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ON COMBINATORIAL CRITERIA FOR ISOLATED SINGULARITIES

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Abstract. In this article we review combinatorial characterizations of isolated singularities. As a new result in two and three-dimensional case we give sufficient and necessary conditions for a nondegenerate singularity to be isolated in terms of its support. We also prove new sufficient conditions in the multidimensional case.

1. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. One of the problems in the theory of singularities is to check effectively that f is an isolated singularity. Many authors give different conditions to deal with this problem. For instance by the local Nullstellensatz f is an isolated singularity if and only if the Milnor number $\mu(f)$ is finite. Similarly the Łojasiewicz exponent $\mathcal{L}_0(f)$ is finite if and only if f is an isolated singularity. In this paper we review combinatorial conditions related to the support of an isolated singularity and give some new results in the nondegenerate class (for definitions see Preliminaries).

Kouchnirenko in [Ko77] gave for a set $M \subset \mathbb{N}^n$ a necessary and sufficient conditions that there exists an isolated singularity f with $\operatorname{supp} f \subset M$ (see Thm. 3.9). Other authors: Wall ([Wa96]), Orlik and Randell ([OR76]), Shcherbak ([Sh79]) obtained similar results. In Remark 3.11 we comment on the history of these results.

The quasihomogeneous case was considered by the authors named above as well as by Saito ([Sa71], [Sa87]), Krezuer and Skarke ([KS92]), Hertling and Kurbel ([HK12]). In this class of singularities we recall the necessary condition for the

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weights so that the singularity is isolated, which turns out sufficient in the two and three-dimensional case (see Thm. 4.2).

In section 5 we examine the problem in the class of nondegenerate singularities and give some new results. For dimension $n \leq 3$ we prove necessary and sufficient conditions for the support of a nondegenerate singularity so that the singularity is isolated (see Thm 5.4). It seems that for $n \geq 4$ Theorem 5.4 is also true (see Conj. 5.5). For higher dimensions we give only sufficient conditions (see Thm. 5.6). Wall considered another type of nondegeneracy than the Kouchnirenko nondegeneracy. He got similar results to the ones obtained in Section 5 (see Lem. 1.2 and Thm. 1.4 in [Wa98]).

In the last section using Remark 1.13 (ii) in [Ko76] we reformulate the results of the previous section in terms of the Newton number (see Cor. 6.2, Prop. 6.3, 6.4).

2. Preliminaries

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a nonzero holomorphic function in an open neighborhood of $0 \in \mathbb{C}^n$. We say that f is *a singularity* if $f(0) = 0$, $\nabla f(0) = 0$, where $\nabla f = (f'_{z_1}, \ldots, f'_{z_n})$. We say that f is *an isolated singularity* if f is a singularity, which has an isolated critical point in the origin i.e. additionally $\nabla f(z) \neq 0$ for $z \neq 0$ near 0. We note $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. We define the set $\operatorname{supp} f = \{\nu \in \mathbb{N}^n : a_\nu \neq 0\}$ and call it *the support of* f. Let $w_1, \ldots, w_n, d$ be positive integer numbers. The polynomial $f \in C[z_1, \ldots, z_n]$ is called *quasihomogeneous with weight system* $(w_1, \ldots, w_n, d)$ if

$$\sum_{i=1}^{n} \nu_i w_i = d \quad \text{for any } \nu \in \operatorname{supp} f.$$

We define

$$\Gamma_+(f) = \operatorname{conv}\{\nu + \mathbb{R}^n_+ : \nu \in \operatorname{supp} f\} \subset \mathbb{R}^n$$

and call it *the Newton diagram of* f. Let $u \in \mathbb{R}^n_+ \setminus \{0\}$. Put

$$l(u, \Gamma_+(f)) = \inf\{\langle u, v\rangle : v \in \Gamma_+(f)\},$$
$$\Delta(u, \Gamma_+(f)) = \{v \in \Gamma_+(f) : \langle u, v\rangle = l(u, \Gamma_+(f))\}.$$

