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ON A GENERIC SYMMETRY DEFECT HYPERSURFACE

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Abstract. We show that symmetry defect hypersurfaces for two generic members of the irreducible algebraic family of $n-$dimensional smooth irreducible subvarieties in general position in $\mathbb{C}^{2n}$ are homeomorphic and they have homeomorphic sets of singular points. In particular symmetry defect curves for two generic curves in $\mathbb{C}^2$ of the same degree have the same number of singular points.

1. Introduction

Let $X^n \subset \mathbb{C}^{2n}$ be a smooth algebraic variety. In [7] we have investigated the central symmetry of X (see also [1], [2], [3]). For $p \in \mathbb{C}^{2n}$ we have introduced a number $\mu(p)$ of pairs of points $x, y \in X$, such that p is the center of the interval $\overline{xy}$. Recall that the subvariety $X^n \subset \mathbb{C}^{2n}$ is *in a general position* if there exist points $x, y \in X^n$ such that $T_xX \oplus T_yX = \mathbb{C}^{2n}$.

We have showed in [7] that if X is in general position, then there is a closed algebraic hypersurface $B \subset \mathbb{C}^{2n}$, called *symmetry defect hypersurface* of X, such that the function μ is constant (non-zero) exactly outside B. Here we prove that the symmetry defect hypersurfaces for two generic members of an irreducible algebraic family of $n-$dimensional smooth irreducible subvarieties in general position in $\mathbb{C}^{2n}$ are homeomorphic.

Moreover, by a version of Sard theorem for singular varieties (see [4]), we have that the symmetry defect hypersurfaces for two generic members of an irreducible

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algebraic family of $n-$dimensional smooth irreducible subvarieties in general position in $\mathbb{C}^{2n}$ have homeomorphic sets of singular points. In particular symmetry defect curves for two generic curves in $\mathbb{C}^2$ of the same degree have the same number of singular points.

2. Bifurcation set

Let X be an irreducible affine variety. Let $Sing(X)$ denote the set of singular points of X. Let Y be another affine variety and consider a dominant morphism $f : X \to Y$. If X is smooth then by Sard's Theorem a generic fiber of f is smooth. In a general case the following theorem holds (see [4]):

Theorem 2.1. *Let $f : X^k \to Y^l$ be a dominant polynomial mapping of affine varieties. For generic $y \in Y$ we have $Sing(f^{-1}(y)) = f^{-1}(y) \cap Sing(X)$.*

Recall the following (see [5], [6]):

Definition 2.2. Let $f : X \to Y$ be a generically-finite (i.e. a generic fiber is finite) and dominant (i.e. $\overline{f(X)} = Y$) polynomial mapping of affine varieties. We say that *f is finite at a point $y \in Y$*, if there exists an open neighborhood U of y such that the mapping $f\mid_{f^{-1}(U)}: f^{-1}(U) \to U$ is proper.

It is well-known that the set S_f of points at which the mapping f is not finite, is either empty or it is a hypersurface (see [5], [6]). We say that the set S_f is *the set of non-properness* of the mapping f.

Definition 2.3. *Let X, Y be smooth affine $n-$dimensional varieties and let $f : X \to Y$ be a generically finite dominant mapping of geometric degree $\mu(f)$. The bifurcation set of the mapping f is the set*

$$B(f) = \{y \in Y : \#f^{-1}(y) \neq \mu(f)\}.$$

We have the following theorem (see [7]):

Theorem 2.4. *Let X, Y be smooth affine complex varieties of dimension n. Let $f : X \to Y$ be a polynomial dominant mapping. Then the set $B(f)$ is either empty (so f is an unramified topological covering) or it is a closed hypersurface.*

3. A super general position

In this section we describe some properties of a variety $X^n \subset \mathbb{C}^{2n}$ which implies that X is in a general position. Recall that the subvariety $X^n \subset \mathbb{C}^{2n}$ is in a general position if there exist points $x, y \in X^n$ such that $T_xX \oplus T_yX = \mathbb{C}^{2n}$.

