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DIVERGENCE-FREE POLYNOMIAL DERIVATIONS

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Abstract. In this paper we present some new and old properties of divergences and divergence-free derivations.

Throughout the paper all rings are commutative with unity. Let k be a ring and let d be a k-derivation of the polynomial ring $k[X] = k[x_1, \dots, x_n]$. We denote by $d^\star$ *the divergence of* d, that is,

$$d^\star = \frac{\partial d(x_1)}{\partial x_1} + \cdots + \frac{\partial d(x_n)}{\partial x_n}.$$

The derivation d is said to be *divergence-free* if $d^\star = 0$.

1. Preliminaries

Let k be a ring, and let R be a k-algebra. A k-linear mapping $d : R \to R$ is said to be a *k-derivation* of R if

$$d(ab) = ad(b) + d(a)b,$$

for all $a, b \in R$. We denote by $\mathrm{Der}_k(R)$ the set of all k-derivations of R. If $d, d_1, d_2 \in \mathrm{Der}_k(R)$ and $x \in R$, then the mappings xd, $d_1 + d_2$ and $[d_1, d_2] = d_1d_2 - d_2d_1$ are also k-derivations of R. Thus, the set $\mathrm{Der}_k(R)$ is an R-module which is also a Lie algebra.

We denote by R^d the kernel of d, that is,

$$R^d = \Big\{a \in R;\ d(a) = 0\Big\}$$

This set is a subring of R, called *the ring of constants of* R (*with respect to* d). If R is a field, then R^d is a subfield of R.

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Now let $k[X] = k[x_1, \dots, x_n]$ be a polynomial ring in n variables over a ring k. For each $i \in \{1, \dots, n\}$ the partial derivative $\frac{\partial}{\partial x_i}$ is a k-derivation of $k[X]$. It is a unique k-derivation d of $k[X]$ such that $d(x_i) = 1$ and $d(x_j) = 0$ for all $j \neq i$. If $f_1, \dots, f_n$ are polynomials belonging to $k[X]$, then the mapping

$$f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$$

is a k-derivation of $k[X]$. It is a k-derivation d of $k[X]$ such that $d(x_j) = f_j$ for all $j = 1, \dots, n$. It is not difficult to show that every k-derivation of $k[X]$ is of the above form. As a consequence of this fact we know that $\mathrm{Der}_k(k[X])$ is a free $k[X]$-module on the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. If $d \in \mathrm{Der}_k(k[X])$ and $f \in k[X]$, then

$$d(f) = \frac{\partial f}{\partial x_1} d(x_1) + \cdots + \frac{\partial f}{\partial x_n} d(x_n).$$

Now assume that k is a domain containing $\mathbb{Q}$ and d is a k-derivation of $k[X]$. We say that $F \in k[X]$ is a *Darboux polynomial of* d if $F \neq 0$ and $d(F) = \Lambda F$, for some $\Lambda \in k[X]$. In this case such Λ is unique and it is said to be the *cofactor* of F. Every nonzero element belonging to the ring of constants $k[X]^d$ is of course a Darboux polynomial. If $F_1, F_2 \in k[X] \setminus \{0\}$ are Darboux polynomials of d then the product $F_1 F_2$ is also a Darboux polynomial of d. The cofactor of $F_1 F_2$ is in this case the sum of the cofactors of F_1 and F_2. If $F \in k[X] \setminus k$ is a Darboux polynomial of d, then all factors of F are also Darboux polynomials of d. Thus, looking for Darboux polynomials of d reduces to looking for irreducible ones.

For a discussion of Darboux polynomial in a more general setting, the reader is referred to [15], [19], [13], [14].

A k-derivation d of $k[X]$ is called *homogeneous of degree* s if all the polynomials $d(x_1), \dots, d(x_n)$ are homogeneous of degree s. In particular, each partial derivative $\frac{\partial}{\partial x_i}$ is homogeneous of degree 0. The zero derivation is homogeneous of every degree. The sum of homogeneous derivations of the same degree s is homogeneous of degree s. Note some basic properties of homogeneous derivations (see [19] for proofs and details).

Proposition 1.1. *Let d be a homogeneous k-derivation of $k[X]$ and let $F \in k[X]$. If $F \in k[X]^d$, then each homogeneous component of F belongs also to $k[X]^d$. In particular, the ring $k[X]^d$, is generated over k by homogeneous polynomials.*

Proposition 1.2. *Let d be a homogeneous k-derivation of $k[X]$, where k is a domain containing $\mathbb{Q}$, and let $0 \neq F \in k[X]$ be a Darboux polynomial of d with the cofactor $\Lambda \in k[X]$. Then Λ is homogeneous, and all homogeneous components of F are also Darboux polynomials with the common cofactor equal to Λ.*

Note that Darboux polynomials of a homogeneous derivation are not necessarily homogeneous. Indeed, let $n = 2$, $d(x_1) = x_1$, $d(x_2) = 2x_2$, and let $F = x_1^2 + x_2$. Then d is homogeneous, F is a Darboux polynomial of d (because $d(F) = 2F$), and F is not homogeneous.

2. Basic properties of divergences

Let k be a ring and let d be a k-derivation of the polynomial ring $k[X] = k[x_1, \dots, x_n]$. Let us recall that we denote by $d^\star$ *the divergence of* d, that is,

$$d^\star = \frac{\partial d(x_1)}{\partial x_1} + \cdots + \frac{\partial d(x_n)}{\partial x_n}.$$

We say that the derivation d is *divergence-free* if $d^\star = 0$. For example, every partial derivative $\frac{\partial}{\partial x_i}$ is a divergence-free k-derivation of $k[X]$. It is clear that $(d+\delta)^\star = d^\star + \delta^\star$ for all $d, \delta \in \mathrm{Der}_k(k[X])$. Thus, the sum of divergence-free derivations is also a divergence-free derivation.

Proposition 2.1. *If $d \in \mathrm{Der}_k(k[X])$ and $r \in k[X]$, then:*

$$(rd)^\star = rd^\star + d(r).$$

Proof. $(rd)^\star = \sum_{p=1}^{n} \frac{\partial rd(x_p)}{\partial x_p} = \sum_{p=1}^{n} \left(r\frac{\partial d(x_p)}{\partial x_p} + \frac{\partial r}{\partial x_p} d(x_p) \right) = r\sum_{p=1}^{n} \frac{\partial d(x_p)}{\partial x_p}$
$+ \sum_{p=1}^{n} \frac{\partial r}{\partial x_p} d(x_p) = rd^\star + d(r).$ □

Thus, if d is a divergence-free k-derivation of $k[X]$ and $r \in k[X]^d$, then the derivation rd is divergence-free.

Proposition 2.2. *Let $d, \delta \in \mathrm{Der}_k(k[X])$ and let $[d, \delta] = d\delta - \delta d$. Then*

$$[d,\delta]^\star = d(\delta^\star) - \delta(d^\star).$$

Proof. Put $f_i = d(x_i)$, $g_i = \delta(x_i)$ for $i = 1, \dots, n$, and observe that

$$\sum_{p=1}^{n}\sum_{i=1}^{n} \frac{\partial g_p}{\partial x_i}\frac{\partial f_i}{\partial x_p} = \sum_{p=1}^{n}\sum_{i=1}^{n} \frac{\partial f_p}{\partial x_i}\frac{\partial g_i}{\partial x_p}.$$

Thus, we have

$$\begin{aligned}
[d,\delta]^* &= \sum_{p=1}^{n} \frac{\partial}{\partial x_p}\Big((d\delta - \delta d)(x_p)\Big) = \sum_{p=1}^{n} \frac{\partial}{\partial x_p}\Big(d(g_p) - \delta(f_p)\Big) \\
&= \sum_{p=1}^{n} \frac{\partial}{\partial x_p}\left(\sum_{i=1}^{n} \frac{\partial g_p}{\partial x_i} f_i - \sum_{i=1}^{n} \frac{\partial f_p}{\partial x_i} g_i\right) \\
&= \sum_{p=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_p}\frac{\partial g_p}{\partial x_i}\cdot f_i + \frac{\partial g_p}{\partial x_i}\frac{\partial f_i}{\partial x_p} - \frac{\partial}{\partial x_p}\frac{\partial f_p}{\partial x_i}\cdot g_i - \frac{\partial f_p}{\partial x_i}\frac{\partial g_i}{\partial x_p}\right) \\
&= \sum_{p=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_p}\frac{\partial g_p}{\partial x_i}\cdot f_i - \frac{\partial}{\partial x_p}\frac{\partial f_p}{\partial x_i}\cdot g_i\right) \\
&= \sum_{p=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}\frac{\partial g_p}{\partial x_p}\cdot f_i - \frac{\partial}{\partial x_i}\frac{\partial f_p}{\partial x_p}\cdot g_i\right) \\
&= \sum_{p=1}^{n}\left(d\left(\frac{\partial g_p}{\partial x_p}\right) - \delta\left(\frac{\partial f_p}{\partial x_p}\right)\right) = d\left(\sum_{p=1}^{n}\frac{\partial g_p}{\partial x_p}\right) - \delta\left(\sum_{p=1}^{n}\frac{\partial f_p}{\partial x_p}\right) \\
&= d\left(\delta^\star\right) - \delta\left(d^\star\right).
\end{aligned}$$

This completes the proof. □

The above propositions imply that the set of all divergence-free derivations of $k[X]$ is closed under the sum and the Lie product.

