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THE CAUCHY-KOWALEVSKI THEOREM

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Abstract. We give a recursive description of polynomials with non-negative rational coefficients, which are coefficients of expansion in a power series solutions of partial differential equations in Cauchy-Kowalevski theorem.

1. Introduction

In recent time we can observe the renewed interest in the algorithms associated with the solution of partial differential equations using power series (see: for example [8]). This study initiated by the famous theorem of Cauchy-Kowalevski[1] (see original work [9], Theorem 2.1 and Proposition 2.1 in this article, compare [2], [3]) were later generalized by Riquier [11] for a wide class of orthonomic passive systems. In both theorems, the proof of the existence and uniqueness consisted of demonstration, in a first step, the existence and uniqueness of formal solutions, and in the second step of its convergence. The work of Riquier for polynomial nonlinear differential equations was complemented by Ritt [12]. The proof used the method of characteristic set. Since that time many algorithms for determining the formal solution of partial differential equations was stated. It is well known that coefficients of such a formal solution are polynomials depending on coefficients occurring in the power series expansion of right-hand side functions in partial differential equations (see: for example [1], [4], [8], see also [13]). Moreover, these polynomials have non-negative rational coefficients. The aim of this paper is to

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[1]After G. B. Folland, [3] the problem of how to spell this name is vexed not only by the usual lack of a canonical scheme for transliterating from the Cyrillic alphabet to the Latin one but also by the question of whether to use the feminine ending (-skaia instead of -ski). The spelling used here is the one preferred by Kowalevski herself in her scientific works.

give a recursive description of these polynomials (Theorem 2.5), which is not given explicitly in textbooks.

Multi-indexes and partial derivatives. The n-element sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers[2] will be called *multi-index of dimension n*. We introduce the following notations:

$$|\alpha| = \sum_{j=1}^{n} \alpha_j, \qquad \alpha! = \alpha_1! \cdots \alpha_n!,$$

and for $x \in \mathbb{R}^n$,

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

In general, we will use the shortcut

$$\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$$

for the partial derivative in $\mathbb{R}^n$. For the partial derivatives of higher order it is more convenient to use multi-index

$$\partial^\alpha = \partial_x^\alpha = \prod_{j=1}^{n} \left(\frac{\partial}{\partial x_j} \right)^{\alpha_j} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

In particular, we note that for $\alpha = 0$, ∂^α it is the identity operator. Let I be any non-empty set containing an element j. Then 1_j designates a system $(\delta_i)_{i \in I}$, where $\delta_i = 1$ for $i = j$ and $\delta_i = 0$ for $i \neq j$. With the above descriptions it is easily to note that the partial derivatives can be defined by induction in the following way:

(1) $\partial^0 = \mathrm{id}$,
(2) $\partial^{\alpha + 1_j} = (\partial^\alpha)^{1_j} = \partial_j \partial^\alpha$ for $j \in \{1, 2, \ldots, n\}$ and all $\alpha \in \mathbb{N}^n$.

Let us order the set of multi-indices. We write that $\alpha \leqslant \beta$, if $\alpha_i \leqslant \beta_i$ for all i. For the given complex numbers a_α for $|\alpha| \leqslant k$, by $(a_\alpha)_{|\alpha| \leqslant k}$ we denote the element of $\mathbb{C}^{N(k)}$ given by ordering the α's in this fashion, where $N(k)$ is the number of elements in the set $\{\alpha \in \mathbb{N}^n : |\alpha| \leqslant k\}$. Similarly, if $A \subset \{\alpha : |\alpha| \leqslant k\}$, then we can consider the elements of space $\mathbb{C}^N$ of the form $(a_\alpha)_{\alpha \in A}$, where $N = \#A$.

2. The Cauchy-Kowalevski Theorem

Let k be a positive integer and let S be an analytic hypersurface of form

$$S = \{(x, t) = (x_1, \ldots, x_{n-1}, t) \in \mathbb{R}^n : t = 0\}.$$

Let $F : \Omega \to \mathbb{R}$ be an analytic function in some neighbourhood $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{N(k)}$ of the origin, where

$$N(k) = \binom{n+k}{k} = \{(\alpha, j) = (\alpha_1, \ldots, \alpha_{n-1}, j) \in \mathbb{N}^n : |\alpha| + j \leqslant k\}.$$

[2]Here the set of non-negative integers is denoted by $\mathbb{N}$.

If $\varphi_0, \dots, \varphi_{k-1}$ are the real analytic functions at the origin of $\mathbb{R}^{n-1}$, then *the analytic Cauchy problem* is to look for the solution u of system (2.1) analytic at the origin of $\mathbb{R}^n$

$$\begin{cases} F\left(x, t, \left(\partial_x^\alpha \partial_t^j u\right)_{|\alpha|+j \leqslant k}\right) = 0, \\ \partial_t^j u(x,0) = \varphi_j(x), \quad 0 \leqslant j < k. \end{cases} \tag{2.1}$$

We assume that the equation $F = 0$ can be solved for $\partial_t^k u$ to yield $\partial_t^k u$ as an analytic function G of the remaining variables. We do this because of the bad behaviour that can occur when this condition is not satisfied (see examples i and ii, page 43 in [3]). The Cauchy problem then takes the form

$$\begin{cases} \partial_t^k u = G\left(x, t, \left(\partial_x^\alpha \partial_t^j u\right)_{|\alpha|+j \leqslant k,\, j<k}\right), \\ \partial_t^j u(x,0) = \varphi_j(x), \quad 0 \leqslant j < k. \end{cases} \tag{2.2}$$

This problem has at most one analytic solution (see [3, Proposition 1,21]):

Proposition 2.1. *Assume that $G, \varphi_0, \dots, \varphi_{k-1}$ are analytic functions near the origin.Then there is at most one analytic function u satisfying* (2.2).

