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A NOTE ON SQUARE-FREE FACTORIZATIONS

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Abstract. We analyze properties of various square-free factorizations in greatest common divisor domains (GCD-domains) and domains satisfying the ascending chain condition for principal ideals (ACCP-domains).

1. Introduction

Throughout this article by a ring we mean a commutative ring with unity. By a domain we mean a ring without zero divisors. By R^* we denote the set of all invertible elements of a ring R. Given elements $a, b \in R$, we write $a \sim b$ if a and b are associated, and $a \mid b$ if b is divisible by a. Furthermore, we write $a \operatorname{rpr} b$ if a and b are relatively prime, that is, have no common non-invertible divisors. If R is a ring, then by $\operatorname{Sqf} R$ we denote the set of all square-free elements of R, where an element $a \in R$ is called square-free if it can not be presented in the form $a = b^2c$ with $b \in R \setminus R^*$, $c \in R$.

In [4] we discuss many factorial properties of subrings, in particular involving square-free elements. The aim of this paper is to collect various ways to present an element as a product of square-free elements and to study the existence and uniqueness questions in larger classes than the class of unique factorization domains. In Proposition 1 we obtain the equivalence of factorizations (ii) – (vii) for GCD-domains. We also prove the existence of factorizations (i) – (iii) in Proposition 1 for ACCP-domains, but their uniqueness we obtain in Proposition 2 for GCD-domains. Recall that a domain R is called a GCD-domain if the intersection of any two principal ideals is a principal ideal. Recall also that a domain R is called an ACCP-domain if it satisfies the ascending chain condition for principal ideals.

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We refer to Clark's survey article [1] for more information about GCD-domains and ACCP-domains.

It turns out that some preparatory properties (Lemma 2) hold in a larger class than GCD-domains, namely pre-Schreier domains. A domain R is called pre-Schreier if every non-zero element $a \in R$ is primal, that is, for every $b, c \in R$ such that $a \mid bc$ there exist $a_1, a_2 \in R$ such that $a = a_1 a_2$, $a_1 \mid b$ and $a_2 \mid c$. Integrally closed pre-Schreier domains are called Schreier domains. The notion of Schreier domain was introduced by Cohn in [2]. The notion of pre-Schreier domain was introduced by Zafrullah in [6], but this property had featured already in [2], as well as in [3] and [5]. The reason why we consider pre-Schreier domains in Lemma 2 is that we were looking for a minimal condition under which a product of pairwise relatively prime square-free elements is square-free. For further information on pre-Schreier domains we refer the reader to [6].

2. Preliminary lemmas

Note the following easy lemma.

Lemma 1. *Let R be a ring. If $a \in \operatorname{Sqf} R$ and $a = b_1 b_2 \dots b_n$, then $b_1, b_2, \dots, b_n \in \operatorname{Sqf} R$ and $b_i \operatorname{rpr} b_j$ for $i \neq j$.*

In the next lemma we obtain the properties we will use in the proofs of Propositions 1 b) and 2 (i). Recall that every GCD-domain is pre-Schreier ([2], Theorem 2.4).

Lemma 2. *Let R be a pre-Schreier domain.*

a) *Let $a, b, c \in R$, $a \neq 0$. If $a \mid bc$ and $a \operatorname{rpr} b$, then $a \mid c$.*

b) *Let $a, b, c, d \in R$. If $ab = cd$, $a \operatorname{rpr} c$ and $b \operatorname{rpr} d$, then $a \sim d$ and $b \sim c$.*

c) *Let $a, b, c \in R$. If $ab = c^2$ and $a \operatorname{rpr} b$, then there exist $c_1, c_2 \in R$ such that $a \sim c_1^2$, $b \sim c_2^2$ and $c = c_1 c_2$.*

d) *Let $a_1, \dots, a_n, b \in R$. If $a_i \operatorname{rpr} b$ for $i = 1, \dots, n$, then $a_1 \dots a_n \operatorname{rpr} b$.*

e) *Let $a_1, \dots, a_n \in R$. If $a_1, \dots, a_n \in \operatorname{Sqf} R$ and $a_i \operatorname{rpr} a_j$ for all $i \neq j$, then $a_1 \dots a_n \in \operatorname{Sqf} R$.*