Let d be a k-derivation of $k[X]$. Given a polynomial $f \in k[X]$, we denote by V_f, the k-submodule of $k[X]$ generated by the set $\{f, d(f), d^2(f), d^3(f), \dots\}$. The derivation d is called *locally finite*, if every module V_f, for all $f \in k[X]$, is a finitely generated over k. The derivation d is called *locally nilpotent*, if for every $f \in k[X]$ there exists a positive integer m such that $d^m(f) = 0$. Every locally nilpotent derivation is locally finite. There exist, of course, locally finite derivations which are not locally nilpotent. Locally finite and locally nilpotent derivations was intensively studied from a long time; see for example [7], [6], [12], [19], where many references on this subject can be found.

The following result is due to H. Bass, G. Meisters [2] and B. Coomes, V. Zurkowski [4]. Another its proof is given in [19] (Theorem 9.7.3).

Theorem 2.3. *Let k be a reduced ring containing $\mathbb{Q}$. If d is a locally finite k-derivation of $k[X] = k[x_1, \dots, x_n]$, then $d^\star$, the divergence of d, is an element of k.*

Recall that a ring k is called *reduced* if k has no nonzero nilpotent elements. If k is non-reduced then the above property does not hold, in general.

Example 2.4. Let $k = \mathbb{Q}[y]/(y^2)$ and let d be the k derivation of $k[x]$ (a polynomial ring in a one variable) defined by $d(x) = ax^2$, where $a = y + (y^2)$. Since $d^2(x) = 2a^2x^3 = 0$, d is locally finite. But $d^\star = 2ax \notin k$.

Note the following important property of locally nilpotent derivations.

Theorem 2.5. ([19], [6]). *If k is a reduced ring containing $\mathbb{Q}$, then every locally nilpotent k-derivation of $k[X]$ is divergence-free.*

The derivation d from Example 2.4 is locally nilpotent. This means that if k is non-reduced then there exist locally nilpotent k-derivations of $k[X]$ with a nonzero divergence.

In the paper of Berson, van den Essen, and Maubach [3] is quoted the following result, which is related to their investigation of the Jacobian Conjecture.

Theorem 2.6. ([3]). *Let k be any commutative $\mathbb{Q}$-algebra, and let d be a k-derivation of $k[x, y]$. If d is surjective and divergence-free, then d is locally nilpotent.*

This result was shown earlier by Stein [21] in the case k is a field.

3. Divergences and jacobians

If $h_1, \dots, h_n$ are polynomials belonging to $k[X] = k[x_1, \dots, x_n]$, then we denote by $[h_1, \dots, h_n]$ the jacobian of $h_1, \dots, h_n$, that is,

$$[h_1, \dots, h_n] = \begin{vmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_1}{\partial x_n} & \frac{\partial h_2}{\partial x_n} & \cdots & \frac{\partial h_n}{\partial x_n} \end{vmatrix} .$$

Proposition 3.1. *Let d be a k-derivation of $k[X]$ and let $h_1, \dots, h_n \in k[X]$. Then*

$$d\Big([h_1, \dots, h_n]\Big) = -[h_1, \dots, h_n]\, d^\star + \sum_{p=1}^{n} [h_1, \dots, d(h_p), \dots, h_n] .$$

Proof. Put $f_i = d(x_i)$, $f_{ij} = \frac{\partial f_i}{\partial x_j}$, $h_{ij} = \frac{\partial h_i}{\partial x_j}$, for all $i, j \in \{1, \dots, n\}$, and let S_n denote the group of all permutations of $\{1, \dots, n\}$. Observe that

$$(a) \qquad d(h_{\sigma(p)p}) = \frac{\partial}{\partial x_p} d(h_{\sigma(p)}) - \sum_{q=1}^{n} h_{\sigma(p)q} f_{qp},$$

for all $\sigma \in S_n$ and $p \in \{1, \dots, n\}$, and

$$(b) \qquad \begin{aligned} &\sum_{\sigma \in S_n} (-1)^{|\sigma|} h_{\sigma(1)1} \cdots h_{\sigma(p-1)(p-1)} h_{\sigma(p)q} h_{\sigma(p+1)(p+1)} \cdots h_{\sigma(n)n} \\ &= [h_1, \dots, h_n]\delta_{pq}, \end{aligned}$$

for all $p, q \in \{1, \dots, n\}$, where $|\sigma|$ is the sign of σ, and δ_{pq} is the Kronecker delta. The above determines that

$$\begin{aligned} d([h_1, \dots, h_n]) &= \sum_{p=1}^{n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} h_{\sigma(1)1} \cdots d(h_{\sigma(p)p}) \cdots h_{\sigma(n)n} \\ &\overset{(a)}{=} \sum_{p=1}^{n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} h_{\sigma(1)1} \cdots \Big(\frac{\partial}{\partial x_p} d(h_{\sigma(p)}) - \sum_{q=1}^{n} h_{\sigma(p)q} f_{pq}\Big) \cdots h_{\sigma(n)n} \\ &\overset{(b)}{=} \sum_{p=1}^{n} [h_1, \dots, d(h_p), \dots, h_n] - \sum_{p=1}^{n} \sum_{q=1}^{n} f_{pq} [h_1, \dots, h_n]\delta_{pq} \\ &= \sum_{p=1}^{n} [h_1, \dots, d(h_p), \dots, h_n] - \sum_{p=1}^{n} f_{pp} [h_1, \dots, h_n] \\ &= \sum_{p=1}^{n} [h_1, \dots, d(h_p), \dots, h_n] - [h_1, \dots, h_n] d^\star . \end{aligned}$$

This completes the proof. □

As a consequence of the above proposition we obtain the following proposition for divergence-free derivations.

Proposition 3.2. *If d is a divergence-free k-derivation of $k[X]$ and $h_1, \ldots, h_n$ are polynomials belonging to $k[X]$, then*

$$d\Big([h_1, \ldots, h_n]\Big) = \sum_{p=1}^{n} [h_1, \ldots, d(h_p), \ldots, h_n] \,.$$

Consider the case $n = 2$. Put $x = x_1$ and $y = x_2$. If $f \in k[x, y]$, then we denote: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$. Observe that for every $f \in k[x, y]$ we have the equality

$$[f_x, x] + [f_y, y] = 0.$$

In fact, $[f_x, x] + [f_y, y] = \begin{vmatrix} f_{xx} & 1 \\ f_{xy} & 0 \end{vmatrix} + \begin{vmatrix} f_{yx} & 0 \\ f_{yy} & 1 \end{vmatrix} = -f_{xy} + f_{yx} = 0.$

In the case $n = 3$ we have a similar equality. If $f, g \in k[x, y, z]$, then

$$[f_x, g, x] + [f_y, g, y] + [f_z, g, z] = 0.$$

Let us check: $[f_x, g, x] + [f_y, g, y] + [f_z, g, z]$

$$\begin{aligned}
&= \begin{vmatrix} f_{xx} & g_x & 1 \\ f_{xy} & g_y & 0 \\ f_{xz} & g_z & 0 \end{vmatrix} + \begin{vmatrix} f_{yx} & g_x & 0 \\ f_{yy} & g_y & 1 \\ f_{yz} & g_z & 0 \end{vmatrix} + \begin{vmatrix} f_{zx} & g_x & 0 \\ f_{zy} & g_y & 0 \\ f_{zz} & g_z & 1 \end{vmatrix} \\
&= \begin{vmatrix} f_{xy} & g_y \\ f_{xz} & g_z \end{vmatrix} - \begin{vmatrix} f_{yx} & g_x \\ f_{yz} & g_z \end{vmatrix} + \begin{vmatrix} f_{zx} & g_x \\ f_{zy} & g_y \end{vmatrix} \\
&= (f_{xy} g_z - f_{xz} g_y) - (f_{yx} g_z - f_{yz} g_x) + (f_{zx} g_y - f_{zy} g_x) \\
&= f_{xy}(g_z - g_z) + f_{xz}(g_y - g_y) + f_{yz}(g_x - g_x) = 0.
\end{aligned}$$

The same we have for every $n \geqslant 2$.

