

INTRODUCTION TO THE LOCAL THEORY OF PLANE ALGEBRAIC CURVES

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ABSTRACT. We consider the algebroid plane curves defined by formal power series of two variables with coefficients in an algebraically closed field. Using quadratic transformations we prove the local normalization theorem. Then we study the intersection multiplicity of algebroid curves and give an introduction to the Newton diagrams.

These notes are intended as a concise introduction to the local theory of plane algebraic curves. We consider the algebroid plane curves defined by formal power series of two variables with coefficients in an algebraically closed field. Using quadratic transformations we prove the local normalization theorem. Then we study the intersection multiplicity of algebroid curves and give an introduction to the Newton diagrams. We assume known the basic theorems on formal power series: the Weierstrass Preparation Theorem, the Implicit Function Theorem and Hensel's Lemma. A standard reference for this material is Abhyankar [1] (see also Hefez [5]). The book [8] by Seidenberg was very helpful when preparing this text. For further study of algebroid curves we refer the reader to Campillo [2].

In what follows \mathbb{K} is an algebraically closed field of arbitrary characteristic. The ring of formal power series in two variables x, y with coefficients in the field \mathbb{K} will be denoted $\mathbb{K}[[x, y]]$ and its field of fractions $\mathbb{K}((x, y))$. If $f = \sum_{i \geq k} f_i$ is a nonzero formal power series represented as the sum of homogeneous forms f_i with $f_k \neq 0$ then we write $\text{ord } f = k$ and in $f = f_k$. Additionally we put $\text{ord } 0 = \infty$ and $\text{in } 0 = 0$. We use the usual conventions on the symbol ∞ . A power series $u \in \mathbb{K}[[x, y]]$ is a unit if $uv = 1$ for a power series $v \in \mathbb{K}[[x, y]]$. Note that u is a unit if and only if its constant term $u(0)$ is nonzero. If $f, g \in \mathbb{K}[[x, y]]$ are such that

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$f = gu$ for a unit u then we write $f \sim g$. The principal ideal of $\mathbb{K}[[x, y]]$ generated by f is denoted $(f)\mathbb{K}[[x, y]]$. The reader will find the description of prime ideals of the ring $\mathbb{K}[[x, y]]$ in Appendix C.

1. ALGEBROID CURVES, QUADRATIC TRANSFORMATIONS

Let $f \in \mathbb{K}[[x, y]]$ be a nonzero formal power series without constant term. The algebroid curve $f = 0$ is by definition the principal ideal $(f)\mathbb{K}[[x, y]]$ generated by f . We also denote $\{f = 0\}$ the algebroid curve of equation $f = 0$. Thus we have $\{f = 0\} = \{g = 0\}$ if and only if $f \sim g$. The curve $\{f = 0\}$ is reduced (resp. irreducible) if the power series f does not have multiple factors (resp. is irreducible). If $f = f_1^{m_1} \dots f_s^{m_s}$ in $\mathbb{K}[[x, y]]$ with f_i irreducible and coprime then the curves $\{f_i = 0\}$ are called irreducible components of $\{f = 0\}$ with multiplicities m_i .

The order (multiplicity) of the curve $\{f = 0\}$ is the number $\text{ord } f$. The definition is correct because from $f \sim g$ it follows $\text{ord } f = \text{ord } g$. The curves of order 1 are called regular or non-singular. The curves of order strictly greater than 1 are called singular. If $f \sim g$ then in $f = c \text{in } g$ for a constant $c \in \mathbb{K} \setminus \{0\}$. The affine curve in $f = 0$ (see Fulton [4]) is called the tangent cone to the curve $f = 0$. From the Factorization Lemma (see Appendix A) we get

Property 1.1. *The tangent cone to the irreducible curve $\{f = 0\}$ is an affine line, i.e. in $f = l^{\text{ord } f}$, where $l = bx - ay$ is a non-zero linear form.*

Let $\Phi(x, y) = (ax + by + \dots, cx + dy + \dots)$ be a pair of formal power series such that $ad - bc \neq 0$. Then $f \mapsto f \circ \Phi$ is an isomorphism of the ring $\mathbb{K}[[x, y]]$ (every \mathbb{K} -isomorphism of $\mathbb{K}[[x, y]]$ is of this form). We have $\text{ord } f = \text{ord}(f \circ \Phi)$ and $\text{in}(f \circ \Phi) = \text{in } f \circ \text{in } \Phi$, where $\text{in } \Phi = (ax + by, cx + dy)$.

The algebroid curves $\{f = 0\}$ and $\{g = 0\}$ are equivalent if $f \circ \Phi = gu$ for a pair Φ satisfying the above conditions and for a unit u . Equivalent curves are of the same orders and their tangent cones are affine isomorphic. Any two regular curves are formally equivalent.

Let $f = f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible power series of order $n > 0$. From Property 1.1 it follows that $\text{ord } f(x, 0) = n$ or $\text{ord } f(0, y) = n$.

Definition 1.2. *Suppose that $f \in \mathbb{K}[[x, y]]$ is a power series such that $\text{ord } f(0, y) = \text{ord } f = n$ (in this case we say that f is y -general). Let y_1 be a new variable. A power series $f_1 \in \mathbb{K}[[x, y_1]]$ is a strict quadratic transformation of $f \in \mathbb{K}[[x, y]]$ if $f_1(0, 0) = 0$ and $f(x, ax + xy_1) = x^n f_1(x, y_1)$ in $\mathbb{K}[[x, y_1]]$ for an $a \in \mathbb{K}$. We write then $f_1 = Q(f)$.*

Let us note the basic properties of quadratic transformations. We keep the notations introduced in Definition 1.2

Lemma 1.3. *Suppose that the irreducible power series $f \in \mathbb{K}[[x, y]]$ is y -general of order n and put $f_1 = Q(f)$. Then*

- (i) the line $y - ax = 0$ is tangent to the curve $f(x, y) = 0$ (so the constant $a \in \mathbb{K}$ is uniquely determined by f) and $\text{ord } f_1(0, y_1) = n$. If $a \neq 0$ then $\text{ord } f(x, 0) = n$.
- (ii) If $f \sim g$ in $\mathbb{K}[[x, y]]$ and $f_1 = Q(f)$, $g_1 = Q(g)$ then $f_1 \sim g_1$ in $\mathbb{K}[[x, y_1]]$.
- (iii) If $f \in \mathbb{K}[[x]][y]$ is a distinguished polynomial in y then $f_1 \in \mathbb{K}[[x]][y_1]$ and f_1 is a distinguished polynomial in y_1 .

Proof. Since f is y -general and irreducible we have $f(x, y) = c(y - a_0x)^n + \dots + (\text{terms of order } > n)$ in $\mathbb{K}[[x, y]]$ for a constant $c \neq 0$ (see Property 1.1). Therefore we get $f(x, ax + xy_1) = x^n f_1(x, y_1)$ in $\mathbb{K}[[x, y_1]]$ with $f_1(x, y_1) = (a - a_0 + y_1)^n + \dots + (\text{terms of order } > n)$. Thus $f_1(0, 0) = 0$ if and only if $a = a_0$ and in this case $\text{ord } f_1(0, y_1) = n$. The remaining properties follow directly from Definition 1.2. □

Lemma 1.4. *If $f \in \mathbb{K}[[x, y]]$ is a y -general irreducible power series then $f_1 = Q(f) \in \mathbb{K}[[x, y]]$ is an irreducible power series.*

Proof. By Lemma 1.3 (iii) we may assume that $f = f(x, y)$ is a y -distinguished polynomial of degree n . Then the power series $f_1 = f_1(x, y_1)$ is a y_1 -distinguished polynomial of degree n and it suffices to check that f_1 is irreducible in the ring $\mathbb{K}[[x]][y_1]$. Suppose the contrary

$$f_1(x, y_1) = (y_1^k + b_1(x)y_1^{k-1} + \dots + b_k(x)) (y_1^l + c_1(x)y_1^{l-1} + \dots + c_l(x))$$

in $\mathbb{K}[[x]][y_1]$, where $k, l > 0$.