Proof. **a)** If $a \mid bc$, then $a = a_1 a_2$ for some $a_1, a_2 \in R \setminus \{0\}$ such that $a_1 \mid b$ and $a_2 \mid c$. If, moreover, $a \operatorname{rpr} b$, then $a_1 \in R^*$. Hence, $a \sim a_2$, so $a \mid c$.

b) Assume that $ab = cd$, $a \operatorname{rpr} c$ and $b \operatorname{rpr} d$. If $a = 0$ and R is not a field, then $c \in R^*$, so $d = 0$ and then $b \in R^*$. Now, let $a, d \neq 0$.

Since $a \mid cd$ and $a \operatorname{rpr} c$, we have $a \mid d$ by a). Similarly, since $d \mid ab$ and $d \operatorname{rpr} b$, we obtain $d \mid a$. Hence, $a \sim d$, and then $b \sim c$.

c) Let $ab = c^2$ and $a \operatorname{rpr} b$. Since $c \mid ab$, there exist $c_1, c_2 \in R \setminus \{0\}$ such that $c_1 \mid a$, $c_2 \mid b$ and $c = c_1 c_2$. Hence, $a = c_1 d$ and $b = c_2 e$ for some $d, e \in R$, and we obtain $de = c_1 c_2$. We have $d \operatorname{rpr} c_2$, because $d \mid a$ and $c_2 \mid b$, analogously $e \operatorname{rpr} c_1$, so $d \sim c_1$ and $e \sim c_2$, by b). Finally, $a \sim c_1^2$, $b \sim c_2^2$.

d) Induction. Let $a_i \operatorname{rpr} b$ for $i = 1, \dots, n+1$. Put $a = a_1 \dots a_n$. Assume that $a \operatorname{rpr} b$. Let $c \in R \setminus \{0\}$ be a common divisor of aa_{n+1} and b. Since $c \mid aa_{n+1}$, there exist $c_1, c_2 \in R \setminus \{0\}$ such that $c_1 \mid a$, $c_2 \mid a_{n+1}$ and $c = c_1 c_2$. We see that $c_1, c_2 \mid b$, so $c_1, c_2 \in R^*$, and then $c \in R^*$.

e) Induction. Take $a_1, \dots, a_{n+1} \in \operatorname{Sqf} R$ such that $a_i \operatorname{rpr} a_j$ for $i \neq j$. Put $a = a_1 \dots a_n$. Assume that $a \in \operatorname{Sqf} R$. Let $aa_{n+1} = b^2 c$ for some $b, c \in R \setminus \{0\}$.

Since $c \mid aa_{n+1}$, there exist $c_1, c_2 \in R \setminus \{0\}$ such that $c = c_1 c_2$, $c_1 \mid a$ and $c_2 \mid a_{n+1}$, so $a = c_1 d$ and $a_{n+1} = c_2 e$, where $d, e \in R$. We obtain $de = b^2$. By d) we have $a \operatorname{rpr} a_{n+1}$, so $d \operatorname{rpr} e$. And then by c), there exist $b_1, b_2 \in R$ such that $d \sim b_1^2$, $e \sim b_2^2$ and $b = b_1 b_2$. Since $a, a_{n+1} \in \operatorname{Sqf} R$, we infer $b_1, b_2 \in R^*$, so $b \in R^*$. $\square$

3. Square-free factorizations

In Proposition 1 below we collect possible presentations of an element as a product of square-free elements or their powers. We distinct presentations (ii) and (iii), presentations (iv) and (v), and presentations (vi) and (vii), because (ii), (iv) and (vi) are of a simpler form, but in (iii), (v) and (vii) the uniqueness will be more natural (in Proposition 2).