Proof. Functions $\varphi_0, \dots, \varphi_{k-1}$ together with (2.2) determine all the partial derivatives of function u of order $\leqslant k$ on S. Since G is analytic, by differentiating (2.2) with respect to t we have

$$\partial_t^{k+1} u = \frac{\partial G}{\partial t} + \sum_{|\alpha|+j \leqslant k,\, j<k} \frac{\partial G}{\partial u_{(\alpha,j)}} \left(x, t, \left(\partial_x^\alpha \partial_t^j u\right)_{|\alpha|+j \leqslant k,\, j<k}\right) \partial_x^\alpha \partial_t^{j+1} u.$$

All the quantities on the right are known on S, so is $\partial_t^{k+1} u$; hence we know all derivatives of u of order $\leqslant k+1$ on S. Applying ∂_t more times, we obtain higher derivatives. All the partial derivatives of the function u at zero are therefore known and determine u uniquely. $\square$

In our article we focus on the following fundamental existence theorem (see [9], compare [2, Theorem 2 in paragraph 4.6.3], [3, Theorem 1.25]).

Theorem 2.2 (The Cauchy-Kowalevski Theorem)**.** *Assume that $G, \varphi_0, \dots, \varphi_{k-1}$ are analytic functions near the origin. Then there is a neighborhood of the origin on which the Cauchy problem (2.2) has a unique analytic solution.*

Uniqueness of solution was proved in Proposition 2.1. It's proof suggests the construction of solution: determine all the derivatives of u at the origin by differentiating

$$\partial_t^k u = G\left(x, t, \left(\partial_x^\alpha \partial_t^j u\right)_{|\alpha|+j \leqslant k,\, j<k}\right)$$

and plug the results into Taylor's formula. The problem is to show that the resulting power series converges. To this end, it is convenient to replace our k-th order equation by a first order system of differential equations.

Theorem 2.3. *The Cauchy problem (2.2) is equivalent to the Cauchy problem for a certain first order quasi-linear system of partial differential equations of the form*

$$\begin{cases} \partial_t Y = \sum_{j=1}^{n-1} A_j(x,t,Y)\partial_{x_j} Y + B(x,t,Y), \\ Y(x,0) = \Phi(x), \end{cases} \tag{2.3}$$

i.e., a solution to one problem can be read off from a solution to the other. Here Y, B, and Φ are vector-valued functions, the A_j's are matrix-valued functions, and A_j, B, and Φ are explicitly determined by the functions in (2.2).

Proof. Let $Y = (y_{\alpha j})_{0\leqslant|\alpha|+j\leqslant k}$, where $y_{\alpha j}$ will stand for $\partial_x^\alpha \partial_t^j u$ as an independent variable in G. Moreover, for multi-index $\alpha \neq 0$, let $i = i(\alpha)$ denote the smallest index i, for which $\alpha_i \neq 0$ and let $1_i = (\delta_1, \dots, \delta_{n-1})$, where

$$\delta_j = \begin{cases} 1 & \text{for} \quad j = i, \\ 0 & \text{for} \quad j \neq i. \end{cases}$$

The first order system we are looking for is

$$\begin{cases} \partial_t y_{\alpha j} = y_{\alpha(j+1)} & \text{for} \quad |\alpha| + j < k, \\ \partial_t y_{\alpha j} = \partial_{x_{i(\alpha)}} y_{(\alpha - 1_{i(\alpha)})(j+1)} & \text{for} \quad |\alpha| + j = k,\ j < k,\ |\alpha| \neq 0, \\ \partial_t y_{0k} = \frac{\partial G}{\partial t} + \sum\limits_{|\alpha|+j<k} \frac{\partial G}{\partial y_{\alpha j}} y_{\alpha(j+1)} & + \sum\limits_{\substack{|\alpha|+j=k \\ j<k}} \frac{\partial G}{\partial y_{\alpha j}} \partial_{x_{i(\alpha)}} y_{(\alpha - 1_{i(\alpha)})(j+1)}, \end{cases} \tag{2.4}$$

and the initial conditions are

$$\begin{cases} y_{\alpha j}(x,0) = \partial_x^\alpha \varphi_j(x) \quad \text{for} \quad j < k, \\ y_{0k}(x,0) = G\left(x, 0, (\partial_x^\alpha \varphi_j(x))_{|\alpha|+j\leqslant k,\, j<k}\right). \end{cases} \tag{2.5}$$

Obviously, if u is a solution of (2.2), then the functions $y_{\alpha j} = \partial_x^\alpha \partial_t^j u$ satisfy (2.4) and (2.5). Conversely, if the $Y = (y_{\alpha j})_{0\leqslant|\alpha|+j\leqslant k}$ is a solution of (2.4) and (2.5), then $u = y_{00}$ satisfies (2.2). This involves the initial conditions in an essential way.

Observe, that the equation $\partial_t y_{\alpha j} = y_{\alpha(j+1)}$ of system (2.4) implies that

$$y_{\alpha(j+l)} = \partial_t^l y_{\alpha j} \quad \text{for} \quad j + l \leqslant k. \tag{2.6}$$

Then the equation $\partial_t y_{\alpha j} = \partial_{x_{i(\alpha)}} y_{(\alpha - 1_{i(\alpha)})(j+1)}$ of system (2.4) implies

$$\partial_t y_{\alpha j} = \partial_t \partial_{x_i} y_{(\alpha - 1_i) j}, \quad \text{for} \quad |\alpha| + j = k,\ j < k.$$

Therefore

$$y_{\alpha j}(x,t) = \partial_{x_i} y_{(\alpha-1_i)j}(x,t) + c_{\alpha j}(x)$$

for some function $c_{\alpha j}$. But by the first equation of (2.5),

$$y_{\alpha j}(x,0) = \partial_x^\alpha \varphi_j(x) = \partial_{x_i} \partial_x^{\alpha - 1_i} \varphi_j(x) = \partial_{x_i} y_{(\alpha-1_i)j}(x,0),$$

hence $c_{\alpha j} = 0$ and we have,

(2.7) $$y_{\alpha j} = \partial_{x_i} y_{(\alpha - 1_i)j} \quad \text{for} \quad |\alpha| + j = k,\ j < k.$$