Clearly $k + l = n$ and consequently

$$\begin{aligned} f(x, ax + xy_1) &= x^n f_1(x, y_1) = \\ &= ((xy_1)^k + b_1(x)x(xy_1)^{k-1} + \dots + b_k(x)x^k) \cdot \\ &\quad \cdot ((xy_1)^l + c_1(x)x(xy_1)^{l-1} + \dots + c_l(x)x^l). \end{aligned}$$

Let z be a new variable. From the above identity it follows that

$$\begin{aligned} f(x, ax + z) &= \\ &= (z^k + xb_1(x)z^{k-1} + \dots + x^k b_k(x)) (z^l + xc_1(x)z^{l-1} + \dots + x^l c_l(x)). \end{aligned}$$

This shows that the power series $f(x, ax + z) \in \mathbb{K}[[x, z]]$ is reducible. We get a contradiction because it is irreducible as the image of the irreducible power series $f(x, y)$ by an isomorphism $\mathbb{K}[[x, y]] \rightarrow \mathbb{K}[[x, z]]$. □

Lemma 1.5. *Let $f = f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible y -general power series of order $n = \text{ord } f > 1$. Then there exists a sequence of power series $f_i = f_i(x, y_i) \in \mathbb{K}[[x, y_i]]$, $i = 0, 1, \dots, m$ such that $f_0 = f$ (and $y_0 = y$), $f_{i+1} = Q(f_i)$, $\text{ord } f_i = n$ for $i < m$ and $\text{ord } f_m < n$.*

Proof. Let $y_0 = y$ and $f_0 = f$ and let us consider $f_1 = Q(f_0)$. If $\text{ord } f_1 < n$ then we put $m = 1$ and the sequence f_0, f_1 verifies the condition. If $\text{ord } f_1 = n$ (we have always $\text{ord } f_1 \leq \text{ord } f$ since $\text{ord } f_1(0, y_1) = n$) then we put $f_2 = Q(f_1)$. If $\text{ord } f_2 < n$ we are done. We have to show that after a finite number of steps

we get a sequence f_0, \dots, f_m such that $f_{i+1} = Q(f_i)$, $\text{ord } f_i = n$ for $i < m$ and $\text{ord } f_m < n$. Otherwise there would exist an infinite sequence f_0, \dots, f_m, \dots such that $f_{i+1} = Q(f_i)$ and $\text{ord } f_i = n$ for all $i \geq 0$. Let $y_i - a_i x = 0$ be the tangent to the curve $f_i(x, y_i) = 0$. It is easy to check that $f(x, y(x)) = 0$, where $y(x) = \sum_{i=1}^{+\infty} a_{i-1} x^i$. We get a contradiction because f is irreducible, $\text{ord } f > 1$ and the condition $f(x, y(x)) = 0$ implies that $y - y(x)$ divides $f(x, y)$ in $\mathbb{K}[[x, y]]$. \square

Now we can construct the transformation reducing the order of an irreducible power series.

Proposition 1.6. *Let $f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible y -general power series of order $n = \text{ord } f > 1$. Let \tilde{y} be a new variable.*

Then there exist an integer $m > 0$ and a polynomial $P(x) = \sum_{i=1}^m a_{i-1} x^i$ of degree $\leq m$ such that

- (i) $f(x, P(x) + x^m \tilde{y}) = x^{mn} \tilde{f}(x, \tilde{y})$ in $\mathbb{K}[[x, \tilde{y}]]$,
- (ii) $\tilde{f} = \tilde{f}(x, \tilde{y}) \in \mathbb{K}[[x, \tilde{y}]]$ is an irreducible power series such that $\text{ord } \tilde{f} < n$,
- (iii) we have $\text{ord } \tilde{f}(0, \tilde{y}) = n$. If $P(x) \neq 0$ then $\text{ord } f(x, 0) = \text{ord } P(x) \cdot n$,
- (iv) if $f \sim W$ and $f \sim \tilde{W}$, where W and \tilde{W} are distinguished polynomials, then $W(x, P(x) + x^m \tilde{y}) = x^{mn} \tilde{W}(x, \tilde{y})$.

Proof. Let f_0, f_1, \dots, f_m be a sequence of power series from Lemma 1.5. Thus we get $f_i(x, a_i x + x y_{i+1}) = x^n f_{i+1}(x, y_{i+1})$ ($i = 0, 1, \dots, m-1$) for some $a_i \in \mathbb{K}$. Let $P(x) = \sum_{i=1}^m a_{i-1} x^i$, $\tilde{y} = y_m$ and $\tilde{f}(x, \tilde{y}) = f_m(x, \tilde{y})$. Since f_{i+1} is the strict transformation of f_i ($i = 0, \dots, m-1$) we get (i) of Proposition 1.6. Part (ii) follows from Lemma 1.4.

To check (iii) suppose that $k = \text{ord } P(x) < \infty$. Hence we have $a_{k-1} \neq 0$ and $a_{i-1} = 0$ for $i < k$. Consequently we get $f_i(x, x y_{i+1}) = x^n f_{i+1}(x, y_{i+1})$ for $i < k-1$ and $f_{k-1}(x, a_{k-1} x + x y_k) = x^n f_k(x, y_k)$. Since $a_{k-1} \neq 0$, from the last identity we obtain $\text{ord } f_{k-1}(x, 0) = n$ by Lemma 1.3 (i). From $\text{ord } f_i(x, 0) = n + \text{ord } f_{i+1}(x, 0)$ for $i < k-1$ we infer that $\text{ord } f(x, 0) = \text{ord } f_0(x, 0) = nk$.

Property (iv) follows from the fact that $f \sim W$ and $f_1 \sim W_1$ imply $W_1 = Q(W)$. \square

Remark 1.7 In the above considerations the power series $f \in \mathbb{K}[[x, y]]$ is y -general and for such a power series we define quadratic transformation. If $f \in \mathbb{K}[[x, y]]$ is x -general then we can easily reformulate the definition. In particular if $\text{ord } f(x, 0) = \text{ord } f = n$ then the quadratic transformation is of the form $f(by + yx_1, y) = y^n f_1(x_1, y)$, $f_1(0, 0) = 0$. If $\text{ord } f(x, 0) = \text{ord } f(0, y) = n$ and $ab \neq 0$ then the obtained strict quadratic transformations of f are equivalent.

2. PARAMETRIZATIONS

Let t be a variable. A parametrization is a pair $(\phi(t), \psi(t)) \in \mathbb{K}[[t]]^2$ such that $\phi(0) = \psi(0) = 0$ and $\phi(t) \neq 0$ or $\psi(t) \neq 0$ in $\mathbb{K}[[t]]$. Two parametrizations $(\phi(t), \psi(t))$ and $(\phi_1(t), \psi_1(t))$ are equivalent if there exists $\tau(t) \in \mathbb{K}[[t]]$, $\text{ord } \tau(t) = 1$ such that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$. A parametrization $(\phi(t), \psi(t))$ is good if there does not exist $\tau(t)$, $\text{ord } \tau(t) > 1$ and a parametrization $(\phi_1(t_1), \psi_1(t_1))$ such that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$.

Theorem 2.1 (Normalization Theorem). *Let $f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible power series. Then there exists a good parametrization $(\phi(t), \psi(t))$ such that $f(\phi(t), \psi(t)) = 0$, $\text{ord } f(x, 0) = \text{ord } \psi(t)$ and $\text{ord } f(0, y) = \text{ord } \phi(t)$. If $(\phi^*(u), \psi^*(u))$ is a parametrization such that $f(\phi^*(u), \psi^*(u)) = 0$ then there exists a series $\sigma(u) \in \mathbb{K}[[u]]$, $\sigma(0) = 0$ such that $\phi^*(u) = \phi(\sigma(u))$ and $\psi^*(u) = \psi(\sigma(u))$.*

A good parametrization $(\phi(t), \psi(t))$ such that $f(\phi(t), \psi(t)) = 0$ is called a normalization of the curve $f(x, y) = 0$. From Theorem 2.1 it follows that every irreducible curve has a normalization unique up to equivalence.

Proof. (of Theorem 2.1) We use induction on $\text{ord } f$.

If $\text{ord } f = 1$ the theorem easily follows from the Implicit Function Theorem. Suppose that $n > 1$ is an integer and that the theorem is true for all irreducible power series of order $< n$. Fix an irreducible power series f such that $\text{ord } f = n$. Without diminishing the generality we may assume that $\text{ord } f(0, y) = n$. Let $\tilde{f}(x, \tilde{y}) \in \mathbb{K}[[x, \tilde{y}]]$ be a power series from Proposition 1.6. Thus we get $f(x, P(x) + x^m \tilde{y}) = x^{mn} \tilde{f}(x, \tilde{y})$, where $P(x)$ is a polynomial of degree $\leq m$, $\text{ord } \tilde{f}(0, \tilde{y}) = n$ and $\text{ord } \tilde{f} < n$. By induction hypothesis there is a normalization $(\phi(t), \tilde{\psi}(t))$ of the curve $\tilde{f}(x, \tilde{y}) = 0$ such that $\text{ord } \phi(t) = \text{ord } \tilde{f}(0, \tilde{y})$ and $\text{ord } \tilde{\psi}(t) = \text{ord } \tilde{f}(x, 0)$. Let us put $\psi(t) = P(\phi(t)) + \phi(t)^m \tilde{\psi}(t)$ and consider the parametrization $(\phi(t), \psi(t))$. Obviously we have $f(\phi(t), \psi(t)) = 0$.

To check that the parametrization $(\phi(t), \psi(t))$ is good suppose that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$ for a parametrization $(\phi_1(t_1), \psi_1(t_1))$ and for a series $\tau(t) \in \mathbb{K}[[t]]$, $\text{ord } \tau(t) \geq 1$. Thus $\psi_1(\tau(t)) - P(\phi_1(\tau(t))) = \phi_1(\tau(t))^m \tilde{\psi}(t)$ and consequently $\text{ord}(\psi_1(t_1) - P(\phi_1(t_1))) \geq \text{ord } \phi_1(t_1)^m$. Let us put $\tilde{\psi}_1(t_1) := \frac{\psi_1(t_1) - P(\phi_1(t_1))}{\phi_1(t_1)^m}$. We get then $\text{ord } \tilde{\psi}_1(t_1) \geq 0$ and $\tilde{\psi}(t) = \tilde{\psi}_1(\tau(t))$. From the equalities $\phi(t) = \phi_1(\tau(t))$ and $\tilde{\psi}(t) = \tilde{\psi}_1(\tau(t))$ it follows that $\text{ord } \tau(t) = 1$ since the parametrization $(\phi(t), \tilde{\psi}(t))$ is good. This proves that $(\phi(t), \psi(t))$ is a normalization of the curve $f(x, y) = 0$.