Proposition 1. *Let R be a ring. Given a non-zero element $a \in R \setminus R^*$, consider the following conditions:*

(i) *there exist $b \in R$ and $c \in \operatorname{Sqf} R$ such that $a = b^2 c$,*

(ii) *there exist $n \geqslant 0$ and $s_0, s_1, \dots, s_n \in \operatorname{Sqf} R$ such that $a = s_n^{2^n} s_{n-1}^{2^{n-1}} \dots s_1^2 s_0$,*

(iii) *there exist $n \geqslant 1$, $s_1, s_2, \dots, s_n \in (\operatorname{Sqf} R) \setminus R^*$, $k_1 < k_2 < \dots < k_n$, $k_1 \geqslant 0$, and $c \in R^*$ such that $a = c s_n^{2^{k_n}} s_{n-1}^{2^{k_{n-1}}} \dots s_2^{2^{k_2}} s_1^{2^{k_1}}$,*

(iv) *there exist $n \geqslant 1$ and $s_1, s_2, \dots, s_n \in \operatorname{Sqf} R$ such that $s_i \mid s_{i+1}$ for $i = 1, \dots, n-1$, and $a = s_1 s_2 \dots s_n$,*

(v) *there exist $n \geqslant 1$, $s_1, s_2, \dots, s_n \in (\operatorname{Sqf} R) \setminus R^*$, $k_1, k_2, \dots, k_n \geqslant 1$, and $c \in R^*$ such that $s_i \mid s_{i+1}$ and $s_i \not\sim s_{i+1}$ for $i = 1, \dots, n-1$, and $a = c s_1^{k_1} s_2^{k_2} \dots s_n^{k_n}$,*

(vi) *there exist $n \geqslant 1$ and $s_1, s_2, \dots, s_n \in \operatorname{Sqf} R$ such that $s_i \operatorname{rpr} s_j$ for $i \neq j$, and $a = s_1 s_2^2 s_3^3 \dots s_n^n$,*

(vii) *there exist $n \geqslant 1$, $s_1, s_2, \dots, s_n \in (\operatorname{Sqf} R) \setminus R^*$, $k_1 < k_2 < \dots < k_n$, $k_1 \geqslant 1$, and $c \in R^*$ such that $s_i \operatorname{rpr} s_j$ for $i \neq j$, and $a = c s_1^{k_1} s_2^{k_2} \dots s_n^{k_n}$.*

a) *In every ring R the following holds:*

$$\text{(i)} \Leftarrow \text{(ii)} \Leftrightarrow \text{(iii)}, \qquad \text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(vi)} \Leftrightarrow \text{(vii)}.$$

b) *If R is a GCD-domain, then all conditions* (ii) – (vii) *are equivalent.*

c) *If R is an ACCP-domain, then conditions* (i) – (iii) *hold.*

d) *If R is a UFD, then all conditions* (i) – (vii) *hold.*

Proof. **a)** Implication (i) $\Leftarrow$ (ii) and equivalencies (ii) $\Leftrightarrow$ (iii), (iv) $\Leftrightarrow$ (v), (vi) $\Leftrightarrow$ (vii) are obvious, so it is enough to prove implication (iv) $\Rightarrow$ (vi).

Assume that $a = s_1 s_2 \dots s_n$, where $s_1, s_2, \dots, s_n \in \operatorname{Sqf} R$ and $s_i \mid s_{i+1}$ for $i = 1, \dots, n-1$. Let $s_{i+1} = s_i t_{i+1}$, where $t_{i+1} \in R$, for $i = 1, \dots, n-1$. Put also $t_1 = s_1$. Then $s_i = t_1 t_2 \dots t_i$ for each i. Since $s_n \in \operatorname{Sqf} R$, by Lemma 1 we obtain that $t_1, t_2, \dots, t_n \in \operatorname{Sqf} R$ and $t_i \operatorname{rpr} t_j$ for $i \neq j$. Moreover, we have $s_1 s_2 \dots s_n = t_1^n t_2^{n-1} \dots t_n$.

b) Let R be a GCD-domain.