Definition 3.1. *Let $X^n \subset \mathbb{C}^{2n}$ be a smooth algebraic variety. We say that X is in very general position if there exists a point $x \in X$ such that the set $T_xX \cap X$ has an isolated point (here we consider T_xX as a linear subspace of $\mathbb{C}^{2n}$).*

We consider also a slightly stronger property:

Definition 3.2. *Let $X^n \subset \mathbb{C}^{2n}$ be a smooth algebraic variety and let $S = \overline{X} \setminus X \subset \pi_\infty$ be the set of points at infinity of X^n. We say that X is in super general position if there exists a point $x \in X$ such that $T_xX \cap S = \emptyset$ (here we consider T_xX as a linear subspace of $\mathbb{P}^{2n} = \mathbb{C}^{2n} \cup \pi_\infty$).*

We have the following:

Proposition 3.3. *If X is in a super general position, then it is in a very general position.*

Proof. Let $x \in X$ be a point such that $T_xX \cap S = \emptyset$. Take $R = T_xX \cap X$. Then the set R is finite, since otherwise the point at infinity of R belongs to $T_xX \cap S = \emptyset$. □

We have also:

Proposition 3.4. *Let $X \subset \mathbb{C}^{2n}$ be in a super general position. Then for a generic point $x \in X$ we have $T_xX \cap S = \emptyset$.*

Proof. It is easy to see that the set $\Gamma = \{(s, x) \in S \times X : s \in T_xX\}$ is an algebraic subset of $S \times X$. Let $\pi : \Gamma \ni (s, x) \to x \in X$ be a projection. It is a proper mapping. Since the variety X is in a very general position, we see that at least one point $x_0 \in X$ is not in the image of π. Thus almost every point of X is not in the image of π, because the image of π is a closed subset of X. □

Finally we have:

Theorem 3.5. *If $X \subset \mathbb{C}^{2n}$ is in a very general position, then it is in a general position, i. e., there exist points $x, y \in X$ such that $T_xX \oplus T_yX = \mathbb{C}^{2n}$. In fact for every generic pair $(x, y) \in X \times X$ we have $T_xX \oplus T_yX = \mathbb{C}^{2n}$.*

Proof. Let $x_0 \in X$ be the point such that the set $T_{x_0}X \cap X$ has an isolated point. The space $T_{x_0}X$ is given by n linear equations $l_i = 0$. Let $F : X \ni x \to (l_1(x), ..., l_n(x)) \in \mathbb{C}^n$. By the assumption the fiber over 0 of F has an isolated point, in particular the mapping F is dominant. Now by the Sard Theorem almost every point $x \in X$ is a regular point of F. This means that T_xX is complementary to $T_{x_0}X$, i.e., $T_{x_0}X \oplus T_xX = \mathbb{C}^{2n}$. If we consider the mapping $\Phi : X \times X \ni (x, y) \to x + y \in \mathbb{C}^{2n}$, we see that it has the smooth point (x_0, x). In particular almost every pair (x, y) is a smooth point of F, which implies that for every generic pair $(x, y) \in X \times X$ we have $T_xX \oplus T_yX = \mathbb{C}^{2n}$. □

We shall use in the sequel the following:

Proposition 3.6. *Let $X^n \subset \mathbb{C}^{2n}$ be a generic smooth complete intersection of multi-degree $d_1, ..., d_n$. If every $d_i > 1$, then X is in a super general position.*

Proof. We can assume that X is given by n smooth hypersurfaces $f_i = a_i + f_{i1} + ... + f_{id_i}$ (where f_{ik} is a homogenous polynomial of degree k), which have independent all coefficients (see section below). The tangent space is described by polynomials $f_{i1}, i = 1, ..., n$ and the set S of points at infinity of X is described by polynomials $f_{id_i}, i = 1, ..., n$. Since these two families of polynomials have independent coefficients, we see that generically the zero sets at infinity of these two families are disjoint. In particular such a generic X is in a super general position. □

4. Algebraic families

Now we introduce the notion of an algebraic family.

Definition 4.1. *Let M be a smooth affine algebraic variety and let Z be a smooth irreducible subvariety of $M \times \mathbb{C}^n$. If the restriction to Z of the projection $\pi : M \times \mathbb{C}^n \to M$ is a dominant map with generically irreducible fibers of the same dimension, then we call the collection $\Sigma = \{Z_m = \pi^{-1}(m)\}_{m \in M}$ an algebraic family of subvarieties in $\mathbb{C}^n$. We say that this family is in a general position if a generic member of Σ is in a general position in $\mathbb{C}^n$.*

We show that the ideals $I(Z_m) \subset \mathbb{C}[x_1, ..., x_n]$ of a generic member of Σ depend in a parametric way on $m \in M$.