Let us recall that $\text{ord } \phi(t) = \text{ord } \tilde{f}(0, \tilde{y}) = n = \text{ord } f(0, y)$. To calculate $\text{ord } \psi(t)$ let us suppose first $P(x) \neq 0$. Then $\text{ord } P(\phi(t)) = (\text{ord } P)(\text{ord } \phi) \leq m(\text{ord } \phi) = \text{ord } \phi^m < \text{ord } \phi^m \tilde{\psi}$ and $\text{ord } \psi(t) = \text{ord}(P(\phi(t)) + \phi(t)^m \tilde{\psi}(t)) = \text{ord } P(\phi(t)) = (\text{ord } P)(\text{ord } \phi) = (\text{ord } P)n = \text{ord } f(x, 0)$ by Proposition 1.6 (iii). If $P(x) = 0$ then

$\text{ord } \psi(t) = \text{ord } \phi(t)^m \tilde{\psi}(t) = mn + \text{ord } \tilde{\psi} = mn + \text{ord } \tilde{f}(x, 0) = \text{ord } f(x, 0)$. Summing up we have checked that $\text{ord } \phi(t) = \text{ord } f(0, y)$ and $\text{ord } \psi(t) = \text{ord } f(x, 0)$.

Now let $(\phi^*(u), \psi^*(u))$ be a parametrization such that $f(\phi^*(u), \psi^*(u)) = 0$. Put $\tilde{\psi}^*(u) = \frac{\psi^*(u) - P(\phi^*(u))}{\phi^*(u)^m} \in \mathbb{K}((u))$. Let $W(x, y)$ be a distinguished polynomial associated with $f(x, y)$. We get

$$\begin{aligned} 0 = W(\phi^*(u), \psi^*(u)) &= W(\phi^*(u), P(\phi^*(u)) + \phi^*(u)^m \tilde{\psi}^*(u)) = \\ &= (\phi^*(u))^{mn} \tilde{W}(\phi^*(u), \tilde{\psi}^*(u)) \end{aligned}$$

and hence $\tilde{W}(\phi^*(u), \tilde{\psi}^*(u)) = 0$.

From the last equality it follows that $\text{ord } \tilde{\psi}^*(u) > 0$ since $\tilde{\psi}^*(u)$ is a root of the distinguished $\tilde{W}(\phi^*(u), y) \in \mathbb{K}[[u]][y]$ (see Remark 2.2 given below). Let $(\phi(t), \tilde{\psi}(t))$ be a normalization of the curve $\tilde{f}(x, \tilde{y}) = 0$. By assumption we get $\phi^*(u) = \phi(\tau(u))$ and $\tilde{\psi}^*(u) = \tilde{\psi}(\tau(u))$, which implies $\phi^*(u) = \phi(\tau(u))$ and $\psi^*(u) = \psi(\tau(u))$. \square

Remark 2.2 If $\zeta(u)^n + \alpha_1(u)\zeta(u)^{n-1} + \dots + \alpha_n(u) = 0$ in $\mathbb{K}((u))$ then it is easy to check that $\text{ord } \zeta(u) \geq \inf_i \{\frac{1}{i} \text{ord } \alpha_i(u)\}$. In particular if the polynomial $y^n + \alpha_1(u)y^{n-1} + \dots + \alpha_n(u)$ is distinguished then $\text{ord } \alpha_i(u) > 0$ for $i = 1, \dots, n$ and consequently $\text{ord } \zeta(u) > 0$.

Corollary 2.3. *If $f(x, y) \in \mathbb{K}[[x, y]]$ with $n = \text{ord } f(0, y) < \infty$ then there exist power series $\alpha(s), \beta_1(s), \dots, \beta_n(s) \in \mathbb{K}[[s]]$ (s is a variable) without constant term such that*

$$f(\alpha(s), y) \sim \prod_{j=1}^n (y - \beta_j(s)) \text{ in } \mathbb{K}[[s, y]].$$

Proof. Using the Weierstrass Preparation Theorem we may assume that $f(x, y) \in \mathbb{K}[[x]][y]$ is a distinguished polynomial of degree n . We prove the corollary by induction on $n = \deg_y f$. If $n = 1$ the corollary is obvious. Suppose that $n > 1$ and the corollary is true for polynomials of degree $n - 1$. Let $f(x, y)$ be a distinguished polynomial of degree n . Using Theorem 2.1 to an irreducible factor of the series $f(x, y)$ we find a parametrization $(\alpha(s), \beta(s))$ such that $f(\alpha(s), \beta(s)) = 0$. We get then $f(\alpha(s), y) = (y - \beta(s))g(s, y)$ in $\mathbb{K}[[s]][y]$, where $g(s, y) = y^{n-1} + \dots$ is a distinguished polynomial of degree $n - 1$. We apply the induction hypothesis to $g(s, y)$. \square

Let us note

Corollary 2.4 (Puiseux Theorem). *Let \mathbb{K} be an algebraically closed field of characteristic l . Let $n > 0$ be an integer such that $n \not\equiv 0 \pmod{l}$. Then for every*

distinguished and irreducible polynomial $P(x, y) = y^n + \sum_{i=1}^n a_i(x)y^{n-i}$ there exists a series $y(s) \in \mathbb{K}[[s]]$, $y(0) = 0$ such that

$$P(s^n, y) = \prod_{\epsilon^n=1} (y - y(\epsilon s)).$$

Proof. Let $(\phi(t), \psi(t))$ be a normalization of the curve $P(x, y) = 0$. Then $\text{ord } \phi(t) = \text{ord } P(0, y) = n$ and there exists a series $\sigma(t)$ such that $\phi(t) = \sigma(t)^n$ in $\mathbb{K}[[t]]$ since $n \not\equiv 0 \pmod{l}$ (we use the Implicit Function Theorem or Hensel's Lemma to the equation $y^n - \phi(t) = 0$). Clearly $\text{ord } \sigma(t) = 1$ and $\psi(t) = y(\sigma(t))$ for a power series $y(s) \in \mathbb{K}[[s]]$. The parametrization $(s^n, y(s))$ is good. Therefore we have $\text{GCD}(\{n\} \cup \text{supp } y(s)) = 1$ and $y(\epsilon_1 s) \neq y(\epsilon_2 s)$ if $\epsilon_1^n = \epsilon_2^n = 1$ and $\epsilon_1 \neq \epsilon_2$. Hence we get the corollary because $P(s^n, y(\epsilon s)) = 0$ for all ϵ such that $\epsilon^n = 1$. \square

Lemma 2.5. *Let $\phi(t) \in \mathbb{K}[[t]]$ be a nonzero power series of order $n > 0$. Then any power series $g(t) \in \mathbb{K}[[t]]$ can be expressed in the following form*

$$g(t) = \sum_{i=0}^{n-1} a_i(\phi(t))t^i, \quad \text{where } a_i = a_i(x) \in \mathbb{K}[[x]] \text{ for } i = 0, \dots, n-1.$$

The coefficients $a_i = a_i(x)$ are uniquely determined by $\phi(t)$ and $g(t)$.

Proof. Let us fix $g(t) \in \mathbb{K}[[t]]$ and put $F(x, t) = \phi(t) - x$. Then we get $\text{ord } F(0, t) = \text{ord } \phi(t) = n$ and the Weierstrass Division Theorem gives $g(t) = q(x, t)F(x, t) + \sum_{i=0}^{n-1} a_i(x)t^i$. Substituting $\phi(t)$ for x we obtain $g(t) = \sum_{i=0}^{n-1} a_i(\phi(t))t^i$. To show the uniqueness it suffices to observe that if we had a relation as above with $g(t) = 0$ and with some nonzero $a_i(x)$, then two terms $a_i(\phi(t))t^i$ and $a_j(\phi(t))t^j$, $i \neq j$ would necessarily have the same finite order. This obviously cannot be the case. \square

Now we can prove a theorem partially converse to Theorem 2.1.