Proposition 3.3. *If $f, g_1, g_2, \ldots, g_{n-2}$ are polynomials belonging to $k[x_1, \ldots, x_n]$, then*

$$\sum_{p=1}^{n} \left[\frac{\partial f}{\partial x_p}, g_1, g_2, \ldots, g_{n-2}, x_p\right] = 0.$$

Proof. Put $f_p = \frac{\partial f}{\partial x_p}$, $f_{p,j} = \frac{\partial f_p}{\partial x_j} = \frac{\partial^2 f}{\partial x_p x_j}$, and

$$A_p = [f_p, g_1, g_2, \ldots, g_{n-2}, x_p] \,, \quad G_j = \left(\frac{\partial g_1}{\partial x_j}, \frac{\partial g_2}{\partial x_j}, \ldots, \frac{\partial g_{n-2}}{\partial x_j}\right),$$

for all $p, j \in \{1, \ldots, n\}$. Note, that A_p is the jacobian of $f_p, g_1, \ldots, g_{n-2}, x_p$, and G_j is a sequence of $n-2$ polynomials from $k[X]$. Observe that, for every $p = 1, \ldots, n$, we have

$$A_p = \begin{vmatrix} f_{p,1} & G_1 & 0 \\ \vdots & \vdots & \vdots \\ f_{p,p-1} & G_{p-1} & 0 \\ f_{p,p} & G_p & 1 \\ f_{p,p+1} & G_{p+1} & 0 \\ \vdots & \vdots & \vdots \\ f_{p,n} & G_n & 0 \end{vmatrix} = (-1)^{n+p} D_p, \text{ where } D_p = \begin{vmatrix} f_{p,1} & G_1 \\ \vdots & \vdots \\ f_{p,p-1} & G_{p-1} \\ f_{p,p+1} & G_{p+1} \\ \vdots & \vdots \\ f_{p,n} & G_n \end{vmatrix}.$$

Consider the $n \times (n-2)$ matrix

$$M = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix}.$$

If p, q are different elements of $\{1, \dots, n\}$, then denote by $B_{p,q}$ the determinant of the $(n-2) \times (n-2)$ matrix that results from deleting the p-th row and the q-th row of the matrix M. It is clear that $B_{p,q} = B_{q,p}$ for all $p \neq q$.

Now consider the Laplace expansions with respect to the first column for all the determinants $D_1, \dots, D_n$. Let $p, q \in \{1, \dots, n\}$, $p < q$. We have

$$\begin{aligned} D_p &= \sum_{i=1}^{p-1} (-1)^{i+1} f_{p,i} B_{p,i} + \sum_{j=p+1}^{n} (-1)^j f_{p,j} B_{p,j}, \\ D_q &= \sum_{i=1}^{q-1} (-1)^{i+1} f_{q,i} B_{q,i} + \sum_{j=q+1}^{n} (-1)^j f_{q,j} B_{q,j}. \end{aligned}$$

In the first equality appears the component $(-1)^q f_{p,q} B_{p,q}$, and in the second equality appears the component $(-1)^{p+1} f_{q,p} B_{q,p}$. But $f_{p,q} = f_{q,p}$, $B_{p,q} = B_{q,p}$, and moreover

$$\sum_{r=1}^{n} A_r = \sum_{r=1}^{n} (-1)^{n+r} D_r.$$

Hence, in the sum $\sum_{r=1}^{n} A_r$ the polynomial $f_{p,q}$ appears exactly two times, and we have

$$\begin{aligned} &(-1)^{p+n}(-1)^q f_{p,q} B_{p,q} + (-1)^{q+n}(-1)^{p+1} f_{p,q} B_{p,q} \\ &= \Big((-1)^{n+p+q} + (-1)^{n+p+q+1} \Big) f_{p,q} B_{p,q} \\ &= 0 \cdot f_{p,q} B_{p,q} = 0. \end{aligned}$$

Therefore, $\sum_{p=1}^{n} \left[\frac{\partial f}{\partial x_p}, g_1, g_2, \dots, g_{n-2}, x_p \right] = \sum_{p=1}^{n} A_p = 0.$ □

4. Jacobian derivations in two variables

Now assume that $n = 2$. If $f \in k[x, y]$, then we denote by Δ_f the k-derivation of $k[x, y]$ defined by

$$\Delta_f(g) = [f, g],$$

for all $g \in k[x, y]$. We say that a k-derivation d of $k[x, y]$ is *jacobian*, if there exists a polynomial $f \in k[x, y]$ such that $d = \Delta_f$. Note, that

$$\Delta_f(x) = -f_y, \quad \Delta_f(y) = f_x.$$

If $f \in k[x, y]$ is a homogeneous polynomial of degree m, then Δ_f is a homogeneous k-derivation of degree $m - 1$.

Proposition 4.1. *Let $f, g \in k[x, y]$, and $a \in k$. Then:*

(1) $\Delta_{f+g} = \Delta_f + \Delta_g$;

(2) $\Delta_{af} = a\Delta_f$;

(3) $\Delta_{fg} = f\Delta_g + g\Delta_f$;

(4) $[\Delta_f, \Delta_g] = \Delta_{[f,g]}$.

Proof. The conditions (1) and (2) are obvious. Let $h \in k[x, y]$. Then we have

$$\begin{aligned} \Delta_{fg}(h) &= [fg, h] = -[h, fg] = -\Delta_h(fg) = -\left(f\Delta_h(g) + g\Delta_h(f)\right) \\ &= -f[h, g] - g[h, f] = f[g, h] + g[f, h] = f\Delta_g(h) + g\Delta_f(h) \\ &= \left(f\Delta_g + g\Delta_f\right)(h). \end{aligned}$$

Thus, we proved (3). We now check (4):

$$\begin{aligned} [\Delta_f, \Delta_g](x) &= \left(\Delta_f\Delta_g - \Delta_g\Delta_f\right)(x) = \Delta_f(-g_y) - \Delta_g(-f_y) \\ &= -g_{yx}(-f_y) - g_{yy}f_x + f_{yx}(-g_y) + f_{yy}g_x \\ &= \left(g_{yx}f_y + g_x f_{yy}\right) - \left(g_{yy}f_x + g_y f_{yx}\right) \\ &= \left(g_x f_y\right)_y - \left(f_x g_y\right)_y = \left(g_x f_y - f_x g_y\right)_y = -[f, g]_y = \Delta_{[f,g]}(x); \\ [\Delta_f, \Delta_g](y) &= \left(\Delta_f\Delta_g - \Delta_g\Delta_f\right)(y) = \Delta_f(g_x) - \Delta_g(f_x) \\ &= -g_{xx}f_y + g_{xy}f_x + f_{xx}g_y - f_{xy}g_x \\ &= \left(g_{xy}f_x + g_y f_{xx}\right) - \left(g_{xx}f_y + g_x f_{xy}\right) \\ &= \left(g_y f_x\right)_x - \left(f_y g_x\right)_x = \left(f_x g_y - f_y g_x\right)_x = [f, g]_x = \Delta_{[f,g]}(y). \end{aligned}$$

Thus, we proved that $[\Delta_f, \Delta_g]$ and $\Delta_{[f,g]}$ are k-derivations of $k[x, y]$ such that

$$[\Delta_f, \Delta_g](x) = \Delta_{[f,g]}(x), \quad [\Delta_f, \Delta_g](y) = \Delta_{[f,g]}(y).$$

This implies that $[\Delta_f, \Delta_g] = \Delta_{[f,g]}$. □

Let us recall the following result of the author [18].

Theorem 4.2. *Let k be a field of characteristic zero, and let $f, g \in k[x, y] \smallsetminus k$. If $[f, g] = 0$, then there exist a polynomial $h \in k[x, y]$ and polynomials $u(t), v(t) \in k[t]$ such that $f = u(h)$ and $g = v(h)$.*

If d and δ are k-derivations of $k[x,y]$, then we write $d \sim \delta$ in the case when $ad = b\delta$, for some nonzero $a, b \in k[x,y]$. It is clear that if $d \sim \delta$, then $k[x,y]^d = k[x,y]^\delta$ and $k(x,y)^d = k(x,y)^\delta$. As a consequence of Theorem 4.2 we get

Proposition 4.3. *Let k be a field of characteristic zero, and let $f, g \in k[x,y] \smallsetminus k$. Then $[f,g] = 0$ if and only if $\Delta_f \sim \Delta_g$.*

Proof. Let us observe that if $u(t) \in k[t] \smallsetminus k$, then $\frac{\partial u}{\partial t}(f) \neq 0$ and $\Delta_f \sim \Delta_{u(f)}$, because

$$\Delta_{u(f)} = \frac{\partial u}{\partial t}(f) \cdot \Delta_f.$$

Assume that $[f,g] = 0$. It follows from Theorem 4.2 that $f = u(h)$ and $g = v(h)$, for some $u, v \in k[t]$ and some $h \in k[x,y]$. Since $f \notin k$ and $g \notin k$, we have $u \notin k$ nad $h \notin k$. Hence, $\Delta_f = \Delta_{u(h)} \sim \Delta_h \sim \Delta_{v(h)} = \Delta_g$, and hence $\Delta_f \sim \Delta_g$.

Now suppose that $\Delta_f \sim \Delta_g$. Let $a\Delta_f = b\Delta_g$, for some nonzero $a, b \in k[x,y]$. Then we have $af_x = a\Delta_f(y) = b\Delta_g(y) = bg_x$ and $af_y = -a\Delta_f(x) = -b\Delta_g(x) = bg_y$. Hence, $f_x = ug_x$ and $f_y = ug_y$, where $u = b/a$. Therefore,

$$[f,g] = f_x g_y - f_y g_x = ug_x g_y - ug_y g_x = 0.$$

This completes the proof. □

Every Δ_f is a divergence-free k-derivation of $k[x,y]$. Indeed:

$$\Delta_f^* = \Delta_f(x)_x + \Delta_f(y)_y = -f_{yx} + f_{xy} = 0.$$

We now show that if k contains $\mathbb{Q}$, then the converse of this fact is also true. The main role in our proof plays the following lemma.

