_n^j$$

for $i = 1, \dots, m$. Let

$$\bar{v}_{\nu}(x) = \sum_{j=0}^D (c_{\nu,j}^0 + \bar{v}_{\nu,j}^0(x)) x_n^j, \quad \bar{v}(x) = (\bar{v}_1(x), \dots, \bar{v}_N(x))$$

where $(\bar{v}_{\nu,j}^0(x'))$ is a family of formal power series without constant term. Thus we get

$$g_i(x, \bar{v}(x)) = Q_i(x, \bar{v}(x)) \delta(x, \bar{v}(x))^2 + \sum_{j=0}^{2p-1} R_{i,j}(x', \bar{v}_{\nu,j}^0(x')) x_n^j \text{ for } i = 1, \dots, m.$$

By the uniqueness of the remainder in the Weierstrass Division Theorem we have that $\bar{v}(x)$ is an approximate solution of the system of analytic equations $f(x, y) = 0$

if and only if $R_{i,j}(x', (\bar{v}_{\nu,j}^0(x'))) = 0$ for $i = 1, \dots, m$ and $j = 0, 1, \dots, 2p - 1$ in $\mathbb{K}[[x']]$. This proves the proposition. \square

5. PROOF OF THE THEOREM (BY INDUCTION ON THE NUMBER n OF VARIABLES x)

The theorem is trivial for $n = 0$. Suppose that $n > 0$ and that the theorem is true for $n - 1$. By Proposition 2.1 we may suppose that $\bar{y}(x)$ is a simple solution of the system $f(x, y) = 0$. Let

$$\delta(x, y) = \det \frac{J(f_1, \dots, f_m)}{J(y_{N-m+1}, \dots, y_N)} .$$

Without diminishing the generality we may suppose that $\delta(x, \bar{y}(x)) \neq 0$. If $\delta(0, 0) \neq 0$ then the theorem follows from the Implicit Function Theorem. Suppose that $\delta(0, 0) = 0$. After a linear change of the variables x_1, \dots, x_n we may assume that $\delta(x, \bar{y}(x))$ is x_n -regular of order $p > 0$. By Proposition 4.1 the system of equations $f(x, y) = 0$ has an approximate solution $\bar{v}(x) = (\bar{v}_1(x), \dots, \bar{v}_N(x)) \in \mathbb{K}[[x']] [x_n]^N$ such that the solution $\bar{y}(x)$ is subordinate to $\bar{v}(x)$. Write

$$\bar{v}_\nu(x) = \sum_{j=0}^D (c_{\nu,j}^0 + \bar{v}_{\nu,j}^0(x')) x_n^j, \quad D \geq 0 \text{ an integer}$$

where $(\bar{v}_{\nu,j}(x'))$ is a family of formal power series without constant term. It is easy to check that

$$\left(\sum_{j=0}^D c_{1,j}^0 x_n^j, \dots, \sum_{j=0}^D c_{N,j}^0 x_n^j \right)$$

is an approximate solution of the system $f(0, x_n, y) = 0$ such that

$$\text{ord } \delta \left(0, x_n, \sum_{j=0}^D c_{1,j}^0 x_n^j, \dots, \sum_{j=0}^D c_{N,j}^0 x_n^j \right) = p .$$

By Proposition 4.2 there exist convergent power series $F(x', V^0) \in \mathbb{K}\{x', V^0\}^M$ such that $F(x', (\bar{v}_{\nu,j}^0(x'))) = 0$. By induction hypothesis there exist convergent power series $(V_{\nu,j}^0(x', s))$ in $\mathbb{K}\{x', s\}$, where $s = (s_1, \dots, s_q)$ are new variables and formal power series $\bar{s}(x') = (\bar{s}_1(x'), \dots, \bar{s}_q(x'))$ without constant term such that

$$F(x', (V_{\nu,j}^0(x', s))) = 0, \quad V_{\nu,j}^0(x', \bar{s}(x')) = \bar{v}_{\nu,j}^0(x') .$$