(vi) $\Rightarrow$ (iv) Assume that $a = s_1 s_2^2 s_3^3 \dots s_n^n$, where $s_1, s_2, \dots, s_n \in \operatorname{Sqf} R$ and $s_i \operatorname{rpr} s_j$ for $i \neq j$. We see that

$$s_1 s_2^2 s_3^3 \dots s_n^n = s_n (s_n s_{n-1})(s_n s_{n-1} s_{n-2}) \dots (s_n s_{n-1} \dots s_2)(s_n s_{n-1} \dots s_2 s_1).$$

Since R is a GCD-domain, $s_n s_{n-1} \dots s_i \in \operatorname{Sqf} R$ for each i by Lemma 2 e).

(vi) $\Rightarrow$ (ii) Let $a = s_1 s_2^2 s_3^3 \dots s_n^n$, where $s_1, s_2, \dots, s_n \in \operatorname{Sqf} R$, and $s_i \operatorname{rpr} s_j$ for $i \neq j$. For every $k \in \{1, 2, \dots, n\}$ put $k = \sum_{i=0}^r c_i^{(k)} 2^i$, where $c_i^{(k)} \in \{0, 1\}$. Then

$$a = \prod_{k=1}^n s_k^k = \prod_{k=1}^n s_k^{\sum_{i=0}^r c_i^{(k)} 2^i} = \prod_{k=1}^n \prod_{i=0}^r s_k^{c_i^{(k)} 2^i} = \prod_{i=0}^r \Big(\prod_{k=1}^n s_k^{c_i^{(k)}} \Big)^{2^i},$$

where $\prod_{k=1}^n s_k^{c_i^{(k)}} \in \operatorname{Sqf} R$ for each i by Lemma 2 e).

(ii) $\Rightarrow$ (vi) Let $a = s_n^{2^n} s_{n-1}^{2^{n-1}} \dots s_1^2 s_0$, where $s_0, s_1, \dots, s_n \in \operatorname{Sqf} R$. For every $k \in \{1, 2, \dots, 2^{n+1} - 1\}$ put $k = \sum_{i=0}^n c_i^{(k)} 2^i$, where $c_i^{(k)} \in \{0, 1\}$. Let $t'_k = \gcd(s_i \colon c_i^{(k)} = 1)$, $t''_k = \operatorname{lcm}(s_i \colon c_i^{(k)} = 0)$ and $t'_k = \gcd(t'_k, t''_k) \cdot t_k$, where $t_k \in R$ (by [2], Theorem 2.1, in a GCD-domain least common multiples exist). Then t_k is the greatest among these common divisors of all s_i such that $c_i^{(k)} = 1$, which are relatively prime to all s_i such that $c_i^{(k)} = 0$. In particular, $t_k \mid s_i$ for every k, i such that $c_i^{(k)} = 1$, and $t_k \operatorname{rpr} s_i$ for every k, i such that $c_i^{(k)} = 0$. In each case, $\gcd(s_i, t_k) = t_k^{c_i^{(k)}}$. Moreover, $t_k \operatorname{rpr} t_l$ for every $k \neq l$.

Since $s_i \mid t_1 t_2 \dots t_{2^{n+1}-1}$, we obtain

$$s_i = \gcd(s_i, \prod_{k=1}^{2^{n+1}-1} t_k) = \prod_{k=1}^{2^{n+1}-1} \gcd(s_i, t_k) = \prod_{k=1}^{2^{n+1}-1} t_k^{c_i^{(k)}},$$

so

$$\prod_{i=0}^n (s_i)^{2^i} = \prod_{i=0}^n \prod_{k=1}^{2^{n+1}-1} \big(t_k^{c_i^{(k)}}\big)^{2^i} = \prod_{k=1}^{2^{n+1}-1} \prod_{i=0}^n t_k^{c_i^{(k)} 2^i} = \prod_{k=1}^{2^{n+1}-1} t_k^{\sum_{i=0}^n c_i^{(k)} 2^i} = \prod_{k=1}^{2^{n+1}-1} t_k^k.$$

Moreover, $t_k \in \operatorname{Sqf} R$, because for $k \in \{1, 2, \dots, 2^{n+1} - 1\}$ there exists i such that $c_i^{(k)} = 1$, and then $t_k \mid s_i$.

c) Let R be an ACCP-domain. In this proof we follow the idea of the second proof of Proposition 9 from [1], p. 7, 8.