Lemma 4.4. *If $\mathbb{Q} \subset k$ and $f, g \in k[x,y]$, then the following conditions are equivalent:*

(*a*) *there exists $H \in k[x,y]$ such that $H_x = f$ and $H_y = g$;*

(*b*) *$f_y = g_x$.*

Proof. (a) ⇒ (b) follows from the equality $\partial_x \partial_y = \partial_y \partial_x$.

(b) ⇒ (a). Let

$$f = \sum_{\alpha,\beta} a(\alpha,\beta) x^\alpha y^\beta, \quad g = \sum_{\alpha,\beta} b(\alpha,\beta) x^\alpha y^\beta,$$

where all $a(\alpha,\beta)$, $b(\alpha,\beta)$ belong to k. If $\alpha \geqslant 1$ and $\beta \geqslant 1$, then $\frac{1}{\alpha} a(\alpha-1,\beta) = \frac{1}{\beta} b(\alpha,\beta-1)$. Put

$$F = \sum_{\alpha,\beta} c(\alpha,\beta) x^\alpha y^\beta,$$

where $c(0,0) = 0$ and, if $\alpha \geqslant 1$ then $c(\alpha,\beta) = \frac{1}{\alpha} a(\alpha-1,\beta)$, and if $\beta \geqslant 1$ then $c(\alpha,\beta) = \frac{1}{\beta} b(\alpha,\beta-1)$. It is easy to check that $H_x = f$ and $H_y = g$. □

Proposition 4.5. *If $\mathbb{Q} \subset k$ and d is a divergence-free k-derivation of $k[x,y]$, then there exists a polynomial $h \in k[x,y]$ such that $d = \Delta_h$.*

Proof. Let $d(x) = P$, $d(y) = Q$ and suppose that $P_x + Q_y = 0$. Put $f = Q$ and $g = -P$. Then $f_y = g_x$ and hence, by Lemma 4.4, there exists a polynomial $h \in k[x,y]$ such that $h_x = f$ and $h_y = g$, that is, $d = \Delta_h$. □

Thus, we have

Proposition 4.6. *Let $\mathbb{Q} \subset k$, and let d be a k-derivation of $k[x,y]$. Then d is jacobian if and only if d is divergence-free .*

Theorem 4.7. *If $\mathbb{Q} \subset k$ and d is a nonzero k-derivation of $k[x,y]$ then the following two conditions are equivalent:*

(1) *$k[x,y]^d \neq k$;*

(2) *$d \sim \delta$, where δ is a divergence-free k-derivation of $k[x,y]$.*

Proof. Since $k[x,y]^d = k[x,y]^{hd}$ for every nonzero polynomial h in $k[x,y]$, we may assume that the polynomials $d(x)$ and $d(y)$ are relatively prime.

(1) $\Rightarrow$ (2). Suppose $k[x,y]^d \neq k$ and let $F \in k[x,y]^d \smallsetminus k$. Put $d(x) = P$, $d(y) = Q$ and $h = \gcd(F_x, F_y)$. Then $PF_x + QF_y = 0$, $h \neq 0$ and there exist relatively prime polynomials $A, B \in k[x,y]$ such that $F_x = Ah$ and $F_y = Bh$. Hence $AP = -BQ$ and hence, $A \mid Q$, $Q \mid A$, $B \mid P$ and $P \mid B$. This implies that there exists an element $a \in k \smallsetminus \{0\}$ such that $aA = Q$ and $aB = -P$. Let $\delta = hd$. Then $d \sim \delta$ and δ is divergence-free . Indeed,

$$\delta^\star = (hP)_x + (hQ)_y = -(ahB)_x + (ahA)_y = -aF_{yx} + aF_{xy} = 0.$$

The implication (2) $\Rightarrow$ (1) is obvious. □

Now it is easy to prove the following theorem (see [19] Theorem 7.2.13).

Theorem 4.8. *Let $\mathbb{Q} \subset k$, and let d and δ be k-derivations of $k[x,y]$ such that $k[x,y]^d \neq k$ and $k[x,y]^\delta \neq k$. Then $k[x,y]^d = k[x,y]^\delta$ if and only if $d \sim \delta$.*

5. Jacobian derivations in n variables

Assume that $n \geqslant 2$. Let $F = (f_1, \dots, f_{n-1})$, where $f_1, \dots, f_{n-1}$ are polynomials belonging to $k[X] = k[x_1, \dots, x_n]$. We denote by Δ_F the mapping from $k[X]$ to $k[X]$ defined by

$$\Delta_F(h) = [f_1, \dots, f_{n-1}, h]\,,$$

for all $h \in k[X]$. This mapping is a k-derivation of $k[X]$. We say that it is a *jacobian derivation* of $k[X]$. If $n = 2$, then $\Delta_F = \Delta_{f_1}$ is the jacobian k-derivation from the previous section. If the polynomials $f_1, \dots, f_{n-1}$ are homogeneous of degrees $m_1, \dots, m_{n-1}$, respectively, then the derivation Δ_F is homogeneous of degree $(m_1 + \cdots + m_{n-1}) - (n-1)$, provided $\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j}\right] = n - 1$.

Now assume that $n = 3$. In this case $F = (f, g)$ is a sequence of two polynomials f, g from $k[X] = k[x, y, z]$, and $\Delta_{(f,g)}$ is a k-derivation of $k[x, y, z]$ such that

$$\Delta_{(f,g)}(x) = f_y g_z - f_z g_y, \quad \Delta_{(f,g)}(x) = f_z g_x - f_x g_z, \quad \Delta_{(f,g)}(x) = f_x g_y - f_y g_x.$$

It is easy to check that $\Delta_{(f,g)}$ is a divergence-free k-derivation of $k[x, y, z]$. In general, for any $n \geqslant 2$, we have the following theorem.

Theorem 5.1. *Every jacobian k-derivation of $k[x_1, \dots, x_n]$ is divergence-free .*

Proof. Consider a jacobian k-derivation Δ_F with $F = (f_1, \dots, f_{n-1})$, where $f_1, \dots, f_{n-1}$ are polynomials belonging to $k[X] = k[x_1, \dots, x_n]$. Since every partial derivative of $k[X]$ is a divergence-free k-derivation, we have (see Proposition 3.2) the equalities of the form

$$\frac{\partial}{\partial x_p}\,[f_1, \dots, f_{n-1}, x_p] = [f_1, \dots, f_{n-1}, 1] + \sum_{i=1}^{n-1} \left[f_1, \dots, \frac{\partial f_i}{\partial x_p}, \dots, f_{n-1}, x_p\right],$$

for all $p = 1, \dots, n$. Note that $[f_1, \dots, f_{n-1}, 1] = 0$. Using Proposition 3.3 we obtain also the equalities of the form

$$\sum_{p=1}^{n} \left[f_1, \dots, \frac{\partial f_i}{\partial x_p}, \dots, f_{n-1}, x_p\right] = 0,$$

for all $i = 1, \dots, n-1$. We now have:

$$\begin{aligned}
(\Delta_F)^\star &= \sum_{p=1}^{n} \tfrac{\partial}{\partial x_p} \Delta_F(x_p) = \sum_{p=1}^{n} \tfrac{\partial}{\partial x_p}\,[f_1, \dots, f_{n-1}, x_p] \\
&= \sum_{p=1}^{n} \left([f_1, \dots, f_{n-1}, 1] + \sum_{i=1}^{n-1} \left[f_1, \dots, \tfrac{\partial f_i}{\partial x_p}, \dots, f_{n-1}, x_p\right] \right) \\
&= \sum_{p=1}^{n} \sum_{i=1}^{n-1} \left[f_1, \dots, \tfrac{\partial f_i}{\partial x_p}, \dots, f_{n-1}, x_p\right] \\
&= \sum_{i=1}^{n-1} \left(\sum_{p=1}^{n} \left[f_1, \dots, \tfrac{\partial f_i}{\partial x_p}, \dots, f_{n-1}, x_p\right] \right) = \sum_{i=1}^{n-1} 0 = 0.
\end{aligned}$$

Therefore, the derivation Δ_F is divergence-free . □

Other proofs of the above theorem appear in Connell and Drost [5], Theorem 2.3; in Makar-Limanow [12]; and in Freudenburg's book [7], Lemma 3.8.

Let k be a field of characteristic zero and let $f_1, \dots, f_n$ be polynomials in $k[X] = k[x_1, \dots, x_n]$. Denote by w the jacobian of $(f_1, \dots, f_n)$, that is, $w = [f_1, \dots, f_n]$. It is well known and easy to be proved that if $k[f_1, \dots, f_n] = k[X]$, then w is a nonzero element of k. The famous *Jacobian Conjecture* states that the converse of this fact is also true: if $w \in k \setminus \{0\}$ then $k[f_1, \dots, f_n] = k[X]$. The problem is still open even for $n = 2$. There exists a long list of equivalent formulations of this conjecture (see for example [22], [1], [6]). One of the equivalent formulations of the Jacobian Conjecture is as follows.