Theorem 2.6. *For every parametrization $(\phi(t), \psi(t))$ there exists an irreducible power series $f = f(x, y)$ such that $f(\phi(t), \psi(t)) = 0$. It is determined uniquely by the parametrization up to a unit of the ring $\mathbb{K}[[x, y]]$.*

Proof. Suppose that $\phi(t) \neq 0$ and put $n = \text{ord } \phi(t)$. By Lemma 2.5 we get that $\mathbb{K}[[t]] = \mathbb{K}[[\phi(t)]] + \mathbb{K}[[\phi(t)]]t + \dots + \mathbb{K}[[\phi(t)]]t^{n-1}$, which implies that the ring $\mathbb{K}[[t]]$ is a finite module over $\mathbb{K}[[\phi(t)]]$. Therefore the ring $\mathbb{K}[[t]]$ is integral over $\mathbb{K}[[\phi(t)]]$. In particular, the series $\psi(t)$ is integral over $\mathbb{K}[[\phi(t)]]$ and there exists $f(x, y) \in \mathbb{K}[[x]][y]$ monic with respect to y such that $f(\phi(t), \psi(t)) = 0$. Replacing $f(x, y)$ by its irreducible factor we get the first part of the theorem. The uniqueness follows from the fact that the ideal I of power series $g(x, y) \in \mathbb{K}[[x, y]]$ such that $g(\phi(t), \psi(t)) = 0$ is a prime ideal and it is not maximal since $(\phi(t), \psi(t)) \neq (0, 0)$ (see Appendix C). \square

Lemma 2.7. *Suppose that the domain A is a subring of the domain B such that B is a free A -module of rank $n > 0$. Let K be the field of fractions of A and L the field of fractions of B . Then $(L : K) = n$.*

Proof. By assumption there exists a sequence e_1, \dots, e_n of elements of B such that every element $b \in B$ can be written uniquely in the form $b = a_1e_1 + \dots + a_n e_n$ for some $a_1, \dots, a_n \in A$. In particular B is a finite A -module and consequently B is integral over A . Therefore for every $b \in B$, $b \neq 0$ there exists $b' \in B$ such that $bb' \in A \setminus \{0\}$. In fact if $b \notin A$ and $b^k + a_1b^{k-1} + \dots + a_k = 0$ is the equation of integral dependence of minimal degree $k > 0$ then $a_k \neq 0$ and $bb' = -a_k$ for $b' = b^{k-1} + a_1b^{k-2} + \dots + a_{k-1}$. Thus every element of the field L may be written in the form $\frac{b}{a}$, where $a \in A \setminus \{0\}$ and $b \in B$. If $b = a_1e_1 + \dots + a_n e_n$ then $\frac{b}{a} = \left(\frac{a_1}{a}\right)e_1 + \dots + \left(\frac{a_n}{a}\right)e_n$ and $(L : K) \leq n$. The equality follows from the fact that e_1, \dots, e_n are linearly independent over K . \square

We denote by $\mathbb{K}((\phi(t)))$ the field of fractions of the domain $\mathbb{K}[[\phi(t)]]$.

Theorem 2.8. *Let $(\phi(t), \psi(t))$ be a good parametrization such that $\phi(t) \neq 0$. Let $n = \text{ord } \phi(t)$. Then*

- (a) $\left(\mathbb{K}((t)) : \mathbb{K}((\phi(t)))\right) = n$,
- (b) $\mathbb{K}((t)) = \mathbb{K}((\phi(t)))(\psi(t))$.

Proof. By Lemma 2.5 the ring $\mathbb{K}[[t]]$ is a free module over $\mathbb{K}[[\phi(t)]]$ of rank n . Therefore Property (a) follows from Lemma 2.7. On the other hand by Theorems 2.6 and 2.1 there exists an irreducible power series $f = f(x, y) \in \mathbb{K}[[x, y]]$ such that $f(\phi(t), \psi(t)) = 0$ and $\text{ord } f(0, y) = \text{ord } \phi(t) = n$. Using the Weierstrass Preparation Theorem we may assume that f is a distinguished polynomial in y of degree n with coefficients in $\mathbb{K}[[x]]$. Furthermore, $f(x, y)$ is irreducible in $\mathbb{K}[[x]][y]$ and consequently in $\mathbb{K}((x))[y]$ since the ring $\mathbb{K}[[x]]$ is normal. Thus $f(\phi(t), y)$ is a minimal polynomial of $\psi(t)$ over $\mathbb{K}((\phi(t)))$ and $\left(\mathbb{K}((\phi(t)))(\psi(t)) : \mathbb{K}((\phi(t)))\right) =$ the degree of $f(\phi(t), y)$ in the indeterminate y , which is equal to $n = \left(\mathbb{K}((t)) : \mathbb{K}((\phi(t)))\right)$. This shows that $\mathbb{K}((\phi(t)))(\psi(t)) = \mathbb{K}((t))$. \square

For any parametrization $(\phi(t), \psi(t)) \in \mathbb{K}[[t]]^2$ we denote by $\mathbb{K}((\phi(t), \psi(t)))$ the field of fractions of the ring $\mathbb{K}[[\phi(t), \psi(t)]]$.

Theorem 2.9. *A parametrization $(\phi(t), \psi(t))$ is good if and only if $\mathbb{K}((\phi(t), \psi(t))) = \mathbb{K}((t))$.*

Proof. Suppose that $\phi(t) \neq 0$. It is easy to see that $\mathbb{K}((\phi(t)))(\psi(t)) \subset \mathbb{K}((\phi(t), \psi(t)))$. Therefore if $(\phi(t), \psi(t))$ is good then $\mathbb{K}((\phi(t), \psi(t))) = \mathbb{K}((t))$ by Theorem 2.8. Suppose that $\mathbb{K}((\phi(t), \psi(t))) = \mathbb{K}((t))$ and let $\tau(t) \in \mathbb{K}[[t]]$ be a power series without constant term such that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$ for a parametrization $(\phi_1(s), \psi_1(s))$. Then $t \in \mathbb{K}((\phi(t), \psi(t))) \subset \mathbb{K}((\tau(t)))$, which implies $\text{ord } \tau(t) = 1$. Therefore $(\phi(t), \psi(t))$ is a good parametrization. \square

Here is another application of Theorem 2.8.

Theorem 2.10. *There exists a nonzero power series $d(t) \in \mathbb{K}[[\phi(t), \psi(t)]]$ (“a universal denominator”) such that $d(t)\mathbb{K}[[t]] \subset \mathbb{K}[[\phi(t), \psi(t)]]$.*

Proof. Suppose that $\phi(t) \neq 0$. Since $\mathbb{K}((t)) = \mathbb{K}((\phi(t)))(\psi(t))$ is an extension of $\mathbb{K}((\phi(t)))$ of degree n , the elements $1, \psi(t), \dots, \psi(t)^{n-1}$ form a linear basis of $\mathbb{K}((t))$ over $\mathbb{K}((\phi(t)))$.

Therefore, we may write

$$(1) \quad t^i = \alpha_{i,0}(\phi(t)) + \alpha_{i,1}(\phi(t))\psi(t) + \dots + \alpha_{i,n-1}(\phi(t))\psi(t)^{n-1},$$

where $i = 0, 1, \dots, n - 1$.

Let $d(t) \in \mathbb{K}[[\phi(t)]]$ be a common denominator of the elements $\alpha_{i,j}(\phi(t))$, where $i = 0, 1, \dots, n - 1$ and $j = 0, 1, \dots, n - 1$. The relation (1) implies

$$(2) \quad d(t)t^i \in \mathbb{K}[[\phi(t)]][\psi(t)] \text{ for } i = 0, 1, \dots, n - 1.$$

Since $\mathbb{K}[[t]] = \mathbb{K}[[\phi(t)]] + \dots + \mathbb{K}[[\phi(t)]]t^{n-1}$ by Lemma 2.5 we get by (2) $d(t)\mathbb{K}[[t]] \subset \mathbb{K}[[\phi(t)]][\psi(t)]$. □

3. INTERSECTION MULTIPLICITY

Let $f = f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible power series. Let us fix a normalization $(\phi(t), \psi(t))$ of the curve $f(x, y) = 0$. For every $g = g(x, y) \in \mathbb{K}[[x, y]]$ we define:

$$v_f(g) = \text{ord } g(\phi(t), \psi(t)) \in \mathbb{N} \cup \{\infty\}.$$

Proposition 3.1. *For any $g, g' \in \mathbb{K}[[x, y]]$ the following properties hold:*

- (i) $v_f(g) = 0$ if and only if $g(0) \neq 0$, $v_f(g) = \infty$ if and only if f divides g in $\mathbb{K}[[x, y]]$,
- (ii) $v_f(g + g') \geq \inf\{v_f(g), v_f(g')\}$. If $v_f(g) \neq v_f(g')$ then the equality holds,
- (iii) $v_f(gg') = v_f(g) + v_f(g')$,
- (iv) $v_f(g + hf) = v_f(g)$ for $h \in \mathbb{K}[[x, y]]$.

Proof. To check part (i) note that the ideal $I = \{h(x, y) \in \mathbb{K}[[x, y]] : h(\phi(t), \psi(t)) = 0\}$ is a prime non-maximal ideal. This implies (see Appendix C) that $I = (f)$ which proves that $v_f(g) = \infty$ if and only if f divides g . The remaining properties follow directly from the definition. □

Remark 3.2 With every irreducible curve $\{f = 0\}$ we associate the field \mathcal{M}_f of meromorphic fractions on $\{f = 0\}$. For this purpose we consider fractions $\frac{g}{h}$, where $g, h \in \mathbb{K}[[x, y]]$ and $h \not\equiv 0 \pmod{f}$. We write $\frac{g}{h} \equiv \frac{g_1}{h_1}$ if f divides $gh_1 - g_1h$. The cosets of the relation \equiv form in a natural way a field denoted \mathcal{M}_f . The function v_f extends to the valuation $v_f : \mathcal{M}_f \rightarrow \mathbb{Z} \cup \{\infty\}$ defined by $v_f\left(\frac{g}{h}\right) = v_f(g) - v_f(h)$.