Then, from the third equation of (2.4), (2.6) and (2.7), we have

$$\partial_t y_{0k} = \frac{\partial G}{\partial t} + \sum_{\substack{|\alpha|+j \leqslant k \\ j<k}} \frac{\partial G}{\partial y_{\alpha j}} \frac{\partial y_{\alpha j}}{\partial t} = \frac{\partial}{\partial t} \left(G \left(x, t, (y_{\alpha j}) \right) \right),$$

whence

$$y_{0k}(x,t) = G\left(x, t, (y_{\alpha j}(x,t))\right) + c_{0k}(x)$$

for some function c_{0k}. But by (2.5),

$$y_{0k}(x,0) = G\left(x, 0, (\partial_x^\alpha \varphi_j(x))\right) = G\left(x, 0, (y_{\alpha j}(x,0))\right),$$

hence again $c_{0k} = 0$ and we have

(2.8) $$y_{0k} = G\left(x, t, (y_{\alpha j})_{|\alpha|+j \leqslant k,\, j<k}\right).$$

Finally, by induction on $p = k - j - |\alpha|$, we will prove that

(2.9) $$y_{\alpha j} = \partial_{x_i} y_{(\alpha - 1_i)j} \quad \text{for} \quad \alpha \neq 0.$$

For $p = 0$, i.e. when $|\alpha| + j = k$ the above is true from (2.7). From the first equation in (2.4), from (2.6) and from the inductive hypothesis we have

$$\partial_t y_{\alpha j} = y_{\alpha(j+1)} = \partial_{x_i} y_{(\alpha-1_i)(j+1)} = \partial_t \partial_{x_i} y_{(\alpha-1_i)j},$$

hence

$$y_{\alpha j}(x,t) = \partial_{x_i} y_{(\alpha-1_i)j}(x,t) + c_{\alpha j}(x).$$

But by the first equation in (2.5),

$$y_{\alpha j}(x,0) = \partial_x^\alpha \varphi_j(x) = \partial_{x_i} \partial_x^{\alpha - 1_i} \varphi_j(x) = \partial_{x_i} y_{(\alpha-1_i)j}(x,0).$$

Therefore $c_{\alpha j} = 0$ and we get (2.9).

Finally, applying (2.6) and (2.9) repeatedly we obtain that

$$y_{\alpha j} = \partial_x^\alpha \partial_t^j y_{00},$$

and then by (2.8) and the first equation in (2.5) we find that $u = y_{00}$ satisfies (2.2). □

We still need a little simplification.

Theorem 2.4. *The Cauchy problem (2.3) is equivalent to another problem of the same form in which $\Phi = 0$ i $A_1, \ldots, A_{n-1}$ and B do not depend on t.*

Proof. To eliminate Φ we set $U(x,t) = Y(x,t) - \Phi(x)$. Then Y satisfies (2.3) if and only if U satisfies:

$$\partial_t U = \sum_{i=1}^{n-1} \tilde{A}_i(x,t,U) \partial_{x_i} U + \tilde{B}(x,t,U), \qquad U(x,0) = 0,$$

where

$$\begin{aligned}\tilde{A}_i(x,t,U) =& A_i(x,t,U+\Phi),\\ \tilde{B}(x,t,U) =& B(x,t,U+\Phi)+\sum_{i=1}^{n-1} A_i(x,t,U+\Phi)\partial_{x_i}\Phi.\end{aligned}$$

To eliminate variable t from $\tilde{A}_i$ and $\tilde{B}$ we add to U an extra component u^0 satisfying the equation $\partial_t u^0 = 1$ and the initial condition $u^0(x,0)=0$. Then we replace t by u^0 in $\tilde{A}_i$ and $\tilde{B}$, by adding the extra equation and initial condition. □

Let us assume the following designations: $(x,Y) \in \mathbb{R}^{n-1}\times\mathbb{R}^N$ and $(x,t) \in \mathbb{R}^{n-1}\times\mathbb{R}$, where $x=(x_1,\dots,x_{n-1})$, $Y=(y_1,\dots,y_N)$. Since the constructions in these theorems preserve analyticity, we have reduced the Cauchy-Kowalevski theorem to the following theorem. This theorem is well known, but we add a recursive description coefficients of solution as polynomials of the coefficients occurring in the series in the partial differential equation.

Theorem 2.5. *Suppose that $B=[b_m]_{m=1}^N$ is a real analytic vector-valued function and $A_i=\left[a^i_{ml}\right]_{m,l=1}^N$, $i\in\{1,\dots,n-1\}$, are real analytic matrix-valued functions defined on a neighborhood of the origin in $\mathbb{R}^{n-1}\times\mathbb{R}^N$. Then there is a neighborhood U of the origin in $\mathbb{R}^n$, on which the Cauchy problem*

$$\begin{cases}\partial_t Y = \sum\limits_{i=1}^{n-1} A_i(x,Y)\partial_{x_i}Y + B(x,Y),\\ Y(x,0)=0\end{cases} \tag{2.10}$$

has a unique analytic solution $Y=(y_1,\dots,y_N): U\ni(x,t)\mapsto Y(x,t)\in\mathbb{R}^N$. Furthermore, if

$$a^i_{ml}(x,y_1,\dots,y_N)=\sum_{\sigma,\tau} a^{i;\sigma\tau}_{ml}x^\sigma Y^\tau, \qquad b_m(x,y_1,\dots,y_N)=\sum_{\sigma,\tau} b^{\sigma\tau}_m x^\sigma Y^\tau,$$

then coefficients $c^{\alpha j}_m$ of $y_m=\sum_{\alpha,j}c^{\alpha j}_m x^\alpha t^j$ depends polynomially on coefficients of a^i_{ml} and b_m. The dependance is defined inductively in the following way:

$$\begin{aligned}c^{\alpha 0}_m &= 0,\\ c^{\alpha j+1}_m &= \frac{1}{j+1}\left(\sum_{i,l}\sum_{\substack{\mu+\nu=\alpha\\ g+h=j}} P^{a^i_{ml}}_{(\mu,g)}\cdot(\nu_i+1)c^{(\nu+1_i)h}_m + P^{b_m}_{(\alpha,j)}\right),\end{aligned}$$

where

$$\begin{aligned}P^{a^i_{ml}}_{(\alpha,j)}(c^{\beta\lambda}_k) &= P_{(\alpha,j)}\left(\left(a^{i;\sigma\tau}_{ml}\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j},\left(c^{\beta\lambda}_k\right)_{\beta\leqslant\alpha,\lambda\leqslant j,k\leqslant N}\right),\\ P^{b_m}_{(\alpha,j)}(c^{\beta\lambda}_k) &= P_{(\alpha,j)}\left(\left(b^{\sigma\tau}_m\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j},\left(c^{\beta\lambda}_k\right)_{\beta\leqslant\alpha,\lambda\leqslant j,k\leqslant N}\right)\end{aligned}$$

and $P_{(\alpha,j)}$ are polynomials in $(X_{(\sigma,\tau)})_{|\sigma|+|\tau|\leqslant|\alpha|+j}$ and $(Y_{k(\beta,\lambda)})_{k\leqslant N,\beta\leqslant\alpha,\lambda\leqslant j}$ defined inductively by the following conditions:

1. $P_{(0,0)} = X_{(0,0)}$,
2. $P_{(\alpha+1_p,j)} =$

$$= \frac{1}{\alpha_p+1}\left[\sum_{\substack{|\sigma|+|\tau|\\ \leqslant|\alpha|+j}} \frac{1}{\sigma!\tau!}\frac{\partial P_{(\alpha,j)}}{\partial X_{(\sigma,\tau)}}\cdot\left(\sum_{k=1}^{n-1} X_{(\sigma+1_k,\tau)}Y_{k(1_p,0)} + \sum_{k=1}^{N} X_{(\sigma,\tau+1_k)}Y_{k(1_p,0)}\right)\right.$$
$$\left. + \sum_{\substack{k\leqslant n-1+N\\ \beta\leqslant\alpha,\lambda\leqslant j}} \frac{1}{\beta!\lambda!}\frac{\partial P_{(\alpha,j)}}{\partial Y_{k(\beta,\lambda)}}Y_{k(\beta+1_p,\lambda)}\right]$$

and

$$P_{(\alpha,j+1)} =$$
$$= \frac{1}{j+1}\left[\sum_{\substack{|\sigma|+|\tau|\\ \leqslant|\alpha|+j}} \frac{1}{\sigma!\tau!}\frac{\partial P_{(\alpha,j)}}{\partial X_{(\sigma,\tau)}}\cdot\left(\sum_{k=1}^{n-1} X_{(\sigma+1_k,\tau)}Y_{k(0,1)} + \sum_{k=1}^{N} X_{(\sigma,\tau+1_k)}Y_{k(0,1)}\right)\right.$$
$$\left. + \sum_{\substack{k\leqslant n-1+N\\ \beta\leqslant\alpha,\lambda\leqslant j}} \frac{1}{\beta!\lambda!}\frac{\partial P_{(\alpha,j)}}{\partial Y_{k(\beta,\lambda)}}Y_{k(\beta,\lambda+1)}\right].$$

The theorem will be preceded by two lemmas.

Lemma 2.6. *Let $f(x) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha$ be an analytic function in a neighbourhood of $0 \in \mathbb{R}^n$ and let $g_k(\xi) = \sum_{\beta\in\mathbb{N}^m} b_{k;\beta}\xi^\beta$, $k = 1,\dots,n$, be an analytic functions in a neighbourhood of $0 \in \mathbb{R}^m$ such that $g_k(0) = 0$. Then the function $F(\xi) = f(g_1(\xi),\dots,g_n(\xi))$ is analytic in a neighbourhood of $0 \in \mathbb{R}^m$, and it's Taylor expansion takes a form*

$$F(\xi) = \sum_{\gamma\in\mathbb{N}^m} P_\gamma\left((a_\alpha)_{|\alpha|\leqslant|\gamma|},(b_{k;\beta})_{\beta\leqslant\gamma,k\leqslant n}\right)\xi^\gamma,$$

where $P_\gamma \in \mathbb{Z}[(X_\alpha)_{|\alpha|\leqslant|\gamma|},(Y_{k\beta})_{\beta\leqslant\gamma,1\leqslant k\leqslant n}]$, $\gamma \in \mathbb{N}^m$ are polynomials with non-negative integer coefficients defined by the following induction conditions:

(1) $P_0(X_0,Y_{10},\dots,Y_{n0}) = X_0$, *where* $X_0 = X_{(0,\dots,0)}$ *and* $0 = (0,\dots,0) \in \mathbb{N}^n$.
(2) *If the polynomial* $P_\gamma = P_\gamma\left((X_\alpha)_{|\alpha|\leqslant|\gamma|},(Y_{k\beta})_{\beta\leqslant\gamma,1\leqslant k\leqslant n}\right)$, *then the polynomial $P_{\gamma+1_j}$ is of the form*

$$P_{\gamma+1_j} = P_{\gamma+1_j}\left((X_\alpha)_{|\alpha|\leqslant|\gamma|+1},(Y_{k\beta})_{\beta\leqslant\gamma+1_j,1\leqslant k\leqslant n}\right),$$

where

$$P_{\gamma+1_j} = \frac{1}{\gamma_j+1}\left(\sum_{|\alpha|\leqslant|\gamma|}\left(\frac{1}{\alpha!}\frac{\partial P_\gamma}{\partial X_\alpha}\cdot\sum_{k=1}^{n} X_{\alpha+1_k}Y_{k1_j}\right) + \sum_{\beta\leqslant\gamma}\sum_{k=1}^{n}\frac{1}{\beta!}\frac{\partial P_\gamma}{\partial Y_{k\beta}}\cdot Y_{k\beta+1_j}\right).$$