Conjecture 5.2. *Let k be a field of characteristic zero, and let $F = (f_1, \dots, f_{n-1})$, where $f_1, \dots, f_{n-1}$ are polynomials belonging to $k[X] = k[x_1, \dots, x_n]$. If there exists $g \in k[X]$ such that $\Delta_F(g) = 1$, then the jacobian derivation Δ_F is locally nilpotent.*

It is difficult to prove that the above Δ_F is locally nilpotent. Let us recall (see Theorem 2.5) that every locally nilpotent derivation is divergence-free. Thus, by theorem 5.1 we already know that Δ_F is divergence-free.

We know that $\operatorname{Der}_k(k[X])$ is a free $k[X]$-module on the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. This basis is commutative. We say that a basis $\{d_1, \dots, d_n\}$ is *commutative*, if $d_i \circ d_j = d_j \circ d_i$ for all $i, j \in \{1, \dots, n\}$. A basis $\{d_1, \dots, d_n\}$ is called *locally finite* (resp. *locally nilpotent*) if each d_i is locally finite (resp. locally nilpotent). Note the following results of the author.

Theorem 5.3. ([17]). *If k is a field of characteristic zero, then the following conditions are equivalent.*

(1) *The Jacobian Conjecture is true in the n-variable case.*

(2) *Every commutative basis of the $k[X]$-module $\operatorname{Der}_k(k[X])$ is locally finite.*

(3) *Every commutative basis of the $k[X]$-module $\operatorname{Der}_k(k[X])$ is locally nilpotent.*

Theorem 5.4. ([19] Theorem 2.5.5). *Let k be a reduced ring containing $\mathbb{Q}$. If $\{d_1, \dots, d_n\}$ is commutative basis of the $k[X]$-module $\operatorname{Der}_k(k[X])$, then each derivation d_i is divergence-free.*

Note also some results of E. Connell, J. Drost [5] and L. Makar-Limanow [12].

Theorem 5.5. ([5]). *Let D be a k-derivation of $k[X] = k[x_1, \dots, x_n]$, where k is a field of characteristic zero. If $\operatorname{tr.deg}_k k[X]^D = n - 1$, then there exists $g \in k[X]$ such that the derivation gD is divergence-free.*

A k-derivation D of $k[X]$ is called *irreducible*, if $\gcd(D(x_1), \dots, D(x_n)) = 1$.

Theorem 5.6. ([12]). *Let D be an irreducible locally nilpotent k-derivation of $k[X] = k[x_1, \dots, x_n]$, where k is a field of characteristic zero. Let $f_1, \dots, f_{n-1}$ be $n - 1$ algebraically independent elements of $k[X]^D$, and set $F = (f_1, \dots, f_{n-1})$. Then there exists $g \in k[X]^D$ such that $\Delta_F = gD$. In particular, the derivation Δ_F is locally nilpotent.*

6. The ideal I(d) for homogeneous derivations

In this section k is a field of characteristic zero, $k[X] = k[x_1, \dots, x_n]$ is a polynomial ring over k, and $d : k[X] \to k[X]$ is a homogeneous k-derivation of degree $s \geqslant 0$. Put

$$g_{ij} = x_i d(x_j) - x_j d(x_i),$$

for all $i, j \in \{1, \dots, n\}$. Each g_{ij} is a homogeneous polynomial of degree $s + 1$. In particular, $g_{ii} = 0$ and $g_{ji} = -g_{ij}$ for all i, j. Moreover, for all $i, j, p \in \{1, \dots, n\}$,

$$x_i g_{jp} + x_j g_{pi} + x_p g_{ij} = 0.$$

We denote by $I(d)$ the ideal in $k[X]$ generated by all the polynomials g_{ij} with $i, j \in \{1, \ldots, n\}$.

Proposition 6.1. *The ideal $I(d)$ is differential, that is, $d(I(d)) \subset I(d)$.*

Proof. Put $f_1 = d(x_1), \ldots, f_n = d(x_n)$. Since $f_1, \ldots, f_n$ are homogeneous polynomials of degree s, we have the Euler formulas:

$$x_1 \frac{\partial f_i}{\partial x_1} + \cdots + x_n \frac{\partial f_i}{\partial x_n} = s f_i$$

for all $i = 1, \ldots, n$. Thus, we have

$$\begin{aligned} d(g_{ij}) &= d(x_i f_j - x_j f_i) \\ &= f_i f_j + x_i d(f_j) - f_j f_i - x_j d(f_i) = x_i d(f_j) - x_j d(f_i) \\ &= x_i \left(\tfrac{\partial f_j}{\partial x_1} f_1 + \cdots + \tfrac{\partial f_j}{\partial x_n} f_n \right) - x_j \left(\tfrac{\partial f_i}{\partial x_1} f_1 + \cdots + \tfrac{\partial f_i}{\partial x_n} f_n \right) \\ &= \left(x_1 \tfrac{\partial f_j}{\partial x_1} + \cdots + x_n \tfrac{\partial f_j}{\partial x_n} \right) f_i - \left(x_1 \tfrac{\partial f_i}{\partial x_1} + \cdots + x_n \tfrac{\partial f_i}{\partial x_n} \right) f_j + a \\ &= (s f_j) f_i - (s f_i) f_j + a = a, \end{aligned}$$

where a is a polynomial belonging to $I(d)$. Thus, $d(g_{ij}) \in I(d)$ for all i, j, and this implies that $d(I(d)) \subset I(d)$. □

We denote by E the *Euler derivation* of $k[X]$, that is,

$$E = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_n \frac{\partial}{\partial x_n}.$$

This derivation is homogeneous of degree 1. If $0 \neq F \in k[X]$ is a homogeneous polynomial of degree s, then $E(F) = sF$. Thus, every nonzero homogeneous polynomial of degree s is a Darboux polynomial of E with cofactor s.

Proposition 6.2. *The ideal $I(d)$ is equal to 0 if and only if $d = h \cdot E$ for some $h \in k[X]$.*

Proof. Suppose that $d = hE$ with $h \in k[X]$, Then $d(x_i) = x_i h$ for $i = 1, \ldots, n$. Thus, $g_{ij} = x_i(x_j h) = x_j(x_i h) = 0$ and so, $I(d) = 0$.

Now let $I(d) = 0$. Put $f_i = d(x_i)$ for all i. Then, for all $i, j \in \{1, \ldots, n\}$, we have the equality $x_i f_j = x_j f_i$ so, each x_i divides f_i. Thus, $f_i = u_i x_i$ for $i = 1, \ldots, n$, where $u_i \in k[X]$. Put $h = u_1$. Observe that $u_i = h$ for all $i = 1, \ldots, n$. Therefore, $d = hE$. □

Proposition 6.3. *Let $d : k[X] \to k[X]$ be a homogeneous k-derivation of degree $s \geqslant 1$ and let $h \in k[X]$ be a homogeneous polynomial of degree $s - 1$. Then $I(d) = I(d - hE)$.*

Proof. Put $\delta = d - hE$. Then, for all $i, j \in \{1, \ldots, n\}$, we have

$$x_i \delta(x_j) - x_j \delta(x_i) = x_i (d(x_j) - x_j h) - x_j (d(x_i) - x_i h) = x_i d(x_j) - x_j d(x_i).$$

Thus, the ideals $I(d)$ and $I(\delta)$ are generated by the same elements. □

Proposition 6.4. *Let $d : k[X] \to k[X]$ be a homogeneous derivation of degree s. Then there exists a homogeneous k-derivation $\delta : k[X] \to k[X]$, of degree s, such that $I(d) = I(\delta)$ and $\delta(x_n) \in k[x_1, \dots, x_{n-1}]$.*

Proof. Let $d(x_n) = Ax_n + B$, where $A \in k[X]$ and $B \in k[x_1, \dots, x_{n-1}]$. Put $\delta = d - AE$. Then $I(d) = I(\delta)$ (by Proposition 6.3) and $\delta(x_n) = d(x_n) - Ax_n = B \in k[x_1, \dots, x_{n-1}]$. □

Let us recall that all the polynomials g_{ij} are homogeneous of degree $s+1$, $g_{ii} = 0$ and $x_i g_{jp} + x_j g_{pi} + x_p g_{ij} = 0$, for all $i, j, p \in \{1, \dots, n\}$.

Proposition 6.5. *Let $\{w_{ij};\ i, j = 1, \dots, n\}$ be a family of polynomials in $k[X]$. Suppose that:*

(1) *all the polynomials w_{ij} are homogeneous of degree $s + 1$;*
(2) *$w_{ii} = 0$ for $i = 1, \dots, n$;*
(3) *$x_i w_{jp} + x_j w_{pi} + x_p w_{ij} = 0$, for all $i, j, p \in \{1, \dots, n\}$.*

Then there exist homogeneous of degree s polynomials $f_1, \dots, f_n \in k[X]$ such that

$$w_{ij} = x_i f_j - x_j f_i,$$

for all $i, j \in \{1, \dots, n\}$.