Lemma 4.2. *Let Σ be an algebraic family given by a smooth variety $Z \subset M \times \mathbb{C}^n$. The ideal $I(Z) \subset \mathbb{C}[M][x_1, ..., x_n]$ is finitely generated, let the polynomials $\{f_1(m, x), ..., f_s(m, x)\}$ form its set of generators. The ideal $I(Z_m) \subset \mathbb{C}[x_1, ..., x_n]$ of a generic member $Z_m := \pi^{-1}(m) \subset \mathbb{C}^n$ of Σ is equal to $I(Z_m) = (f_1(m, x), ..., f_s(m, x))$.*

Proof. Let dim $Z = p$ and dim $M = q$. Thus the variety $M \times \mathbb{C}^n$ has dimension $n + q$. Choose local holomorphic coordinates on M. Since the variety Z is smooth we have

$$\operatorname{rank} \begin{bmatrix} \frac{\partial f_1}{\partial m_1}(m,x) & \dots & \frac{\partial f_1}{\partial m_q}(m,x) & \frac{\partial f_1}{\partial x_1}(m,x) & \dots & \frac{\partial f_1}{\partial x_n}(m,x) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_s}{\partial m_1}(m,x) & \dots & \frac{\partial f_s}{\partial m_q}(m,x) & \frac{\partial f_s}{\partial x_1}(m,x) & \dots & \frac{\partial f_s}{\partial x_n}(m,x) \end{bmatrix} = n + q - p$$

on Z. Let us consider the projection $\pi : Z \ni (m, x) \mapsto m \in M$. By Sard's theorem a generic $m \in M$ is a regular value of the mapping π. For such a regular value m we have that dim ker $d_{(m,x)}\pi \cap T_{(m,x)}Z = p - q$ for every x such that $(m, x) \in Z$. In local coordinates on M this is equivalent to

$$\operatorname{rank}\begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ * & \dots & * & \frac{\partial f_1}{\partial x_1}(m,x) & \dots & \frac{\partial f_1}{\partial x_n}(m,x) \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \dots & * & \frac{\partial f_s}{\partial x_1}(m,x) & \dots & \frac{\partial f_s}{\partial x_n}(m,x) \end{bmatrix} = n+2q-p.$$

Consequently for $(m,x) \in Z$ and m a regular value of π we have

$$\operatorname{rank}\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(m,x) & \dots & \frac{\partial f_1}{\partial x_n}(m,x) \\ \vdots & & \vdots \\ \frac{\partial f_s}{\partial x_1}(m,x) & \dots & \frac{\partial f_s}{\partial x_n}(m,x) \end{bmatrix} = n+q-p.$$

Note that $n+q-p = \operatorname{codim} Z_m$ (in $\mathbb{C}^n$). This means that the ideal $(f_1(m,x),...,f_s(m,x))$ locally coincide with $I(Z_m)$, because it contains local equations of Z_m. Hence it also coincides globally, i.e., $(f_1(m,x),...,f_s(m,x)) = I(Z_m)$. □

Remark 4.3. This can be also obtained by a computation of a scheme theoretic fibers of π and using the fact that such generic fibers are reduced.

Example 4.4. a) Let $N := \binom{n+d}{d}$ and let $Z \subset \mathbb{C}^N \times \mathbb{C}^n$ be given by equations $Z = \{(a,x) \in \mathbb{C}^N \times \mathbb{C}^n : \sum_{|\alpha|\le d} a_\alpha x^\alpha = 0\}$. The projection $\pi : Z \ni (a,x) \to a \in \mathbb{C}^N$ determines an algebraic family of hypersurfaces of degree d in $\mathbb{C}^n$. If $n=2$ and $d>1$ this family is in general position in $\mathbb{C}^2$.

