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MULTIPLE ZETA VALUES AND THE WKB METHOD

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Abstract. The multiple zeta values $\zeta(d_1,\dots,d_r)$ are natural generalizations of the values $\zeta(d)$ of the Riemann zeta functions at integers d. They have many applications, e.g. in knot theory and in quantum physics. It turns out that some generating functions for the multiple zeta values, like $f_d(x) = 1-\zeta(d)x^d+\zeta(d,d)x^{2d}-\dots$, are related with hypergeometric equations. More precisely, $f_d(x)$ is the value at $t=1$ of some hypergeometric series ${}_dF_{d-1}(t) = 1-x^dt+\dots$, a solution to a hypergeometric equation of degree d with parameter x. Our idea is to represent $f_d(x)$ as some connection coefficient between certain standard bases of solutions near $t=0$ and near $t=1$. Moreover, we assume that $|x|$ is large. For large complex x the above basic solutions are represented in terms of so-called WKB solutions. The series which define the WKB solutions are divergent and are subject to so-called Stokes phenomenon. Anyway it is possible to treat them rigorously. In the paper we review our results about application of the WKB method to the generating functions $f_d(x)$, focusing on the cases $d=2$ and $d=3$.

1. Introduction

We study the following hypergeometric equations

$$(1-t)\partial(t\partial)^{d-1}g + x^dg = 0, \tag{1.1}$$

where $\partial = \partial_t = \partial/\partial t$, with one solution in form of the hypergeometric series (see [BE1])[1]

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[1] Recall the standard formula ${}_pF_q(\alpha_1,\dots\alpha_p;\beta_1,\dots,\beta_q;t) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}t^n$ where $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$ is the known Pochhammer symbol. Eq. (1.1) can be found in [Zud1] and [Zo2]

$$\begin{aligned}(1.2)\qquad \varphi_1(t;x) &= {}_dF_{d-1}(-\varsigma^0 x,\dots,-\varsigma^{d-1}x;1,\dots,1;t)\\ &= 1-x^d t+(-x^d)\left(1-x^d\right)t^2/(2!)^d+\dots\end{aligned}$$

here

$$\varsigma = e^{2\pi i/d} \tag{1.3}$$

is the primitive root of unity of degree d (other solutions $\varphi_2,\dots,\varphi_d$ are given in Section 3.1). For $d=1$ we have the simple (and unique solution) $\varphi_1=(1-t)^x$, so this case is not interesting.

But when the degree of the equation is greater, $d\geq 2$, then something interesting happens. It turns out that the solution (1.2) evaluated at $t=1$ is a generating function for so-called **multiple zeta values** (MZV's, see [Zag1])[2]

$$\zeta(d_1,\dots,d_k)=\sum_{0<n_1<\dots<n_k}\frac{1}{n_1^{d_1}\dots n_k^{d_k}},\quad d_j\geq 1,\quad d_k\geq 2. \tag{1.4}$$

Namely,

$$\varphi_1(1;x)=f_d(x) \tag{1.5}$$

where f_d is the following **generating function**:

$$f_d(x)=1-\zeta(d)x^d+\zeta(d,d)x^{2d}-\dots \tag{1.6}$$

(see [Zo2] and Section 3 below).

It is easy to show the formula

$$f_d(x)=\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n}\right)^d\right) \tag{1.7}$$

which implies, in particular, that

$$f_2(x)=\frac{\sin\pi x}{\pi x}. \tag{1.8}$$

But for odd degrees we do not have similar formulas. Since the R. Apery's work [Ap] we know that the number $\zeta(3)$ is irrational, but it is not known whether it is algebraic or not. Due to formula (1.8) below we assume that:

$$d=2 \text{ or } d>2 \text{ is odd.} \tag{1.9}$$

The idea of this paper and of [Zo2, ZZ1, ZZ2, ZZ3] is to express the solution (1.2) in suitable basis $(\theta_1,\dots,\theta_d)$ of solutions near $t=1$;

$$\varphi_1=A_1(x)\theta_1+\dots+A_d(x)\theta_d.$$

The basis near $t=1$ is such that $\theta_j|_{t=1}=0$ for $j=1,\dots,d-1$ and $\theta_d|_{t=1}$ is a known nonzero number. Therefore it is enough to find the coefficient $A_d(x)$ before θ_d. The coefficients $A_j(x)$ are analytic functions in $x\in\mathbb{C}\setminus 0$, with only possible

[2]In some sources the sum in Eq. (1.4) is denoted $\zeta(d_k,\dots d_1)$.

singularities at $x=0$ and at $x=\infty$ (see Sections 3). So there appears an idea to consider behavior of the solutions when the parameter x becomes large.

For large $|x|$ there exist some special solutions of the form

$$g\sim x^{\gamma}e^{xS(t)}\left\{\chi_0(t)+\chi_1(t)x^{-1}+\ldots\right\},$$

known as the WKB solutions. Here the 'action' $S(t)$ and the amplitudes $\chi_j(t)$ satisfy some ODEs which are easy to integrate. There exist basic WKB solutions $g^{\sigma}(t;x)\sim\exp(\sigma x S_d(t))$ with $S_d(t)=\int_0^t\tau^{1/d-1}(1-\tau)^{-1/d}d\tau$ and $\sigma=\varsigma^{j+1/2}$ $(j=0,\ldots,d-1)$ to Eq. (1.1) (see Section 4). One would like to represent the solutions φ_1 and θ_j in the WKB basis. To this aim one could use some integral representations of the solutions φ_1 and θ_j and then to evaluate the corresponding integrals, which are of oscillatory type, using the stationary phase formula (see [Fed, He]).

This approach is tempting but it encounters serious obstacles. One of them is the question of uniqueness of the series defining the WKB solutions. The functions $\chi_j(t)$ satisfy an infinite series of ODEs and an infinite number of constants of integration of these equations has to be determined. In Definition 1 (in Section 4.1) we define so-called testing WKB solutions $g^{\sigma}_{\mathrm{test}}$ by choosing some arbitrary procedure of fixing the integration constants. But it is not the right choice. In Section 4.2 we define so-called normal WKB solutions $g^{\sigma}_{\mathrm{norm}}$ which are more natural, because they are obtained via some normalization procedure (i.e. a diagonalization) of a corresponding linear first order differential system and this procedure is unique.

But the main difficulty arises from the fact that the series defining the WKB solutions are divergent. It turns out that one can define analytic WKB solutions by applying an analytic version of the normalization procedure (see Section 4.3), but the domains of definition of the latter solutions are quite small: for $0<t<1$ the parameter x lies in a sector in $\mathbb{C}$ with vertex at $x=\infty$. Moreover, the analytic normalization requires solving some integral equation and the solutions obtained are not unique.

In Section 5 we develop a new approach in the asymptotic analysis of linear differential equations like Eq. (1.1). For t near 0 we approximate Eq. (1.1) with so-called Bessel type equation $\partial_y\left(y\partial_y\right)^{d-1}G+G=0$ for $G(y)$ where $y=x^3t$ (see Eq. (5.3)). Similarly, for $s=1-t$ close to 0 we have an approximation by another Bessel type equation (Eq. (5.5)) for $H(z)$, where $z=x^ds^{d-1}$. These Bessel type equations have only two singular points: regular at $y=0$ (respectively at $z=0$) and irregular at $y=\infty$ (respectively at $z=\infty$). In Theorem 1 we prove that the hypergeometric equation (1.1) for $g(t;x)$ near $t=0$ is analytically equivalent with the corresponding Bessel type equation for $G(y)$ and that the corresponding equation for $h(s;x)=g(1-s;x)$ near $s=0$ is analytically equivalent with the Bessel type equation for $H(z)$. The Bessel type equations admit uniquely defined WKB type solutions $G^{\sigma}(y)\sim e^{d\sigma y^{1/d}}$ for $y\to\infty$ and $H^{\sigma}\sim e^{(d/(1-d))\sigma z^{1/d}}$ for $z\to\infty$. In Section 5.3 we define so-called principal WKB solutions $g^{\sigma}_{\mathrm{princ}}$ and $h^{\sigma}_{\mathrm{princ}}$ as images of the WKB solutions G^{σ} and H^{σ} using the above analytic equivalences.

To represent the solution $\varphi_1(t;x)$ (defined by the hypergeometric series (1.2)) in the basis $\left(g^\sigma_{\text{princ}}\right)$ one expresses this hypergeometric function via a contour integral (in Section 6.1). This is an oscillatory type integral (or a mountain pass integral). It is evaluated asymptotically as $x\to\infty$ using well known stationary phase formula (or the mountain pass formula).

For the degree $d=2$ one can write down suitable integral representations for the basic solutions $\theta_1(s;x)$ and $\theta_2(s;x)$ near $s=1-t=0$. The corresponding stationary phase formula allows to represent θ_j in the basis $\left(h^\sigma_{\text{princ}}\right)$. Because the relation between the bases $\left(g^\sigma_{\text{princ}}\right)$ and $\left(h^\sigma_{\text{princ}}\right)$ is given by a diagonal matrix (at least formally) it is possible to give new proofs of the formula (1.8). We give two proofs, one in Section 6.3 and another one in Section 7.2.1.

However, here we must underline that the existence of the integral formulas for $\theta_{1,2}$ in the case $d=2$ follows from the formula $\theta_j(s)=-s\partial_s\varphi_j(s)$, which is a consequence of so-called self-duality for the MZV's $\zeta(2,\dots,2)$ (see Eqs. (2.8)–(2.9) and Lemma 3 below).

In the case of odd $d>2$ there are no integral formulas for the basic solutions θ_j, $j=1,\dots,d$. But we can find such formulas for corresponding solutions $\Theta_j(z)$ (to the Bessel type equation) which approximate the solutions θ_j. Evaluating these integrals, using the mountain pass formula for large $|z|$, one finds expansions of the functions Θ_j in the basis (H^σ). Next, one uses the equivalence of the hypergeometric and the Bessel equations near $s=0$ to expand θ_j in the principal WKB basis $\left(h^\sigma_{\text{princ}}\right)$. We do it for the case $d=3$.

The WKB solutions G^σ (respectively H^σ) are subject to so-called Stokes phenomenon. It relies upon the property that the formal solutions G^σ are asymptotic expansions of some genuine analytic solutions G^σ_j, defined in some sectors $\mathcal{S}_j$, but in intersection of two adjacent sectors the relation between the corresponding bases is given by so-called Stokes matrix (which is not identical). This explains the divergence of the series defining G^σ and is responsible for the unpleasant fact that the coefficients in the expansion of the function $\Phi_1(y)$ (approximating φ_1) given by the stationary phase formula are not exact. More precisely, only the dominating terms const$\cdot e^{d\sigma y^{1/d}}$, as $|y|\to\infty$ and $\arg y$ is fixed, are correct. Other terms are determined by an analysis leading to computation of the Stokes matrices. The same is true for the WKB solutions H^σ and representations of $\Theta_j(z)$ in terms of (H^σ) for $|z|\to\infty$ and fixed $\arg z$. This is done in Section 7.1.

In Section 7.2 we apply the above theory to get a representation

$$A_d(x)=\sum a_\sigma\cdot F^\sigma(x)$$

for the connection coefficient before θ_d in the representation of φ_1 in the basis (θ_j). Here $F^\sigma(x)$ are functions of WKB type. For $d=2$ we prove that the functions F^σ are single valued, i.e. the corresponding Stokes operators are trivial.

For $d = 3$ we have

$$F^{\sigma} = \pm x^{-3/2} e^{2\pi\sigma x/\sqrt{3}} \omega^{\sigma}(x^{-1/2})$$

which are subject to a nontrivial Stokes phenomenon. Moreover, their monodromy, as x makes a turn around ∞, is nontrivial (due to the factor $x^{-3/2}$). This implies that the function $A_3(x)$ is a solution of a meromorphic sixth order linear equation with irregular singularity at $x = \infty$ (Theorem 2).

Since the function $A_3(x)$ is entire (and holomorphic at $x = 0$) it is quite plausible that the equation satisfied by F^{σ}'s has regular singularity at $x = 0$. Then this equation should take the following form

$$f^{(VI)} + c_1 x^{-1} f^{(V)} + c_2 x^{-2} f^{(IV)} + \left(c_3 + c_4 x^{-3}\right) f^{(III)} + \left(c_5 x^{-1} + c_6 x^{-4}\right) f^{(II)} \\ + \left(c_7 x^{-2} + c_8 x^{-5}\right) f^{(I)} + \left(c_9 + c_{10} x^{-3} + c_{11} x^{6}\right) f = 0$$

where $c_3 = 2\left(2\pi/\sqrt{3}\right)^3$, $c_9 = \left(2\pi\sqrt{3}\right)^6$ and other coefficients c_j are computable (most probably are expressed in an algebraic way via π and $\sqrt{3}$). But then the coefficients $b_k = (-1)^k \zeta(3, \ldots, 3)$ in the expansion $f_3 = \sum b_k x^{3k}$ should satisfy a recurrent relation, hence all the zeta values $\zeta(3, \ldots., 3)$ are expressed via $\zeta(3)$ and $\zeta(3)$ would satisfy an algebraic equation with coefficients depending on the c_j's. We plan to calculate the coefficients c_j in a separate paper.

Sections 2 of the paper is devoted to presentation of some basic facts about MZV's and about their relations with hypergeometric series.

2. MZV's, polylogarithms and hypergeometric series

The Multiple Zeta Values (MZV's) $\zeta(d_1, \ldots, d_k)$ are defined in Eq. (1.4). Any such quantity has its **weight** $d = d_1 + \ldots + d_k$, **depth** equal k and **height** $h = \sharp\{i : d_i > 1\}$.

They form a graded algebra, where the grading is defined by the weight. Indeed, we can rewrite the product of two infinite sums

$$\left(\sum_{n_1 < \ldots < n_k}\right)\left(\sum_{m_1 < \ldots < m_l}\right)$$

in the product $\zeta(d_1, \ldots, d_k)\zeta(e_1, \ldots, e_l)$ as a finite sum corresponding to different orderings of the index set $\{n_1, \ldots, n_k, m_1, \ldots, m_l\}$. The corresponding identity is sometimes called the **first shuffle product**. For example, we have

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4) \tag{2.1}$$

which implies $\zeta(4) = \pi^4/90$. It was Euler who used this sort of shuffle relations to prove that $\zeta(2k) = \pi^{2k} \times$(rational number).

Important is the problem of calculation of the dimension D_d of the space $\mathfrak{Z}_d$ (over the field $\mathbb{Q}$) generated by the MZV's of weight d. There exists a conjecture (see [Zag1]) that these dimensions satisfy the recursion $D_d = D_{d-2} + D_{d-3}$ (with $D_0 = 1$ and $D_d = 0$ for $d < 0$). This is equivalent to the property

$$\sum D_d t^d = \frac{1}{1 - t^2 - t^3}.$$

M. Hoffman [Hof] conjectured that the algebra of MZV's is generated by special values of the form $\zeta(d_1, \dots, d_k)$ with $d_j \in \{2, 3\}$. This conjecture was recently proved by F. Brown [Bro]; in the proof some explicit relations between the values $\zeta(2, \dots, 2)$, $\zeta(2r+1)$ and $\zeta(2, \dots, 2, 3, 2, \dots, 2)$ (proved by D. Zagier [Zag2]) are used.

There exists the following **Kontsevich–Drinfeld formula** ([KoZa]) for the MZV's. Let

$$\omega_0(t) = dt/t, \quad \omega_1(t) = dt/(1-t) \tag{2.2}$$

be two 1−forms. For given $d_1, \dots, d_k$ we define the d−form

$$\begin{aligned} \Omega_{d_1,\dots,d_k} \;=\;& \omega_0(t_{d_1+\dots+d_k}) \dots \omega_0(t_{d_1+\dots+d_{k-1}+2}) \omega_1(t_{d_1+\dots+d_{k-1}+1}) \\ & \dots \omega_0(t_{d_1}) \dots \omega_0(t_2) \omega_1(t_1); \end{aligned} \tag{2.3}$$

there are k forms ω_1 with arguments $t_1, t_{d_1+1}, \dots, t_{d_1+\dots+d_{k-1}+1}$. Next, we integrate it over the simplex $\{0 \le t_1 \le \dots \le t_d \le 1\}$:

$$\zeta(d_1, \dots, d_k) = \int_{0 \le t_1 \le \dots \le t_d \le 1} \Omega_{d_1,\dots,d_k}. \tag{2.4}$$

For example, we have[3]

$$\int_{0 \le t_1 \le t_2 \le 1} \frac{dt_2}{t_2} \frac{dt_1}{1 - t_1} = \sum_{n \ge 1} \frac{1}{n} \int_0^1 t_2^{n-1} dt_2 = \sum \frac{1}{n^2} = \zeta(2). \tag{2.5}$$

The latter formula is generalized to the **generalized polylogarithms**

$$\begin{aligned} \mathrm{Li}_{d_1,\dots,d_k}(t) &= \textstyle\sum_{0 < n_1 < n_2 < \dots < n_k} t^{n_k} / n_1^{d_1} \dots n_k^{d_k} \\ &= \textstyle\int_{0 \le t_1 \le \dots \le t_d \le t} \Omega_{d_1,\dots,d_k}. \end{aligned} \tag{2.6}$$

It implies another shuffle multiplication. The product

$$\left(\int_{t_1 \le \dots \le t_d \le t} \right) \left(\int_{s_1 \le \dots \le s_e \le t} \right)$$

[3]Such integrals appear as coefficients in some knot invariants and in evaluation of some Feynmann integrals in quantum physics.

of integrals is represented as a finite sum of integrals according to the ordering of the variables set $\{t_1, \ldots, t_d, s_1, \ldots, s_d\}$. For example, we have

$$
\begin{aligned}
\text{(2.7)} \quad \mathrm{Li}_2(t)\mathrm{Li}_1(t) &= \left(\int_{0\le t_1\le t_2\le t} \frac{dt_2 dt_1}{t_2(1-t_1)}\right)\left(\int_0^t \frac{dt_3}{1-t_3}\right) \\
&= \left(2\int_{0\le t_1\le t_3\le t_2\le t} + \int_{0\le t_1\le t_2\le t_3\le t}\right) \frac{dt_2 dt_3 dt_1}{t_2(1-t_3)(1-t_1)} \\
&= 2\mathrm{Li}_{1,2}(t) + \mathrm{Li}_{2,1}(t).
\end{aligned}
$$

The **second shuffle formula** leads to an interesting shuffle algebra (see [MPH, Zud1]), but there is no place to describe its details.

The Drinfeld–Kontsevich formula (2.4) leads to the following **MZV duality**. Namely, we put $s_1 = 1 - t_d, \ldots, s_d = 1 - t_1$; thus $\omega_{\varepsilon_j}(t_j) = \omega_{1-\varepsilon_j}(1 - s_{d-j+1})$ and we get

$$
\text{(2.8)} \quad \zeta(1,\ldots 1, m_1+2, \ldots, 1, \ldots, 1, m_r+2) = \zeta(1, \ldots 1, n_r+2, \ldots, 1, \ldots, 1, n_1+2)
$$

where the sequences of 1's have lengths n_j in the left-hand side and m_{r-j+1} in the right hand side. We observe that the quantities

$$
\text{(2.9)} \qquad \zeta(2, \ldots, 2) \quad \text{and} \quad \zeta(1, 3, \ldots, 1, 3)
$$

are invariant with respect to the MZV duality. We have also the formula

$$
\text{(2.10)} \qquad \zeta(3) = \zeta(1, 2)
$$

which is proved in many ways in the literature.