(i) If $a \notin \operatorname{Sqf} R$, then $a = b_1^2 c_1$, where $b_1 \in R \setminus R^*$, $c_1 \in R$. If $c_1 \notin \operatorname{Sqf} R$, then $c_1 = b_2^2 c_2$, where $b_2 \in R \setminus R^*$, $c_2 \in R$. Repeating this process, we obtain a strongly ascending chain of principal ideals $Ra \subsetneqq Rc_1 \subsetneqq Rc_2 \subsetneqq \dots$, so for some k we will have $c_{k-1} = b_k^2 c_k$, $b_k \in R \setminus R^*$, and $c_k \in \operatorname{Sqf} R$. Then $a = (b_1 \dots b_k)^2 c_k$.

(iii) If $a \notin \operatorname{Sqf} R$, then by (i) there exist $a_1 \in R \setminus R^*$ and $s_0 \in \operatorname{Sqf} R$ such that $a = a_1^2 s_0$. If $a_1 \notin \operatorname{Sqf} R$, then again, by (i) there exist $a_2 \in R \setminus R^*$ and $s_1 \in \operatorname{Sqf} R$ such that $a_1 = a_2^2 s_1$. Repeating this process, we obtain a strongly ascending chain of principal ideals $Ra \subsetneqq Ra_1 \subsetneqq Ra_2 \subsetneqq \dots$, so for some k we will have $a_{k-1} = a_k^2 s_{k-1}$, $a_k \in (\operatorname{Sqf} R) \setminus R^*$, $s_{k-1} \in \operatorname{Sqf} R$. Putting $s_k = a_k$ we obtain:

$$a = a_1^2 s_0 = a_2^{2^2} s_1^2 s_0 = \dots = s_n^{2^n} \dots s_2^{2^2} s_1^2 s_0.$$

d) This is a standard fact following from the irreducible decomposition. □

4. The uniqueness of factorizations

The following proposition concerns the uniqueness of square-free decompositions from Proposition 1. In (i) – (iii) we assume that R is a GCD-domain, in (iv) – (vii) R is a UFD.

Proposition 2. (i) *Let $b, d \in R$ and $c, e \in \operatorname{Sqf} R$. If*

$$b^2 c = d^2 e,$$

then $b \sim d$ and $c \sim e$.

(ii) *Let $s_0, s_1, \dots, s_n \in \operatorname{Sqf} R$ and $t_0, t_1, \dots, t_m \in \operatorname{Sqf} R$, $n \leqslant m$. If*

$$s_n^{2^n} s_{n-1}^{2^{n-1}} \dots s_1^2 s_0 = t_m^{2^m} t_{m-1}^{2^{m-1}} \dots t_1^2 t_0,$$

then $s_i \sim t_i$ for $i = 0, \dots, n$ and, if $m > n$, then $t_i \in R^*$ for $i = n+1, \dots, m$.

(iii) *Let $s_1, s_2, \dots, s_n \in (\operatorname{Sqf} R) \setminus R^*$, $t_1, t_2, \dots, t_m \in (\operatorname{Sqf} R) \setminus R^*$, $k_1 < k_2 < \dots < k_n$, $l_1 < l_2 < \dots < l_m$ and $c, d \in R^*$. If*

$$c s_n^{2^{k_n}} s_{n-1}^{2^{k_{n-1}}} \dots s_2^{2^{k_2}} s_1^{2^{k_1}} = d t_m^{2^{l_m}} t_{m-1}^{2^{l_{m-1}}} \dots t_2^{2^{l_2}} t_1^{2^{l_1}},$$

then $n = m$, $s_i \sim t_i$ and $k_i = l_i$ for $i = 1, \dots, n$.