b) More generally let $N_1 := \binom{n+d_1}{d_1}$, $N_2 := \binom{n+d_2}{d_2}$, $N_n := \binom{n+d_n}{d_n}$ and let $Z \subset \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} ... \times \mathbb{C}^{N_n} \times \mathbb{C}^{2n}$ be given by equations $Z = \{(a_1,a_2,...,a_n,x) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} ... \times \mathbb{C}^{N_n} \times \mathbb{C}^n : \sum_{|\alpha|\le d_1} a_{1\alpha}x^\alpha = 0, \sum_{|\alpha|\le d_2} a_{2\alpha}x^\alpha = 0, ..., \sum_{|\alpha|\le d_n} a_{n\alpha}x^\alpha = 0\}$. The projection $\pi : Z \ni (a_1,a_2,...,a_n,x) \to (a_1,a_2,...,a_n) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} ... \times \mathbb{C}^{N_n}$ determines an algebraic family $\Sigma(d_1,d_2,...,d_n,2n)$ of complete intersections of multi-degree $d_1,d_2,....,d_n$ in $\mathbb{C}^{2n}$. If $d_1,d_2,...,d_n > 1$, then this family is in general position in $\mathbb{C}^{2n}$. This follows from Proposition 3.6.

5. Defect of symmetry

Let us recall that a following result is true (see e.g. [7]):

Lemma 5.1. *Let X, Y be complex algebraic varieties and $f : X \to Y$ a polynomial dominant mapping. Then two generic fibers of f are homeomorphic.*

Proof. Let X_1 be an algebraic completion of X. Take $X_2 = \overline{graph(f)} \subset X_1 \times \overline{Y}$, where $\overline{Y}$ is a smooth algebraic completion of Y. We can assume that $X \subset X_2$. Let $Z = X_2 \setminus X$. We have an induced mapping $\overline{f} : X_2 \to \overline{Y}$, such that $\overline{f}_X = f$.

There is a Whitney stratification $\mathcal{S}$ of the pair (X_2, Z). For every smooth strata $S_i \in \mathcal{S}$ let B_i be the set of critical values of the mapping $f|_{S_i}$.Take $B = \overline{\bigcup B_i}$. Take $X_3 = X_2 \setminus f^{-1}(B)$ and $Z_1 = Z \setminus f^{-1}(B)$. The restriction of the stratification $\mathcal{S}$ to X_3 gives a Whitney stratification of the pair (X_3, Z_1). We have a proper mapping $f_1 : X_3 \to \overline{Y} \setminus B$ which is submersion on each strata. By the Thom first isotopy theorem there is a trivialization of f_1, which preserves the strata. It is an easy observation that this trivialization gives a trivialization of the mapping $f : X \setminus f^{-1}(B) \to Y \setminus B$. □

Definition 5.2. Let X be an affine variety. Let us define $Sing^k(X) := Sing(X)$ for $k := 1$ and inductively $Sing^{k+1}(X) := Sing(Sing^k(X))$.

As a direct application of the Lemma 5.1 and Theorem 2.1 we have:

Theorem 5.3. *Let $f : X^n \to Y^l$ be a dominant polynomial mapping of affine varieties. If y_1, y_2 are sufficiently general then $f^{-1}(y_1)$ is homeomorphic to $f^{-1}(y_2)$ and $Sing(f^{-1}(y_1))$ is homeomorphic to $Sing(f^{-1}(y_2))$. More generally, for every k we have $Sing^k(f^{-1}(y_1))$ is homeomorphic to $Sing^k(f^{-1}(y_2))$.*

Now we are ready to prove:

Theorem 5.4. *Let Σ be an algebraic family of $n-$dimensional algebraic subvarieties in $\mathbb{C}^{2n}$ in general position. Symmetry defect hypersurfaces B_1, B_2 for generic members $C_1, C_2 \in \Sigma$ are homeomorphic and they have homeomorphic singular parts i.e., $Sing(B_1) \cong Sing(B_2)$. More generally, for every k we have $Sing^k(B_1)$ is homeomorphic to $Sing^k(B_2)$.*

Proof. Let Σ be given by a variety $Z \subset M \times \mathbb{C}^{2n}$. The ideal $I(Z) \subset \mathbb{C}[M][x_1, ..., x_{2n}]$ is finitely generated. Choose a finite set of generators $\{f_1(m,x), ..., f_s(m,x)\}$.