There exist interesting generating functions which imply series of relations between MZV's. One of them is following (see [BBB]):

$$
\text{(2.11)} \quad \sum_{m,n\ge 0} x^{m+1}y^{n+1}\zeta(m+2, 1, \ldots, 1) = 1 - \exp\left\{\sum_{k\ge 2} \frac{x^k + y^k - (x+y)^k}{k}\zeta(k)\right\}
$$

where the sequence of 1's has length n.

Some of the generating series are expressed via hypergeometric functions. In the next example we put

$$
G(d, k, h) = \sum \zeta(d_1, \ldots, d_k),
$$

where in the sum the weight $d = d_1 + \ldots + d_k$, the depth k and the height $h = \sharp\{i : d_i > 1\}$ are fixed and $d_k \ge 2$. Let also α and β satisfy

$$
\alpha + \beta = x + y, \quad \alpha\beta = z.
$$

Then we have the following identity for

$$
\Phi(x, y, z) = \sum G(d, k, h) x^{d-k-h} y^{k-h} z^{h-1}
$$

(see [OhZa]):

$$\Phi = \frac{1}{xy-z}\left\{1 - {}_2F_1(\alpha - x, \beta - x; 1-x; 1)\right\} \tag{2.12}$$

$$= \frac{1}{xy-z}\left\{1 - \exp\left(\sum_{n\geq 2}\frac{x^n+y^n-\alpha^n-\beta^n}{n}\zeta(n)\right)\right\}. \tag{2.13}$$

This result was generalized in [AOW] and [Li]. Specializing Eq. (2.13) to $xy=z$ one obtains the formula

$$\sum_{d,k,h} G(d,k,h)x^{d-k-1}y^{k-1} = \sum \zeta(d)x^{d-k-1}y^{k-1}. \tag{2.14}$$

In particular,

$$\sum_{d_1+\ldots+d_k=d} \zeta(d_1,\ldots,d_k) = \zeta(d) \tag{2.15}$$

where the depth k is fixed. For $k=2$ the latter identity is known as the **Euler formula**.

We note also the following Borwein formula for the generating function $f_{1,3}(x) = 1 - \zeta(1,3)x^4 + \zeta(1,3,1,3)x^8 - \ldots$:

$$f_{1,3}(x) = f_4\left(x/\sqrt{2}\right) \tag{2.16}$$

which follows from a corresponding identity for generating functions for polylogarithms (see [KoZa], [BBBL]). This formula was conjectured by D. Zagier in [Zag1].

It was conjectured in [BBB] and proved in [Zhao] that

$$\zeta(3,\ldots,3) = 8^k \cdot \zeta(1,\bar{2},\ldots,1,\bar{2}) \tag{2.17}$$

where

$$\zeta(1,\bar{2},\ldots,1,\bar{2}) = \sum_{0<m_1<n_1<\ldots<m_k<n_k} \frac{(-1)^{n_1+\ldots+n_k}}{m_1n_1^2\ldots m_kn_k^2} \tag{2.18}$$

is so-called **alternating Euler sum**. The generating function for the latter values

$$f_{1,\bar{2},\ldots,1,\bar{2}}(x) = \sum \zeta(1,\bar{2},\ldots,1,\bar{2})\cdot(-x^3)^k \tag{2.19}$$

is related with the following sixth order equation:

$$(1-t)\partial(1-t)\partial t\partial(1+t)\partial(1+t)\partial_t t\partial_t g - x^6 g = 0.$$

Namely, this equation has two solutions analytic near $t=0$ and of the form $\varphi_1 = 1 + O(x^6)$ and $\varphi_2 = \sum_{0<m<n}\frac{(-t)^n}{mn^2} + O(x^6)$. Then $f_{1,\bar{2},\ldots,1,\bar{2}}(x) = \varphi_1(1;x) - x^3\varphi_2(1;x)$. The Zhao's result implies that $f_{1,\bar{2},\ldots,1,\bar{2}}(x) = f_3(x/2) = \prod\left(1 - \left(\frac{x}{2n}\right)^3\right)$.

Some hypergeometric series are also used in irrationality proofs of some zeta values. Here we refer the reader to the exemplary papers [CFR, Zud2, Hut].

We finish this section by noticing that some third order linear differential equations, similar to Eq. (1.1) for $d = 3$ were considered by F. Beukers with C. Peters in [BePe] and by S.-T. Yau with B. Lian in [LYau]. In [BePe] the equation

$$(t^4 - 34t^3 + t^2)\partial^3 z + (6t^3 - 153t^2 + 3t)\partial^2 z + (7t^2 - 112t + 1)\partial z + (t-5)z = 0,$$

which is directly related with the recurrence used by R. Apéry in his proof of irrationality of $\zeta(3)$ (see [Ap], [vPo]), turns out to be a Picard–Fuchs equation for periods of some K3 surface. In [LYau] the authors consider equations of the form

$$\left((t\partial)^3 - t\left(\sum_{i=1}^{3} r_i\,(t\partial)^i\right)\right) z = 0;$$

they are Picard–Fuchs equations for a one-parameter deformations of K3 surfaces and are used in the mirror symmetry property for K3 surfaces. However the choice of parameters r_j used in [LYau] is different than in Eq. $(1.1)_{d=3}$.

3. Two bases of solutions

3.1. **Basic solutions near $t = 0$.** Recall that we consider Eq. (1.1). The hypergeometric function (1.2) is one of the basic solutions. We may represent it as a series in powers of x^d with coefficients depending on t. Also other solutions can be written in the form $g = \phi(t;x) = \phi_0(t) - \phi_1(t)x^d + \phi_2(t)x^{2d} - \dots$, where the coefficient functions satisfy the series of equations: $(t\partial)^d\,\phi_0 = 0$ and $(t\partial)^d\,\phi_k = \frac{t}{1-t}\phi_{k-1}$ for $k \geq 1$. The first equation has d independent solutions which we can choose in the following form:

$$\varphi_{1,0}(t) = 1, \quad \varphi_{2,0} = \ln\left(x^d t\right), \dots, \varphi_{d,0} = \frac{1}{(d-1)!}\ln^{d-1}\left(x^d t\right) \tag{3.1}$$

(this special choice is justified in Section 5). The other equations are solved as follows:

$$\phi_k(t) = \int_{0<t_d\dots<t_1<t} \frac{dt_1}{t_1}\dots\frac{dt_{d-1}}{t_{d-1}}\frac{dt_d}{1-t_d}\phi_{k-1}(t_d). \tag{3.2}$$

It is easy to see that the coefficients ϕ_k decrease very fast with k (like $1/k!$), so the obtained solutions are analytic functions in $x^d \in \mathbb{C}\setminus 0$ with known singularities at $x = 0$.

The above implies that the basic solutions to Eq. (1.1) are of the form

$$\varphi_j(t;x) = \varphi_{j,0}(t) - \varphi_{j,1}(t)x^d + \varphi_{j,2}(t)x^{2d} - \dots, \qquad j = 1,\dots,d, \tag{3.3}$$

with $\varphi_{j,k}$ given by the integral recurrence (3.2). They can be rewritten as follows:

$$\begin{aligned}
&\varphi_1 = 1 + O(t),\\
&\varphi_2 = \varphi_1 \ln\left(x^d t\right) + \psi_2,\\
&\varphi_3 = \tfrac{1}{2!}\varphi_1 \ln^2(x^d t) + \psi_2\ln(x^d t) + \psi_3,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\varphi_{d-1} = \tfrac{1}{(d-1)!}\varphi_1 \ln^{d-1}\left(x^d t\right) + \dots + \psi_{d-1}\ln(x^d t) + \psi_d
\end{aligned} \tag{3.4}$$

where $\varphi_1, \psi_2, \ldots, \psi_d$ are analytic in t near $t = 0$. (The above form of the basic solutions can be explained by the defining equation $\lambda^d = 0$ for the leading exponents in the solutions $\phi = t^\lambda + \ldots$.)

Of course, for us the principal is the first of these solutions. Using the Drinfeld–Kontsevich formula (2.6) we find

$$\begin{aligned}\varphi_{1,2}(t) &= \int_{0<t_d\ldots<t_1<t} \frac{dt_1}{t_1}\cdots\frac{dt_{d-1}}{t_{d-1}}\frac{dt_d}{1-t_d} \\ &= \sum_{n=1}^{\infty}\int_{0<t_d\ldots<t_1<t}\frac{dt_1}{t_1}\cdots\frac{dt_{d-1}}{t_{d-1}}t_d^{n-1}dt_d = \sum\frac{t^n}{n^d} = \mathrm{Li}_d(t),\end{aligned}$$

i.e. a polylogarithm. Other coefficient functions $\varphi_{1,k}$ are also expressed via polylogarithms and we have

$$\varphi_1 = 1 - \mathrm{Li}_d(t)x^d + \mathrm{Li}_{d,d}(t)x^{2d} - \ldots,$$

which implies formula (1.5).[4]

Remark 1. *Other solutions $\varphi_2, \ldots, \varphi_d$ also admit expressions in terms of hypergeometric series. For example, in the case $d = 2$ we can take the following perturbation of Eq. (1.1): $t\left\{(1-t)\partial_t t\partial_t g + x^2 g\right\} - \mu^2 g = 0$ with small parameter μ (see* [ZZ1]*). It has the solutions η_μ and $\eta_{-\mu}$, where $\eta_\mu = \frac{\Gamma(1+x+\mu)}{\Gamma(1+x-\mu)\Gamma(1+2\mu)}\cdot t^\mu\cdot F(\mu+x,\mu-x;1+2\mu;t)$, and therefore*

$$\widehat{\varphi}_2 = \lim_{\mu\to 0}\left(\eta_\mu - \eta_{-\mu}\right)/2\mu$$

is a solution to Eq. $(1.1)_{d=2}$ with the logarithmic term (arising from $t^\mu \approx 1+\mu\ln t$).

Since $\frac{\Gamma(1+x+\mu)}{\Gamma(1+x-\mu)\Gamma(1+2\mu)} \approx 1 + 2\mu(\Psi(1+x) - \Psi(1))$, where Ψ denotes the Euler Psi function and $\Psi(1) = -\gamma$ is the Euler–Mascheroni constant, it follows that $\widehat{\varphi}_2 = \varphi_2 + 2(\Psi(1+x)+\gamma-\ln x)\cdot\varphi_1$ and the analytic part of the solution φ_2 equals $\psi_2 = \frac{\partial}{\partial\mu}F(\mu+x,\mu-x;1+2\mu;t)|_{\mu=0}$.

Moreover, from the expansions $\Psi(1+x) = -\gamma + \zeta(2)x - \zeta(3)x^2 + \zeta(4)x^3 - \ldots$ (see [BE1, Eq. 1.17(5)]*) and $\frac{\pi}{\tan\pi x} = \frac{1}{x} - 2\zeta(2)x - 2\zeta(4)x^3 - \ldots$ (compare* [BE1, Eq. 1.20(3)] *we get $\widehat{\varphi}_2(1;x) = -\frac{\cos\pi x}{x} + \frac{1}{x}f_2(x)$. It implies that the function*

$$\check{\varphi}_2 = \widehat{\varphi}_2 - x^{-1}\cdot\varphi_1$$

is a solution to Eq. (1.1), independent with φ_1 and such that

$$\check{\varphi}_2(1;x) = -\frac{\cos\pi x}{x}.$$

[4] Also other series ψ_j appearing in the formulas for φ_j are generating functions for some polylogarithms. For instance, in [ZZ1] it is proved that in the case $d = 2$ we have $\varphi_{2,k} = \mathrm{Li}_{2,\ldots,2}(t)\ln(x^2 t) - 2\sum_{j-1}^{k}\mathrm{Li}_{2,\ldots,3,\ldots,2}(t)$, where only one index in Li equals 3. After a simple resummation one finds $\varphi_2(1;x) = 2f_2(x)\ln x + 2x^2 f_2(x)\left\{\zeta(3)+\zeta(5)x^2+\zeta(7)x^4+\ldots\right\}$. However we should not regard the latter identity as something important.

Also the below solutions θ_j are expressed via the polylogarithms and $\ln s$.

In the case of higher order equations ($d > 2$) the perturbation relies on adding a differential operator of lower order with $d-1$ small parameters.

3.2. **Basic solutions near $t = 1$.** With the variable $s = 1 - t$ Eq. (1.1) takes the form

$$s\partial_s(1-s)\partial_s \dots (1-s)\partial_s g + (-1)^d x^d g = 0. \tag{3.5}$$

Analogously as in Section 3.1 we consider solutions of the form $g(1-s) = \theta_j(s;x)$ such that

$$\begin{aligned} \theta_j &= (-x^{d/(d-1)})^j \left\{\theta_{j,0}(s) + \theta_{j,1}(s)x^d + \dots\right\}, \quad (j = 1, \dots, d-1), \\ \theta_d &= \theta_{d,0}(s) + \theta_{d,1}(s)x^d + \dots \end{aligned} \tag{3.6}$$

where

$$\theta_{j,0} = \frac{1}{j!}\ln^j(1-s) = \mathrm{Li}_{1,\dots,1}(s), \quad (j = 1, \dots, d-1), \quad \theta_{d,0} = 1-d+\theta_{d-1,0}\ln x^d \tag{3.7}$$

and

$$\theta_{j,k}(s) = \int_{0<s_d\dots<s_1<s} \frac{ds_1}{1-s_1} \cdots \frac{ds_2}{1-s_{d-1}} \frac{ds_d}{s_d} \theta_{j,k-1}. \tag{3.8}$$

It is clear that these solutions are analytic in $x \in \mathbb{C}\setminus 0$ with known singularities at the origin.

Their behavior near $s = 0$ is following:

$$\begin{aligned} \theta_j(s;x) &= \tfrac{1}{j!}\left(x^{d/(d-1)}s\right)^j + O(s^d) \quad (j = 1, \dots, d-1), \\ \theta_d(s;x) &= \theta_{d-1}\ln\left(x^d s^{d-1}\right) + (1-d) + O(s). \end{aligned} \tag{3.9}$$

(compare [ZZ1, ZZ3]).

3.3. **Some relations between the two bases.** Firstly, we underline the following property which follows directly from independence of the two systems $\varphi = (\varphi_1, \dots, \varphi_d)^\top$ and $\theta = (\theta_1, \dots, \theta_d)^\top$ of solutions (see [ZZ3]).

Lemma 1. *The matrix $M = M(x)$ defined by $\theta = M\varphi$ is an analytic function of $x \in \mathbb{C}\setminus 0$ with regular singularity at $x = 0$.*

Also the following obvious statement is important in this paper.

Lemma 2. *Let*

$$\varphi_1(t;x) = A_1(x)\cdot\theta_1(1-t;x) + \dots + A_d(x)\cdot\theta_d(1-t;x)$$

be the representation of $\varphi_1(t;x)$ near $t = 1$ in the basis θ (with the connection coefficients A_j). Then the generating function (1.6) is expressed via the last connection coefficient,

$$f_d(x) = (1-d)\cdot A_d(x).$$

In the case of standard hypergeometric equation of second order we have the following property which is proved by direct checking.

Lemma 3. *Let $d = 2$. Then, if $\varphi(t; x)$ is a solution to Eq. (1.1), then $\theta(s; x) = -s\partial_s\varphi(s; x)$ is a solution to Eq. (3.5). In particular, we have*

$$\theta_{1,2}(s; x) = -s\partial_s\varphi_{1,2}(s; x).$$

This lemma will be used below in explanation of the formula (1.8) for $f_2(x)$. On the other side, it has simple explanation in terms of the MZV duality relations.

Together with Eq. (1.1) one can consider the following equation:

$$[(1-t)\partial_t]^{d-1}\, t\partial_t g + x^d g = 0. \tag{3.10}$$

It has one solution of the form

$$\phi_1(t; x) = 1 - \mathrm{Li}_{1,\dots,1,2}(t)x^d + \mathrm{Li}_{1,\dots,1,2,1,\dots,,2}(t)x^{2d} - \dots$$

(where each sequence of 1's is of length $d-1$) and hence $\phi_1(1; x) = f_{1,\dots,1,2}(x) = 1-\zeta(1,\dots,1,2)x^d+\dots$ is a generating function for multiple zeta values $\zeta(1,\dots,1,2 \dots 1,\dots,1,2)$. But the MZV duality (see Eq. (2.8)) implies that the latter numbers equal $\zeta(d,\dots,d)$. Therefore

$$\phi(1; x) = f_d(x)$$

is the generating function for $\zeta(d,\dots,d)$ from Eq. (1.6). Of course, for $d = 2$ it is nothing new, because the values $\zeta(2,\dots,2)$ are fixed under the duality transformation.

There exists another relation between Eqs. (1.1) and (3.10). Namely,

if $\varphi(t; x)$ is a solution to Eq. (1.1) near $t = 0$ then for $s = 1 - t \approx 0$ the function $\vartheta(s; x) = (s\partial_s)^{d-1}\varphi(s; -x)$ is a solution to Eq. (3.10) near $t = 1$ but for the parameter x replaced with $-x$, i.e. to the equation

$$(s\partial_s)^{d-1}(1-s)\partial_s g + (-x)^d g = 0.$$

4. WKB solutions

Theoretically Eq. (1.1) for large parameter x can be solved using the WKB method. This means that one represents a solution as a finite sum of terms of the form

$$x^\gamma e^{xS(t)}\left\{\chi_0(t) + \chi_1(t)x^{-1} + \dots\right\}. \tag{4.1}$$

In general the series in the above formula are divergent, but this divergence can be somehow controlled. Below we present three approaches to the WKB solutions to Eq. (1.1): formal, via normal forms and using the stationary phase formula (in Section 6).

The name of the method comes from the names of its authors G. Wentzel [Wen], H. Kramers [Kr] and L. Brillouin [Bri]. Originally it was used to solve approximately the Schrödinger equation [Sch], but here we use it to the hypergeometric equation.

4.1. **Testing WKB solutions.** These are solution of the form

$$g(t;x)=x^{\gamma}e^{xS(t)}\chi(t;x^{-1}), \tag{4.2}$$

where χ is a power series in x^{-1}. Substituting it into equation (1.1) we get

$$x^d\left\{(1-t)t^{d-1}\left(\dot{S}\right)^d+1\right\}\chi+x^{d-1}\frac{1-t}{t}\mathcal{P}_1\chi+\ldots+\frac{1-t}{t}\mathcal{P}_d\chi=0, \tag{4.3}$$

where $\dot{S}=dS/dt$ and $\mathcal{P}_j$ are some differential operators and the first of them is following:

$$\mathcal{P}_1\chi=d\cdot\left(t\dot{S}\right)^{d-2}\cdot\left\{t\partial S\cdot t\partial\chi+\frac{d-1}{2}(t\partial)^2S\cdot\chi\right\}. \tag{4.4}$$

It follows that the '*action*' $S(t)$, the solution to the '*Hamilton–Jacobi equation*'

$$(1-t)t^{d-1}\left(\dot{S}\right)^d+1=0, \tag{4.5}$$

equals

$$S=\sigma S_d(t):=\sigma\int_0^t\frac{d\tau}{\tau^{(d-1)/d}(1-\tau)^{1/d}},\quad \sigma=\varsigma^{j+1/2},\quad j=0,\ldots,d-1, \tag{4.6}$$

where ς is the root of unity from Eq. (1.3). These d possibilities correspond to d solutions, which can be expanded as follows

$$g^{\sigma}_{\text{test}}(t;x)=(\sigma x)^{\gamma}\,e^{\sigma xS_d(t)}\left\{\chi_0(t)-\frac{\chi_1(t)}{\sigma x}+\frac{\chi_2(t)}{(\sigma x)^2}\cdots\right\},\quad \gamma=-\frac{d-1}{2}. \tag{4.7}$$

The functions χ_j satisfy the '*transport equations*'

$$\mathcal{P}_1\chi_0=0,\quad \mathcal{P}_1\chi_1=\mathcal{P}_2\chi_0,\ldots$$

where in definition of $\mathcal{P}_j$ we use $S=S_d$. The first transport equation is easy: we have $\chi_0=$const$\cdot\left(t\dot{S}_d\right)^{(1-d)/2}$. We choose it in the form

$$\chi_0(t)=\left(\frac{1-t}{t}\right)^{(d-1)/2d}. \tag{4.8}$$

To solve the other equations one introduces the new variable

$$u=\left(\frac{t}{1-t}\right)^{1/4}\text{ for }d=2\text{ and }u=\left(\frac{t}{1-t}\right)^{1/d}\text{ for odd }d\geq 3; \tag{4.9}$$

thus $\chi_0(t)=u^{-1}$ $(d=2)$ or $\chi_0(t)=u^{(1-d)/2}$ (odd $d\geq 3$). The following result was proved in[ZZ1] for $d=2$ and in [ZZ3] for $d=3$ but it holds in general case.