Proposition 3.3 (Basic Inequality). *We have $v_f(g) \geq (\text{ord } f)(\text{ord } g)$. The equality holds if and only if $\{f = 0\}$ and $\{g = 0\}$ don't have a common tangent.*

We need

Lemma 3.4. *Let $(\phi(t), \psi(t))$ be a parametrization, $n = \inf\{\text{ord } \phi(t), \text{ord } \psi(t)\} < \infty$, $\phi(t) = at^n + \dots$, $\psi(t) = bt^n + \dots$, where $a \neq 0$ or $b \neq 0$. Then for every power series $g = g(x, y) \in \mathbb{K}[[x, y]]$: $\text{ord } g(\phi(t), \psi(t)) \geq (\text{ord } g)n$ with equality if and only if $(\text{in } g)(a, b) \neq 0$.*

Proof. (of Lemma 3.4) Let us write $g(x, y) = \sum_{\alpha+\beta=m} g_{\alpha\beta}(x, y)x^\alpha y^\beta$, where $m = \text{ord } g$ and $\sum_{\alpha+\beta=m} g_{\alpha\beta}(0, 0)x^\alpha y^\beta = \text{in } g$ (“Hadamard’s Lemma”).

We get $g(\phi(t), \psi(t)) = t^{mn} \sum_{\alpha+\beta=m} g_{\alpha\beta}(\phi(t), \psi(t)) \left(\frac{\phi(t)}{t^n}\right)^\alpha \left(\frac{\psi(t)}{t^n}\right)^\beta = t^{mn} ((\text{in } g)(a, b) + \text{terms of order } > 0)$ which proves the lemma. \square

Proof. (of Proposition 3.3) Let $(\phi(t), \psi(t))$ be a normalization of the irreducible curve $f(x, y) = 0$. Then $\inf\{\text{ord } \phi(t), \text{ord } \psi(t)\} = \inf\{\text{ord } f(0, y), \text{ord } f(x, 0)\} = \text{ord } f$ since $f = 0$ has exactly one tangent. Let $n = \text{ord } f$, $\phi(t) = at^n + \dots$, $\psi(t) = bt^n + \dots$. Thus $a \neq 0$ or $b \neq 0$. Since $\text{ord } f(\phi(t), \psi(t)) = \text{ord } 0 = \infty$ we get from Lemma 3.4 that $(\text{in } f)(a, b) = 0$ and consequently the unique tangent to $f = 0$ is given by the equation $bx - ay = 0$.

Now we get $v_f(g) = \text{ord } g(\phi(t), \psi(t)) \geq (\text{ord } g) \inf\{\text{ord } \phi(t), \text{ord } \psi(t)\} = (\text{ord } g)(\text{ord } f)$ by the first part of Lemma 3.4. The equality $v_f(g) = (\text{ord } g)(\text{ord } f)$ holds if and only if $(\text{in } g)(a, b) \neq 0$, which takes place exactly when the system of equations in $g = \text{in } f = 0$ has the unique solution $x = 0, y = 0$ that is if $f = 0$ and $g = 0$ don’t have a common tangent. \square

Proposition 3.5. *For any irreducible $f, g \in \mathbb{K}[[x, y]]$ we get $v_f(g) = v_g(f)$.*

To prove Proposition 3.5 we check the following lemma.

Lemma 3.6. *Suppose that f is irreducible, $n = \text{ord } f(0, y) < \infty$ and $f(\alpha(s), y) \sim \prod_{j=1}^n (y - \beta_j(s))$ in $\mathbb{K}[[s]][y]$. Then for any $g(x, y) \in \mathbb{K}[[x, y]]$:*

$$\sum_{j=1}^n \text{ord } g(\alpha(s), \beta_j(s)) = (\text{ord } \alpha(s))v_f(g).$$

Proof. (of Lemma 3.6) Let $(\phi(t), \psi(t))$ be a normalization of the curve $f(x, y) = 0$. Then $\alpha(s) = \phi(\sigma_j(s))$, $\beta_j(s) = \psi(\sigma_j(s))$ for a power series $\sigma_j(s)$, $\sigma_j(0) = 0$.

We get then

$$\sum_{j=1}^n \text{ord } g(\alpha(s), \beta_j(s)) = \sum_{j=1}^n \text{ord } g(\phi(t), \psi(t)) \text{ord } \sigma_j(s) = v_f(g) \sum_{j=1}^n \text{ord } \sigma_j(s).$$

To calculate the last sum let us note that $\text{ord } \alpha(s) = \text{ord } \phi(t) \text{ord } \sigma_j(s) = n \text{ord } \sigma_j(s)$ and consequently $\sum_{j=1}^n \text{ord } \sigma_j(s) = \text{ord } \alpha(s)$, which proves the lemma. \square

Proof. (of Proposition 3.5) Let $f, g \in \mathbb{K}[[x, y]]$ be irreducible. Suppose that f, g are y -general; $n = \text{ord } f(0, y)$, $p = \text{ord } g(0, y)$. By Corollary 2.3 we get

$$f(\alpha(s), y) \sim \prod_{j=1}^n (y - \beta_j(s)),$$

$$g(\alpha(s), y) \sim \prod_{j=1}^p (y - \gamma_j(s)).$$

Using Lemma 3.6 twice we get:

$$\begin{aligned} \text{ord } \alpha(s) v_f(g) &= \sum_{j=1}^n \text{ord } g(\alpha(s), \beta_j(s)) = \sum_{j=1}^n \text{ord } \prod_{k=1}^p (\beta_j(s) - \gamma_k(s)) = \\ &= \sum_{j=1}^n \sum_{k=1}^p \text{ord } (\beta_j(s) - \gamma_k(s)) = \sum_{k=1}^p \text{ord } f(\alpha(s), \gamma_k(s)) = (\text{ord } \alpha(s)) v_g(f). \end{aligned}$$

Then $v_f(g) = v_g(f)$.

Suppose that $\text{ord } f(0, y) = n < \infty$ and $\text{ord } g(0, y) = \infty$. The last conditions imply that $g \sim x$ and $v_f(g) = v_f(x) = \text{ord } \phi(t) = \text{ord } f(0, y) = v_x(f) = v_g(f)$.

Similarly we check the proposition when $\text{ord } f(0, y) = \infty$ and $\text{ord } g(0, y) = p < \infty$. If $\text{ord } f(0, y) = \text{ord } g(0, y) = \infty$ then f and g are divisible by x and $v_f(g) = \infty = v_g(f)$.

Let us note the formula for the order of the resultant of two polynomials.

Proposition 3.7. *Let $R_{f,g}(x)$ be the resultant of two polynomials $f(x, y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ and $g(x, y) = b_0(x)y^p + b_1(x)y^{p-1} + \dots + b_p(x)$. Assume that f is irreducible and distinguished. Then*

$$\text{ord } R_{f,g}(x) = v_f(g).$$

Proof. By Corollary 2.3 there exist power series $\alpha(s), b_1(s), \dots, \beta_n(s) \in \mathbb{K}[[s]]$ without constant term such that $f(\alpha(s), y) = \prod_{j=1}^n (y - \beta_j(s))$. From the definition of resultant we get $R_{f,g}(\alpha(s)) = \pm \prod_{j=1}^n g(\alpha(s), \beta_j(s))$ and consequently $\text{ord } R_{f,g}(\alpha(s)) = \sum_{j=1}^n \text{ord } g(\alpha(s), \beta_j(s)) = (\text{ord } \alpha(s)) v_f(g)$ by Lemma 3.6 and $\text{ord } R_{f,g} = v_f(g)$ since $\text{ord } R_{f,g}(\alpha(s)) = (\text{ord } R_{f,g}) \text{ord } \alpha(s)$. \square

Now let $f \in \mathbb{K}[[x, y]]$ be an arbitrary non-zero power series without constant term and let $f = \prod_{i=1}^r f_i$ be the decomposition of f into irreducible factors. We define $i_0(f, g) = \sum_{i=1}^r v_{f_i}(g)$. Moreover if $f(0) \neq 0$ then we put $i_0(f, g) = 0$ and if $f \equiv 0$: $i_0(f, g) = \infty$. From the properties of v_f (Propositions 3.1, 3.3, 3.5) we

get the fundamental properties of $i_0(f, g)$ (if $f(0) = g(0) = 0$ then $i_0(f, g)$ is called intersection multiplicity of the curves $f = 0$ and $g = 0$).

Proposition 3.8. *For any $f, g, g' \in \mathbb{K}[[x, y]]$:*

- (i) $0 \leq i_0(f, g) \leq \infty$, $i_0(f, g) = 0$ if and only if $f(0) \neq 0$ or $g(0) \neq 0$; $i_0(f, g) = \infty$ if and only if f, g have a common factor in $\mathbb{K}[[x, y]]$,
- (ii) $i_0(f, gg') = i_0(f, g) + i_0(f, g')$,
- (iii) $i_0(f, g + hf) = i_0(f, g)$ for every $h \in \mathbb{K}[[x, y]]$,
- (iv) $i_0(f, g) = i_0(g, f)$,
- (v) $i_0(f, g) \geq (\text{ord } f)(\text{ord } g)$; the equality holds if and only if the curves $f = 0$ and $g = 0$ do not have a common tangent.