Let

$$v_\nu(x, s) = \sum_{j=0}^D (c_{\nu,j}^0 + V_{\nu,j}^0(x', s)) x_n^j \text{ for } \nu = 1, \dots, N$$

and $v(x, s) = (v_1(x, s), \dots, v_N(x, s))$. Thus $\bar{v}_\nu(x) = v_\nu(x, \bar{s}(x'))$ for $\nu = 1, \dots, N$. Again by Proposition 4.2 $v(x, s)$ is an approximate solution of the system

$f(x, y) = 0$. By the Bourbaki-Tougeron Implicit Function Theorem the system $f(x, y) = 0$ has the parametric solution determined by $v(x, s)$:

$$y_\nu(x, s, t) = v_\nu(x, s) + \delta(x, v(x, s))^2 t_\nu \text{ for } \nu = 1, \dots, N - m$$

$$y_\nu(x, s, t) = v_\nu(x, s) + \delta(x, v(x, s)) u_\nu(x, s, t) \text{ for } \nu = N - m + 1, \dots, N .$$

On the other hand

$$\bar{y}_\nu(x, t) = \bar{v}_\nu(x) + \delta(x, \bar{v}(x))^2 t_\nu \text{ for } \nu = 1, \dots, N - m$$

$$\bar{y}_\nu(x, t) = \bar{v}_\nu(x) + \delta(x, \bar{v}(x)) \bar{u}_\nu(x, t) \text{ for } \nu = N - m + 1, \dots, N$$

is the parametric solution determined by $\bar{v}(x)$. Since the formal solution $\bar{y}(x)$ is subordinate to the approximate solution $\bar{v}(x)$ there exist formal power series $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_{N-m}(x))$, $\bar{t}(0) = 0$, such that $\bar{y}(x) = \bar{y}(x, \bar{t}(x))$. We have

$$y_\nu(x, \bar{s}(x'), t) = \bar{v}_\nu(x) + \delta(x, \bar{v}(x))^2 t_\nu \text{ for } \nu = 1, \dots, N - m$$

$$y_\nu(x, \bar{s}(x'), t) = \bar{v}_\nu(x) + \delta(x, \bar{v}(x)) u_\nu(x, \bar{s}(x'), t) \text{ for } \nu = N - m + 1, \dots, N$$

By the uniqueness of the parametric solution determined by the approximate solution $\bar{v}(x)$ we get

$$y(x, \bar{s}(x'), \bar{t}(x)) = \bar{y}(x, \bar{t}(x)) = \bar{y}(x).$$

□

REFERENCES

[1] S.S. Abhyankar, *Local analytic geometry*, Academic Press 1964.
 [2] M. Artin, *On the solutions of analytic equations*, Invent. math. 5 (1968), 277–291.
 [3] M. Artin, *Algebraic approximations of structures over complete local rings*, Inst. Hautes Etudes Sci. Publ. Math. 36 (1969), 23–58.
 [4] N. Bourbaki, *Algèbre commutative*, Fasc. XXVIII, Chap. III, §4, No. 5 (1961).
 [5] T. de Jong and G. Pfister, *Local Analytic Geometry*, vieweg 2000.
 [6] S. Lefschetz, *Algebraic Geometry*, Princeton NJ, 1953.
 [7] A. Płoski, *Rozwiązania formalne i zbieżne równań analitycznych*, Ph D thesis, Institute of Mathematics, Polish Academy of Sciences (1973) (in Polish).
 [8] A. Płoski, *Note on a theorem of M. Artin*, Bull. Acad. Polonaise Sci. Ser. Math. 22 (1974), 1107–1109.
 [9] B. Teissier, *Résultats récents sur l’approximation des morphismes en algèbre commutative* (d’après André, Artin, Popescu et Spivakovsky). Séminaire BOURBAKI, 16 ème année (1993-94), n°784, 259–281.
 [10] J.C. Tougeron, *Idéaux de fonctions différentiables*, New York: Springer 1972.
 [11] John J. Wavrik, *A Theorem on Solutions of Analytic Equations with Applications to Deformations of Complex Structures*, Math. Ann. 216 (1975), 127–142.
 [12] O. Zariski and P. Samuel, *Commutative Algebra*, vol. II, Van Nostrand, Princeton, New Jersey, 1960.

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