Lemma 4. *The functions $\chi_j(t)$, $j > 1$, can be chosen as Laurent polynomials in u, such that the term with u^{-1} (respectively $u^{(1-d)/2}$) is absent.*[5]

For example, when $d = 2$ we have

$$\chi_{k+1}(t) = (T\chi_k)(u) = \frac{1}{8u}\int^u \frac{1}{v}\partial_u\left(v(1+v^4)\,\partial_u\chi_k\right)dv.$$

This gives

$$\chi_1 = -\left(u^{-3} + 3u\right)/16, \quad \chi_2 = 3(3u^{-5} - 5u^3)/8^3. \tag{4.10}$$

A general algebraic formula can be obtained using the functions $\omega_k(u) = (2k-1)\ u^{-2k-1} + (-1)^{k+1}(2k+1)\cdot u^{2k-1}$, $k = 1, 2, \ldots$, which satisfy the recurrent relations: $T\omega_1 = -\frac{3\cdot 1}{8\cdot 4}\omega_2$, $T\omega_k = -\frac{4k^2-1}{8}\left\{\frac{\omega_{k+1}}{k+1} - \frac{\omega_{k-1}}{k-1}\right\}$. It follows that $\chi_k(t) = a_{k,k}\omega_k(u) + a_{k,k-2}\omega_{k-2}(u) + \ldots$, for some coefficients $a_{k,l}$ which are calculated inductively. The latter coefficients grow very fast with k; for instance, we have $a_{k,k} = (2k-1)\left(-1/8\right)^{k-1}((2k-3)!!)^2/(2k-2)!!$.

Definition 1. *The formal expressions*

$$g^\sigma_{\text{test}}(t;x) \sim \frac{e^{\sigma x S_d(t)}}{(\sigma x)^{(d-1)/2}}\cdot\left(\frac{1-t}{t}\right)^{(d-1)/2d},$$

*$\sigma = \varsigma^{j+1/2}$, $j = 0, \ldots, d-1$, defined in equation (4.7) with the coefficients $\chi_j(t)$ defined as above (without u^{-1} or $u^{(1-d)/2}$ for $j > 1$) are called the **testing WKB solutions associated with** $t = 0$.*

*We introduce also another system of **testing WKB solutions associated with** $s = 1 - t = 0$:*

$$\begin{aligned} h^\sigma_{\text{test}}(s;x) &= \xi_\sigma(\sigma x)^{d/2}e^{-\sigma x S_d(1)}\cdot g^\sigma_{\text{test}}(1-s;x) \\ &\sim \sqrt{-\sigma x}\cdot e^{-\sigma x(S_d(1)-S_d(1-s))}\cdot\left(\frac{s}{1-s}\right)^{(d-1)/2d}, \end{aligned} \tag{4.11}$$

where $\xi_\sigma \in \mathbb{S}^1$.

Above we agree that for $0 < t < 1$ and $\arg x = 0$ we take:[6]

$$g^\pm \sim \frac{\exp}{\sqrt{\pm ix}} = e^{\mp i\pi/4}\frac{\exp}{\sqrt{x}}, \quad h^\pm \sim \sqrt{\frac{x}{\pm i}}\exp = e^{\mp i\pi/4}\sqrt{x}\exp$$

[5] The general solution to the system of transport equations contains infinitely many constants, to each particular solution $\chi_j(t)$ we can add $c_j\chi_0(t)$ for a constant c_j. It the case of Schrödinger equation one avoids analogous problem of arbitrary constants of integration by assuming that the wave functions (representing bound states of a quantum system) vanish at infinity; that restriction leads to so-called Born–Sommerfeld quantization condition (see [Sch]).

[6] In [ZZ1] the notations g_0^+ and g_0^- for g^i_{test} and g^{-i}_{test}, $i = e^{i\pi/2}$, are used. In [ZZ3] one uses the notations $g_0^-, g_0^\epsilon, g_0^{\bar\epsilon}$ for g^σ_{test}, $\sigma = -1, \epsilon = e^{i\pi/3}, \bar\epsilon$. Also for h^σ_{test} analogous notations are used.

for $d=2$ and

$$g^\sigma \sim \frac{\exp}{\sigma x}, \quad h^- \sim \sqrt{x}\exp, \quad h^\epsilon = \bar\epsilon\sqrt{x}\exp, \quad h^{\bar\epsilon} \sim \epsilon\sqrt{x}\exp,$$

$(\sigma = -1, \epsilon, \bar\epsilon)$ for $d=3$.

4.2. **Formal reduction to normal form.** Here we present an alternative way to derive WKB type solutions to equations with a parameter like Eq. (1.1). The obtained basic WKB solutions g^σ_{norm} differ from the testing WKB solutions g^σ_{test} from Definition 1 by factors which depends on x. There are reasons to regard the new solutions are more natural than the testing solution.

In the presentation we describe only the simplest case $d=2$. Here we will use the notations $g^\pm$ (see Note 6).

Putting

$$g_1 = g, \quad g_2 = \dot g/x \tag{4.12}$$

we rewrite Eq. (1.1) in form of the following first order system

$$\frac{d}{dt}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A(t;x)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

where

$$A = xA_1(t) + A_0(t), \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1/t(t-1) & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1/t \end{pmatrix}.$$

The normal form of such system is a diagonal (or independent) system obtained by means of a formal linear change which depends on t.

The first step is the diagonalization of the matrix $A_1(t)$ with the eigenvalues

$$\lambda_1^\pm(t) = \pm i/\sqrt{t(1-t)} = \pm i\cdot \dot S_2(t). \tag{4.13}$$

We put

$$X^+ = \lambda_1^+(t)g_1 + g_2, \quad X^- = \lambda_1^-(t)g_1 + g_2 \tag{4.14}$$

and we get

$$\begin{aligned} \dot X^+ &= \lambda_1^+(t)xX^+ - \tfrac14\left(\tfrac3t - \tfrac1{1-t}\right)X^+ - \tfrac14\left(\tfrac1t + \tfrac1{1-t}\right)X^-, \\ \dot X^- &= \lambda_1^-(t)xX^- - \tfrac14\left(\tfrac1t + \tfrac1{1-t}\right)X^+ - \tfrac14\left(\tfrac3t - \tfrac1{1-t}\right)X^-. \end{aligned} \tag{4.15}$$

The general theory says that such system can be diagonalized by means of an infinite series of 'shearing' transformations. Let us apply some initial changes, in order to compare the obtained (partial) normal form with the results of the previous and next subsections. We put

$$X^+ = X_1^+ + \left(\frac{b_1}{x} + \frac{b_2}{x^2} + \dots\right)X_1^-, \quad X^- = \left(\frac{c_1}{x} + \frac{c_2}{x^2} + \dots\right)X_1^+ + X_1^-, \tag{4.16}$$

where b_j, c_j depend on t, and we expect to obtain the following separated system

$$\dot X_1^+ = \lambda^+(t;x)X_1^+, \quad \dot X_1^- = \lambda_1^-(t;x)X_1^-, \tag{4.17}$$

$$\lambda^{\pm}(t;x)=\lambda_1^{\pm}(t)x+\lambda_0^{\pm}(t)+\lambda_{-1}^{\pm}(t)x^{-1}+\dots$$

The resulted system of equations onto b_j, c_j, $\lambda_j^{\pm}$ is easily solved; moreover, in algebraic way. Using the variable $u=(t/(1-t))^{1/4}$ (see Eq. (4.9)) we get $b_1=-c_1=-i/8\,(t(1-t))^{1/2}=-i(1+u^4)/8u^2$, $b_2=c_2=(1-2t)\,/32t(1-t)=\left(1-u^8\right)/32u^4$ and $\lambda_0^{\pm}=\mp\frac{1}{4}\left(\frac{3}{t}-\frac{1}{1-t}\right)$, $\lambda_{-1}^{\pm}=\mp i/32\,(t(1-t))^{3/2}=\mp i(1+u^4)^3/32u^6$, $\lambda_{-2}^{\pm}=(2t-1)/128t^2(1-t)^2=(u^4-1)(1+u^4)^4/128u^8$.

General solutions to the system (4.17) are of the form

$$\begin{aligned} X_1^{+}&=K_{+}\tfrac{e^{ixS(t)}}{t^{3/4}(1-t)^{1/4}}\exp\left\{\tfrac{-i}{16x}\left(u^2-\tfrac{1}{u^2}\right)-\tfrac{1}{512x^2}\left(u^4+2+\tfrac{1}{u^4}\right)+\dots\right\},\\ X_1^{-}&=K_{-}\tfrac{e^{-ixS(t)}}{t^{3/4}(1-t)^{1/4}}\exp\left\{\tfrac{i}{16x}\left(u^2-\tfrac{1}{u^2}\right)-\tfrac{1}{512x^2}\left(u^4+2+\tfrac{1}{u^4}\right)+\dots\right\}, \end{aligned}\tag{4.18}$$

with arbitrary constants $K_{\pm}$ (which may depend on x). Substituting this to Eq. (4.16) and then to $g=\frac{1}{2\lambda}\left(X^{+}-X^{-}\right)$ (see Eq. (4.14)) one finds a general solution to Eq. (1.1) in the form

$$g=K_{+}g_{\text{norm}}^{+}(t;x)+K_{-}g_{\text{norm}}^{-}(t;x),$$

where

$$g_{\text{norm}}^{\pm}(t;x)=\left(1+(5/256)x^{-2}+\dots\right)\cdot g_{\text{test}}^{\pm}(t;x)\tag{4.19}$$

and $g_{\text{test}}^{\pm}$ are the testing WKB solutions (see Definition 1 and Eq. (4.7)).

For general degree $d\geq 2$ we have $g_1=g, g_2=\partial g/x,\dots,g_d=\partial^{d-1}g/x^{d-1}$ in an analogue of Eqs. (4.12), $\lambda_1^{\sigma}=\sigma\dot{S}_d(t)$, $\sigma=\varsigma^{j+1/2}$, $j=0,\dots,d-1$, in Eq. (4.13) and we finally obtain the diagonal system

$$\dot{X}_1^{\sigma}=\lambda^{\sigma}(t;x)X_1^{\sigma},\quad \lambda^{\sigma}=\lambda_1^{\sigma}(t)x+\lambda_0^{\sigma}(t)+\lambda_{-1}^{\sigma}(t)x^{-1}+\dots,\tag{4.20}$$

with solutions $X_1^{\sigma}=K_{\sigma}\cdot\exp\int_0^t\lambda^{\sigma}(\tau;x)d\tau$, which imply the formula

$$g=\sum_{\sigma}K_{\sigma}\cdot g_{\text{norm}}^{\sigma}(t;x)\tag{4.21}$$

for a general (formal) solution to the hypergeometric equation (1.1).

Definition 2. *The solutions g^{σ} are called the* ***normal WKB solutions associated with the point*** $t=0$*. Corresponding* ***normal WKB solutions associated with the point*** $s=1-t=0$ *are* $h_{\text{norm}}^{\sigma}(s;x)=\xi_d\,(\sigma x)^{d/2}\,e^{-\sigma x S_d(1)}g^{\sigma}(1-s;x)$ *(where ξ_d is the same as in Definition 1).*

The normal WKB solutions are also defined uniquely, because the reduction to the normal form is unique and essentially algebraic. They seem to be more important than the testing WKB solutions g_{test}^{σ}, because we can show that they are represented by analytic functions in some sectorial domains (due to some Birkhoff's theorem discussed below).

Note also that the normal form system (4.20) is more natural than the WKB solutions g_{norm}^{σ}, because the latter involve the initial condition $S_d(0)=0$.

Remark 2. *The relation between* g^σ_{norm} *and* g^σ_{test} *is of the form*

$$g^\sigma_{\text{norm}}(t;x) = C^\sigma_{\text{norm}}(x^{-1}) \cdot g^\sigma_{\text{test}}(t;x),$$

where $C^\sigma_{\text{norm}}(x^{-1}) = 1 + O(x^{-1})$ *are formal series. It seems that all the series* $C^\sigma_{\text{norm}}(x^{-1})$ *are the same for any index* σ *and depend on* x^{-d}. *This is proved for* $d = 2$ *in* [ZZ1]. *Also from Eq. (4.19) it follows that these series are nontrivial.*

4.3. **Analytic normalization.** We have seen that the process (which is standard) of successive reduction of Eq. (4.15) to the normal (diagonal) form is essentially algebraic. It is also unique. Unfortunately, it is divergent.

The problem of analytic interpretation of the WKB method is highly nontrivial. There exist known results about WKB functions which are analytic in some rather special domains and have the same asymptotic expansions as the formal WKB series. But those analytic functions undergo dramatic changes when the domains are changed; this is the famous Stokes phenomenon studied in Section 7.

Additional complication arises from the dependence of two variables: x (which is large) and t (which is bounded). In a traditional approach, used mostly by the physicists [He, BNR], the parameter x is real and the variable t may vary in some complex domain. In that domain there exist so-called Stokes lines which separate domains of uniqueness of the WKB functions. Several Stokes lines meet at so-called turning points, which are the ramification points of the derivative $\dot{S}(t) = dS/dt$ of the 'action' (like $\dot{S}(t) = \sqrt{q(t)}$ for the Schrödinger equation $\ddot{\psi} = -x^2 q(t)\psi$). In our situation, the fact that $\dot{S}(t)$ is infinite at $t = 0$ and $t = 1$ causes additional complication.

Since our principal aim is to study analytic properties of the connection coefficient $A_d(x)$ in Lemma 2, we should rather consider complex x, while t can stay real. When one allows $\arg x$ to vary the Stokes lines also should vary in a controllable way (see [DePh]). But this controlling is rather troublesome and we prefer to use our own method.

One ingredient of this method is exemplified in Theorem 1 below (we refer the reader to our original work [ZZ2]). It allows to treat analytically WKB functions in two domains in $\mathbb{C} \times \mathbb{C} = \{(t,x)\} : \mathcal{U}_0 \times \mathcal{V}_\infty$ and $\mathcal{U}_1 \times \mathcal{V}_\infty$, where $\mathcal{U}_{0,1}$ are neighborhoods of $t = 0, 1$ and $\mathcal{V}_\infty = (\mathbb{C}, \infty)$. In these domains we are able to control perfectly the Stokes lines and their $x-$dependence (see Section 7).

Another ingredient (realized in this section) is an analogue of a theorem due to G. D. Birkhoff [Bir] about WKB functions analytic in domains like $\mathcal{W} \times \mathcal{S}$ where $\mathcal{W}$ is a neighborhood of the 'interior' of the segment $[0,1]$ in the $t-$plane and $\mathcal{S}$ is a sector in the $x-$plane. The above domains have non-empty suitable intersections which allows to provide an analytic realization of formal WKB type series for solutions of differential equations and of the connection coefficient $A_d(x)$.

The reduction (4.16) is divergent (as a power series in x^{-1}) and the WKB solutions $g^{\pm}$ are only formal solutions. G. Birkhoff [Bir] was the first who proved that such a system can be diagonalized analytically in some sectorial domains. Below we present a scheme of the Birkhoff's proof in the case $d = 2$.

We apply a change

$$X^{+} = X_1^{+} + V^{12}(t) X_1^{-}, \quad X^{-} = V^{21}(t) X_1^{+} + X_1^{-} \tag{4.22}$$

which should transform system (4.15), i.e.

$$\frac{d}{dt}\begin{pmatrix} X^{+} \\ X^{-} \end{pmatrix} = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix} \begin{pmatrix} X^{+} \\ X^{-} \end{pmatrix},$$

to the diagonal form

$$\dot{X}_1^{+} = D_{+}(t) X_1^{+}, \quad \dot{X}_1^{-} = D_{-}(t) X_1^{-}. \tag{4.23}$$

We get $D_{+} = B^{11} + B^{12} V^{21}$, $D_{-} = B^{21} V^{12} + B^{22}$ and two independent Riccati equations

$$\begin{aligned} \dot{V}^{12} &= B^{11} V^{12} - V^{12} B^{22} + B^{12} - V^{12} B^{21} V^{12}, \\ \dot{V}^{21} &= B^{22} V^{21} - V^{21} B^{11} + B^{21} - V^{21} B^{12} V^{21}. \end{aligned}$$

The latter differential equations are rewritten in form of the following integral equations:

$$V^{12}(t) = \int_{\Gamma_1(t)} e^{P(t)-P(\tau)} \left\{ B^{12}(\tau) - V^{12}(\tau) B^{21}(\tau) V^{12}(\tau) \right\} d\tau, \tag{4.24}$$

$$V^{21}(t) = \int_{\Gamma_2(t)} e^{P(\tau)-P(t)} \left\{ B^{21}(\tau) - V^{21}(\tau) B^{12}(\tau) V^{21}(\tau) \right\} d\tau, \tag{4.25}$$

$P(t) = \int_0^t (B^{11}(\iota) - B^{22}(\iota)) d\iota = 2ix S_2(t) + \ldots$. Here $\Gamma_1(t)$ and $\Gamma_2(t)$ are some well chosen paths in the $\tau-$plane.

One would like to treat Eqs. (4.24)–(4.25) as fixed point equations in suitable functional spaces. For this the nonlinear operators defined by the right-hand sides should be contracting, at least bounded (see [Was, Zo3]).

The crucial element in the proof of the latter property is the possibility to estimate the factors $e^{\pm(P(t)-P(\tau))} \approx \exp\left\{\pm 2ix(S_2(t) - S_2(\tau))\right\}$. Thus, if $t \in (0,1)$ is real, then for $\operatorname{Im} x > 0$ we take the integration paths as segments $\Gamma_1 = [0,t]$ and $\Gamma_2 = [1,t]$; when $\operatorname{Im} x < 0$ we take $\Gamma_1 = [1,t]$ and $\Gamma_2 = [0,t]$.

But the entries $B^{ij}(t)$ of the matrix B have poles at $t = 0$ and $t = 1$. Moreover, we want to extend the range of $\arg x$ and to allow complex values of t. We choose three small constants $\alpha > 0$, $\beta > 0$ and $0 < \tau_0 << \beta$ and define the following domains: $\mathcal{W} = \{t = t_1 + it_2 : \ \beta < t_1 < 1 - \beta, \ \ |t_2| < \beta t_1(1 - t_1)\} \subset \mathbb{C}$ (a neighborhood of the open segment $(\beta, 1-\beta) \subset \mathbb{R}$) and $\mathcal{D}_u, \mathcal{D}_d \subset \mathbb{C}^2$ ('up' and 'down') by the conditions

$$\begin{aligned} \operatorname{Im} x S_2(t), \operatorname{Im} x (S_2(1) - S_2(t)) &> -\alpha, \ t \in \mathcal{W} \quad (\text{for } \mathcal{D}_u), \\ \operatorname{Im} x S_2(t), \operatorname{Im} x\,(S_2(1) - S_2(t)) &< \alpha, \ t \in \mathcal{W} \quad (\text{for } \mathcal{D}_d). \end{aligned}$$

If $(t,x)\in\mathcal{D}_u$ then the contour Γ_1 begins at $\tau=\tau_0$ and ends at $\tau=t$ and the path Γ_2 begins at $\tau=1-\tau_0$ and ends at $\tau=t$ and with $\operatorname{Im} x(S(t)-S(\tau))<0$. For $(t,x)\in\mathcal{D}_d$ the choice of the contours is opposite.