We say that $S \subset \mathbb{R}^n$ is *a face of* $\Gamma_+(f)$ if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbb{R}^n_+ \setminus \{0\}$. The vector u is called *the primitive vector of* S. It is easy to see that S is a closed and convex set and $S \subset \operatorname{Fr}(\Gamma_+(f))$, where $\operatorname{Fr}(A)$ denotes the boundary of A. One can prove that a face $S \subset \Gamma_+(f)$ is compact if and only if all coordinates of its primitive vector u are positive. We call the family of all compact faces of $\Gamma_+(f)$ *the Newton boundary of* f and denote by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k = 0, \ldots, n-1$. For every compact face $S \in \Gamma(f)$ we define quasihomogeneous polynomial $f_S = \sum_{\nu \in S} a_\nu z^\nu$. We say that f is *nondegenerate on the face* $S \in \Gamma(f)$ if the system of equations

$$\frac{\partial f_S}{\partial z_1} = \ldots = \frac{\partial f_S}{\partial z_n} = 0$$

has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that f is *nondegenerate in the sense of Kouchnirenko* (shortly *nondegenerate*) if it is nondegenerate on each face of $\Gamma(f)$. We say that f is *convenient* if $\Gamma_+(f)$ has nonempty intersection with every coordinate axis. We say that f is *nearly convenient* if the distance of $\Gamma_+(f)$ to every coordinate axis does not exceed 1. Denote by $\mathcal{O}^n$ the local ring of germs of holomorphic functions in n-variables at $0 \in \mathbb{C}^n$. Let us recall that *the Milnor Number* $\mu(f)$ and *the Newton number* $\nu(f)$ are defined as

$$\mu(f) = \dim \mathcal{O}^n/(f'_{z_1}, \ldots, f'_{z_n}), \quad \nu(f) = n!V_n - (n-1)!V_{n-1} + \ldots + (-1)^n V_0,$$

where V_i denotes the sum of i-dimensional volumes of the intersection of the cone spanned by $\Gamma_+(f)$ with the coordinate subspace of dimension i.

3. Generic case

In this section we recall some known results dealing with support of isolated singularities. Kouchnirenko in [Ko77, Thm 1] gave for a set $M \subset \mathbb{N}^n$ necessary and sufficient conditions so that there exists an isolated singularity f with $\operatorname{supp} f \subset M$. Moreover, every singularity f with $\operatorname{supp} f \subset M$ and generic coefficients is isolated. Before giving his result we start with some notions and definitions.

Let $M \subset \mathbb{N}^n$. Define the sets $M_i = \{\nu \in \mathbb{N}^n : \nu + e_i \in M\}$, where $e_i, i = 1, \ldots, n$, is the standard basis in $\mathbb{R}^n$. Notice that if we take $f_M = \sum_{m \in M} z^m$ then $M_i = \operatorname{supp} \partial f_M / \partial z_i$ for every $i = 1, 2, \ldots, n$. Let $I \subset \{1, \ldots, n\}$. Set

$$OX_I = \{x \in \mathbb{R}^n \colon x_i = 0, i \notin I\}.$$

Observe that OX_I is the hyperplane spanned by axes $OX_i, i \in I$.

Let $I \subset \{1, 2, \ldots, n\}$. We say that M *satisfies the Kouchnirenko condition for* I if there exist at least $|I|$ nonempty sets among the sets $M_1 \cap OX_I, \ldots, M_n \cap OX_I$. We say that M *satisfies the Kouchnirenko condition* if M satisfies the Kouchnirenko condition for every $I \subset \{1, 2, \ldots, n\}$.

Remark 3.1. It is easy to check that M satisfies the Kouchnirenko condition if and only if a finite subset of M satisfies the Kouchnirenko condition.

Remark 3.2. If M satisfies the Kouchnirenko condition, it can happen that the singularity f_M is not an isolated singularity. For example let $f_M = (z_1+z_2)(z_3+z_1)$. It is easy to check that f is not isolated singularity and is degenerate on the face S determined by $f_S = z_3(z_1 + z_2)$.

Example 3.3. a) Let $f(z_1, z_2) = z_1^2 + z_1 z_2$. We show that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Put $M = \operatorname{supp} f$. Then $M_1 = \{(0,1), (1,0)\}$, $M_2 = \{(1,0)\}$. If $I = \{1,2\}$ or $I = \emptyset$ we easily check that M satisfies the Kouchnirenko condition. If $I = \{1\}$, then $M_2 \cap OX_2 \neq \emptyset$. If $I = \{2\}$, then $M_1 \cap OX_1 \neq \emptyset$.

b) Let $f(z_1, z_2, z_3) = z_1(z_1 + z_2 + z_3)$. We show that $\operatorname{supp} f$ does not satisfy the Kouchnirenko condition. Indeed, take $I = \{2, 3\}$ then $|I| = 2$ but only $M_1 \cap OX_I \neq \emptyset$.