From Proposition 3.7 we get easily the following:

Proposition 3.9. *If $f(x, y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ is distinguished, $g(x, y) = b_0(x)y^p + b_1(x)y^{p-1} + \dots + b_p(x)$ and $R_{f,g}(x)$ is their y -resultant, then $\text{ord } R_{f,g}(x) = i_0(f, g)$.*

We can give here an axiomatic characterization of the intersection multiplicity (see Kałużny-Spodzieja [6]).

Theorem 3.10. *Let $I : \mathbb{K}[[x, y]] \times \mathbb{K}[[x, y]] \rightarrow \mathbb{N} \cup \{\infty\}$ be a function with properties*

- (1) $I(f, g) = I(g, f)$,
- (2) $I(f, g_1g_2) = I(f, g_1) + I(f, g_2)$,
- (3) $I(f, g) = I(f, g + hf)$,
- (4) $I(x, y) \neq 0, \infty$

Then $I(f, g) = i_0(f, g)I(x, y)$.

Clearly properties (1) and (2) imply

$$(2') \quad I(f_1f_2, g) = I(f_1, g) + I(f_2, g).$$

To prove Theorem 3.10 we need the following lemma.

Lemma 3.11. *If I is a function such as in Theorem 3.10 then the following properties hold:*

- (5) *if f or g is a unit then $I(f, g) = 0$,*
- (6) *if f and g have a common divisor of positive order then $I(f, g) = \infty$.*

Proof. (of Lemma 3.11) To check property (5) note that using properties (2') and (3) we get

$$I(x, y) = I(1, y) + I(x, y) = I(1, y + (-y)1) + I(x, y) = I(1, 0) + I(x, y)$$

and

$$I(1, 0) + I(x, y) = I(1, g + (-g)1) + I(x, y) = I(1, g) + I(x, y).$$

Using the above equalities we get $I(x, y) = I(1, g) + I(x, y)$ hence $I(1, g) = 0$ since $I(x, y) \neq 0, \infty$.

If $f(0) \neq 0$ then we have

$$0 = I(1, g) = I\left(f\left(\frac{1}{f}\right), g\right) = I\left(g, f\left(\frac{1}{f}\right)\right) = I(g, f) + I\left(g, \frac{1}{f}\right).$$

Hence $I(g, f) = 0$ and consequently $I(f, g) = 0$.

To check (6) consider a power series h such that $h(0) = 0$. We can write $h = xh_1 + yh_2$ in $\mathbb{K}[[x, y]]$ and

$$I(h, 0) = I(h, 0 \cdot x) = I(h, 0) + I(h, x) = I(h, 0) + I(xh_1 + yh_2, x).$$

From properties (1) and (3) we get that $I(xh_1 + yh_2, x) = I(yh_2, x)$ and

$$\begin{aligned} I(h, 0) &= I(h, 0) + I(yh_2, x) = \\ &= I(h, 0) + I(y, x) + I(h_2, x) = I(h, 0) + I(x, y) + I(h_2, x). \end{aligned}$$

Hence $I(h, 0) = \infty$ since $I(x, y) \neq 0, \infty$.

Now suppose that f and g have a common divisor h , $h(0) = 0$. So we have $f = f_1h$, $g = g_1h$ in $\mathbb{K}[[x, y]]$ and we get

$$I(f, g) = I(f_1, g_1h) + I(h, g_1h) = I(f_1, g_1h) + I(h, 0) = \infty.$$

□

Remark 3.12 From property (5) it follows that $I(f, g) = I(uf, vg)$ for any units u, v .

Now we can give the proof of Theorem 3.10.

Proof. (of Theorem 3.10.) If $i_0(f, g) = \infty$ then f and g have a common factor of positive order and $I(f, g) = \infty$ by property (6).

It suffices to check that if f, g are coprime then $I(f, g) = i_0(f, g)I(x, y)$. We will prove this equality by induction with respect to $i_0(f, g)$. If $i_0(f, g) = 0$ then f or g is a unit and $I(f, g) = 0$ by property (5).

Let $k > 0$ be an integer and suppose that the equality $I(f, g) = i_0(f, g)I(x, y)$ is true for every pair f, g such that $i_0(f, g) < k$. If the series f or g is reducible then the equality $I(f, g) = i_0(f, g)I(x, y)$ is true: we use properties (2) and (2') of function I and the induction hypothesis. Thus it suffices to consider the case where f, g are irreducible and $i_0(f, g) = k$. If a power series h is irreducible then $h \sim x$ or $h \sim y^n + a_1(x)y^{n-1} + \dots + a_n(x)$, where $y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ is a distinguished polynomial. We have to consider three cases:

- (1) $f(x, y) = x$, $g(x, y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ is a distinguished polynomial. Then $i_0(f, g) = n$ and $I(f, g) = I(x, y^n) = nI(x, y) = i_0(f, g)I(x, y)$.
- (2) $f(x, y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$, $g(x, y) = x$. We use the first case and symmetry of I , i_0 .
- (3) $f(x, y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$, $g(x, y) = y^p + b_1(x)y^{p-1} + \dots + b_p(x)$ are distinguished polynomials of degrees $n, p > 0$. Without diminishing the

generality we may suppose that $p \geq n$. Then we may write $g = y^{p-n}f + xh$ in $\mathbb{K}[[x, y]]$ and consequently

$$I(f, g) = I(f, y^{p-n}f + xh) = I(f, x) + I(f, h) = nI(x, y) + I(f, h)$$

since $I(f, x) = nI(x, y)$ by Case 2.

To finish the proof it suffices to check the formula $I(f, h) = i_0(f, h)I(x, y)$.

If $h(0) = 0$ then this equality follows from the induction hypothesis since $i_0(f, h) < i_0(f, g) = k$. If $h(0) \neq 0$ then the both sides of this equality are 0. \square

As the first application of the theorem proved above we give the following property.

Proposition 3.13. *Let f, g be coprime power series without constant term. Then for any power series $\Phi, \Psi \in \mathbb{K}[[u, v]]$ we have:*

$$i_0(\Phi(f, g), \Psi(f, g)) = i_0(\Phi, \Psi)i_0(f, g).$$

Proof. Let us consider the function I given by formula $I(\Phi, \Psi) = i_0(\Phi(f, g), \Psi(f, g))$. It is easy to see that the function I satisfies the conditions (1), (2), (3) and (4) of Theorem 3.10. Thus $I(\Phi, \Psi) = i_0(\Phi, \Psi)I(u, v) = i_0(\Phi, \Psi)i_0(f, g)$. \square

For any power series $f, g \in \mathbb{K}[[x, y]]$ the ideal (f, g) generated by f and g is a \mathbb{K} -linear subspace of the algebra $\mathbb{K}[[x, y]]$.

Theorem 3.14 (Macaulay's Formula). *For every $f, g \in \mathbb{K}[[x, y]]$:*

$$i_0(f, g) = \dim_{\mathbb{K}} \mathbb{K}[[x, y]] / (f, g).$$

Proof. Let us denote by $I(f, g)$ the right side of the above equality (the codimension of the ideal generated by f, g). It is easy to see that the function I satisfies (1), (3) and (4) of Theorem 3.10 and $I(x, y) = 1$. Thus to check the theorem it suffices to prove property (2): $I(f, g_1g_2) = I(f, g_1) + I(f, g_2)$. If $I(f, g_1g_2) = \infty$ then f, g_1g_2 have a common prime divisor (see Appendix B). Then f, g_1 or f, g_2 have a common divisor and consequently $I(f, g_1) = \infty$ or $I(f, g_2) = \infty$.

Suppose that $I(f, g_1g_2) < \infty$ i.e. f, g_1g_2 are coprime. Recall the following fact of Linear Algebra. If U, V, W are \mathbb{K} -linear spaces such that $W \subset V \subset U$ and W have a finite codimension in U then

$$\dim_{\mathbb{K}} U/W = \dim_{\mathbb{K}} U/V + \dim_{\mathbb{K}} V/W.$$

Applying the above formula to $W = (f, g_1g_2), V = (f, g_1)$ and $U = \mathbb{K}[[x, y]]$ we get $I(f, g_1g_2) = I(f, g_1) + I(f, g_2)$ since $\dim_{\mathbb{K}} V/W = I(f, g_2)$. \square

Let $f, g \in \mathbb{K}[[x, y]]$ be power series without constant term. Let $\mathbb{K}((f, g))$ be the field of fractions of the ring $\mathbb{K}[[f, g]]$. Then $\mathbb{K}((f, g))$ is a subfield of the field $\mathbb{K}((x, y))$.