Proof. Let $Y_i = \displaystyle\sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j}$, for $i = 1, \dots, n$. Then, for $i, j, \in \{1, \dots, n\}$, we have:

$$\begin{aligned}
x_i Y_j - x_j Y_i &= x_i \sum_{p=1}^{n} \frac{\partial w_{jp}}{\partial x_p} - x_j \sum_{p=1}^{n} \frac{\partial w_{ip}}{\partial x_p} \\
&= x_i \frac{\partial w_{ji}}{\partial x_i} - x_j \frac{\partial w_{ij}}{\partial x_j} + x_i \sum_{p \neq i} \frac{\partial w_{jp}}{\partial x_p} - x_j \sum_{p \neq j} \frac{\partial w_{ip}}{\partial x_p} \\
&= x_i \frac{\partial w_{ji}}{\partial x_i} - x_j \frac{\partial w_{ij}}{\partial x_j} + x_i \sum_{p \neq i,\, p \neq j} \frac{\partial w_{jp}}{\partial x_p} - x_j \sum_{p \neq j,\, p \neq i} \frac{\partial w_{ip}}{\partial x_p} \\
&= x_i \frac{\partial w_{ji}}{\partial x_i} - x_j \frac{\partial w_{ij}}{\partial x_j} + \sum_{p \neq i,\, p \neq j} \frac{\partial}{\partial x_p} \left(x_i w_{jp} - x_j w_{ip} \right) \\
&= x_i \frac{\partial w_{ji}}{\partial x_i} - x_j \frac{\partial w_{ij}}{\partial x_j} + \sum_{p \neq i,\, p \neq j} \frac{\partial}{\partial x_p} \left(-x_p w_{ij} \right) \\
&= x_i \frac{\partial w_{ji}}{\partial x_i} + x_j \frac{\partial w_{ji}}{\partial x_j} - \sum_{p \neq i,\, p \neq j} x_p \frac{\partial w_{ij}}{\partial x_p} - \sum_{p \neq i,\, p \neq j} w_{ij} \\
&= -\textstyle\sum_{p=1}^{n} x_p \frac{\partial w_{ij}}{\partial x_p} - (n-2) w_{ij} = -(s+1) w_{ij} - (n-2) w_{ij} \\
&= -(s+n-1) w_{ij}.
\end{aligned}$$

Thus, $x_i Y_j - x_j Y_i = -(s+n-1) w_{ij}$. Let $f_i = -\frac{1}{s+n-1} Y_i$, for $i = 1, \dots, n$. Then we have

$$w_{ij} = x_i f_j - x_j f_i,$$

for all $i, j \in \{1, \dots, \}$. It is clear that the polynomials $f_1, \dots, f_n$ are homogeneous of degree s. □

Proposition 6.6. *Let $\{w_{ij};\ i, j = 1, \dots, n\}$ be a family of polynomials in $k[X]$ such as in Proposition* 6.5*, and let $Y_i = \sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j}$, for $i = 1, \dots, n$. Then $\sum_{i=1}^{n} \frac{\partial Y_i}{\partial x_i} = 0$.*

Proof. Put $A = \sum_{i=1}^{n} \frac{\partial Y_i}{\partial x_i}$. Then we have:

$$A = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 w_{jp}}{\partial x_i \partial x_j} = - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 w_{ji}}{\partial x_j \partial x_i} = -A.$$

Thus, $A = 0$. □

Theorem 6.7. *Let k be a field of characteristic zero, and let $d : k[X] \to k[X]$ be a homogeneous k-derivation of degree s. Then there exists a divergence-free k-derivation $\delta : k[X] \to k[X]$ such that δ is homogeneous of degree s and $I(d) = I(\delta)$.*

Proof. Let $w_{ij} = x_i d(x_j) - x_j d(x_i)$ for $i, j \in \{1, \dots, n\}$. The polynomials w_{ij} satisfy the properties $(1) - (3)$ of Proposition 6.5. Put

$$Y_i = \sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j}, \quad f_i = -\frac{1}{s + n - 1} Y_i,$$

for $i = 1, \dots, n$. Then $w_{ij} = x_i f_j - x_j f_i$ (see the proof of Proposition 6.5). Let $\delta : k[X] \to k[X]$ be the k-derivation defined by $\delta(x_i) = f_i$, for $i = 1, \dots, n$. Then δ is homogeneous of degree s and $I(d) = I(\delta)$. Moreover, it follows from Proposition 6.6 that the divergence δ^* is equal to zero. □

7. Polynomials M_d in two variables

In this section we assume that $n = 2$ and k is a field of characteristic zero. Given a homogeneous k-derivation d of $k[X]$ we studied in the previous section the differential ideal generated by all polynomials of the form $x_i d(x_j) - x_j d(x_i)$. In the case $n = 2$ this ideal is generated only by one polynomial

$$M_d = xd(y) - yd(x).$$

If d is homogeneous derivation of degree s, then M_d is a homogeneous polynomial and $\deg M_d = s + 1$. If d is the Euler derivation E, then $M_d = 0$. It is easy to check that $M_d = 0$ if and only if $d = h \cdot E$ for some $h \in k[x, y]$.

Proposition 7.1. *If d is a homogeneous k-derivation of $k[x, y]$ and $M_d \neq 0$, then M_d is a Darboux polynomial of d and its cofactor is equal to the divergence d^*, that is,*

$$d(M_d) = d^* M_d.$$

Proof. Put $f = d(x), g = d(y)$. Since d is homogeneous, we have $xf_x + yf_y = sf$ and $xg_x + yg_y = sg$, where s is the degree of d. So, we have,

$$\begin{aligned} d(M_d) - d^* M_d &= d(xg - yf) - (f_x + g_y)(xg - yf) \\ &= fg + x(g_x f + g_y g) - gf - y(f_x f + f_y g) - (f_x + g_y)(xg - yf) \\ &= xg_x f + xg_y g - yf_x f - yf_y g - xf_x g + yf_x f - xg_y g + yg_y f \\ &= (xg_x + yg_y)f - (xf_x + yf_y)g \\ &= sgf - sfg = 0, \end{aligned}$$

and hence, M_d is a Darboux polynomial with cofactor d^* □

The above property does not hold when $d(x), d(y)$ are homogeneous of different degrees. Let for example, $d(x) = 1$, $d(y) = x$. Then $M_d = x^2 - y$, $d^* = 0$ and $d(M_d) = d(x^2 - y) = 2x - x = x \neq 0 \cdot (x^2 - y)$. The above property also does not hold when $\deg d(x) = \deg d(y)$ and the polynomials $d(x)$, $d(y)$ are not homogeneous. Let $d(x) = x + 1$, $d(y) = y$. Then $M_d = -y$, $d^* = 2$, $d(M_d) = -y \neq -2y$.

We say that a Darboux polynomial f is said to be *essential* if $f \notin k$.

Proposition 7.2. *Every homogeneous k-derivation of $k[x, y]$ has an essential Darboux polynomial $f \in k[x, y] \setminus k$.*

Proof. If $M_d \neq 0$ then, by the previous proposition, M_d is a Darboux polynomial. If $M_d = 0$, then $x - y$ is a Darboux polynomial. □

The following examples show that the above property does not hold when d is not homogeneous, and when d is a homogeneous derivations in three variables. Let us recall that k is a field of characteristic zero.

Example 7.3. ([10], [19], [20]). *The derivation $\partial_x + (xy + 1)\partial_y$ has no essential Darboux polynomial.*

Example 7.4. ([8]). *The derivation $(1 - xy)\partial_x + x^3\partial_y$ has no essential Darboux polynomial.*

Example 7.5. ([9]). *Let d be the k-derivation of $k[x, y, z]$ defined by:*

$$d(x) = y^2, \quad d(y) = z^2, \quad d(z) = x^2.$$

Then d is homogeneous, divergence-free , and d has no essential Darboux polynomial.

Proposition 7.6. *Let $d : k[x, y] \to k[x, y]$ be a homogeneous k-derivation, and let $f = d(x)$, $g = d(y)$. If $h, \lambda \in k[x, y]$ are homogeneous polynomials such that $d(h) = \lambda h$, then*

$$M_d h_x = (y\lambda - mg)h, \quad M_d h_y = (mf = x\lambda)h,$$

where $m = \deg h$.