Proof. Obviously,

$$a_\alpha = \frac{\partial^\alpha f(0)}{\alpha!}, \quad b_{k;\beta} = \frac{\partial^\beta g_k(0)}{\beta!} \quad \text{for} \quad 1 \leqslant k \leqslant n. \tag{2.11}$$

Let $F(\xi) = \sum_\gamma c_\gamma \xi^\gamma$. Then

$$c_\gamma = \frac{\partial^\gamma F(0)}{\gamma!}. \tag{2.12}$$

Let $1_\alpha = (\delta_\kappa)_{\kappa \in \mathbb{N}^n}$, where $\delta_\alpha = 1$ and $\delta_\kappa = 0$ for $\kappa \neq \alpha$. Clearly

$$\partial^\gamma = \prod_{j=1}^{m} \left(\partial^{1_j}\right)^{\gamma_j}.$$

The above lemma arises from the fact that

$$\partial^{1_j} F = \sum_{k=1}^{n} \partial^{1_k} f \cdot \partial^{1_j} g_k$$

by induction on $|\gamma|$. Indeed, it suffices to show that, for every γ there exists a polynomial Q_γ with variables X_α, $|\alpha| \leqslant |\gamma|$ and $Y_{k\beta}$, $1 \leqslant k \leqslant n$, $\beta \leqslant \gamma$ with non-negative integer coefficients such that

$$\partial^\gamma F(\xi) = Q_\gamma \left(\left(\partial^\alpha f(g(\xi))\right)_{|\alpha| \leqslant |\gamma|}, \left(\partial^\beta g_k(\xi)\right)_{\beta \leqslant \gamma, 1 \leqslant k \leqslant n} \right) \tag{2.13}$$

and $\deg Q_\gamma \leqslant |\gamma| + 1$. For $|\gamma| = 0$ that is, for $\gamma = 0$ we have

$$\partial^0 F(\xi) = f(g_1(\xi), \ldots, g_n(\xi)),$$

hence we set $Q_0(X_0, Y_{10}, \ldots, Y_{n0}) = X_0$, where $\deg Q_0 = 1$. If (2.13) holds for $|\gamma| = p$, then for $|\gamma| = p+1$ multi-index γ can be written as $\gamma = \overline{\gamma} + 1_j$, where $|\overline{\gamma}| = p$ for some $j \in \{1, \ldots, m\}$. Therefore, induction hypothesis implies

$$\partial^\gamma F(\xi) = \partial^{1_j} \partial^{\overline{\gamma}} F(\xi) = \sum_{|\alpha| \leqslant |\bar{\gamma}|} \left(\partial^{1_\alpha} Q_{\overline{\gamma}} \cdot \sum_{k=1}^{n} \partial^{\alpha + 1_k} f\left(g(\xi)\right) \partial^{1_j} g_k(\xi) \right) +$$
$$+ \sum_{\beta \leqslant \bar{\gamma}} \sum_{k=1}^{n} \left(\partial^{1_\beta} Q_{\overline{\gamma}} \cdot \partial^{\beta + 1_j} g_k(\xi) \right).$$

The right-hand side of this equation is a polynomial with non-negative integer coefficients of variables $\left(\partial^\alpha f(g(\xi))\right)_{|\alpha| \leqslant |\gamma|}$ and $\left(\partial^\beta g_k(\xi)\right)_{\beta \leqslant \gamma, k \leqslant n}$. It's degree is $\leqslant |\gamma| + 1$. Thus, it is a searched polynomial Q_γ for $|\gamma| = p+1$. Induction ends the above reasoning. By (2.13),(2.11) and (2.12) we obtain

$$c_\gamma = \frac{1}{\gamma!} Q_\gamma \left(\left(\partial^\alpha f(0)\right)_{|\alpha| \leqslant |\gamma|}, \left(\partial^\beta g_k(0)\right)_{\beta \leqslant \gamma, k \leqslant n} \right) =$$
$$= \frac{1}{\gamma!} Q_\gamma \left((\alpha! a_\alpha)_{|\alpha| \leqslant |\gamma|}, (\beta! b_{k;\beta})_{\beta \leqslant \gamma, k \leqslant n} \right).$$

Then the right-hand side of the above formula is a searched polynomial P_γ. Moreover, from the above formula, we can easily read the inductive conditions describing polynomial P_γ of variables $(X_\alpha)_{|\alpha|\leqslant|\gamma|}, (Y_{k\beta})_{\beta\leqslant\gamma,k\leqslant n}$:

$$P_0 = X_0,$$

$$P_{\gamma+1_j} = \frac{1}{\gamma_j+1}\left(\sum_{|\alpha|\leqslant|\gamma|}\left(\frac{1}{\alpha!}\frac{\partial P_\gamma}{\partial X_\alpha}\cdot\sum_{k=1}^{n} X_{\alpha+1_k}Y_{k1_j}\right) + \sum_{\substack{k\leqslant n\\ \beta\leqslant\gamma}}\frac{1}{\beta!}\frac{\partial P_\gamma}{\partial Y_{k\beta}}\cdot Y_{k\beta+1_j}\right)$$

because $P_\gamma\left((X_\alpha)_{|\alpha|\leqslant|\gamma|}, (Y_{k\beta})_{\beta\leqslant\gamma,k\leqslant n}\right) = \frac{1}{\gamma!}Q_\gamma\left((\alpha!X_\alpha)_{|\alpha|\leqslant|\gamma|}, (\beta!Y_{k;\beta})_{\beta\leqslant\gamma,k\leqslant n}\right)$ and

$$Q_{\gamma+1_j}\left((\alpha!X_\alpha)_{|\alpha|\leqslant|\gamma|+1}, (\beta!Y_{k\beta})_{\beta\leqslant\gamma+1_j,k\leqslant n}\right) = \sum_{|\alpha|\leqslant|\gamma|}\left(\frac{\gamma!}{\alpha!}\frac{\partial P_\gamma}{\partial X_\alpha}\cdot\sum_{k=1}^{n} X_{\alpha+1_k}Y_{k1_j}\right) + \\ + \sum_{\substack{k\leqslant n\\ \beta\leqslant\gamma}}\frac{\gamma!}{\beta!}\frac{\partial P_\gamma}{\partial Y_{k\beta}}\cdot Y_{k\beta+1_j}.$$