Theorem 3.15 (Weil’s Formula). *If power series f, g without constant term are coprime then*

$$i_0(f, g) = \left(\mathbb{K}((x, y)) : \mathbb{K}((f, g)) \right).$$

Proof. By Palamodov’s Theorem (see Appendix D) the extension $\mathbb{K}[[x, y]] \supset \mathbb{K}[[f, g]]$ is a free module of rank $\dim_{\mathbb{K}} \mathbb{K}[[x, y]] / (f, g)$. Thus Theorem 3.15 follows from Theorem 3.14 and Lemma 2.7. □

4. NEWTON DIAGRAMS AND PARAMETRIZATIONS OF ALGEBROID CURVES

In this section we sketch an approach to Newton’s study of plane curve singularities valid in arbitrary characteristic. A lucid and interesting introduction to Newton’s method is due to Teissier [9]. See also Teissier [10] where a systematic treatment of the subject is given and Cassou-Noguès, Płoski [3] for applications to invariants of singularities.

Let $\mathbb{R}_+ = \{a \in \mathbb{R} : a \geq 0\}$. For any subsets $\Delta, \Delta' \subset \mathbb{R}_+^2$ we consider the Minkowski sum $\Delta + \Delta' = \{u + v : u \in \Delta \text{ and } v \in \Delta'\}$. For any subset $E \subset \mathbb{N}^2$ we denote by $\Delta(E)$ the convex hull of the set $E + \mathbb{R}_+^2$. The sets of the form $\Delta(E)$, where $E \subset \mathbb{N}^2$ are called Newton diagrams. We use Teissier’s notation: $\left\{ \frac{k}{l} \right\} = \Delta(\{(k, 0), (0, l)\})$, $\left\{ \frac{k}{\infty} \right\} = \Delta(\{(k, 0)\}) = (k, 0) + \mathbb{R}_+^2$, $\left\{ \frac{\infty}{l} \right\} = \Delta(\{(0, l)\}) = (0, l) + \mathbb{R}_+^2$ for any integers $k, l > 0$. For any power series $f = \sum c_{\alpha\beta} x^\alpha y^\beta \in \mathbb{K}[[x, y]]$ we put $\text{supp } f = \{(\alpha, \beta) \in \mathbb{N}^2 : c_{\alpha, \beta} \neq 0\}$. It is easy to check that $\text{supp } fg \subset \text{supp } f + \text{supp } g$. The Newton diagram $\Delta_{x,y}(f)$ of a power series f is by definition $\Delta(\text{supp } f)$. Note that if the coordinates (x, y) are generic i.e. $\text{ord } f(x, 0) = \text{ord } f(0, y) = \text{ord } f$ then $\Delta_{x,y}(f) = \left\{ \frac{\text{ord } f}{\text{ord } f} \right\}$. The property of order: $\text{ord } fg = \text{ord } f + \text{ord } g$ may be generalized as follows:

Lemma 4.1. $\Delta_{x,y}(fg) = \Delta_{x,y}(f) + \Delta_{x,y}(g)$.

Proof. The rule of multiplication of formal power series implies the following two properties:

- (a) if $(\alpha, \beta) \in \text{supp } fg$ then $(\alpha, \beta) = (\alpha_1, \beta_1) + (\alpha_2, \beta_2)$, where $(\alpha_1, \beta_1) \in \text{supp } f$ and $(\alpha_2, \beta_2) \in \text{supp } g$,
- (b) if $(\alpha, \beta) \in \mathbb{N}^2$ has a unique representation $(\alpha, \beta) = (\alpha_1, \beta_1) + (\alpha_2, \beta_2)$ for some $(\alpha_1, \beta_1) \in \text{supp } f$ and $(\alpha_2, \beta_2) \in \text{supp } g$ then $(\alpha, \beta) \in \text{supp } fg$.

To abbreviate the notation we write Δ instead of $\Delta_{x,y}$. Note first that the set $\Delta(f) + \Delta(g)$ being the sum of two convex subsets of \mathbb{R}_+^2 is convex. From (a) we get $\text{supp } fg + \mathbb{R}_+^2 \subset (\text{supp } f + \mathbb{R}_+^2) + (\text{supp } g + \mathbb{R}_+^2) \subset \Delta(f) + \Delta(g)$ and consequently $\Delta(fg) \subset \Delta(f) + \Delta(g)$ since $\Delta(fg)$ is the smallest convex subset which contains $\text{supp } fg + \mathbb{R}_+^2$.

On the other hand if (α, β) is a vertex of $\Delta(f) + \Delta(g)$ then (α, β) has property (b) and $(\alpha, \beta) \in \text{supp } fg \subset \Delta(fg)$. Since the vertices of $\Delta(f) + \Delta(g)$ belong to $\Delta(fg)$ we get $\Delta(f) + \Delta(g) \subset \Delta(fg)$.

Summing up, we have $\Delta(fg) = \Delta(f) + \Delta(g)$. \square

Proposition 4.2. *Let $f \in \mathbb{K}[[x, y]]$ be an irreducible power series. Then*

$$\Delta_{x,y}(f) = \left\{ \frac{i_0(f, y)}{i_0(f, x)} \right\}.$$

Proof. If $f \sim x$ or $f \sim y$ then the proposition is obvious. Let $f(x, 0)f(0, y) \neq 0$ and put $m = \text{ord } f(x, 0)$, $n = \text{ord } f(0, y)$. Since $\Delta_{x,y}(f) = \Delta_{x,y}(fu)$ for any unit u we may assume by the Weierstrass Preparation Theorem that $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ is a distinguished polynomial. Let $(\phi(t), \psi(t))$ be a normalization of the branch $f = 0$. Then $\text{ord } \phi(t) = i_0(f, x) = n$ and $\text{ord } \psi(t) = i_0(f, y) = m$. By Corollary 2.3 there are nonzero power series $\alpha(s), \beta_1(s), \dots, \beta_n(s) \in \mathbb{K}[[s]]$ without constant term such that

$$y^n + a_1(\alpha(s))y^{n-1} + \dots + a_n(\alpha(s)) = (y - \beta_1(s)) \cdots (y - \beta_n(s)).$$

We have $\alpha(s) = \phi(\sigma_j(s))$, $\beta_j(s) = \psi(\sigma_j(s))$ for a $\sigma_j(s)$ without constant term. Thus we get $\text{ord } \beta_j(s) = \frac{\text{ord } \psi}{\text{ord } \phi} \text{ord } \alpha = \frac{m}{n} \text{ord } \alpha$ for $j = 1, \dots, n$. Let $k \in [1, n]$ be such that $a_k(x) \neq 0$. Then $a_k(\alpha(s)) = (-1)^k (\beta_1(s) \cdots \beta_k(s) + \dots)$ and $\text{ord } a_k(\alpha(s)) \geq \inf \{ \text{ord } \beta_{j_1} \cdots \beta_{j_k} : 1 \leq j_1 < \dots < j_k \leq n \} = k \frac{m}{n} \text{ord } \alpha$, which implies $\frac{\text{ord } a_k}{k} \geq \frac{m}{n} = \frac{i_0(f, y)}{i_0(f, x)}$ with equality for $k = n$. This proves the proposition. \square

Now we can pass to the main result of this section

Theorem 4.3. *Let $f \in \mathbb{K}[[x, y]]$ be a nonzero formal power series without constant term and let $f = f_1 \cdots f_r$ in $\mathbb{K}[[x, y]]$ with irreducible f_i , $i = 1, \dots, r$. Let $(\phi_i(t_i), \psi_i(t_i))$ be a normalization of the branch $f_i = 0$ for $i = 1, \dots, r$. Then*

$$\Delta_{x,y}(f) = \sum_{i=1}^r \left\{ \frac{\text{ord } \psi_i}{\text{ord } \phi_i} \right\}.$$

Proof. By Lemma 4.1 we get $\Delta_{x,y}(f) = \sum_{i=1}^r \Delta_{x,y}(f_i)$. On the other hand by Proposition 4.2 and the Normalization Theorem we have $\Delta_{x,y}(f_i) = \left\{ \frac{\text{ord } \psi_i}{\text{ord } \phi_i} \right\}$ for $i = 1, \dots, r$. \square

APPENDIX

Let \mathbb{K} be an arbitrary field not necessarily algebraically closed.