Proof. We have the following sequences of equalities:

$$\begin{array}{rcl} fh_x + gh_y &=& \lambda h, \\ yfh_x + ygh_y &=& y\lambda h, \\ yfh_x + g(mh - xh_x) &=& y\lambda h, \\ (xg - yf)h_x &=& (y\lambda - mg)h, \\ M_d h_x &=& (y\lambda - mg)h. \end{array}$$

$$\begin{array}{rcl} fh_x + gh_y &=& \lambda h, \\ xfh_x + xgh_y &=& x\lambda h, \\ f(mh - yh_y) + xgh_y &=& x\lambda h, \\ (xg - yf)h_y &=& (mf - x\lambda)h, M_d h_y = (mf = x\lambda)h. \end{array}$$

We used the Euler formula. □

Proposition 7.7. *If $d : k[x, y] \to k[x, y]$ is a nonzero homogeneous k-derivation, then every irreducible Darboux polynomial of d is a divisor of the polynomial M_d.*

Proof. Let $h \in k[x, y] \smallsetminus k$ be an irreducible Darboux polynomial of d, and let λ be its cofactor. Thus, $d(h) = \lambda h$. We know, by Proposition 1.2, that λ is homogeneous. Since $h \notin k$, we have either $h_x \neq 0$ or $h_y \neq 0$. Let us suppose that $h_x \neq 0$. Then the polynomials h_x and h are relatively prime and (by Proposition 7.6) $M_d h_x = (y\lambda - mg)h$. Thus, h divides M_d. In the case $h_y \neq 0$ we do the same procedure, □

The Euler derivation $E : k[x, y] \to k[x, y]$ is a nonzero homogeneous derivation, and every nonzero homogeneous polynomial from $k[x, y]$ is a Darboux polynomial of E. Thus, E has infinitely many homogeneous irreducible Darboux polynomials, The same property has every derivation hE with a nonzero homogeneous $h \in k[x, y]$. Let us recall that in this case the polynomial M_d is equal to zero. The following proposition states that other homogeneous derivations have only finitely many homogeneous irreducible Darboux polynomials.

Theorem 7.8. *Let k be a field of characteristic zero, and let $d : k[x, y] \to k[x, y]$ be a nonzero homogeneous k-derivation of degree s such that $M_d \neq 0$. Then d has at most $s + 1$ pairwise nonassociated irreducible homogeneous Darboux polynomials.*

Proof. It follows from Proposition 7.7, because M_d is a nonzero homogeneous polynomial of degree $s + 1$. □

In the above theorem we were interested in irreducible homogeneous Darboux polynomials. Without the word "homogeneous" such property does not hold, in general. Let for example, $d = x\partial_x + 2y\partial_y$. Then $d(x^2 + ay) = 2(x^2 + ay)$ for every $a \in k$ and hence, d is a nonzero homogeneous k-derivation and d has infinitely many, pairwise nonassociated, irreducible Darboux polynomials,

8. Sums of jacobian derivations

In this section k is always a commutative ring containing $\mathbb{Q}$.

We know (see Proposition 4.6) that every divergence-free k-derivation of $k[x,y]$ is a jacobian derivation. A similar property for $n \geqslant 3$ variables does not hold in general. Let, for example, d be the k-derivation of $k[x,y,z]$, defined by: $d(x) = y^2$, $d(y) = z^2$, $d(z) = x^2$ (as in Example 7.5). Then d is divergence-free . It is known that $k[x,y,z]^d = k$ (see [9] or [15], [19]) so, d is not jacobian. There exist many similar examples for arbitrary $n \geqslant 3$ (see [11], [23], [19]). In this section we will show that every divergence-free k-derivation of $k[X] = k[x_1,\dots,x_n]$ is a finite sum of some jacobian derivation.

Let f be a polynomial from $k[X]$, and let $i,j \in \{1,\dots,n\}$. We denote by $\Omega^f_{i,j}$ the k-derivation of $k[X]$ defined by

$$\Omega^f_{i,j}(g) = \begin{vmatrix} \frac{\partial f}{\partial x_i} & \frac{\partial g}{\partial x_i} \\ \frac{\partial f}{\partial x_j} & \frac{\partial g}{\partial x_j} \end{vmatrix} = f_{x_i} g_{x_j} - f_{x_j} g_{x_i}$$

for all $g \in k[X]$. It is clear that $\Omega^f_{i,i} = 0$ and $\Omega^f_{j,i} = -\Omega^f_{i,j}$ for all $i,j \in \{1,\dots,n\}$. If $i \neq j$, then we have

$$\Omega^f_{i,j}(x_p) = \begin{cases} 0, & \text{if } p \neq i,\ p \neq j, \\ -\frac{\partial f}{\partial x_j}, & \text{if } p = i, \\ \frac{\partial f}{\partial x_i}, & \text{if } p = j, \end{cases}$$

for all $p = 1,\dots,n$. Note the following obvious proposition.

Proposition 8.1. *Every derivation of the form $\Omega^f_{i,j}$ is divergence-free .*

Another common notation for $\Omega^f_{i,j}$, is $\Omega^f_{x_i,x_j}$. If $n = 2$ and $f \in k[x,y]$, then $\Omega^f_{x,y} = \Delta_f$, where Δ_f is the jacobian derivation of $k[x,y]$ from a previous section. If $n = 3$ and $f \in k[x,y,z]$, then we have three k-derivations of the above forms: $\Omega^f_{x,y}$, $\Omega^f_{x,z}$ and $\Omega^f_{y,z}$.

Proposition 8.2. *Let d be a k-derivation of $k[x,y,z]$, where k is a commutative ring containing $\mathbb{Q}$. If d is divergence-free , then there exist polynomials $u,v \in k[x,y,z]$ such that*

$$d = \Omega^u_{x,y} + \Omega^v_{y,z}.$$

Proof. Put $f = d(x)$, $g = d(y)$, $h = d(z)$ and $R = k[x,y,z]$. Since d is divergence-free , we have the equality $f_x + g_y + h_z = 0$. Since the partial derivative $\frac{\partial}{\partial y}$ is a surjective mapping from R to R, there exists a polynomial $H \in R$ such that $h = H_y$. Let

$$\overline{f} = f, \quad \overline{g} = g + H_z,$$

and consider the $k[z]$-derivation $\overline{d}$ of $R = k[z][x,y]$ defined by $\overline{d}(x) = \overline{f}$ and $\overline{d}(y) = \overline{g}$. Observe that the derivation $\overline{d}$ is divergence-free . Indeed,

$$\left(\overline{d}\right)^* = \overline{f}_x + \overline{g}_y = f_x + g_y + H_{zy} = f_x + g_y + H_{yz} = f_x + g_y + h_z = 0.$$

It follows from Proposition 4.5, that there exists a polynomial $F \in R$ such that $\overline{d} = \Delta_F$. Hence, $\overline{d}(x) = -F_y$ and $\overline{d}(y) = F_x$ and hence, $f = -F_y,\ g = F_x - H_z$. Put $u = F$, $v = H$ and $\delta = \Omega^u_{x,y} + \Omega^v_{y,z}$. Then we have:

$$\delta(x) = \begin{vmatrix} u_x & 1 \\ u_y & 0 \end{vmatrix} = -u_y = -F_y = f,$$

$$\delta(y) = \begin{vmatrix} u_x & 0 \\ u_y & 1 \end{vmatrix} + \begin{vmatrix} v_y & 1 \\ v_z & 1 \end{vmatrix} = u_x - v_z = F_x - H_z = g,$$

$$\delta(z) = \begin{vmatrix} v_y & 0 \\ v_z & 1 \end{vmatrix} = v_y = H_y = h.$$

Therefore, $d = \delta = \Omega^u_{x,y} + \Omega^v_{y,z}$. □

Example 8.3. *Let* $d = y^s \frac{\partial}{\partial x} + z^s \frac{\partial}{\partial y} + x^s \frac{\partial}{\partial z}$, *where* $s \geqslant 1$. *Then* $d = \Omega^u_{x,y} + \Omega^v_{y,z}$ *for* $u = z^s x - \frac{1}{s+1} y^{s+1}$ *and* $v = x^s y$.

Proposition 8.4. *Let d be a k-derivation of $k[x,y,z]$, where k is a commutative ring containing $\mathbb{Q}$. If d is divergence-free , then there exist polynomials $A, B, C \in k[x,y,z]$ such that*

$$d = \Omega^A_{x,y} + \Omega^B_{y,z} + \Omega^C_{z,x}.$$

In other words, there exist polynomials $A, B, C \in k[x,y,z]$ such that

$$d(x) = C_z - A_y, \quad d(y) = A_x - B_z, \quad d(z) = B_y - C_x.$$

Proof. Let $u, v \in k[x,y,z]$ as in Proposition 8.2. Put $A = u$, $B = v$ and $C = 0$. Then $d = \Omega^A_{x,y} + \Omega^B_{y,z} + \Omega^C_{z,x}$. □

Example 8.5. *Let* $d = y^s \frac{\partial}{\partial x} + z^s \frac{\partial}{\partial y} + x^s \frac{\partial}{\partial z}$, *where* $s \geqslant 1$. *Then* $d = \Omega^A_{x,y} + \Omega^B_{y,z} + \Omega^C_{z,x}$ *where* $A = \frac{1}{2}\left(z^s x - \frac{1}{s+1} y^{s+1}\right)$, $B = \frac{1}{2}\left(x^s y - \frac{1}{s+1} z^{s+1}\right)$ *and* $C = \frac{1}{2}\left(y^s z - \frac{1}{s+1} x^{s+1}\right)$.

Example 8.6. *If $f, g \in k[x,y,z]$, then $\Delta_{(f,g)} = \Omega^A_{x,y} + \Omega^B_{y,z} + \Omega^C_{z,x}$, where*

$$A = f_z g, \quad B = f_x g, \quad C = f_y g.$$

Quite recently, Piotr Jędrzejewicz generalizes Propositions 8.2 and 8.4 for arbitrary $n \geqslant 3$. Such generalizations seem to be well-known, although we could not find a reference.