Solving the integral equations in the domains $\mathcal{D}_u$ and $\mathcal{D}_u$ one obtains analytic solutions $g_u^{\pm}(t;x)$ and $g_d^{\pm}(t;x)$ respectively. They have the same formal asymptotic expansions as the principal WKB solutions $g^{\pm}(t;x)$.

We note the conjugation symmetry of the above construction:

$$\overline{g_u^+(t;x)}=g_d^-(\bar{t};\bar{x}),\quad \overline{g_u^-(t;x)}=g_d^+(\bar{t};\bar{x}).$$

In the case of general degree $d\geq 2$ the corresponding system of Riccati type equations consists of $d(d-1)$ equations for the off-diagonal entries $V^{\sigma\rho}(t)$ of the matrix $V(t)$ (with 1's on the diagonal) such that $X=VX_1$. The corresponding integral equations take the form

$$V^{\sigma\rho}(t)=\int_{\Gamma^{\sigma\rho}}e^{(\sigma-\rho)x(S_d(t)-S_d(\tau))}F^{\sigma\rho}(\tau,V(\tau))d\tau. \tag{4.26}$$

Here there are $2d$ domains $\mathcal{D}_{1,2},\mathcal{D}_{2,3},\ldots,\mathcal{D}_{2d,1}$ being neighborhoods of the sectorial sets $[\beta,1-\beta]\times\overline{S_{k,k+1}}$, where $\overline{S_{k,k+1}}$, $k=1,\ldots 2d$ (and $2d+1=1$), are closed sectors defined by division of a neighborhood of $x=\infty$ by the lines $\arg x=j\pi/d$, $j=0,\ldots,d-1$. One obtains solutions $g^{\sigma}_{k,k+1}(t;x)$ analytic in the domains $\mathcal{D}_{k,k+1}$. From the construction they satisfy the following symmetry properties:

$$g^{\sigma}_{k+2,k+3}(t,\varsigma x) = g^{\varsigma\sigma}_{k,k+1}(t;x), \tag{4.27}$$

$$\overline{g^{\sigma}_{k,k+1}(t;x)} = g^{\bar{\sigma}}_{2d-k+1,2d-k+2}(\bar{t};\bar{x}), \tag{4.28}$$

$\varsigma=e^{2\pi i/d}$.

Let us summarize the results of this subsection in the following

Proposition 1. *For $d>2$ there exist $2d$ systems of solutions $(g^{\sigma}_{k,k+1})$, $k=1,\ldots,2d$, analytic in the domains $\mathcal{D}_{k,k+1}$ (defined above) whose formal expansions are the same as for the normal WKB solutions $g^{\sigma}_{\mathrm{norm}}$ from Definition 2. They satisfy relations (4.27) and (4.28).*

For $d=2$ there exist two such systems $(g_u^{\sigma})=\left(g^{\sigma}_{1,2}\right)$ and $(g_d^{\sigma})=\left(g^{\sigma}_{2,1}\right)$ analytic in the domains $\mathcal{D}_u=\mathcal{D}_{1,2}$ and $\mathcal{D}_d=\mathcal{D}_{2,1}$.

5. Bessel approximations

5.1. **Bessel type equations and their basic solutions.** Consider series (1.2) when $x\to\infty$ and

$$y=x^d t$$

is finite. Then we get

$$\varphi_1(t;x) \approx \Phi_1(y) := \sum_{n=0}^{\infty} \frac{(-y)^n}{(n!)^d} =_0 F_{d-1}(1,\dots,1;-y), \tag{5.1}$$

i.e. a confluent hypergeometric function. For $d = 2$ the function Φ_1 is expressed via a Bessel function:[7]

$$\Phi_1(y)|_{d=2} = J_0(2\sqrt{y}). \tag{5.2}$$

The function Φ_1 satisfies a special confluent hypergeometric equation, which we call the **Bessel type equation**:

$$\partial_y (y\partial_y)^{d-1} G + G = 0. \tag{5.3}$$

The other independent solutions to Eq. (5.3) are

$$\begin{array}{l} \Phi_2(t) = \Phi_1(y)\ln y + \Psi_2(y), \\ \Phi_3(t) = \frac{1}{2!}\Phi_1 \ln^2 y + \Psi_2 \ln y + \Psi_3(y), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \Phi_d(y) = \frac{1}{(d-1)!}\Phi_1 \ln^{d-1} y + \frac{1}{(d-2)!}\Psi_2 \ln^{d-2} y + \dots + \Psi_d(y) \end{array} \tag{5.4}$$

(where Ψ_j are some entire functions), they approximate the solutions φ_j.

Of course, Eq. (5.3) is obtained from Eq. (1.1) by the change $t = y/x^d$, $\partial_t = x^d \partial_y$ and taking limit as $x \to \infty$. We shall do analogous change with Eq. (3.5) by taking x large and

$$z = x^d s^{d-1}$$

finite. The obtained **Bessel type equation** is following:

$$(1-d)^d \cdot z^{\frac{1}{d-1}} \left(z^{\frac{d-2}{d-1}} \partial_z \right)^d H + H = 0. \tag{5.5}$$

It has basic solutions of the form

$$\begin{array}{l} \Theta_j(z) = \frac{1}{j!} z^{j/(d-1)} F_j(z) = \frac{1}{j!} z^{j/(d-1)} \cdot (1 + O(z)), \quad (j = 1,\dots,d-1), \\ \Theta_d(z) = \Theta_{d-1}(z)\ln z + \Xi_d(z), \end{array} \tag{5.6}$$

where $F_j(z)$ are some concrete confluent hypergeometric series and Ξ_d is an entire function.

For $d = 2$ we have

$$\Theta_1|_{d=2} = \sqrt{z} J_1(2\sqrt{z}) \tag{5.7}$$

[7]Recall that the Bessel function with index μ equals $J_\mu(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\mu+n-1)n!} \left(\frac{w}{2}\right)^{2n+\mu}$.

and for $d = 3$ we have

$$\Theta_1|_{d=3} = \sqrt{z}\left(1 + \sum_{n=1}^{\infty} \frac{z^n}{(2n+1)!(2n-1)!!}\right) = \sqrt{z} \cdot {}_0F_2\left(\alpha, \beta; \frac{z}{8}\right), \tag{5.8}$$

$$\Theta_2|_{d=3} = 2\sum_{n=1}^{\infty} \frac{z^n}{(2n)!(2n-2)!!} = z \cdot {}_0F_2\left(\gamma, \delta; \frac{z}{8}\right), \tag{5.9}$$

where $\alpha = \delta = n + 1/2$, $\beta = n - 1/2$, $\gamma = n + 1$.

5.2. **Formal and analytic WKB solutions.** The Bessel type equation (5.3) has irregular singular point at $y = \infty$ and equation (5.5) has irregular singular point at $z = \infty$. Any linear meromorphic differential equation with an irregular singular point has uniquely defined (up to a multiplicative constants) formal solution which we call the WKB solutions.

For Eq. (5.3) the **WKB solutions** are of the form

$$G^\sigma(y) = \left(\sigma y^{1/d}\right)^\gamma e^{d\sigma y^{1/d}} \left\{1 - \frac{a_1}{\sigma y^{1/d}} + \frac{a_2}{\left(\sigma y^{1/d}\right)^2} - \dots\right\}, \quad \gamma = -\frac{d-1}{2}, \tag{5.10}$$

and the **WKB solutions** for Eq. (5.5) are following:

$$H^\sigma(z) = \sqrt{-\sigma z^{1/d}} e^{(d/(1-d)\sigma z^{1/d}} \left\{1 + \frac{b_1}{\sigma z^{1/d}} + \frac{b_2}{\left(\sigma z^{1/d}\right)^2} + \dots\right\}, \tag{5.11}$$

where $\sigma = \varsigma^{j+1/2}$, $j = 0, \dots, d-1$, (as usual), the choice of the square root $\sqrt{-\sigma z^{1/d}}$ is defined in Definition 1 and the coefficients are computed recursively.

The dependence of the above functions on the roots $y^{1/d}$ and $z^{1/d}$ is not useful in calculations. Often we will use the variables

$$v = y^{1/d}, \quad w = z^{1/d} \tag{5.12}$$

and denote corresponding WKB solutions as

$$\widetilde{G}^\sigma(v) = -G^\sigma(v^3), \quad \widetilde{H}^\sigma(w) = H^\sigma(w^d). \tag{5.13}$$

They satisfy the following Bessel type equations:

$$(v\partial_v)^d \widetilde{G} + d^d \cdot v^d \widetilde{G} = 0, \tag{5.14}$$

$$(1/d-1)^d \cdot w^{\frac{d}{d-1}} \left(w^{\frac{-1}{d-1}} \partial_w\right)^d \widetilde{H} + d^d \cdot \widetilde{H} = 0. \tag{5.15}$$

Like in Section 4.2 we can transform each of the Eqs. (5.14)–(5.15) to a corresponding linear system which is next diagonalized using shearing transformations. The obtained diagonal system has basic solutions which must equal the WKB solutions from Eqs. (5.13). This formal reduction of the Bessel type equations to the normal form is in complete agreement with the analogous reduction of the hypergeometric equation.

But when we want to obtain analytic normal forms, then one encounters some differences with what is done in Section 4.3. For example, in the case of Eq. (5.14) one arrives to an analogue of Eq. (4.26), i.e.

$$V^{\sigma\rho}(v)=\int_{\Gamma^{\sigma\rho}} e^{d(\sigma-\rho)(v-\tau)}F^{\sigma\rho}(\tau,V(\tau))d\tau,$$

but now the paths $\Gamma^{\sigma\rho}=\Gamma^{\sigma\rho}(v)$ of integration are chosen rather differently.

Consider sectors $\mathcal{S}_1,\ldots,\ \mathcal{S}_{2d}$ with angles $2\pi/d-\delta$ ($\delta>0$ small) and with the bisectrices $\arg v=0,\pi/d,\ldots,(d-1)\pi/d$. These bisectrices $\mathcal{R}_j$ correspond to the situations when $\operatorname{Im}(\sigma-\rho)\,v=0$ (for some σ and ρ) and are called the **rays of division** associated with the pair (σ,ρ).

With given unordered pair $\{\sigma,\rho\}$ two rays of division $\mathcal{R}_j$ and $\mathcal{R}_{j+d}$ are associated (here $j+d$ is taken mod $2d$). Consider larger sectors $\mathcal{S}_{j-[d/2]}\cup\ldots\cup\mathcal{S}_j\cup\ldots\cup\mathcal{S}_{j+[d/2]}$ and $\mathcal{S}_{j+d-[d/2]}\cup\ldots\cup\mathcal{S}_{j+d}\cup\ldots\cup\mathcal{S}_{j+d+[d/2]}$ with the above rays as their bisectrices; they cover a neighborhood of $v=\infty$. For $v\in\ldots\cup\mathcal{S}_j\cup\ldots$ (respectively $v\in\ldots\cup\mathcal{S}_{j+3}\cup\ldots$) the path $\Gamma^{\sigma\rho}(v)$ runs parallel to the ray $\mathcal{R}_j$ from $\tau=\infty$ to $\tau=v$. Due to the fact that the factors $e^{d(\sigma-\rho)\tau}$ in the corresponding integral equations are bounded for $\tau\in\Gamma^{\sigma\rho}(v)$ the solutions to the integral equations exist and are analytic in the sectors $\mathcal{S}_k$.

We denote the analytic solutions in the sectors $\mathcal{S}_j$ obtained above by

$$\widetilde{G}_j^{\sigma}(v),\ v\in\mathcal{S}_j,\quad j=1,\ldots,6. \tag{5.16}$$

They are formally equivalent to the formal WKB solutions form Eqs. (5.10)–(5.13). (But for $d=2$ we have only two sectors $\mathcal{S}_1=\mathcal{S}_r$ (right)and $\mathcal{S}_2=\mathcal{S}_l$ (left) with bisectrices $\mathcal{R}_1=\{\arg v=0\}$ and $\mathcal{R}_2=\{\arg v=\pi\}$ and angles $2\pi-\delta$ and two sets of solutions $\widetilde{G}^{\pm}_{r,l}(v)$.

Analogously we obtain systems of analytic solutions to Eq. (5.15):

$$\widetilde{H}_j^{\sigma}(w),\quad w\in\mathcal{S}_j,\quad j=1,\ldots,2d. \tag{5.17}$$

Remark 3. *Functions (5.16) and (5.17) were constructed by solving corresponding integral equations. But there exist explicit integral formulas for analytic WKB solutions to Bessel type equations (and to general hypergeometric confluent equations) due to A. Duval and C. Mitschi* [DuMi] *(see also* [ZZ3]*). For example, for $d=3$ the following* ***Mellin–Barnes integral***

$$G^-_{DM}(y)=\frac{1}{2\pi i}\int_\gamma\Gamma^3(-\tau)y^\tau d\tau,$$

where γ is a path from $\tau=-i\infty$ to $\tau=+i\infty$ which leaves the poles $\tau=1,2,\ldots$ of the Gamma function from the right, defines a solution to the Bessel type equation (5.3) for $d=3$. (The function G^-_{DM} is a particular case of the so-called Meijer G–functions, [Me] *and* [BE1]*). It turns out that $G^-_{DM}(y)$ is analytic in the sector $\left\{-\pi-\varepsilon<\arg y^{1/3}<\pi+\varepsilon\right\}$ and has the form $G^-_{DM}=e^{-3y^{1/3}}y^{-1/3}\Omega_0(y^{-1/3})$ (like G^-).*

Moreover other WKB solutions can be taken in the form

$$G^{\epsilon}_{DM}(y) = e^{3\epsilon y^{1/3}} y^{-1/3}\Omega_0(\bar{\epsilon} y^{-1/3}), \quad G^{\bar{\epsilon}}_{DM}(y) = e^{3\bar{\epsilon} y^{1/3}} y^{-1/6}\Omega_0(\epsilon y^{-1/3})$$

(where the notations $-, \epsilon, \bar{\epsilon}$ are like in Note 6). The new WKB solutions H^-_{DM}, H^{ϵ}_{DM}, $H^{\bar{\epsilon}}_{DM}$ to the Bessel type equation (3.7) are defined similarly, via the following Mellin–Barnes integral:

$$\begin{aligned} H^-_{DM}(z) &= \frac{1}{2\pi i}\int_\gamma \Gamma(1-\tau)\Gamma(1/2-\tau)\Gamma(-\tau)(-z/8)^\tau d\tau \\ &= e^{\frac{3}{2}z^{1/3}} z^{1/6}\Omega_1(z^{-1/3}). \end{aligned}$$

Also for other degrees $d \neq 3$ Duval and Mitschi define WKB solutions G^σ_{DM} and H^σ_{DM} analytic in suitable sectors about infinity.

Finally, we note that analyticity of the WKB solutions in sectors can be proved in still another way, using the fact that the formal WKB solutions are defined via Gevrey type series, by applying corresponding Borel and Laplace transforms. We refer the reader to the books of W. Balser [Bal] *and J.-P. Ramis* [Ram].

5.3. **Equivalences of hypergeometric equation and its Bessel approximations.** Importance of the above approximations can be seen from the following result, which is a special case of a more general theorem proved in [ZZ2, Theorem 2]. Let $\Phi = (\Phi_1, \dots, \Phi_d)$, $\Theta = (\Theta_1, \dots, \Theta_d)$ denote the bases (5.1)–(5.4) and (5.6) and φ, θ be corresponding bases from Section 3.

Theorem 1. *There exist matrix-valued functions $\mathcal{H}_0(t) = I + O(t)$ and $\mathcal{H}_1(s) = I + O(s)$, defined in a neighborhood of $t = 0$ and $s = 1 - t = 0$ in $\mathbb{C}$ and analytic there, such that*

$$\varphi\mathcal{H}_0 = \Phi, \quad \theta\mathcal{H}_1 = \Theta.$$

Proof. Let

$$\mathcal{F}_0 = \begin{bmatrix} \varphi_1 & \dots & \varphi_d \\ \dots & \dots & \dots \\ \partial_t^{d-1}\varphi_1 & \dots & \partial_t^{d-1}\varphi_d \end{bmatrix}, \quad \mathcal{G}_0 = \begin{bmatrix} \Phi_1 & \dots & \Phi_d \\ \dots & \dots & \dots \\ \partial_t^{d-1}\Phi_1 & \dots & \partial_t^{d-1}\Phi_d \end{bmatrix}$$

be the fundamental matrices associated with the bases φ (see Eq. (3.4)) and Φ and $\partial_t\Phi_j = x^d\partial_y\Phi_j$ means differentiation with respect to the time t. Then we have

$$\mathcal{H}_0(t; x) = \mathcal{F}_0^{-1}\mathcal{G}_0.$$

Analogously the fundamental matrices $\mathcal{F}_1$ and $\mathcal{G}_1$ associated with the fundamental systems θ and Θ define the matrix-valued function

$$\mathcal{H}_1(s; x) = \mathcal{F}_1^{-1}\mathcal{G}_1.$$

It is clear from Section 3 that the matrices $\mathcal{F}_0(t, x)$ and $\mathcal{G}_0(t, x)$ are analytic in (t, x) for $t \in (\mathbb{C}\setminus 0, 0)$ and $x \in \mathbb{C}\setminus 0$. It was observed in [ZZ2] that the matrices $\mathcal{F}_0$

and $\mathcal{G}_0$ have the same monodromy properties as t turns around 0 and as x turns around 0 (or around ∞) and have the same singularities at $t = 0$ and at $x = 0$. Moreover, from the analysis in Sections 6 and 7 it follows that these matrices have almost the same asymptotic as $x \to \infty$, i.e. in sectorial domains. Therefore the matrix valued function $\mathcal{H}_0$ is single valued in the both variables and is bounded at possible singularities: $t = 0$, $x = 0$ and $x = \infty$. It follows that it is analytic in $t \in (\mathbb{C}, 0)$ and constant in $x \in \mathbb{C}$.

The same arguments prove that $\mathcal{H}_1(s; x)$ is holomorphic in $s \in (\mathbb{C}, 0)$ and constant in $x \in \mathbb{C}$. ∎

Theorem 2 from [ZZ2] is a generalization of a theorem of W. Wasow from [Was] about reduction of equations of the form $d^2x/dt^2 = \left\{\lambda^2 ta(t) + \lambda b(t, 1/\lambda)\right\} x$, $a(0) = 1$ (with analytic germs a and b and large λ) to the *Airy equation* $\partial_T^2 y = Ty$, $T = t\lambda^{2/3}$, which is also of the Bessel type. In [ZZ2] a slightly weaker result was proved; namely, it was stated that $\mathcal{H}_0(t, x)$ is analytic in $t \in (\mathbb{C}, 0)$ and $x^{-1} \in (\mathbb{C}, 0)$.