(iv) *Let $s_1, s_2, \dots, s_n \in \operatorname{Sqf} R$, $t_1, t_2, \dots, t_m \in \operatorname{Sqf} R$, $n \leqslant m$, $s_i \mid s_{i+1}$ for $i = 1, \dots, n-1$, and $t_i \mid t_{i+1}$ for $i = 1, \dots, m-1$. If*

$$s_1 s_2 \dots s_n = t_1 t_2 \dots t_m,$$

then $s_i \sim t_{i+m-n}$ for $i = 1, \dots, n$ and, if $m > n$, then $t_i \in R^*$ for $i = 1, \dots, m-n$.

(v) *Let $s_1, s_2, \dots, s_n \in (\operatorname{Sqf} R) \setminus R^*$, $t_1, t_2, \dots, t_m \in (\operatorname{Sqf} R) \setminus R^*$, $k_1, k_2, \dots, k_n \geqslant 1$, $l_1, l_2, \dots, l_m \geqslant 1$, $c, d \in R^*$, $s_i \mid s_{i+1}$ and $s_i \not\sim s_{i+1}$ for $i = 1, \dots, n-1$, $t_i \mid t_{i+1}$ and $t_i \not\sim t_{i+1}$ for $i = 1, \dots, m-1$. If*

$$c s_1^{k_1} s_2^{k_2} \dots s_n^{k_n} = d t_1^{l_1} t_2^{l_2} \dots t_m^{l_m},$$

then $n = m$, $s_i \sim t_i$ and $k_i = l_i$ for $i = 1, \dots, n$.

(vi) *Let* $s_1, s_2, \ldots, s_n \in \operatorname{Sqf} R$, $t_1, t_2, \ldots, t_m \in \operatorname{Sqf} R$, $n \leqslant m$, $s_i \operatorname{rpr} s_j$ *for* $i \neq j$ *and* $t_i \operatorname{rpr} t_j$ *for* $i \neq j$. *If*

$$s_1 s_2^2 s_3^3 \ldots s_n^n = t_1 t_2^2 t_3^3 \ldots t_m^m,$$

then $s_i \sim t_i$ *for* $i = 1, \ldots, n$ *and, if* $m > n$, *then* $t_i \in R^*$ *for* $i = n+1, \ldots, m$.

(vii) *Let* $s_1, s_2, \ldots, s_n \in (\operatorname{Sqf} R) \setminus R^*$, $t_1, t_2, \ldots, t_m \in (\operatorname{Sqf} R) \setminus R^*$, $1 \leqslant k_1 < k_2 < \ldots < k_n$, $1 \leqslant l_1 < l_2 < \ldots < l_m$, $c, d \in R^*$, $s_i \operatorname{rpr} s_j$ *for* $i \neq j$, *and* $t_i \operatorname{rpr} t_j$ *for* $i \neq j$. *If*

$$c s_1^{k_1} s_2^{k_2} \ldots s_n^{k_n} = d t_1^{l_1} t_2^{l_2} \ldots t_m^{l_m},$$

then $n = m$, $s_i \sim t_i$ *and* $k_i = l_i$ *for* $i = 1, \ldots, n$.

Proof. (i) Assume that $b^2 c = d^2 e$. Put $f = \gcd(b, d)$, $g = \gcd(c, e)$, $b = f b_0$, $d = f d_0$, $c = g c_0$, and $e = g e_0$, where $b_0, c_0, d_0, e_0 \in R$. We obtain $b_0^2 c_0 = d_0^2 e_0$, $\gcd(c_0, e_0) = 1$ and $\gcd(b_0, d_0) = 1$, so also $\gcd(b_0^2, d_0^2) = 1$. By Lemma 2 b), we infer $b_0^2 \sim e_0$ and $c_0 \sim d_0^2$, but $c_0, e_0 \in \operatorname{Sqf} R$ by Lemma 1, so $b_0, d_0 \in R^*$, and then $c_0, e_0 \in R^*$.

Statements (ii), (iii) follow from (i).

Statements (iv) – (vii) are straightforward using the irreducible decomposition. □

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