□

We say that a power series $\sum a_\alpha(x-x^0)^\alpha$ with non-negative coefficients *majorize* power series $\sum b_\alpha(x-x^0)^\alpha$, if $|b_\alpha| \leqslant a_\alpha$ for every multi-index α. In this case the series $\sum b_\alpha(x-x^0)^\alpha$ is absolutely convergent everywhere the series $\sum a_\alpha(x-x^0)^\alpha$ is absolutely convergent. We say that the series $a = \sum a_\alpha(x-x^0)^\alpha$ is *a majorant* of series $b = \sum b_\alpha(x-x^0)^\alpha$ and we write $a \ll b$, after Poincaré. Similarly, for $A = [a_i]_{i\in I}$ and $B = [b_i]_{i\in I}$ symbol $A \ll B$ means that $a_i \ll b_i$ for every $i \in I$.

Lemma 2.7. *Suppose that the series $\sum a_\alpha x^\alpha$ is convergent in*

$$T_R = \{x:\ \max_{j=1}^{n}|x_j| < R\}.$$

Then for every positive number $r < R$ end every $M \geqslant \sup\{|a_\alpha|r^{|\alpha|}:\ \alpha\in\mathbb{N}^n\}$, the geometric series

$$\sum_{\alpha\in\mathbb{N}^n}\frac{M|\alpha|!}{\alpha!r^{|\alpha|}}x^\alpha$$

is convergent in $T_{r/n} = \{x:\ \max|x_j| < r/n\}$ to the function

$$T_{r/n}\ni x\mapsto\frac{Mr}{r-(x_1+\ldots+x_n)}\in\mathbb{R}$$

and majorize series $\sum a_\alpha x^\alpha$.

Proof. Let r be a positive number less than R. Then the series $\sum a_\alpha r^{|\alpha|}$ is convergent and for every $M > 0$ such that $|a_\alpha r^{|\alpha|}| \leqslant M$ for all α we have

$$|a_\alpha| \leqslant \frac{M}{r^{|\alpha|}} \leqslant \frac{M|\alpha|!}{\alpha!r^{|\alpha|}}.$$

On the other hand, function

$$f(x) = \frac{Mr}{r - (x_1 + \ldots + x_n)}$$

is analytic in $T_{r/n}$ and for $x \in T_{r/n}$

$$f(x) = M \sum_{k=0}^{\infty} \frac{(x_1 + \ldots + x_n)^k}{r^k} = \sum_{|\alpha| \geqslant 0} \frac{M|\alpha|!}{\alpha! r^{|\alpha|}} x^\alpha.$$

This ends the proof. $\square$

Let's move on to the proof of the Theorem 2.5.

Proof of Theorem 2.5. We are looking for the solution $Y = (y_1, \ldots, y_N)$ of the Cauchy problem (2.10), where

$$y_m = \sum_{\alpha,j} c_m^{\alpha j} x^\alpha t^j \quad \text{for} \quad 1 \leqslant m \leqslant N. \tag{2.14}$$

Obviously,

$$c_m^{\alpha j} = \frac{\partial_x^\alpha \partial_t^j y_m(0,0)}{\alpha! j!}.$$

The initial conditions implies that $c_m^{\alpha 0} = 0$ for every α and m. In order to determine the coefficients $c_m^{\alpha j}$ for $j > 0$, we substitute (2.14) to the differential equations

$$\partial_t y_m = \sum_{i,l} a_{ml}^i(x, y_1, \ldots, y_N) \partial_{x_i} y_l + b_m(x, y_1, \ldots, y_N). \tag{2.15}$$

Let

$$\begin{aligned} a_{ml}^i(x, y_1, \ldots, y_N) &= \sum_{\sigma,\tau} a_{ml}^{i;\sigma\tau} x^\sigma Y^\tau, \\ b_m(x, y_1, \ldots, y_N) &= \sum_{\sigma,\tau} b_m^{\sigma\tau} x^\sigma Y^\tau, \\ \partial_{x_i} y_l &= \sum_{\alpha,j} (\alpha_i + 1) c_l^{(\alpha + 1_i)j} x^\alpha t^j. \end{aligned} \tag{2.16}$$

Lemma 2.6 implies that a_{ml}^i is a power series in x and t, whose coefficients of $x^\alpha t^j$ are polynomials with non-negative rational coefficients in $(a_{ml}^{i;\sigma\tau})_{|\sigma|+|\tau| \leqslant |\alpha|+j}$ and $(c_k^{\beta\lambda})_{\beta \leqslant \alpha, \lambda \leqslant j, k \leqslant N}$. Moreover, the coefficients of the terms in which t occurs to the j-th power only involve the $c_k^{\beta\lambda}$ with $\lambda \leqslant j$. The same is true for the series obtained from b_m and $\partial_{x_i} y_l$, and multiplying a_{ml}^i by $\partial_{x_i} y_l$ still preserves these properties.