By Sard Theorem we can assume that all fibers of $\pi : Z \to M$ are smooth and for every $m \in M$ we have $I(Z_m) = \{f_1(m,x), ..., f_s(m,x)\}$ (see Lemma 4.2). Let us define

$$R = \{(m,x,y) \in M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} : f_i(m)(x) = 0, i = 1, ..., s \quad \& \quad f_i(m)(y) = 0, \\ i = 1, ..., s\}.$$

The variety R is a smooth irreducible subvariety of $M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n}$ of codimension $2n$. Indeed, for given $(m,x,y) \in M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n}$ choose polynomials $f_{i_1}, ..., f_{i_n}$ and $f_{j_1}, ..., f_{j_n}$ such that rank $[\frac{\partial f_{i_l}}{\partial x_s}(m,x)]_{l=1,...,n;s=1,...,n} = n$ and rank $[\frac{\partial f_{j_l}}{\partial x_s}(m,x)]_{l=1,...,n;s=1,...,n} = n$. Since Z is a smooth variety of dimension $\dim M + n$, we have that Z locally near (m,x) is given by equations $f_{i_1}, ..., f_{i_n}$ and near (m,y)

is given by equations $f_{j_1}, ..., f_{j_n}$. Hence the variety R near the point (m, x, y) is given as

$$\{(m, x, y) \in M \times \mathbb{C}^{2n} \times \mathbb{C}^{2n} : f_{i_l}(m)(x) = 0, l = 1, ..., n \quad \& \quad f_{j_l}(m)(y) = 0, \\ l = 1, ..., s\}.$$

In particular R is locally a smooth complete intersection, i.e., R is smooth.

Moreover we have a projection $R \to M$ with irreducible fibers which are products $Z_m \times Z_m, \quad m \in M$. This means that R is irreducible. Note that R is an affine variety. Consider the following morphism

$$\Psi : R \ni (m, x, y) \mapsto (m, \frac{x+y}{2}) \in M \times \mathbb{C}^{2n}.$$

By the assumptions the mapping Ψ is dominant. Indeed for every $m \in M$ the fiber Z_m is in a general position in $\mathbb{C}^{2n}$ and consequently the set $\Psi(R) \cap m \times \mathbb{C}^{2n}$ is dense in $m \times \mathbb{C}^{2n}$.

We know by Theorem 2.4 that the mapping Ψ has constant number of points in the fiber outside the bifurcation set $B(\Psi) \subset M \times \mathbb{C}^{2n}$. This implies that $B(Z_m) = m \times \mathbb{C}^{2n} \cap B(\Psi)$. In particular the symmetry defect hypersurface of the variety Z_m coincide with the fiber over m of the projection $\pi : B(\Psi) \ni (m, x) \mapsto m \in M$. Now we conclude the proof by Theorem 5.3. □

Corollary 5.5. *Symmetry defect sets B_1, B_2 for generic curves $C_1, C_2 \subset \mathbb{C}^2$ of the same degree $d > 1$ are homeomorphic and they have the same number of singular points.*

Corollary 5.6. *Let C_1, C_2 be two smooth varieties, which are generic complete intersection of multi-degree $d_1, d_2, ..., d_n$ in $\mathbb{C}^{2n}$ (where all $d_i > 1$). Then symmetry defect hypersurfaces B_1, B_2 of C_1, C_2, are homeomorphic and they have homeomorphic singular parts (i.e., $Sing(B_1) \cong Sing(B_2)$). More generally, for every k we have $Sing^k(B_1)$ is homeomorphic to $Sing^k(B_2)$.*

References

[1] P.J. Giblin and P.A. Holtom, *The centre symmetry set*, in: Geometry and Topology of Caustics (Warsaw 1998), Banach Center Publ. Vol. 50, ed. S. Janeczko and V.M. Zakalyukin, Warsaw, 1999, 91–105.

[2] P.J. Giblin and S. Janeczko, *Geometry of curves and surfaces through the contact map*, Topology Appl. 159 (2012), 466–475.

[3] S. Janeczko, *Bifurcations of the Center of Symmetry*, Geom. Dedicata 60, (1996), 9–16.

[4] Z. Jelonek, *On semi-equivalence of generically finite polynomial mappings,* to appear.

[5] Z. Jelonek, *The set of points at which a polynomial map is not proper,* Ann. Polon. Math. 58 (1993), 259–266.

[6] Z. Jelonek, *Testing sets for properness of polynomial mappings,* Math. Ann. 315 (1999), 1–35.

[7] S. Janeczko, Z. Jelonek and M.A.S. Ruas, *Symmetry defect of algebraic varieties*, Asian J. Math. 18 (2014), 525–544.

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