Theorem 8.7 (Jędrzejewicz)**.** *Let d be a k-derivation of $k[X] = k[x_1, \dots, x_n]$, where $n \geqslant 3$ and k is a commutative ring containing $\mathbb{Q}$. If d is divergence-free, then there exist polynomials $u_1, \dots, u_{n-1} \in k[X]$ such that*

$$d = \Omega^{u_1}_{1,2} + \Omega^{u_2}_{2,3} + \cdots + \Omega^{u_{n-1}}_{n-1,n}.$$

In particular, we have the following equalities

$$(*)\qquad \begin{cases} d(x_1) &= -(u_1)_{x_2}, \\ d(x_2) &= (u_1)_{x_1} - (u_2)_{x_3}, \\ d(x_3) &= (u_2)_{x_2} - (u_3)_{x_4}, \\ &\vdots \\ d(x_{n-1}) &= (u_{n-2})_{x_{n-2}} - (u_{n-1})_{x_n}, \\ d(x_n) &= (u_{n-1})_{x_{n-1}}. \end{cases}$$

Proof. By induction on n. For $n = 3$ it follows from Proposition 8.2. Let $n \geqslant 3$ and suppose that our assertion is true for this n. Let d be a divergence-free k-derivation of $R = k\,[x_1, \dots, x_{n+1}]$. Put $f_i = d(x_i)$ for all $i = 1, \dots, n+1$. We have the equality $\sum_{i=1}^{n+1} (f_i)_{x_i} = 0$. Since the partial derivative $\frac{\partial}{\partial x_n}$ is a surjective mapping from R to R, there exists a polynomial $P \in R$ such that $f_{n+1} = P_{x_n}$. Let

$$g_1 = f_1,\ g_2 = f_2,\ \dots,\ g_{n-1} = f_{n-1},\ g_n = f_n + P_{x_{n+1}},$$

and consider the $k\,[x_{n+1}]$-derivation $\overline{d}$ of R defined by $\overline{d}(x_i) = g_i$ for all $i = 1, \dots, n$. Observe that the derivation $\overline{d}$ is divergence-free . Indeed,

$$\left(\overline{d}\right)^* = \sum_{i=1}^{n} (g_i)_{x_i} = \sum_{i=1}^{n-1} (f_i)_{x_i} + (f_n)_{x_n} + P_{x_n x_{n+1}} = \sum_{i=1}^{n+1} (f_i)_{x_i} = 0,$$

because $P_{x_n x_{n+1}} = (f_{n+1})_{x_{n+1}}$. By induction there exist polynomials $v_1, \dots, v_{n-1} \in R$ satisfying the equalities $(*)$ for the derivation $\overline{d}$, that is,

$$g_1 = \overline{d}(x_1) = -(v_1)_{x_2}, \quad g_n = \overline{d}(x_n) = (v_{n-1})_{x_{n-1}}$$

and $g_i = \overline{d}(x_i) = (v_{i-1})_{x_{i-1}} - (v_i)_{x_{i+1}}$ for $i = 2, \dots, n-1$. Let us recall that $g_n = f_n + P_{x_{n+1}}$ Put $u_i = v_i$ for $i = 1, \dots, n-1$, and $u_n = P$. Then $d(x_1) = f_1 = -(u_1)_{x_2}$, and $d(x_i) = -(u_{i-1})_{x_{i-1}}$ for $i = 2, \dots, n-1$. Moreover,

$$d(x_n) = f_n = g_n - P_{x_{n+1}} = (v_{n-1})_{x_{n-1}} - P_{x_{n+1}} = (u_{n-1})_{x_{n-1}} - (u_n)_{x_{n+1}}$$

and $d\,(x_{n+1}) = f_{n+1} = P_{x_n} = u_{x_n}$. This means that $d = \Omega^{u_1}_{1,2} + \Omega^{u_2}_{2,3} + \dots + \Omega^{u_n}_{n,n+1}$, and this completes the proof. □

Theorem 8.8. *Let d be a k-derivation of $k[x_1, \dots, x_n]$, where $n \geqslant 3$ and k is a commutative ring containing $\mathbb{Q}$. If d is divergence-free, then there exist polynomials $A_1, \dots, A_n \in k[x_1, \dots, x_n]$ such that*

$$d = \Omega^{A_1}_{1,2} + \Omega^{A_2}_{2,3} + \dots + \Omega^{A_{n-1}}_{n-1,n} + \Omega^{A_n}_{n,1}.$$

In particular, $d(x_i) = (A_{i-1})_{x_{i-1}} - (A_i)_{x_{i+1}}$ for all $i \in \mathbb{Z}_n$.

Proof. Let $u_1, \dots, u_{n-1} \in k[x_1, \dots, x_n]$ be as in Theorem 8.7. Put $A_i = u_i$ for $i = 1, \dots, n-1$ and $A_n = 0$. Then our assertion follows from Theorem 8.7. □

Example 8.9. *Let d be the k-derivation of $k[x_1, \dots, x_n]$ defined by $d(x_i) = x_{i+1}^s$ for $i = 1, \dots, n$, where k is a commutative ring containing $\mathbb{Q}$, $s \geqslant 0$, and $x_{n+1} = x_1$, $x_0 = x_n$. Then d is divergence-free, and $d = \Omega_{1,2}^{A_1} + \Omega_{2,3}^{u_2} + \cdots + \Omega_{n-1,n}^{A_{n-1}} + \Omega_{n,1}^{A_n}$. with*

$$A_i = \frac{1}{2}\left(x_{i+2}^s x_i - \frac{1}{s+1} x_{i+1}^{s+1}\right)$$

for all $i = 1, \dots, n$.

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References

[1] H. Bass, E.H. Connell and D. Wright, *The jacobian conjecture: Reduction of degree and formal expansion of the inverse* Bull. Amer. Math. Soc., **7** (1982), 287–330.

[2] H. Bass and G.H. Meisters, *Polynomial flows in the plane*, J. Algebra, **55** (1985), 173–208.

[3] J. Berson, A. van den Essen and S. Maubach, *Derivations having divergence zero on $R[x, y]$*, Israel J. Math., **124** (2001), 115–124.

[4] B.A. Coomes and V. Zurkowski, *Linearization of polynomial flows and spectra of derivations*, J. Dynamics and Diff. Equations, **3** (1991), 29–66.

[5] E. Connell and J. Drost, *Conservative and divergence free algebraic vector fields*, Proc. Amer. Math. Soc., **87** (1983), 607–612.

[6] A. van den Essen, *Polynomial automorphisms and the Jacobian Conjecture*, Progress in Mathematics 190, 2000.

[7] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopedia of Mathematical Sciences, Springer, 2006.

[8] D.A. Jordan, *Differentially simple rings with no invertible derivatives*, Quart. J. Math. Oxford, **32** (1981), 417–424.

[9] J.-P. Jouanolou, *Equations de Pfaff algébriques*, Lect. Notes in Math. **708**, Springer-Verlag, Berlin, 1979.

[10] I. Kaplansky, *An Introduction to Differential Algebra*, Hermann, Paris, 1976.

[11] A. Maciejewski, J.M. Ollagnier, A. Nowicki and J.-M. Strelcyn, *Around Jouanolou non-integrability theorem*, Indag. Mathem., **11**(2) (2000), 239–254.

[12] L. Makar-Limanov, *Locally nilpotent derivations, a new ring invariant and applications*, preprint 1998.

[13] J. Moulin Ollagnier and A. Nowicki, *Derivations of polynomial algebras without Darboux polynomials*, J. Pure Appl. Algebra, **212** (2008), 1626–1631.

[14] J. Moulin Ollagnier and A. Nowicki, *Monomial derivations*, Comm. Algebra, **39**(9) (2011), 3138–3150.

[15] J. Moulin Ollagnier, A. Nowicki and J.-M. Strelcyn, *On the non-existence of constants of derivations: The proof of a theorem of Jouanolou and its development,* Bull. Sci. Math., **119** (1995), 195–233.

[16] P. Nousiainen and M.E. Sweedler, *Automorphisms of polynomial and power series rings,* J. Pure Appl. Algebra **29** (1983), 93–97.

[17] A. Nowicki, *Commutative basis of derivations in polynomial and power series rings*, J. Pure Appl. Algebra, **40** (1986), 279–283.

[18] A. Nowicki, *On the jacobian equation $J(f, g) = 0$ for polynomials in two variables*, Nagoya J. Math., **109** (1988), 151–157.

[19] A. Nowicki, *Polynomial derivations and their rings of constants*, Nicolaus Copernicus University Press, Toruń, 1994.

[20] A. Nowicki, *An example of a simple derivation in two variables*, Colloq. Math., **113** (2008), 25–31.

[21] Y. Stein, *On the density of image of differential operators generated by polynomials*, J. Analyse Math., **52** (1989), 291–300.

[22] S.S-S. Wang, *A jacobian criterion for separability,* J. Algebra **65** (1980), 453–494.

[23] H. Żołądek, *Multidimensional Jouanolou system,* J. Reine Angew. Math **556** (2003), 47–78.

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