Definition 3. *The functions $g^\sigma_{\mathrm{princ}} = G^\sigma \mathcal{H}_0^{-1}$ are called the **principal WKB solutions near** $t = 0$ to hypergeometric equations (1.1) and the functions $h^\sigma_{\mathrm{princ}} = H^\sigma \mathcal{H}_1^{-1}$ are called the **principal WKB solutions near** $s = 1 - t = 0$ to the same equation.*

Remark 4. *Since the WKB solutions G^σ to Eq. (5.3) and H^σ to Eq. (5.5) are formal the principal WKB solutions $g^\sigma_{\mathrm{princ}}$ and $h^\sigma_{\mathrm{princ}}$ are also only formal. Their relations with the formal and normal WKB solutions from Definition 1 and Definition 2 are of the form*

$$g^\sigma_{\mathrm{princ}} = K^\sigma_{\mathrm{princ}}(x^{-1}) \cdot g^\sigma_{\mathrm{test}}, \quad h^\sigma_{\mathrm{princ}} = L^\sigma_{\mathrm{princ}}(x^{-1/(d-1)}) \cdot h^\sigma_{\mathrm{test}} \tag{5.18}$$

for some series $K^\sigma_{\mathrm{princ}}(x^{-1}) = 1 + O(x^{-1})$ and $L^\sigma_{\mathrm{princ}}(x^{-1/(d-1)}) = 1 + O(x^{-1/(d-1)})$. Here $L^\sigma_{\mathrm{princ}}$ is a series in powers of $x^{-1/(d-1)}$ because the hypergeometric equation (1.1) is a perturbation of the Bessel type equation (5.5) and in the perturbation we encounter powers of $s = z^{1/(d-1)} x^{-d/(d-1)}$; in fact we solve it by solving a system of equations in variations (see [ZZ3]*).*

Therefore

$$g^\sigma_{\mathrm{princ}}(1 - s) = \xi_d^{-1} \frac{K^\sigma_{\mathrm{princ}}}{L^\sigma_{\mathrm{princ}}} (\sigma x)^{-d/2} e^{\sigma x S_d(1)} \cdot h^\sigma_{\mathrm{princ}}(s). \tag{5.19}$$

We have not calculated the series $K^\sigma_{\mathrm{princ}}(x^{-1})$ and $L^\sigma_{\mathrm{princ}}(x^{-1})$, but there is no reason to expect that they are equal. But Eq. (4.19) above and Lemma 5 below suggest that probably $K^\sigma_{\mathrm{princ}}(x^{-1}) = L^\sigma_{\mathrm{princ}}(x^{-1}) = C_{\mathrm{norm}}(x^{-2}) = 1 + (5/256)x^{-2} + \ldots$ for $d = 2$.

On the other hand, if we choose analytic versions (i.e. in some sectors) of the formal WKB solutions to Eqs. (5.3) and (5.5), like in Section 5.2, then by applying *the operators $\mathcal{H}_0^{-1}$ and $\mathcal{H}_1^{-1}$ to them we obtain analytic principal WKB solutions in corresponding domains.*

Moreover, the domain of definition of $\mathcal{H}_0(t)$ is not limited to a small neighborhood of $t=0$. $\mathcal{H}_0$ is analytic in a disc $\{|t|<1-\varepsilon_0\}$ for small ε_0. Similarly $\mathcal{H}_1(s)$ is analytic in $\{|s|<1-\varepsilon_0\}$. These two domains have quite big intersection.

Finally, because there exist analytic (in sectors) versions G_j^σ and H_j^σ of the formal WKB functions, application of $\mathcal{H}_0^{-1}$ and $\mathcal{H}_1^{-1}$ to them gives corresponding analytic principal WKB solution to the hypergeometric equation.

Definition 4. *We introduce the following WKB type formal functions*

$$F^\sigma(x)=\frac{g^\sigma_{\text{princ}}(1-s;x)}{h^\sigma_{\text{princ}}(s;x)}=\xi_d^{-1}\,(\sigma x)^{-d/2}\,e^{\sigma x S_d(1)}\omega^\sigma(x^{-1/(d-1)}).$$

Here $\omega^\sigma(x^{-1/(d-1)})=K^\sigma_{\text{princ}}(1/x)/L^\sigma_{\text{princ}}(1/x^{1/(d-1)})$ *and*

$$S_2(1)=\pi \text{ and } S_3(1)=2\pi/\sqrt{3}.$$

We have

$$F^\pm = \frac{1}{x}e^{\pm ix\pi}\omega^\pm(1/x)\,, \tag{5.20}$$

$$F^\sigma = \pm\frac{e^{-2x\sigma\pi/\sqrt{3}}}{x^{3/2}}\omega^\sigma\left(x^{-1/2}\right), \tag{5.21}$$

for $d=2$ and $d=3$ respectively; in Eq. (5.21) $\pm=+$ for $\sigma=\epsilon,\bar\epsilon$ and $=-$ for $\sigma=-1$.

In the case $d=3$ the series $\omega^\sigma(x^{-1/2})$ are not single valued. We can write instead

$$x^{-3/2}\omega^\sigma_\pm=\pm\sqrt{x}\cdot x^{-2}\omega_0^\sigma(x^{-1})+x^{-2}\omega_1^\sigma(x^{-1}).$$

Then we have six WKB type functions

$$F^\sigma_\pm=x^{-3/2}e^{2\sigma x\pi/\sqrt{3}}\omega^\sigma_\pm. \tag{5.22}$$

In the case of odd $d>3$ there are $d(d-1)$ similar WKB functions.

6. Integral representations and stationary phase formula

6.1. **Integral formulas.** Some of the series defining solutions of hypergeometric and Bessel type equations have integral representations. We begin with the standard representation of the Bessel functions:

$$\begin{aligned} J_n(w) &= \frac{1}{2\pi i}\oint_{|u|=1}\exp\left(\frac{w}{2}(u-1/u)\right)\frac{du}{u^{n+1}} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(iw\sin\alpha\right)e^{-in\alpha}d\alpha. \end{aligned} \tag{6.1}$$

This formula was obtained by Bessel and can be found in the literature (see [BE2, GM]). Let us recall its simple proof whose argumentation can be used in more general situations. The series $\sum_{m=0}^{\infty}(-1)^m(w^2/4)^{m+n/2}/(m+n)!m!$ which defines $J_n(w)$ admits the following residue representation:

$$\operatorname{res}_{u=0}\frac{1}{u^{n+1}}\left(\sum\frac{(wu/2)^m}{m!}\right)\left(\sum\frac{(-w/2u)^m}{m!}\right).$$

Next we use the Cauchy formula.

For a non-integer index μ we have the following *Schläfli representation:*

$$\begin{aligned} J_\mu(w) = & \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(i(w\sin\alpha-\mu\alpha)\right)d\alpha \\ & -\frac{\sin\pi\mu}{\pi}\int_0^\infty\exp\left(-w\sinh\beta-\mu\beta\right)d\beta. \end{aligned} \tag{6.2}$$

This follows from some generalization of the residuum formula for J_n with integer n. We have $J_\mu(w)=\frac{1}{2\pi i}\int_C\exp\left(\frac{1}{2}w\left(u-1/u\right)\right)u^{-\mu-1}du$ where C is a contour which begins and ends at $u=-\infty$ and surrounds $u=0$ in positive direction. Next the contour C is deformed to two half-lines along $(-\infty,-1)$ (parametrized by $-e^\beta$) and the circle $|u|=1$. For more details we refer reader to [BE2, Eq. 7.3(9)]. (In the original Schläfli formula the first integral in Eq. (6.2) is replaced with $\frac{1}{\pi}\int_0^\pi\cos\left(w\sin\alpha-\mu\alpha\right)d\alpha$.)

Now we are ready to present a multidimensional contour integrals. We have

$$\Phi_1=\left(\frac{1}{2\pi i}\right)^{d-1}\int\cdots\int_{|Q_0|=\ldots=|Q_{d-2}|=1}\exp\left\{-y^{1/d}\sum_{j=0}^{d-1}\varsigma^jP_j\right\}\prod_{j=0}^{d-2}\frac{dQ_j}{Q_j} \tag{6.3}$$

for the generalized Bessel function (5.1). Here and below $\varsigma=e^{2\pi i/d}$ and

$$\begin{aligned} & P_0=Q_0, P_1=Q_1Q_0^{-1/(d-1)},\ldots,P_{d-2}=Q_{d-2}Q_{d-3}^{-1/2}\ldots Q_0^{-1/(d-1)}, \\ & P_{d-1}=Q_{d-2}^{-1}Q_{d-3}^{-1/2}\ldots Q_0^{-1/(d-1)}; \end{aligned} \tag{6.4}$$

thus $\prod P_j=1$.

For the hypergeometric function (1.2) we get the following formula:

$$\varphi_1=\left(\frac{1}{2\pi i}\right)^{d}\int\cdots\int_{|Q_0|=\ldots=|Q_{d-2}|=1}\left\{\prod_{j=0}^{d-1}\left(1-t^{1/d}P_j\right)^{\varsigma^j}\right\}^x\prod_{j=0}^{d-1}\frac{dQ_j}{Q_j}. \tag{6.5}$$

In the proof one uses the expansions

$$(1-z)^{-a}=\sum\frac{\Gamma(a+n)}{\Gamma(a)n!}z^n$$

and

$${}_dF_{d-1}(a_1,\ldots,a_d;1,\ldots,1;t)=\sum\frac{\Gamma(a_1+n)}{\Gamma(a_1)n!}\cdots\frac{\Gamma(a_d+n)}{\Gamma(a_d)n!}t^n.$$

Using the Schläfli formula (6.2) we can prove the formula (with the Euler–Mascheroni constant γ)

$$(6.6)\qquad \begin{aligned}(\Phi_2+2\gamma\Phi_1)\,|_{d=2} \;=\; & \tfrac{1}{i\pi}\int_{-\pi}^{\pi}\alpha\exp\left(2i\sqrt{y}\sin\alpha\right)d\alpha \\ & -2\int_0^{\infty}\exp\left(-2\sqrt{y}\sinh\beta\right)d\beta\end{aligned}$$

for another solution $\lim_{\nu\to 0}\frac{1}{\nu}\left\{J_\nu(2\sqrt{y})-J_{-\nu}(2\sqrt{y})\right\}$ to the Bessel type equation (5.3) for $=2$.

The Schläfli formula admits a generalization to the case of hypergeometric integrals (see [ZZ1]). It allows to prove the following formula for the solution $\widehat{\varphi}_2$ (for $d=2$) from Remark 1:

$$(6.7)\qquad \begin{aligned}\widehat{\varphi}_2|_{d=2} \;=\; & \tfrac{1}{2\pi i}\int_{|v|=1}\left(\tfrac{1-\sqrt{t}v}{1-\sqrt{t}/v}\right)^x\ln\left(\tfrac{1-\sqrt{t}v}{v^2(1-\sqrt{t}/v)}\right)\tfrac{dv}{v} \\ & -\int_1^{1/\sqrt{t}}\left(\tfrac{1-\sqrt{t}v}{1-\sqrt{t}/v}\right)^x\left\{\tfrac{\sin\pi x}{\pi}\ln\left(\tfrac{1-\sqrt{t}v}{v^2(1-\sqrt{t}/v)}\right)+3\cos\pi x\right\}\tfrac{dv}{v}.\end{aligned}$$

Unfortunately, we do not have integral formulas for the basic solutions θ_j to the hypergeometric equation near $1-t=0$ for odd $d>2$. (For $d=2$ we can use the duality formula from Lemma 3.) The reason for this is that the recurrence relations for the coefficients in the series defining θ_j are of length greater than two.

Fortunately, we can find such formulas for the solutions Θ_j to the Bessel type equation (5.5).

In the case $d=2$ the duality relation implies

$$\Theta_j(z)|_{d=2} = -z\partial_z\Phi_j(z), \quad j=1,2,$$

and, in particular,

$$\Theta_1(z)|_{d=2} = \sqrt{z}J_1(2\sqrt{z}).$$

For $d=3$ we have the following formulas (for the proofs see [ZZ3]):

$$(6.8)\qquad \begin{aligned}\Theta_1|_{d=3} \;=\; & -\tfrac{z^{1/6}}{8\pi} \\ & \cdot\int_{C'}\tfrac{d\tau}{(1-\tau)^{3/2}}\int_{-\pi}^{\pi}d\alpha\sinh\left(z^{1/3}e^{i\alpha/2}\right)\exp\left(\tfrac{1}{2}z^{1/3}e^{-i\alpha}\tau\right)e^{-i\alpha/2},\end{aligned}$$

$$(6.9)\qquad \Theta_2|_{d=3} = \frac{z^{1/3}}{2\pi}\int_{-\pi}^{\pi}\cosh\left(z^{1/3}e^{i\alpha/2}\right)\exp\left(\frac{1}{2}z^{1/3}e^{-i\alpha}\right)e^{-i\alpha}d\alpha.$$

In Eq. (6.8) C' is a contour which begins and ends at $\tau=0$ and surrounds $\tau=1$ in positive direction. (The third solution $\Theta_3|_{d=3}$ to the Bessel like equation (5.5) can be found by taking the perturbation $8\left\{z^2\partial_z\sqrt{z}\partial_z\sqrt{z}\partial_z-\nu(\nu-1/2)(\nu-1)\right\}H-zH=0$ and passing to the limit as $\nu\to 0$ with suitable combination of the basic solutions.)

6.2. **The stationary phase formula.** Recall (see [He]) that the stationary phase formula concerns integrals of the type

$$I(\lambda) = \int e^{\lambda\phi(\alpha)}\chi(\alpha)d^k\alpha \tag{6.10}$$

over a $k-$dimensional manifold when $|\lambda| \to \infty$. Assuming that the 'phase' $\phi(\alpha)$ has finitely many critical points $\alpha_1, \ldots, \alpha_n$, which are Morsean, one has the following asymptotic **stationary phase formula**:

$$I(\lambda) \sim \sum_i \chi(\alpha_i)\frac{1}{\sqrt{\det(-D^2\phi(\alpha_i))}}e^{\lambda\phi(\alpha_i)}\left(\frac{2\pi}{\lambda}\right)^{k/2}. \tag{6.11}$$

Usually, in applications, the large parameter λ is imaginary and the phase ϕ is a real function; then the integral in Eq. (6.10) is called the **oscillating integral**. Otherwise the name **mountain pass integral** is sometimes used; with such case we deal in this paper. In the case of real x and t the integrals (6.3), $(6.5)_{d=2}$, (6.6) and (6.7) are oscillating integrals and for $d > 2$ we deal with mountain pass integrals.

We want to apply formula (6.11) to the above integrals with large $|y|$ or $|z|$. However here the large parameter λ is not purely imaginary and the phase ϕ is not a real function. So we shall assume that λ lies in some sector S (in the complex plane) with vertex at ∞. Then the sum in Eq. (6.11) becomes restricted to those critical points α_i for which the function

$$z \to \exp\left\{\lambda D^2\phi(\alpha_i)(z,z)\right\}$$

is integrable, i.e. the eigenvalues μ_j of the Hessian $D^2\phi(\alpha_i)$ satisfy

$$\mathrm{Re}(\lambda\mu_j) \leq 0.$$

We shall also deal with integrals of the type

$$J(\lambda) = \int_{\beta_0}^{\beta_1} e^{\lambda\varphi(\beta)}\chi(\beta)d\beta, \tag{6.12}$$

where the 'phase' function φ is noncritical. Assume that

$$\varphi' < 0, \quad \chi(\beta) = (\beta - \beta_0)^{\sigma-1}(D + l.o.t.), \tag{6.13}$$

where the function $\chi_1(\beta) = D + l.o.t.$ is analytic near β_0. In this case, for large λ, with $\mathrm{Re}\,\lambda \geq 0$, and $\mathrm{Re}\,\sigma > 0$ we have

$$J(\lambda) \sim D \cdot \Gamma(\sigma) \cdot \exp\{\lambda\varphi(\beta_0)\} \cdot (-\lambda\varphi'(\beta_0))^{-\sigma} \tag{6.14}$$

(see [ZZ3, Lemma 3.7]). Moreover, this formula holds also when $\mathrm{Re}\,\sigma < 0$ and is not integer, but the integral in Eq. (6.12) is replaced by $(1 - e^{-2\pi i\sigma})^{-1}$ times an integral along a contour which surrounds the point β_0 in negative direction.

The aim of this subsection is to derive initial terms of the asymptotic expansions of the functions expressed via the above contour integrals.

Let us consider firstly the simplest case of the oscillating integral $\Phi_1(y)|_{d=2} = \frac{1}{2\pi}\int \exp\left(2i\sqrt{y}\sin\alpha\right)d\alpha$. The phase function $\phi(\alpha) = 2i\sin\alpha$ has two critical points $\alpha_1 = \frac{\pi}{2}$ with $\phi(\alpha_1) = 2i$, $\phi''(\alpha_1) = -2i$ and $\alpha_2 = -\frac{\pi}{2}$ with $\phi(\alpha_2) = -2i$, $\phi''(\alpha_2) = 2i$. Therefore we obtain the following (well known) asymptotic formula for $y \to \infty$:

$$\Phi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}y^{1/4}}\left(e^{i(2\sqrt{y}-\pi/4)} + e^{-i(2\sqrt{y}-\pi/4)}\right). \tag{6.15}$$

In the right-hand side of Eq. (6.6) the second integral can be ignored, because it decreases like $y^{-1/2}$ (without any exponent). The first integral in that formula is an oscillating integrals and standard application of Eq. (6.11) gives (for $y \to \infty$)

$$\left(\Phi_2 + 2\gamma\Phi_1\right)|_{d=2} \sim \frac{\sqrt{\pi}}{2iy^{1/4}}\left(e^{i(2\sqrt{y}-\pi/4)} - e^{-i(2\sqrt{y}-\pi/4)}\right). \tag{6.16}$$

In the case of the oscillating integral $(6.3)_{d\geq 3}$ the phase equals

$$\phi(Q) = \sum \varsigma^j P_j.$$

Its critical points are calculated using a Lagrange multiplier κ corresponding to the restriction $\prod P_j = 1$. One finds $P_j = \kappa\varsigma^{-j}$, where $\kappa^d = -1$. This gives d points $P^{(k)}$, $k = 0,\ldots,d-1$, $P_j^{(k)} = \varsigma^{k-j+1/2}$, and to $d!$ critical points $Q^{(l)}$ (when we take into account choices of the roots $Q_0^{1/(d-1)},\ldots,Q_{d-2}^{1/2}$. Next, one substitutes $P_j = P_j^{(k)}e^{ip_j}$ and $Q_j = Q_j^{(l)}e^{iq_j}$, where p_j and q_i satisfy definite linear relations (see Eqs. (6.4)). The Taylor expansion of the phase at $Q^{(l)}$ takes the form $\phi(q) = \phi(Q^{(l)}) + \frac{1}{2}\sum a_{mn}^{(l)}q_m q_n + \ldots$ and the corresponding contribution in the stationary phase formula takes the form

$$(2\pi)^{(1-d)/2}\left(\det\mathcal{A}^{(l)}\right)^{-1/2}\cdot e^{-y^{1/d}\phi(Q^{(l)})}\cdot y^{(1-d)/2d}, \quad \mathcal{A}^{(l)} = \left(a_{mn}^{(l)}\right)$$

In the case $d = 3$ we obtain, as $y \to \infty$,

$$\Phi_1|_{d=3} \sim \frac{1}{\pi\sqrt{3}y^{1/3}}\left(\frac{e^{3\epsilon y^{1/3}}}{\epsilon} + \frac{e^{3\bar{\epsilon}y^{1/3}}}{\bar{\epsilon}} + \frac{e^{-3y^{1/3}}}{-1}\right), \quad \epsilon = e^{i\pi/3}. \tag{6.17}$$

(We have not finished calculations for $d > 3$.)