Roughly speaking, on the right side of (2.15) we obtain an expression of the form

$$\sum_{\alpha,j} P_m^{\alpha j} \left(\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau} \right)_{|\sigma|+|\tau| \leqslant |\alpha|+j, i \leqslant n-1, l \leqslant N}, \left(c_k^{\beta\lambda} \right)_{\beta \leqslant \alpha, \lambda \leqslant j, k \leqslant N} \right) x^\alpha t^j,$$

where $P_m^{\alpha j}$ is a polynomial with non-negative coefficients. On the left side, we have

$$\partial_t y_m = \sum_{\alpha,j} (j+1) c_m^{\alpha(j+1)} x^\alpha t^j.$$

Hence,

$$c_m^{\alpha(j+1)} = \frac{P_m^{\alpha j}\left(\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau}\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j, i\leqslant n-1, l\leqslant N}, \left(c_k^{\beta\lambda}\right)_{\beta\leqslant\alpha,\lambda\leqslant j,k\leqslant N}\right)}{j+1},$$

so if we know that $c_k^{\beta\lambda}$ with $\lambda \leqslant j$, we can determine the $c_k^{\beta\lambda}$ with $\lambda = j+1$. Proceeding inductively, we determine all the $c_m^{\alpha j}$ and we find that

$$\begin{aligned} c_m^{\alpha j} &= Q_m^{\alpha j}\left(\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau}\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j, i\leqslant n-1, l\leqslant N}, \left(c_k^{\beta\lambda}\right)_{\beta\leqslant\alpha,\lambda<j,k\leqslant N}\right) = \\ &= Q_m^{\alpha j}\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau}, c_k^{\beta\lambda}\right), \end{aligned}$$

where $Q_m^{\alpha j}$ is a polynomial with nonnegative coefficients in $c_k^{\beta\lambda}$, where $\lambda < j$. More precisely the coefficients of $x^\alpha t^j$ of power series a_{ml}^i i b_m are polynomials with non-negative rational coefficients of the form:

$$\begin{aligned} P_{(\alpha,j)}^{a_{ml}^i}(c_k^{\beta\lambda}) &= P_{(\alpha,j)}\left(\left(a_{ml}^{i;\sigma\tau}\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j}, \left(c_k^{\beta\lambda}\right)_{\beta\leqslant\alpha,\lambda\leqslant j,k\leqslant N}\right), \\ P_{(\alpha,j)}^{b_m}(c_k^{\beta\lambda}) &= P_{(\alpha,j)}\left(\left(b_m^{\sigma\tau}\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j}, \left(c_k^{\beta\lambda}\right)_{\beta\leqslant\alpha,\lambda\leqslant j,k\leqslant N}\right), \end{aligned}$$

where $P_{(\alpha,j)}$ is a polynomial in $(X_{(\sigma,\tau)})_{|\sigma|+|\tau|\leqslant|\alpha|+j}$ and $(Y_{k(\beta,\lambda)})_{k\leqslant N,\beta\leqslant\alpha,\lambda\leqslant j}$ defined inductively by the following conditions:

1. $P_{(0,0)} = X_{(0,0)}$,
2. $P_{(\alpha+1_p,j)} =$

$$\begin{aligned} &= \frac{1}{\alpha_p+1}\left[\sum_{\substack{|\sigma|+|\tau|\\ \leqslant|\alpha|+j}} \frac{1}{\sigma!\tau!}\frac{\partial P_{(\alpha,j)}}{\partial X_{(\sigma,\tau)}} \cdot \left(\sum_{k=1}^{n-1} X_{(\sigma+1_k,\tau)} Y_{k(1_p,0)} + \sum_{k=1}^{N} X_{(\sigma,\tau+1_k)} Y_{k(1_p,0)}\right)\right. \\ &\left.+ \sum_{\substack{k\leqslant n-1+N\\ \beta\leqslant\alpha,\lambda\leqslant j}} \frac{1}{\beta!\lambda!}\frac{\partial P_{(\alpha,j)}}{\partial Y_{k(\beta,\lambda)}} Y_{k(\beta+1_p,\lambda)}\right] \end{aligned}$$

and

$$P_{(\alpha,j+1)} =$$
$$= \frac{1}{j+1}\Bigg[\sum_{\substack{|\sigma|+|\tau|\\ \leqslant|\alpha|+j}} \frac{1}{\sigma!\tau!}\frac{\partial P_{(\alpha,j)}}{\partial X_{(\sigma,\tau)}}\cdot\left(\sum_{k=1}^{n-1} X_{(\sigma+1_k,\tau)}Y_{k(0,1)} + \sum_{k=1}^{N} X_{(\sigma,\tau+1_k)}Y_{k(0,1)}\right)$$
$$+ \sum_{\substack{k\leqslant n-1+N\\ \beta\leqslant\alpha,\lambda\leqslant j}} \frac{1}{\beta!\lambda!}\frac{\partial P_{(\alpha,j)}}{\partial Y_{k(\beta,\lambda)}} Y_{k(\beta,\lambda+1)}\Bigg].$$

Therefore,

$$P_m^{\alpha j}\left(\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau}\right)_{|\sigma|+|\tau|\leqslant|\alpha|+j, i\leqslant n-1, l\leqslant N}, \left(c_k^{\beta\lambda}\right)_{\beta\leqslant\alpha,\lambda\leqslant j,k\leqslant N}\right) =$$
$$= \sum_{i,l}\sum_{\substack{\mu+\nu=\alpha\\ g+h=j}} P_{(\mu,g)}^{a_{ml}^{i}}\cdot(\nu_i+1)c_m^{(\nu+1_i)h} + P_{(\alpha,j)}^{b_m},$$

thereby $c_m^{\alpha 0} = Q_m^{\alpha 0} = 0$ and

$$c_m^{\alpha j+1} = Q_m^{\alpha j+1}\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau}, c_k^{\beta\lambda}\right) =$$
$$= \frac{1}{j+1}\left(\sum_{i,l}\sum_{\substack{\mu+\nu=\alpha\\ g+h=j}} P_{(\mu,g)}^{a_{ml}^{i}}\cdot(\nu_i+1)c_m^{(\nu+1_i)h} + P_{(\alpha,j)}^{b_m}\right).$$

Now, to show convergence of Y, it suffices to find the Cauchy problem

$$\begin{cases} \partial_t\tilde{Y} = \sum\limits_{i=1}^{n-1}\tilde{A}_i(x,\tilde{Y})\partial_{x_i}\tilde{Y} + \tilde{B}(x,\tilde{Y}),\\ \tilde{Y}(x,0) = 0,\end{cases}$$

(where $\tilde{A}_i$ and $\tilde{B}$ are analytic equivalents of A_i and B respectively), for which:

a) there exists the analytic solution $\tilde{Y}$ nearby $(0,0)$;

b) $A_i \ll \tilde{A}_i$ and $B \ll \tilde{B}$.