A. Factorization Lemma. *Suppose that a power series $f \in \mathbb{K}[[x, y]]$ satisfies the condition in $f = \phi\psi$, where ϕ, ψ are coprime homogeneous forms of positive degree. Then there exist $g, h \in \mathbb{K}[[x, y]]$ such that $f = gh$ in $\mathbb{K}[[x, y]]$, where $\text{in } g = \phi$, $\text{in } h = \psi$.*

The proof of the lemma is based on the following property:

Macauley’s property *If $\phi, \psi \in \mathbb{K}[x, y]$ are coprime homogeneous forms of degree $m > 0$ and $n > 0$ then every homogeneous form of degree $\geq m + n - 1$ can be written as $\alpha\phi + \beta\psi$, where α, β are homogeneous forms.*

Proof. Every homogeneous form χ of degree $\geq m + n - 1$ can be written as $\sum_{i+j=m+n-1} \chi_{ij}x^i y^j$, so it suffices to check Macauley’s property for forms of degree $m + n - 1$. Let H_k be the \mathbb{K} -linear space of homogeneous forms of degree k (by convention the zero is a homogeneous form of degree k for all k). The mapping

$$H_{n-1} \times H_{m-1} \ni (\alpha, \beta) \mapsto \alpha\phi + \beta\psi \in H_{m+n-1}$$

is a linear mapping of vector spaces of the same dimension $m + n$. Since the forms ϕ, ψ are coprime the mapping is injective. Hence, the mapping is also surjective. \square

Proof of Factorization Lemma. Write $f = f_{m+n} + f_{m+n+1} + \dots$. We are looking for power series g and h in the form $g = \phi_m + \phi_{m+1} + \dots$ and $h = \psi_n + \psi_{n+1} + \dots$, where $\phi_m = \phi$ and $\psi_n = \psi$. The equality $f = gh$ holds if and only if the following conditions are fulfilled

$$\begin{aligned} \phi_m \psi_n &= f_{m+n} \\ \phi_{m+1} \psi_n + \phi_m \psi_{n+1} &= f_{m+n+1} \\ \phi_{m+2} \psi_n + \phi_{m+1} \psi_{n+1} + \phi_m \psi_{n+2} &= f_{m+n+2} \\ \dots\dots\dots \end{aligned}$$

Applying Macauley’s property to the given $\phi_m = \phi, \psi_n = \psi$ and utilizing the above equations, first we find the forms ϕ_{m+1}, ψ_{n+1} , then the forms $\phi_{m+2}, \psi_{n+2}, \dots$. Proceeding in this way we get step by step the homogeneous components of g and h . \square

B. Elimination Lemma. *Let $f, g \in \mathbb{K}[[x, y]]$ be non-zero power series without constant term. Then f, g are coprime if and only if the following condition holds*

- (*) *there exist integers $d, d' > 0$ such that the monomials $x^d, y^{d'}$ lie in the ideal (f, g) generated by f and g in $\mathbb{K}[[x, y]]$.*

Proof. If $x^d, y^{d'} \in (f, g)$ then every divisor of f and g divides x^d and $y^{d'}$ so f, g are coprime. Suppose that f and g are coprime. Then $f(0, y) \neq 0$ or $g(0, y) \neq 0$

since if $f(0, y) = g(0, y) = 0$ in $\mathbb{K}[[y]]$ then x divides f and g . Suppose that $f(0, y) \neq 0$. Using the Weierstrass Preparation Theorem we may assume that $f = y^n + a_1(x)y^{n-1} + \dots + a_n(x)$ is a distinguished polynomial. Replacing g by the remainder of division by f , we get $g = b_0(x)y^{n-1} + \dots + b_{n-1}(x)$. Let $R(x)$ be the y -resultant of polynomials f, g . Then f, g are coprime as elements of $\mathbb{K}[[x]][y]$ and consequently $R(x) \neq 0$. Let $d = \text{ord } R(x)$. We get $x^d \in (f, g)$ since the resultant lies in the ideal generated by f and g . Similarly we check that $y^{d'} \in (f, g)$ for an integer $d' > 0$. \square

C. Prime ideals in the ring $\mathbb{K}[[x, y]]$. *Prime ideals in the ring $\mathbb{K}[[x, y]]$ are: (0) , maximal ideal $\mathcal{M} = (x, y)$ and principal ideals (f) generated by irreducible power series $f \in \mathbb{K}[[x, y]]$.*

Proof. Let I be a non-zero prime ideal of the ring $\mathbb{K}[[x, y]]$. Since the ring of power series is a unique factorization domain there exists an irreducible power series $f \in I$. If $I \neq (f)$ then there exists a power series $g \in I$ such that f does not divide g and hence the power series f, g are coprime. By the Elimination Lemma we get $x^d, y^{d'} \in (f, g) \subset I$ which implies $x, y \in I$ i.e. $I = (x, y)$ and we are done. \square

From the description of prime ideals it follows that the Krull dimension of $\mathbb{K}[[x, y]]$ is equal to 2.

D. Parameters of the ring $\mathbb{K}[[x, y]]$. Every ideal I of the ring $\mathbb{K}[[x, y]]$ is a \mathbb{K} -linear subspace of $\mathbb{K}[[x, y]]$ and its codimension $\text{codim } I = \dim_{\mathbb{K}} \mathbb{K}[[x, y]]/I$ is defined. The powers of the maximal ideal $\mathcal{M}^k = (x^k, x^{k-1}y, \dots, xy^{k-1}, y^k)$ have a finite codimension $\text{codim } \mathcal{M}^k = \frac{1}{2}k(k+1)$. It is easy to see that $\text{codim } I < \infty$ if and only if $I \supset \mathcal{M}^k$ for some $k \geq 0$ i.e. if I contains all monomials of degree big enough. A pair of power series f, g without constant term is a system of parameters (s.p.) of the ring $\mathbb{K}[[x, y]]$ if the ideal (f, g) has a finite codimension. This takes place if and only if $x^d, y^{d'} \in (f, g)$ for some $d, d' > 0$. Hence, from the Elimination Lemma it follows that a pair of power series f, g without constant term is a s.p. if and only if the series f, g are coprime.

Palamodov's Theorem *Let f, g be a s.p. of the ring $\mathbb{K}[[x, y]]$. Then $\mathbb{K}[[x, y]]$ is a finitely generated free module over $\mathbb{K}[[f, g]]$ whose rank is equal to the codimension of the ideal (f, g) .*

Proof. Let m be the codimension of the ideal $I = (f, g)$ and let e_1, \dots, e_m be a sequence of power series such that the images of e_1, \dots, e_m under the natural epimorphism $\mathbb{K}[[x, y]] \rightarrow \mathbb{K}[[x, y]]/I$ form a \mathbb{K} -linear basis of $\mathbb{K}[[x, y]]/I$. For any $h \in \mathbb{K}[[x, y]]$ there exist constants $c_1, \dots, c_m \in \mathbb{K}$ such that $h \equiv c_1e_1 + \dots + c_me_m \pmod{I}$. We put $A_i^0(u, v) = c_i$ for $i = 1, \dots, m$. We get

then

$$h = \sum_{i=1}^m c_i e_i + h_1 f + h_2 g \text{ in } \mathbb{K}[[x, y]]$$

and

$$h_1 \equiv \sum_{i=1}^m c_{1i} e_i \pmod{(f, g)},$$

$$h_2 \equiv \sum_{i=1}^m c_{2i} e_i \pmod{(f, g)}.$$

From the above relations we get:

$$h \equiv \sum_{i=1}^m c_i e_i + \sum_{i=1}^m (c_{1i} f) e_i + \sum_{i=1}^m (c_{2i} g) e_i \pmod{(f, g)^2}.$$

Let $A_i^1(u, v) = c_i + c_{1i}u + c_{2i}v$; so we get

$$h \equiv \sum_{i=1}^m A_i^1(f, g) e_i \pmod{(f, g)^2}.$$

In this way we define by induction the sequences of polynomials $A_i^k = A_i^k(u, v)$ ($i = 1, \dots, m, k = 0, 1, \dots, m$) such that:

- (1) $h \equiv \sum_{i=1}^m A_i^k(f, g) e_i \pmod{(f, g)^{k+1}},$
- (2) A_i^k is a polynomial of degree $\leq k$; $A_i^{k+1} - A_i^k$ is a homogeneous form of degree $k + 1$.

Let us put $A_i = \sum_{k \geq 0} (A_i^{k+1} - A_i^k) + c_i$ for $i = 1, \dots, m$. It is easy to show that

$$h = \sum_{i=1}^m A_i(f, g) e_i.$$

It remains to check that the above representation is unique. It suffices to prove that

$$\sum_{i=1}^m A_i(f, g) e_i = 0 \quad \Rightarrow \quad A_i(u, v) = 0 \text{ in } \mathbb{K}[[u, v]] \text{ for } i = 1, \dots, m.$$

Let us suppose, to get a contradiction, that the set $I_0 = \{i : A_i(u, v) \neq 0\}$ is not empty. We get

$$\sum_{i \in I_0} A_i(0, 0) e_i \equiv 0 \pmod{(f, g)}$$

hence $A_i(0, 0) = 0$ for $i \in I_0$. Dividing $A_i(u, v)$ by a sufficiently large power of u we may assume that $r = \inf\{\text{ord } A_i(0, v)\} < \infty$. We get $A_i(u, v) = A_i(0, v) + uq_i(u, v) = v^r c_i(v) + uq_i(u, v)$, where not all $c_i(0)$ are equal zero.

So we have

$$\sum_{i=1}^m g^r c_i(g) e_i + \sum_{i=1}^m f q_i(f, g) e_i = 0$$

and

$$g^r \left(\sum_{i=1}^m c_i(g) e_i \right) \equiv 0 \pmod{(f)}.$$

The power series f, g are coprime because they form a s.p. Therefore from the last relation we obtain

$$\sum_{i=1}^m c_i(g) e_i \equiv 0 \pmod{(f)}$$

and

$$\sum_{i=1}^m c_i(0) e_i \equiv 0 \pmod{(f, g)}$$

so we get $c_i(0) = 0$ for all $i = 1, \dots, m$, which is a contradiction.

An elementary treatment of parameters in power series ring in n variables is given in [7].

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