For the integral (6.5) the phase

$$\phi(Q) = \sum \varsigma^j \ln(1 - t^{1/d}P_j)$$

also has $d!$ critical points.

For $d = 2$ the critical points in Eq. $(6.5)_{d=2}$ are $Q^{\pm} = \sqrt{t} \pm i\sqrt{s}$, $s = 1-t$, and $\phi(Q^{\pm}e^{iq}) = \pm iS_2(t) \mp iu^2q^2$, $u = \sqrt[4]{t/s}$. Therefore the leading term of the oscillatory integral corresponding to the critical point $\alpha_{\pm}$ equals

$$e^{\pm ixS(t)}\frac{1}{2\pi}\int \exp(\mp ixu^2q^2)dq \sim \frac{1}{2u\sqrt{\pm i\pi x}}e^{\pm ixS_2(t)}.$$

We obtain

$$\varphi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}} \left\{ \frac{e^{ixS_2(t)}}{u\sqrt{ix}} + \frac{e^{-ixS_2(t)}}{u\sqrt{-ix}} \right\}. \tag{6.18}$$

For $d = 3$ the critical points are $Q^{\sigma,\pm}$, $\sigma = -1, \epsilon, \bar{\epsilon}$, such that

$$Q_1^{\sigma,\pm} = \frac{1}{t^{1/3} - \bar{\sigma} s^{1/3}}, \quad Q_2^{\sigma,\pm} = \pm\sqrt{\frac{u + \bar{\epsilon}\bar{\sigma}}{u + \epsilon\bar{\sigma}}}, \quad u = \left(\frac{t}{s}\right)^{1/3}, \quad s = 1 - t.$$

Here the absolute values of $Q_j^{\sigma,\pm}$ are different from 1, so it is rather a mountain pass integral than an oscillating integral. We deform the initial integration contour, the torus $\mathbb{T}_0 = \{Q_1 = e^{i\alpha},\ Q_2 = e^{i\beta} : 0 \le \alpha, \beta \le 2\pi\}$, to another contour $\mathbb{T}_1$ such that it passes through the critical points and near these points we can write $Q_1 = Q_1^{\sigma,\pm} e^{iq_1}$, $Q_2 = Q_2^{\sigma,\pm} e^{iq_2}$ (see [ZZ3] for details).

One has $\phi(Q^{\sigma,\pm}) = \sigma S_3(t)$ and the corresponding matrix $\mathcal{A}^\sigma$ defining the quadratic terms equals

$$-\sigma u \begin{pmatrix} \frac{3}{4}(2 - \sigma u) & i\frac{\sqrt{3}}{2}\sigma u \\ i\frac{\sqrt{3}}{2}\sigma u & 2 + \sigma u \end{pmatrix},$$

with the determinant $3\,(\sigma u)^2$.

The leading part of the hypergeometric function $(6.3)_{d=3}$ arising from a neighborhood of the point $Q^{\sigma,\pm}$ for large $|x|$ equals $e^{\sigma x S_3(t)}$ times

$$\left(\frac{1}{2\pi}\right)^2 \int\int e^{-x(\mathcal{A}q,q)/2} d^2q = \frac{1}{2\pi\sqrt{3}} \times \left\{ \left(\frac{1-t}{t}\right)^{1/3} \frac{1}{\sigma x} \right\}.$$

It agrees, up to a constant, with the first term in the testing WKB solution $g^\sigma_{\mathrm{test}}(t;x)$ given in Definition 1. We get the following formal expansion as $x \to \infty$:

$$\varphi_1|_{d=3} \sim \frac{1}{2\pi\sqrt{3}} \left\{ \frac{e^{-xS_3(t)}}{-ux} + \frac{e^{\epsilon x S_3(t)}}{\epsilon u x} + \frac{e^{\bar{\epsilon} x S_3(t)}}{\bar{\epsilon} u x} \right\}. \tag{6.19}$$

Let us present the corresponding stationary phase expansions for the functions $\Theta_j(z)|_{d=2,3}$. For $d = 2$ we have the following expansions, as $z \to \infty$,

$$\begin{aligned} \Theta_1|_{d=2} &\sim \tfrac{-1}{2\sqrt{\pi}} \left\{ \sqrt{\tfrac{z^{1/2}}{i}} e^{-2i\sqrt{z}} + \sqrt{\tfrac{z^{1/2}}{-i}} e^{2i\sqrt{z}} \right\}, \\ (\Theta_2 + 2\gamma\Theta_1)|_{d=2} &\sim \tfrac{\sqrt{\pi}}{2i} \left\{ \sqrt{\tfrac{z^{1/2}}{i}} e^{-2i\sqrt{z}} - \sqrt{\tfrac{z^{1/2}}{-i}} e^{2i\sqrt{z}} \right\}. \end{aligned} \tag{6.20}$$

In [ZZ3] it was found that the integrals (6.8) and (6.9) have the following expansions:

$$
\begin{aligned}
\Theta_1|_{d=3} &\sim \sqrt{1/3}\cdot\left\{z^{1/6}e^{\frac{3}{2}z^{1/3}} - \epsilon z^{1/6}e^{-\frac{3}{2}\bar{\epsilon}z^{1/3}} - \bar{\epsilon}z^{1/6}e^{-\frac{3}{2}\epsilon z^{1/3}}\right\},\\
\Theta_2|_{d=3} &\sim \sqrt{2/3\pi}\left\{z^{1/6}e^{\frac{3}{2}z^{1/3}} + \epsilon z^{1/6}e^{-\frac{3}{2}\bar{\epsilon}z^{1/3}} + \bar{\epsilon}z^{1/6}e^{-\frac{3}{2}\epsilon z^{1/3}}\right\},\\
\Theta_3|_{d=3} &\sim -2i\sqrt{2\pi/3}z^{1/6}\left\{\epsilon e^{-\frac{3}{2}\bar{\epsilon}z^{1/3}} - \bar{\epsilon}e^{-\frac{3}{2}\epsilon z^{1/3}}\right\}+\\
&\qquad \sqrt{6/\pi}\ln 2\cdot z^{1/6}\left\{e^{\frac{3}{2}z^{1/3}} + \epsilon e^{-\frac{3}{2}\bar{\epsilon}z^{1/3}} + \bar{\epsilon}e^{-\frac{3}{2}\epsilon z^{1/3}}\right\}.
\end{aligned}
\tag{6.21}
$$

Remark 5. *The formulas (6.15)–(6.21) cannot be treated rigorously and the reason for this is not the fact that the corresponding series are divergent. In fact, only one or two leading terms are correct when* $\arg y$ *or* $\arg x$ *or* $\arg z$ *is fixed. This is related with the Stokes phenomenon discussed in detail in Section 7. Also there the correct coefficients in the expansions (6.15)–(6.21) are computed.*

6.3. **Applications.**

6.3.1. *Expansion in the principal WKB solutions.* The first application is the correct WKB expansion of the analytic solution φ_1 to our hypergeometric equation.

Proposition 2. *(a) For* $d=2$ *and* $0<t<1$, $x>0$ *we have*

$$\varphi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}}\left\{g^{+}_{\text{princ}} + g^{-}_{\text{princ}}\right\}.$$

(b) For $d=3$ *and* $0<t<1$, $x>0$ *we have*

$$\varphi_1|_{d=3} \sim \frac{1}{2\pi\sqrt{3}}\left\{g^{\epsilon}_{\text{princ}} + g^{\bar{\epsilon}}_{\text{princ}} - 2g^{-}_{\text{princ}}\right\}.$$

Here $g^{\sigma}_{\text{princ}}$ *are the principal WKB solutions from Definition 3. Of course, these expansions are subject to the limitation from Remark 5.*

This follows from Definition 3 and the fact that the solution $\Phi_1(y)|_{d=2,3}$ has the same representation as in Proposition 2 with g^{σ} replaced with G^{σ}. In the point (b) the coefficient before g^{-}_{princ} is different than in Eq. (6.19); but by Remark 5 this coefficient is not determined in that formula. It is calculated in Section 7.

We can formulate a result like Proposition 2 but with respect to the basic solutions θ_j. The formulas (6.20) for $d=2$ and (6.21) (for $d=3$) give representation of the solutions Θ_j to a Bessel type equations in the WKB bases H^{σ}. By Theorem 1 the same relations connect the solutions θ_j and $h^{\sigma}_{\text{princ}}$. But for us important is the coefficient before θ_d in the representation of the WKB solutions $h^{\sigma}_{\text{princ}}$ in the basis θ. We have the following result (where F^{σ} are defined in Definition 4).

Proposition 3. *(a) If $d = 2$ and $0 < t < 1$, $x > 0$ then we have*

$$h^+_{\mathrm{princ}} = -h^-_{\mathrm{princ}} = \frac{-1}{\sqrt{\pi}} \cdot \theta_2 \mod \theta_1.$$

This implies that

$$\varphi_1 = \frac{i}{2\pi} \left\{F^+ - F^-\right\} \cdot \theta_2 \mod \theta_1.$$

(b) If $d = 3$ and $0 < t < 1$, $x > 0$ then we have

$$h^-_{\mathrm{princ}} = 0 \cdot \theta_3, \quad h^\epsilon_{\mathrm{princ}} = -h^{\bar\epsilon}_{\mathrm{princ}} = \frac{-i}{4}\sqrt{\frac{3}{2\pi}} \cdot \theta_3 \mod(\theta_1, \theta_2).$$

This implies that

$$\varphi_1 = \frac{i}{(2\pi)^{3/2}} \left\{F^{\bar\epsilon} - F^\epsilon\right\} \cdot \theta_3 \mod(\theta_1, \theta_2).$$

In other sectors the relations are different than in item (b), but always we have something like $h^\sigma_{\mathrm{princ}} = \mathrm{const}\cdot\frac{i}{4}\sqrt{\frac{3}{2\pi}} \cdot \theta_3$, where the constant is either 0 or 1 or -1 (see the next section).

6.3.2. *Gaussian type integrals for $d = 2$.* In the case $d = 2$ in [ZZ1] we continued further the stationary phase expansion. We have $Q = Q^\pm e^{iq}$ (as above). We put $q = A/(u\sqrt{x_\pm})$, $x_\pm = \pm ix$, and we expand $ix\Delta_\pm\phi := ix(\phi - \phi_\pm)$ in powers of $x_\pm^{-1/2}$. We get

$$ix\Delta_\pm\phi = \pm ix_\pm \ln(1 \mp iu^2 \left(e^{iA/u\sqrt{x_\pm}} - 1\right)) \mp ix_\pm \ln\left(1 \mp iu^2\left(e^{-iA/u\sqrt{x_\pm}} - 1\right)\right).$$

The $x^0_\pm$-term of this expression equals $-A^2$ and other terms, denoted by $\Omega(A)$, can be grouped as follows:

$$x_\pm u^2 \left[\sum_{m\geq 0, n\geq 2} c_{m,n} u^{4m} \left(\frac{A^2}{u^2 x_\pm}\right)^n\right] + (\pm i\sqrt{x_\pm}u^3 A)\left[\sum_{m\geq 0, n\geq 1} d_{m,n} u^{4m}\left(\frac{A^2}{u^2 x_\pm}\right)^n\right]$$

for some real coefficients $c_{m,n}$ and $d_{m,n}$ (which do not depend on the sign $\pm$). We get an integral of the form $\frac{1}{2\pi u\sqrt{x_\pm}}\int e^{-A^2} \times e^\Omega dA$, where $e^{\Omega(A)}$ is expanded in powers of A and integrated. By analogy with the Gaussian integrals we can assume that

$$\langle A^n\rangle := \frac{1}{\sqrt{\pi}}\int e^{-A^2} A^n dA = (n-1)!! \cdot \left(\frac{1}{2}\right)^{n/2}$$

if n is even and zero otherwise. Our computations lead to the following properties of the basic solutions to the hypergeometric equation.

Lemma 5. *(a) We have*

$$\varphi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}} K_{\text{princ}}(x^{-2}) \left(g^+_{\text{test}} + g^-_{\text{test}} \right),$$

where $K_{\text{princ}}(x^{-2})$ is a formal series with real coefficients such that $K_{\text{princ}}(x^{-2}) = 1 + \frac{5}{256}x^{-2} + \ldots \not\equiv 1$ (compare Eq. (4.19)).

(b) We have

$$\widehat{\varphi}_2 \sim \frac{\sqrt{\pi}}{2i} \left\{ D_+(x^{-1}) g^+_{\text{test}} - D_-(x^{-1}) g^-_{\text{test}} \right\},$$

where $\widehat{\varphi}_2$ is defined in Remark 1 and $D_\pm(x^{-1})$ are formal series satisfying

$$D_+(x^{-1}) + D_-(x^{-1}) = 2K_{\text{princ}}(x^{-2}).$$

First proof of formula (1.8). By Remark 1, Proposition 2 and Lemma 5 we have

$$\theta_1(s) = -\frac{K_{\text{princ}}}{2\sqrt{\pi}} \left\{ s\partial_s g^+_{\text{test}} + s\partial_s g^-_{\text{test}} \right\}$$

and a second solution can be taken in the form

$$\widehat{\theta}_2(s) = -s\partial_s \hat{\varphi}_2 \sim -\frac{\sqrt{\pi}}{2i} \left\{ D_+ s\partial_s g^+_{\text{test}} - D_- s\partial_s g^-_{\text{test}} \right\}.$$

Since $\widehat{\varphi}_2 = \varphi_2 + \text{const}\cdot\varphi_1$, also $\widehat{\theta}_2 = \theta_2 + \text{const}\cdot\theta_1$, and hence Eq. (2.9) gives $\widehat{\theta}_2(0) = \theta_2(0) = -1$.

For the WKB functions $g^\pm_{\text{test}}$ we find the identity (see [ZZ1])

$$s\partial_s g^\pm_{\text{test}}(s) = xe^{\pm i\pi x} g^\mp_{\text{test}}(t) = \mp i h^\mp_{\text{test}}(s), \quad t = 1-s,$$

where $\pi = S_2(1)$. This, together with the results of the previous, yields the following:

$$\begin{aligned} \theta_1(s) &\sim -x\frac{K_{\text{princ}}}{2\sqrt{\pi}} \{ e^{i\pi x} g^-_{\text{test}}(t) + e^{-i\pi x} g^+_{\text{test}}(t) \}, \\ \widehat{\theta}_2(s) &\sim x\frac{\sqrt{\pi}}{2i} \{ -D_+ e^{i\pi x} g^-_{\text{test}}(t) + D_- e^{-i\pi x} g^+_{\text{test}}(t) \}. \end{aligned} \tag{6.22}$$

It implies that the formula

$$\varphi_1(t) = -\frac{2K_{\text{princ}}}{D_+ + D_-} \frac{\sin \pi x}{\pi x} \cdot \widehat{\theta}_2(s) \mod \theta_1,$$

This and the equalities $\hat{\theta}_2(0) = -1$, $D_+ + D_- = 2K_{\text{princ}}$ (see Lemma 5(b)) imply the formula $f_2(x) = -A_2(x) = \sin \pi x / \pi x$.

Finally, we note that Eq. (6.22) implies the equality $K^\pm_{\text{princ}} = L^\pm_{\text{princ}}$ and hence $F^\pm = e^{\pm ix}/x$ (see Definition 4). Then the formula $\varphi_1 = -\frac{\sin \pi x}{\pi x} \cdot \theta_2 \mod \theta_1$ follows also from Proposition 3 (but it needs the analysis from Section 7). ∎

7. The Stokes phenomenon

The Stokes phenomenon is related with 'jumps' of constants in the asymptotic expansions of solutions of linear meromorphic differential equations near irregular critical point. Here we define the Stokes operators as acting on the basic WKB solutions. For precise informations about Stokes operators (in the case of a linear equation near an irregular singularity) we refer the reader to [Was], [Zo3] and to [ZZ2], where the Stokes phenomenon for the genuine WKB solutions of equations with large parameter is discussed.

The Stokes phenomenon [St] is related with normalization of a linear system $\dot{z} = A(t)z$ in a neighborhood of an irregular singular point, say at $t = 0$. The neighborhood of $t = 0$ is divided into sectors $\mathcal{S}_j$, such that there exist changes $z = \mathcal{B}_j(t)y$ holomorphic with respect to $t \in \mathcal{S}_j$ which lead to a diagonal system $\dot{y} = \mathrm{diag}(d_1(t), \ldots, d_n(t))y$. But the matrix-valued functions $\mathcal{B}_j$ are different in different sectors. The difference between $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$ is measured via so-called Stokes matrices (see [Zo3]).

In the context of WKB solutions, e.g. for $t \in (0,1)$ and large parameter x, usually the Stokes matrices are related with solutions near one of the endpoints of the time interval, $t = 0$ or $t = 1$ (see [He]). One would like to define analogues of the Stokes operators for the WKB solutions, but when the time $t \in (0,1)$ is real and the large parameter x varies in some sectors near $x = \infty$, i.e. in $(\mathbb{C}, \infty)$. However, a rather detailed analysis performed in [ZZ2] demonstrates that it is not possible to do this in uniform way with respect to t. Moreover, calculations of the Stokes operators associated with the third order hypergeometric equation $(1.1)_{d=3}$ demonstrate that the Stokes operators at the two endpoints of the interval $(0,1)$ are essentially different.

When studying the Stokes phenomenon in [He] and [Fed] greater attention is focused on analytic properties of the WKB solutions with respect to the time t, while the parameter $x \approx +\infty$ is usually real. The so called Stokes lines are drawn in the complex $t-$plane near the 'turning points' points $t = 0$ and $t = 1$. In this section we focus our attention on the parameter x, which will vary in whole sectors near infinity, and the time t will vary in a small neighborhood of the interval $(\beta, 1-\beta) \subset \mathbb{C}$ (like in Section 4.3).

Below we firstly calculate the Stokes operators for the Bessel type equations $(5.14)_{d=2,3}$ and $(5.15)_{d=2,3}$, i.e. in the WKB bases $\widetilde{G}^\sigma$ and $\widetilde{H}^\sigma$ in Eqs. (5.13). We use essentially two methods: one from the book of J. Heading [He] and using perturbation of the Bessel type equations to equations with regular singularities and then considering corresponding monodromy matrices. An alternative approach is to use results of the paper [DuMi] which imply that the principal Stokes matrix differs from the identity only at one place.

It is worth to underline the fact that the Heading's method is sufficient only in the case $d = 2$. In the case $d \geq 3$ it is insufficient.

Finally, in the second part of this section, we apply the results about the Bessel type equations to analysis of the Stokes phenomenon for the principal WKB solutions $g^{\sigma}_{\rm princ}$ and $h^{\sigma}_{\rm princ}$ the hypergeometric equation (1.1). We show that the connection coefficient $A_d(x)$ from Lemma 3.2 is a sum of WKB type the formal summands F^{σ}, they are subject to Stokes phenomenon which is trivial in the case $d=2$ and nontrivial in the case $d=3$.

7.1. **Stokes operators for the Bessel type equations.**

7.1.1. *The case* $d=2$. We begin with Eq. $(5.14)_{d=2}$. By a sectorial normalization theorem the solutions $\widetilde{G}^{\pm}(v)$ from Eq. $(5.13)_{d=2}$ represent asymptotic series for solutions $\widetilde{G}^{\pm}_{r,l}(v)$ which are analytic in some sectors about $v=\infty$ (in the complex $v-$plane).