Indeed, the solution $\tilde{Y} = (\tilde{y}_1,\ldots,\tilde{y}_N)$ of this problem has the form $\tilde{y}_m = \sum \tilde{c}_m^{\alpha j}x^\alpha t^j$, $m = 1,\ldots,N$, where

$$\tilde{c}_m^{\alpha j} = Q_m^{\alpha j}\left(\tilde{a}_{ml}^{i;\sigma\tau}, \tilde{b}_m^{\sigma\tau}, \tilde{c}_k^{\beta\lambda}\right),$$

and $Q_m^{\alpha j}$ are polynomials defined for the preceding Cauchy problem. Since $Q_m^{\alpha j}$ has non-negative coefficients and depends only on $c_k^{\beta\lambda}$, where $\lambda < j$, then we can easily show by induction that:

$$
\begin{aligned}
|c_m^{\alpha j}| = \left|Q_m^{\alpha j}\left(a_{ml}^{i;\sigma\tau}, b_m^{\sigma\tau}, c_k^{\beta\lambda}\right)\right| &\leqslant Q_m^{\alpha j}\left(|a_{ml}^{i;\sigma\tau}|, |b_m^{\sigma\tau}|, |c_k^{\beta\lambda}|\right) \\
&\leqslant Q_m^{\alpha j}\left(\tilde{a}_{ml}^{i;\sigma\tau}, \tilde{b}_m^{\sigma\tau}, \tilde{c}_k^{\beta,\lambda}\right) = \tilde{c}_m^{\alpha j}.
\end{aligned}
$$

Therefore $\tilde{Y}$ majorize Y which gives convergence of Y in some neighbourhood of $(0,0)$.

We will construct such a majorizing system. Let $M > 0$ be sufficiently large and $r > 0$ sufficiently small so that by Lemma 2.7 series for A_i and B are all majorized by the series for

$$
\frac{Mr}{r - (x_1 + \ldots + x_{n-1}) - (y_1 + \ldots + y_N)}.
$$

Thus we consider the following Cauchy problem: for $m = 1, \ldots, N$,

$$
\begin{cases} \partial_t y_m = \frac{Mr}{r - \sum x_i - \sum y_l}\left(\sum\limits_i \sum\limits_l \partial_{x_i} y_l + 1\right), \\ y_m(x, 0) = 0. \end{cases} \tag{2.17}
$$

To determine the solution of this Cauchy problem it is enough to solve the Cauchy problem consisting of one equation

$$
\begin{cases} \partial_t u = \frac{Mr}{r - s - Nu}\left(N(n-1)\partial_s u + 1\right), \\ u(s, 0) = 0, \end{cases} \tag{2.18}
$$

for if we will put

$$
y_j(x, t) = u(x_1 + \ldots + x_{n-1}, t) \qquad (j = 1, \ldots, N),
$$

we obtain that $Y = (y_1, \ldots, y_N)$ satisfies (2.17). We will transform (2.18) to

$$
(r - s - Nu)\partial_t u - MrN(n-1)\partial_s u = Mr,
$$

and will solve this by method of characteristics:

$$
\frac{dt}{d\tau} = r - s - Nu, \qquad \frac{ds}{d\tau} = -MrN(n-1), \qquad \frac{du}{d\tau} = Mr
$$

with the initial conditions:

$$
t(0) = 0, \qquad s(0) = \sigma, \qquad u(0) = 0.
$$

The solution of the above is given by the formulas:

$$
t = \frac{1}{2}MrN(n-2)r^2 + (r - \sigma)\tau, \qquad s = -MrN(n-1)\tau + \sigma, \qquad u = Mr\tau.
$$

The elimination of σ and τ yields

$$
u(s, t) = \frac{r - s - \sqrt{(r-s)^2 - 2MrNnt}}{Mn}.
$$

Clearly this is analytic for s and t near 0, so the proof is complete. $\square$

There remains the question of whether the Cauchy problem (2.2) might admit non-analytic solutions as well. In the linear case, the answer is negative: this is the Holmgren uniqueness theorem. The proof can be found in John [5], Hörmander [6], [7, vol I], or Treves [14].

A major drawback of the Cauchy-Kowalevski theorem is that it gives little control over the dependence of the solution on the Cauchy data.

Example 2.8. Consider the following example in $\mathbb{R}^2$, due to Hadamard:

$$\begin{cases} \partial_1^2 u + \partial_2^2 u = 0 \\ u(x_1, 0) = 0, \qquad \partial_2 u(x_1, 0) = ke^{-\sqrt{k}} \sin kx_1, \end{cases}$$

where $k > 0$. One easily checks that the solution is

$$u(x_1, x_2) = e^{-\sqrt{k}} (\sin kx_1)(\sinh kx_2).$$

As $k \to \infty$, the Cauchy data and their derivatives of all orders tend uniformly to zero since $e^{-\sqrt{k}}$ decays faster than polynomially. But if $x_2 > 0$, then

$$\lim_{k\to\infty} e^{-\sqrt{k}} \sinh kx_2 = +\infty.$$

The solution for the limiting case $k = +\infty$ is of course $u \equiv 0$. This example shows that the solution of the Cauchy problem may not depend continuously on the Cauchy data in most of the usual topologies on functions.

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