There are two such sectors: $\mathcal{S}_r$ (right) and $\mathcal{S}_l$ (left) with vertex at ∞ of angle $2\pi-2\delta$ ($\delta>0$ and small) and with the rays $\arg v=0$ and $\arg v=\pi$ as their bisectrices. The latter rays are called the **rays of division**. Then the sectors $\mathcal{S}_u=\mathcal{S}_r\cap\mathcal{S}_l\cap\{\operatorname{Im} v>0\}$, and $\mathcal{S}_d=\mathcal{S}_r\cap\mathcal{S}_l\cap\{\operatorname{Im} v<0\}$ have angle $\pi-2\delta$. The sectors $\mathcal{S}_u$ and $\mathcal{S}_d$ are 'transitional' sectors; their bisectrices are called the **Stokes lines**. $\widetilde{G}^{\pm}_r$ and $\widetilde{G}^{\pm}_l$ are the corresponding solutions in the sectors $\mathcal{S}_r$ and $\mathcal{S}_l$ respectively obtained from the sectorial normalization theorem.

We note the following relations (where $f\prec h$ means that the function f is much smaller than the functions h) :

$$\widetilde{G}^{+}_{r,l}\prec\widetilde{G}^{-}_{r,l}\ \text{in}\ \mathcal{S}_u,\qquad \widetilde{G}^{-}_{r,l}\prec\widetilde{G}^{+}_{r,l}\ \text{in}\ \mathcal{S}_d. \tag{7.1}$$

The solutions $\widetilde{G}^{\pm}_r$ (respectively $\widetilde{G}^{\pm}_l$) are analytic in the adjacent sectors $\mathcal{S}_u$ (up) and $\mathcal{S}_d$ (down). Therefore they are expressed as linear linear combinations of the corresponding solutions $\widetilde{G}^{\pm}_l$ (respectively $\widetilde{G}^{\pm}_r$). The corresponding matrices C_u and C_d of changes between the basic solutions are called the **Stokes matrices**.

Each Stokes matrix is triangular with 1 on the diagonal. We have

$$C_u=\begin{pmatrix}1 & c_{12}\\ 0 & 1\end{pmatrix},\quad C_d=\begin{pmatrix}1 & 0\\ c_{21} & 1\end{pmatrix}. \tag{7.2}$$

This means that, after passing from the sector $\mathcal{S}_r$ to the sector $\mathcal{S}_l$, the basic solutions undergo the following changes:

$$\widetilde{G}^{+}_r=\widetilde{G}^{+}_l,\quad \widetilde{G}^{-}_r=\widetilde{G}^{-}_l+c_{12}\widetilde{G}^{+}_l\quad (\text{in}\quad \mathcal{S}_u), \tag{7.3}$$

$$\widetilde{G}^{+}_l=\widetilde{G}^{+}_r+c_{21}\widetilde{G}^{-}_l,\quad \widetilde{G}^{-}_l=\widetilde{G}^{-}_r\quad (\text{in}\quad \mathcal{S}_d). \tag{7.4}$$

The rule is that to a given solution one can add a solution with smaller asymptotic at infinity. We shall calculate the coefficients c_{12} and c_{21} using the method from [He], where Stokes matrices associated with the Bessel equation were computed (see also [Zo3]).

We note also the following *symmetry property*:

$$\widetilde{G}_l^+(e^{i\pi}v) = -\widetilde{G}_r^-(v), \quad \widetilde{G}_l^-(e^{i\pi}v) = \widetilde{G}_r^+(v), \quad v > 0. \tag{7.5}$$

Let $\widetilde{G}_r^+(v)$ on the ray $\arg v = 0$ (in the sector $\mathcal{S}_r$) be represented by the following combination of the basic solutions $\widetilde{\Phi}_1(v) = \Phi_1(v^2)$, $\widetilde{\Phi}_2(v) = \Phi_2(v^2) = \widetilde{\Phi}_1 \ln v^2 + \widetilde{\Psi}_2(v^2)$:

$$\widetilde{G}_r^+(v) = K_1\widetilde{\Phi}_1(v) + K_2\widetilde{\Phi}_2(v), \quad v > 0, \tag{7.6}$$

for some coefficients K_1 and K_2. After passing to the ray $\arg v = \pi$ (in $\mathcal{S}_l$) and the substitution $v \to -v$ (using Eqs. (7.5) and the logarithmic singularity of $\widetilde{\Phi}_2$) we get

$$-\widetilde{G}_r^-(v) = (K_1 + 2\pi i K_2)\widetilde{\Phi}_1(v) + K_2\widetilde{\Phi}_2(v), \quad v > 0. \tag{7.7}$$

Analogously, after passing to the ray $\arg x = 2\pi$ and using an analogue of the relations (7.5), we get

$$-\widetilde{G}_r^+(v) - c_{21}\widetilde{G}_r^-(v) = (K_1 + 4\pi i K_2)\widetilde{\Phi}_1(v) + K_2\widetilde{\Phi}_2(v), \quad v > 0. \tag{7.8}$$

Eqs. (7.6)–(7.8) imply the representation (on $\arg v = 0$)

$$\widetilde{\Phi}_1(v) = \frac{i}{2\pi K_2}(\widetilde{G}_r^+ + \widetilde{G}_r^-), \quad \widetilde{\Phi}_2(v) = \left(\frac{1}{K_2} - \frac{iK_1}{2\pi K_2^2}\right)\widetilde{G}_r^+ - \frac{iK_1}{2\pi K_2^2}\widetilde{G}_r^-,$$

and that

$$c_{21} = 2.$$

Moreover, the asymptotic formula (6.18) implies that $K_2 = i/\sqrt{\pi}$.

In the same way one proves that $c_{12} = -2$ and obtains the representation

$$\widetilde{\Phi}_1(v) = \frac{1}{2\sqrt{\pi}}(\widetilde{G}_l^- - \widetilde{G}_l^+), \quad \arg v = \pi.$$

Calculation of the Stokes matrices associated with the Bessel type equation $(5.15)_{d=2}$ runs practically in the same way as above. The formal WKB solutions

$$\widetilde{H}^\pm(w) = \sqrt{-w_\pm}e^{-2w_\pm}\left\{1 + \frac{b_1}{w_\pm} + \frac{b_2}{w_\pm^2} - \dots\right\}, \quad w_\pm = \pm iw.$$

satisfy the Bessel type equation $(5.15)_{d=2}$ with another pair of solutions

$$\widetilde{\Theta}_1(w) = w - \frac{1}{2}w^2 + \dots, \quad \widetilde{\Theta}_2(w) = \widetilde{\Theta}_1(w)\cdot\ln w + \widetilde{\Xi}_3(w) \tag{7.9}$$

(with analytic $\widetilde{\Theta}_1$ and $\widetilde{\Xi}_3$).

Now we have the same sectors $\mathcal{S}_{r,l}$, with analytic solutions $\widetilde{H}_{r,l}^\pm$, and $\mathcal{S}_{u,d}$ about $w = \infty$, but with domination relations different than in Eq. (7.1). Therefore the corresponding Stokes matrices take the following form

$$D_u = \begin{pmatrix} 1 & 0 \\ d_{21} & 1 \end{pmatrix}, \quad D_d = \begin{pmatrix} 1 & d_{12} \\ 0 & 1 \end{pmatrix}. \tag{7.10}$$

Anyway (using also Eqs. (6.20)) we arrive to the following result, where Eq. (7.17) is a consequence of the factor $\sqrt{-w_\pm}$ in definition of $\widetilde{H}^\pm$: we have $\widetilde{H}_l^\pm(e^{2\pi i}w) = -\widetilde{H}_l^\pm(w)$.

We summarize this in the following

Proposition 4. *(a) We have $c_{12} = -2$ and $c_{21} = 2$ in Eqs (7.2). Moreover,*

$$\widetilde{\Phi}_1(v) = \frac{1}{2\sqrt{\pi}}\left(\widetilde{G}_r^+ + \widetilde{G}_r^-\right), \quad \widetilde{\Phi}_2 = -i\sqrt{\pi}\cdot\widetilde{G}_r^+ \mod \widetilde{\Phi}_1, \quad \arg v = 0; \tag{7.11}$$

$$\widetilde{\Phi}_1(v) = \frac{1}{2\sqrt{\pi}}\left(\widetilde{G}_l^- - \widetilde{G}_l^+\right), \quad \widetilde{\Phi}_2 = -i\sqrt{\pi}\cdot\widetilde{G}_l^+ \mod \widetilde{\Phi}_1, \quad \arg v = \pi. \tag{7.12}$$

(b) We have $d_{12} = -2$ and $d_{21} = 2$ in Eqs (7.10). Moreover,

$$\widetilde{\Theta}_1 = \frac{1}{2\sqrt{\pi}}\left(\widetilde{H}_r + \widetilde{H}_r^-\right), \quad \widetilde{\Theta}_2 = -i\sqrt{\pi}\cdot\widetilde{H}_r^+ \mod \widetilde{\Theta}_1, \quad \arg w = 0; \tag{7.13}$$

$$\widetilde{\Theta}_1 = \frac{1}{2\sqrt{\pi}}\left(\widetilde{H}_l^- - \widetilde{H}_l^+\right), \quad \widetilde{\Theta}_2 = i\sqrt{\pi}\cdot\widetilde{H}_l^- \mod \widetilde{\Theta}_1, \quad \arg w = \pi. \tag{7.14}$$

In particular, we get

$$\widetilde{H}_r^+(w) = -\widetilde{H}_r^- = \left(i/\sqrt{\pi}\right)\cdot\widetilde{\Theta}_2 \mod \widetilde{\Theta}_1, \quad \arg w = 0; \tag{7.15}$$

$$\widetilde{H}_l^+(w) = \widetilde{H}_l^- = \left(-i/\sqrt{\pi}\right)\cdot\widetilde{\Theta}_2 \mod \widetilde{\Theta}_1, \quad \arg w = \pi; \tag{7.16}$$

$$\widetilde{H}_l^+(w) = -\widetilde{H}_l^- = \left(i/\sqrt{\pi}\right)\cdot\widetilde{\Theta}_2 \mod \widetilde{\Theta}_1, \quad \arg w = -\pi. \tag{7.17}$$

Above we give the representation of the function $\widetilde{\Phi}_1(v)$ for v on the two rays of division. But, in fact, these formulas hold true in the whole sectors $\mathcal{S}_{r,l}$ which contains the corresponding ray of division. The same remark applies in other expansions.

7.1.2. *The case $d = 3$.* Eq. $(5.14)_{d=3}$ has the following independent solutions

$$\widetilde{\Phi}_1(v) = \Phi_1(v^3), \quad \widetilde{\Phi}_2(v) = \widetilde{\Phi}_1\ln v^3 + \widetilde{\Psi}_2(v), \quad \widetilde{\Phi}_3 = \frac{1}{2}\Phi_1\ln^2(v^3) + \widetilde{\Psi}_2\ln v^3 + \widetilde{\Psi}_3,$$

where $\widetilde{\Phi}_1$, $\widetilde{\Psi}_2$ and $\widetilde{\Psi}_3$ are entire functions and depend on v^3. We have also the system $\widetilde{G}_j^\sigma$ of WKB type solutions defined in the sectors $\mathcal{S}_j$ about $v = \infty$ (see Eq. (5.16) and Figure 1 below).

The **rays of division** $\mathcal{R}_j$ (or the *anti-Stokes lines*) are given by $\arg v = 0$, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, i.e. they are the bisectrices of the sectors $\mathcal{S}_j$. Then the sectors $\mathcal{S}_{12} = \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathcal{S}_{23}$, $\mathcal{S}_{34}$, $\mathcal{S}_{45}$, $\mathcal{S}_{56}$, $\mathcal{S}_{61}$ have angle $\pi/3 - \delta$ (see Figure 1); their bisectrices are known as the **Stokes lines**. The corresponding Stokes matrices C_{ji} are the matrices of changes between the basic solutions $\left\{\widetilde{G}_i^\sigma\right\}$ and $\left\{\widetilde{G}_j^\sigma\right\}$ in the sectors $\mathcal{S}_i$ and $\mathcal{S}_j$.

Each matrix C_{ji}, after suitable ordering of the basic solutions, becomes upper triangular with 1's on the diagonal. For example, in the sector $\mathcal{S}_{12}$ we have

$$\widetilde{G}_j^- \prec \widetilde{G}_j^{\epsilon} \prec \widetilde{G}_j^{\bar{\epsilon}}, \quad j = 1, 2.$$

The Stokes matrix associated with the sector $\mathcal{S}_{12}$ equals

$$C_{21} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.18}$$

where the parameters a, b, c are to be determined.

Other Stokes matrices can be obtained from the matrix C_{21} using the fact that Eq. (5.16) is invariant with respect to:

— the rotation $v \to \epsilon^2 v$ (where $\epsilon = e^{i\pi/3}$),

— the complex conjugation $v \to \bar{v}$.

Formally the rotation $\epsilon^2 v$ is reflected in the cyclic permutation of solutions, $\widetilde{G}_{j+2}^{\sigma}(\epsilon^2 v) = \widetilde{G}_j^{\epsilon^2\sigma}(v)$. The double rotation results in the change $\widetilde{G}_{j+4}^{\sigma}(\epsilon^4 v) = \widetilde{G}_j^{\epsilon^4\sigma}(v)$. The complex conjugation induces the change $\widetilde{G}_j^{\sigma}(v) = \widetilde{G}_{7-j}^{\bar{\sigma}}(\bar{v})$; but here also the orientation of the $v-$plane is reversed. Compare also Eqs. (4.27)–(4.28).

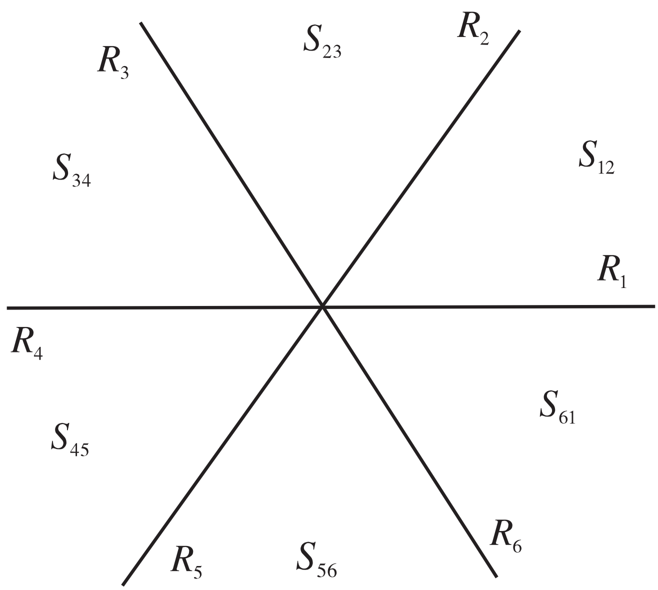

Figure 1. Rays of division

Therefore the Stokes matrices C_{43} and C_{65} are obtained from C_{21} by application of conjugation with suitable permutation matrices. The matrix C_{16} is obtained from C_{21} by: complex conjugation, taking the inverse and conjugation with the permutation $(1)(23)$. The matrices C_{32} and C_{54} are obtained from the matrix C_{16} by permutations.

In the calculation of the Stokes matrix C_{21} we follow the Heading method described in the previous section for the case $d=2$. We represent the function $\widetilde{G}^{-}(v)$ in the ray $\mathcal{R}_1 = \{\arg v = 0\}$ in the basis $\left\{\widetilde{\Phi}_j\right\}$,

$$\widetilde{G}_1^{-} = K_1\widetilde{\Phi}_1 + K_2\widetilde{\Phi}_2 + K_3\widetilde{\Phi}_3$$

(with coefficients K_j), and we pass to the rays $\mathcal{R}_3$, $\mathcal{R}_5$ and $\mathcal{R}_1$, using actions of the matrices $C_{31} = C_{32}C_{21}$, $C_{53} = C_{54}C_{43}$ and $C_{15} = C_{16}C_{65}$ and substitutions $\epsilon^2 v$, $\epsilon^4 v$ and $\epsilon^6 v$ in the argument. We arrive at the following relation

$$b = 3 + \bar{a} + \bar{c}, \tag{7.19}$$

but the parameters a and c are not determined.

We repeat the same analysis, but starting from the ray $\mathcal{R}_6 = \{\arg v = -\pi/3\}$ and use the matrices $C_{26} = C_{21}C_{16}$, C_{42} and C_{64}. Again we get relation (7.19).

In order to calculate the constants a and c we use the known property (see [Gl] or [Zo1]) that Stokes operators are limits of monodromy operators of a perturbed equation which has regular singularities.

An obvious perturbation of Eq. (5.3) is the our initial hypergeometric equation, i.e. $(1 - yx^{-3})\partial_y y \partial_y y \partial_y G + G = 0$, and the corresponding perturbation of Eq. (5.14) is

$$\left(1 - (v/x)^3\right)\partial_v v \partial_v v \partial_v \widetilde{G} + 27v^2\widetilde{G} = 0. \tag{7.20}$$

Together with perturbation (7.20) we shall consider the following one:

$$\left(1 + (v/x)^3\right)\partial_v v \partial_v v \partial_v \widetilde{G} + 27v^2\widetilde{G} = 0, \tag{7.21}$$

i.e. with change of the sign before $(v/x)^3$.

Eq. (7.20) has three additional singular points $v_1 = x, \quad v_2 = \epsilon^2 x, \quad v_3 = \epsilon^{-2}x$ which tend to infinity as $x \to \infty$ and where we assume that x is real positive. The latter singular points lie in the division rays $\mathcal{R}_1$, $\mathcal{R}_3$ and $\mathcal{R}_5$ and the monodromy matrices M_1, M_2 and M_3 (in some basis of solutions) defined by prolongation of solutions along curves around these points (in the clockwise direction) should tend (as $x \to \infty$) to matrices equivalent to C_{26}^{-1}, C_{42}^{-1} and C_{64}^{-1} respectively.

On the other hand, each monodromy matrix M_j, $j = 1,2,3$, is equivalent to some monodromy matrix $\mathcal{M}_1$ related with the hypergeometric equation $(1.1)_{d=3}$ and corresponding to the singular point $t=1$. Since the basic solutions of the latter equation near $s = 1 - t = 0$ are $s + \ldots$, $s^2 + \ldots$, and $(s^2 + \ldots)\ln x^3 s + \alpha + \ldots$ the corresponding monodromy matrix $\mathcal{M}_1$ has all eigenvalues equal to 1 and its Jordan decomposition consists of two cells; anyway, the characteristic polynomial is $P(\lambda) = \det(\mathcal{M}_1 - \lambda) = (1-\lambda)^3$. Looking at the matrix C_{26} in [ZZ3] one finds that its characteristic polynomial is $\left(1-\lambda\right)(\lambda^2-\left(2-|c|^2\right)\lambda+1\Big)$. It follows that $c=0$.

Equation (7.21) is related with the modified hypergeometric equation $(1+t)\partial t\partial t\partial g+x^3g=0$, where one checks that the basic solutions near $s=1+t=0$ are $s+\ldots$, $s^2+\ldots$ and $\left(s^2+\ldots\right)\ln s+\ldots$. Here also the corresponding monodromy matrix has eigenvalues 1 and two Jordan cells. On the other hand, the monodromy matrices related with the singular points $v=-x,\epsilon x,\bar{\epsilon}x$ of equation (7.21) tend to the matrices C_{53}^{-1}, C_{31}^{-1}, C_{16}^{-1}. The same arguments as above show that $a=0$.

From the above we get the following result.

Proposition 5. *The principal Stokes matrix associated with the WKB bases* $\left(\widetilde{G}_1^\sigma\right)$ *and* $\left(\widetilde{G}_2^\sigma\right)$, $\sigma=-1,\epsilon,\bar{\epsilon}$, *takes the form*

$$C_{21}=\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}.$$

Moreover we have the following representations:

$$\begin{aligned}\widetilde{\Phi}_1 &= \tfrac{1}{\pi\sqrt{3}}(\widetilde{G}_1^\epsilon+\widetilde{G}_1^{\bar{\epsilon}}-2\widetilde{G}_1^-),\\ \widetilde{\Phi}_2 &= \tfrac{i}{\sqrt{3}}(\widetilde{G}_1^{\bar{\epsilon}}-\widetilde{G}_1^\epsilon) \mod \widetilde{\Phi}_1,\\ \widetilde{\Phi}_3 &= -\tfrac{4\pi}{\sqrt{3}}\widetilde{G}_1^- \mod \left(\widetilde{\Phi}_1,\widetilde{\Phi}_2\right)\end{aligned}\tag{7.22}$$

(for $v\in\mathcal{R}_1$). Analogous representations hold in other rays of division:

$$\begin{aligned}\widetilde{\Phi}_1 &= \tfrac{1}{\pi\sqrt{3}}(\widetilde{G}_j^-+\widetilde{G}_j^\epsilon+\widetilde{G}_j^{\bar{\epsilon}}),\ v\in\mathcal{R}_j,\ j=2,4,6,\\ &= \tfrac{1}{\pi\sqrt{3}}(\widetilde{G}_j^-+\widetilde{G}_j^\epsilon+\widetilde{G}_j^{\bar{\epsilon}}-3\widetilde{G}_j^*),\ v\in\mathcal{R}_j,\ (j,*)=(1,-),(3,\epsilon),(5,\bar{\epsilon}).\end{aligned}\tag{7.23}$$

Note that in the ray $\mathcal{R}_1$ two dominating WKB solutions $\widetilde{G}^\epsilon$ and $\widetilde{G}^{\bar{\epsilon}}$ are of the the same order. So the coefficients between them in Eq. (7.22) are determined by the asymptotic of the oscillating integral (via the stationary phase formula). The coefficients before $\widetilde{G}^\epsilon$ and $\widetilde{G}^{\bar{\epsilon}}$ in Eq. (7.22) agree with Proposition 2, but the coefficient before $\widetilde{G}^-$ is different.

From the proof of Proposition 5 it is seen that using only the method from the Heading's book [He] we are not able to compute all the Stokes matrices, we obtain only one relation (7.19). On the other hand, only the knowledge of the Jordan decomposition of the composed Stokes matrices, like C_{31}, does not allow to obtain relation (7.19). Therefore the both methods should be used. Probably this fact is true in more general high order linear meromorphic ODE's.

Of course, the relative simplicity of the principal Stokes matrix can be explained by the fact that the domains of analyticity of the functions $\widetilde{G}_j^\sigma$ are larger than the sectors $\mathcal{S}_j$ (compare Section 5.2).

As we have mentioned, the Stokes matrices associated with the WKB solutions G^{σ}_{DM} from Remark 3 were calculated by A. Duval and C. Mitschi [DuMi]. Their calculations rely upon properties of the Mellin–Barnes integrals proved by C. Meijer [Me]. Anyway, their result completely agrees with ours.[8]

The analysis leading to Stokes operators associated with formal WKB solutions $\widetilde{H}^{\sigma}(w) \sim \sqrt{-\sigma w}e^{-3\sigma w/2}$ (see Eq. (5.11)) which are asymptotic series for analytic WKB solutions $\widetilde{H}^{\sigma}_j$ defined in sectors $\mathcal{S}_j$ about $w = \infty$ (see Eq. (5.17)) leads to the following result. Below the constants

$$L_1 = \sqrt{3}/2 \ \text{ and } \ L_3 = (-i/4)\sqrt{3/2\pi}$$

appear in the representation

$$H_4^- = L_1\widetilde{\Theta}_1 + L_2\widetilde{\Theta}_2 + L_3\widetilde{\Theta}_3, \quad w \in \mathcal{R}_4,$$

and are taken from Eq. (6.21).

Proposition 6. *The principal Stokes matrix associated with the WKB bases* $\left(\widetilde{H}^{\sigma}_4\right)$ *and* $\left(\widetilde{H}^{\sigma}_5\right)$, $\sigma = -1, \epsilon, \bar{\epsilon}$, *takes the form*

$$C_{54} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover we have the following representation:

$$\begin{aligned} 4L_1\widetilde{\Theta}_1 &= 2\widetilde{H}_4^- - \widetilde{H}_4^{\epsilon} - \widetilde{H}_4^{\bar{\epsilon}}, \\ 4\pi i L_3\widetilde{\Theta}_2 &= -\widetilde{H}_4^{\epsilon} + \widetilde{H}_4^{\bar{\epsilon}}, \\ 4L_3\widetilde{\Theta}_3 &= 2(\widetilde{H}_4^- + \widetilde{H}_4^{\epsilon}) \qquad \bmod \quad \widetilde{\Theta}_2 \end{aligned} \tag{7.24}$$

for $w \in \mathcal{R}_4$ *and* $0 < t < 1$. *The representations in other rays* $\mathcal{R}_j$ *(and* $0 < t < 1$*) are presented in* [ZZ3, Prop. 5.5]. *This implies the following relations* mod $\left(\widetilde{\Theta}_1, \widetilde{\Theta}_2\right)$:

$$\begin{gathered} \widetilde{H}_1^- = 0, \ \ -\widetilde{H}_1^{\epsilon} = \widetilde{H}_1^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_1 \\ \widetilde{H}_2^- = -\widetilde{H}_2^{\epsilon} = \widetilde{H}_2^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_2 \\ \widetilde{H}_3^{\epsilon} = 0, \ \ \widetilde{H}_3^- = \widetilde{H}_3^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_3 \\ \widetilde{H}_4^- = \widetilde{H}_4^{\epsilon} = \widetilde{H}_4^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_4 \\ \widetilde{H}_5^{\bar{\epsilon}} = 0, \ \ \widetilde{H}_5^- = \widetilde{H}_5^{\epsilon} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_5 \\ \widetilde{H}_6^- = \widetilde{H}_6^{\epsilon} = -\widetilde{H}_6^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_6 \\ \widetilde{H}_1^- = 0, \ \ \widetilde{H}_1^{\epsilon} = -\widetilde{H}_1^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \ w \in \mathcal{R}_1. \end{gathered} \tag{7.25}$$

[8]In the sequent paper [Mit] Mitschi applied the results of [DuMi] to compute the differential Galois groups of some confluent hypergeometric equations. Previously these groups were calculated in algebro–geometrical way (which avoids calculation of the Stokes constants) by N. Katz [Ka1] and [Ka2]; the method of Katz was initiated in the paper [BBH].

Note that for $z > 0$, i.e. $w > 0$, the value of $\widetilde{\Theta}_3 \bmod \widetilde{\Theta}_2$ agrees with Eq. (6.21), which was obtained by calculation of corresponding mountain pass integrals.

Note also the difference between the data of the latter tables for the ray $\mathcal{R}_1$ (in the first and in the last row in Eq. (7.25)). It corresponds to the turning $w \longmapsto e^{2\pi i} w$. Here $\widetilde{\Theta}_1$ changes to $-\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$ is unchanged, $\widetilde{\Theta}_3$ acquires a term proportional to $\widetilde{\Theta}_2$ and $\widetilde{H}^\sigma$ change to $-\widetilde{H}^\sigma$; all is OK.

7.2. **Stokes operators for the hypergeometric equation.** We deal with formal WKB solutions for the hypergeometric equation as well as for the corresponding Bessel type equations. By results of Section 5.2 the reductions to the normal (diagonal) form for associated with them systems are compatible. Recall that these formal solutions are of Gevrey type and in suitable domains are represented by analytic functions, but the above analytic constructions are not quite compatible. In the other hand, the analytic equivalences with corresponding Bessel type equations (using the matrices $\mathcal{H}_0$ and $\mathcal{H}_1$ in Section 5.3) imply compatibility of analytic and of formal solutions.

So, in order to avoid technicalities, we limit ourselves to the formal case. This is the way chosen in [ZZ3] for $d = 3$. In [ZZ1] the case $d = 2$ is done with complete details.

7.2.1. *The case* $d = 2$. Let $0 < t < 1$. Using Theorem 1 and Definition 3 we can replace in Proposition 4 $\widetilde{\Phi}_j$ and $\widetilde{\Theta_j}$ with φ_j and θ_j and the WKB solutions $\widetilde{G}_j^\pm$and $\widetilde{H}_j^\pm$ with $g_{\text{princ}}^\pm$ and $h_{\text{princ}}^\pm$. Therefore, for $\arg x = 0$, we have

$$\begin{aligned} \varphi_1 &= \frac{1}{2\sqrt{\pi}} \left\{ g_{\text{princ}}^+ + g_{\text{princ}}^- \right\} \\ &= \frac{1}{2\sqrt{\pi}} \left\{ F^+(x) h_{\text{princ}}^+ + F^-(x) h_{\text{princ}}^- \right\} \\ &= \frac{1}{2\sqrt{\pi}} \cdot \frac{i}{\sqrt{\pi}} \left\{ F^+ - F^- \right\} \cdot \theta_2 \mod \theta_1 . \end{aligned}$$

For $\arg x = \pi$ we have

$$\begin{aligned} \varphi_1 &= \frac{1}{2\sqrt{\pi}} \left\{ g_{\text{princ}}^- - g_{\text{princ}}^+ \right\} \\ &= \frac{1}{2\sqrt{\pi}} \cdot \frac{-i}{\sqrt{\pi}} \left\{ F^- - F^+ \right\} \cdot \theta_2 \mod \theta_1 . \end{aligned}$$

Here $F^\pm(x) = \frac{1}{x} e^{\pm i x \pi} \omega^\pm(1/x)$, $\omega^\pm = 1 + O(1/x)$ are defined in Definition 4 (compare also Propositions 2 and 3). The above pattern repeats as $\arg x$ increases by 2π.

We arrive at the following.

Proposition 7. *The connection coefficient $A_2(x)$ from Lemma 2 equals*

$$A_2(x) = \frac{i}{2\pi}\left\{F^+(x) - F^-(x)\right\}, \quad x \to \infty,$$

where the functions $F^\pm(x)$ are single valued.

Second proof of the formula (1.8). We note that the function $f_2(x) = -A_2(x)$ vanishes at the points $x = \pm 1, \pm 2, \dots$. Since the function $\sin \pi x/x$ has simple zeroes at these points, we find that the function

$$f_2(x)/\left(\sin \pi x/x\right)$$

is entire on $\mathbb{C}$. By Proposition 7 it is bounded at infinity. Therefore it is a constant function equal $1/\pi$ (since $f(0) = 1$). ∎

7.2.2. *The case $d = 3$.* Here we follow the previous case with use of Propositions 5 and 6. For $0 < t < 1$, we have

$$\begin{aligned} \pi\sqrt{3}\varphi_1(t;x) = \quad & g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} - 2g^{-}_{\text{princ}}, && x \in \mathcal{R}_1,\\ & g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} + g^{-}_{\text{princ}}, && x \in \mathcal{R}_2,\\ & g^{-}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} - 2g^{\epsilon}_{\text{princ}}, && x \in \mathcal{R}_3,\\ & g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} + g^{-}_{\text{princ}}, && x \in \mathcal{R}_4,\\ & g^{-}_{\text{princ}} + g^{\epsilon}_{\text{princ}} - 2g^{\bar\epsilon}_{\text{princ}}, && x \in \mathcal{R}_5,\\ & g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} + g^{-}_{\text{princ}}, && x \in \mathcal{R}_6, \end{aligned}$$

where $g^\sigma_{\text{princ}} = F^\sigma h^\sigma_{\text{princ}}$.

We have also the following relation modulo (θ_1, θ_2) :

$$h^{-}_{\text{princ}} = 0, \quad h^{\bar\epsilon}_{\text{princ}} = -h^{\epsilon}_{\text{princ}} = L_3\theta_3, \quad x \in \mathcal{R}_1,$$

and other relations like in Eqs. (7.25), where $L_3 = -\frac{i}{8}\sqrt{3/2\pi}$.

This implies the following representations of the generating function $f_3(x) = -2A_3(x)$:

$$\begin{aligned} -i\,(2\pi)^{3/2} f_3(x) = \quad & F^{\bar\epsilon} - F^{\epsilon}, && x \in \mathcal{R}_1,\\ & F^{\bar\epsilon} - F^{\epsilon} - F^{-}, && x \in \mathcal{R}_2,\\ & F^{-} + F^{\bar\epsilon}, && x \in \mathcal{R}_3,\\ & F^{\epsilon} + F^{\bar\epsilon} + F^{-}, && x \in \mathcal{R}_4,\\ & F^{-} + F^{\epsilon}, && x \in \mathcal{R}_5,\\ & F^{-} + F^{\epsilon} - F^{\bar\epsilon}, && x \in \mathcal{R}_6, \end{aligned} \tag{7.26}$$

where $F^\sigma = \frac{\pm 1}{x^{3/2}} e^{2\pi\sigma x/\sqrt{3}} \omega^\sigma(x^{-1/2})$ are the WKB type functions from Definition 4.

Since $F^\sigma(x) = F^\sigma_\pm(x^{1/2})$ depend on $x^{1/2}$ (see Eq. 5.22)), table (7.26) should be continued in order to turn twice around $x = \infty$. The corresponding formulas are related with compositions of the changes from Eqs. (7.26) with the monodromy of the functions $F_{\pm}^{\sigma}$:

$$\mathcal{M}_{\infty}: F_{\pm}^{\sigma} \longmapsto -F_{\mp}^{\sigma}. \tag{7.27}$$

We also see that the functions $F_{\pm}^{\sigma}$ are subject to Stokes phenomenon with the principal Stokes matrix relating solutions at the rays $\mathcal{R}_1$ and $\mathcal{R}_2$ of the form

$$C_{21} = \begin{bmatrix} 1 & p & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p - q = 1. \tag{7.28}$$

We can state the fundamental result of the whole paper.

Theorem 2. *The collection $\left\{F_{\pm}^{\sigma}\right\}$ of WKB type functions is subject to the monodromy (7.27) around $x = \infty$ and the Stokes phenomenon with the constant principal matrix (7.28) (other Stokes matrices are obtained from this by applying the conjugation and rotation symmetries). The generating function $f_3(x)$, which is entire function of x, in each sector $\mathcal{S}_j$ near infinity is a linear combination with constant coefficients of the functions $F_{\pm}^{\sigma}$.*

Moreover, the functions $F_{\pm}^{\sigma}$ are WKB solutions to a sixth order differential equation near $x = \infty$ of the form

$$\partial_x^6 f + a_1 \partial_x^5 f + a_2 \partial_x^4 f + a_3 \partial_x^3 f + a_4 \partial_x^2 f + a_1 \partial_x f + a_6 f = 0 \tag{7.29}$$

with analytic coefficients

$$a_j(x) = \sum_{k \geq 0} a_{j,k} x^{-j} \tag{7.30}$$

such that

$$a_{3,0} = 2S_3(1)^3, \quad a_{6,0} = S_3(1)^6, \quad a_{1,1} = a_{4,1}, \quad a_{2,1} = a_{5,1}, \quad a_{3,1} = a_{6,1}. \tag{7.31}$$

Also the generating function $f_3(x)$ satisfies Eq. (7.29).

Proof. The first statement of the theorem (about the monodromy and the Stokes matrices) is already proved. From this it follows that the space generated by the functions $F_{\pm}^{\sigma}(x)$ near $x = \infty$ (or their analytic representatives) is invariant with respect to monodromy around $x = \infty$ and with respect to passing from one sector to an adjacent sector. Since the monodromy matrix $\mathcal{M}_{\infty}$ and the Stokes matrices have constant coefficients, also the spaces generated by the successive derivatives $\partial_x^i F_{\pm}^{\sigma}$ are invariant. As in other similar situations (see [Zo3]), we arrive to the determinant equation

$$\det \begin{bmatrix} f & \partial_x f & \dots & \partial_x^6 f \\ F_1 & \partial_x F_1 & \dots & \partial^6 F_1 \\ \dots & \dots & \dots & \dots \\ F_6 & \partial_x F_6 & \dots & \partial^6 F_6 \end{bmatrix} = 0$$

which is satisfied by the functions F_j (where we have ordered the functions $F_{\pm}^{\sigma} = F_j$). This equation is equivalent to Eq. (7.29), where the coefficients $a_j(x)$ are ratios of some minors of sixth dimension and are holomorphic and single valued functions of x.

The form (7.30) of the coefficients $a_j(x)$ and the relations (7.31) follow from the fact that the WKB solutions have the form $\sim e^{\sigma x S_3(1)} x^{-3/2}$. When we assume a solution $f \sim e^{\kappa x} x^{\gamma}$, then we should get the 'Hamilton–Jacobi equation' $\sum_j a_{j,0}\kappa^{6-j} = \left(\kappa^3 + S_3(1)\right)^2 = 0$ and the value $\gamma = -3/2$ implies the equation

$$6 \cdot (\sigma S_3(1))^5 \cdot \left(\frac{-3}{2}\right) + a_{3,0} \cdot 3 \cdot (\sigma S_3(1))^2 \cdot \left(\frac{-3}{2}\right) + \sum_j a_{j,1} \cdot (\sigma S_3(1))^j = 0,$$

which is satisfied for any $\sigma = -1, \epsilon, \bar{\epsilon}$. ∎

Remark 6. *It is highly interesting whether Eq. (7.29) can be prolonged to the whole $x-$plane with the other singularity at $x = 0$. Indeed, the function $f_3(x)$ is its solution and has very regular behavior at $x = 0$. So, maybe Eq. (7.29) has regular singularity at $x = 0$.*

But then each its coefficient $a_j(x)$ should be rational with pole at $x = 0$ of order $\leq j$. Moreover, since f_3 depends on x^3, our equation should be of the form

$$\begin{gathered} f^{(VI)} + c_1 x^{-1} f^{(V)} + c_2 x^{-2} f^{(IV)} + \left(c_3 + c_4 x^{-3}\right) f^{(III)} \\ + \left(c_5 x^{-1} + c_6 x^{-4}\right) f^{(II)} + \left(c_7 x^{-2} + c_8 x^{-5}\right) f^{(I)} \\ + \left(c_9 + c_{10} x^{-3} + c_{11} x^{6}\right) f = 0. \end{gathered} \tag{7.32}$$

Then we get the following recurrence for the coefficients in $f_3 = \sum b_k x^{3k}$:

$$\begin{gathered} \{c_{11} + 3kc_8 + 3k(3k-1)c_6 + 3k(3k-1)(3k-2)c_4 + 3k \dots (3k-3)c_2 \\ +3k \dots (3k-4)c_1 + 3k \dots (3k-5)\} b_k + \\ \{c_{10} + (3k-3)c_7 + (3k-3)(3k-4)c_5 + (3k-3) \dots (3k-5)c_3\} b_{k-1} \\ +c_9 b_{k-2} = 0. \end{gathered}$$

In a particular, for $k = 2$ we get an equation relating $b_0 = 1$, $b_1 = -\zeta(3)$ and $b_2 = \zeta(3,3) = \frac{1}{2}\left(\zeta(3)^2 - \zeta(6)\right)$ (where $\zeta(6) = \pi^6/945$). Since the coefficients c_j are potentially calculable, we could arrive at a quadratic equation for $\zeta(3)$ with coefficients which most probably belong to the field $\mathbb{Q}(\pi, \sqrt{3})$.

Recall that R. Apéry [Ap] *was the first who proved the irrationality of $\zeta(3)$. If our speculations turned out correct it would be quite spectacular achievement.*

Another question is about the values of the constants p, q in the principal Stokes matrix in Eq. (7.28). Probably $p = 0$ and $q = -1$.

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