Now we explain the Kouchnirenko condition for I in the border cases $|I| = 1$ and $|I| = n$.

Property 3.4. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a singularity. We have the following properties:*

(i) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition for every $I = \{i\}, i = 1, 2, \ldots, n$ if and only if f is nearly convenient,*
(ii) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition for $I = \{1, 2, \ldots, n\}$ if and only if $f'_{z_i} \neq 0$, $i = 1, 2, \ldots, n$.*

Proof.

(i) Put $M = \operatorname{supp} f$. Suppose that M satisfies the Kouchnirenko condition for every $I = \{i\}, i = 1, 2, \ldots, n$. It is equivalent to saying that for every $i = 1, 2, \ldots, n$, there exists j_i such that $M_{j_i} \cap OX_i \neq \emptyset$. This condition is equivalent to the condition that there exists a vertex of $\Gamma_+(f)$ lying on the plane $OX_{j_i}X_i$ at most at distance 1 to OX_i.

(ii) It is a direct consequence of the definition of the Kouchnirenko condition. ■

The following property shows that the Kouchnirenko condition for $\operatorname{supp} f$ implies that the Newton diagram of a singularity f has non-empty intersection with every coordinate hyperplane in $\mathbb{R}^n$, $n \geq 3$.

Property 3.5. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0), n \geq 3$, be a singularity. If $\operatorname{supp} f$ satisfies the Kouchnirenko condition then $\Gamma_+(f) \cap OX_I \neq \emptyset$ for every set $I \subset \{1, 2, \ldots, n\}, |I| = n - 1$.*

Proof. Put $M = \operatorname{supp} f$. Suppose that M satisfies the Kouchnirenko condition. Without loss of generality it suffices to show $\Gamma_+(f) \cap OX_I \neq \emptyset$ for $I = \{2, 3, \ldots, n\}$. Indeed, by the Kouchnirenko condition there exist at least $n - 1$ nonempty sets among the sets $M_1 \cap OX_I, \ldots, M_n \cap OX_I$. Since $n \geq 3$ there exists $i \neq 1$ such that $M_i \cap OX_I \neq \emptyset$. Let $A \in M_i \cap OX_I$ for some $i \neq 1$. Since $i \neq 1$ then $A - e_i \in M \cap OX_I$. Hence $\Gamma_+(f) \cap OX_I \neq \emptyset$. It ends the proof. ■

The two following propositions give conditions equivalent to the Kouchnirenko condition for $\operatorname{supp} f$ in terms of the Newton diagram of singularity f in two and three variables.

Proposition 3.6. *Let $f : \left(\mathbb{C}^2, 0\right) \longrightarrow (\mathbb{C}, 0)$ be a singularity. Then the following conditions are equivalent:*

(i) *f is nearly convenient,*
(ii) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Property 3.4(i). Now let us suppose that the condition (i) is satisfied. Let $I \subset \{1, 2\}$. For $I = \emptyset$ or $I = \{1, 2\}$ then it is easy to see that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. If $I = \{1\}$

or $I = \{2\}$ then by Property 3.4(i) we get that $\operatorname{supp} f$ satisfies the Kouchnirenko condition for such I. ■

Proposition 3.7. *Let $f : \left(\mathbb{C}^3, 0\right) \longrightarrow (\mathbb{C}, 0)$ be a singularity. Then the following conditions are equaivalent:*

(i) *f is nearly convenient and $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$ for every $i, j \in \{1, 2, 3\}$, $i \neq j$,*
(ii) *$\operatorname{supp} f$ satisfies the Kouchnirenko condition.*

Proof. Put $M = \operatorname{supp} f$. The implication $(ii) \Rightarrow (i)$ follows from Properties 3.4(i) and 3.5. Now let us suppose that the condition (i) is satisfied and take $I \subset \{1, 2, 3\}$. If $I = \emptyset$ or $I = \{1, 2, 3\}$ then it is easy to check that M satisfies the Kouchnirenko condition for such I. If $I = \{i\}$ for some $i \in \{1, 2, 3\}$ then by Property 3.4(i) M satisfies the Kouchnirenko condition for such I. Now let $I = \{1, 2, 3\} \setminus \{i\}$ for some $i \in \{1, 2, 3\}$. Without loss of generality we may assume that $i = 1$. Since f is nearly convenient we can choose points $A, B \in \operatorname{supp} f$ such that $\operatorname{dist}(A, OX_2) \leq 1$ and $\operatorname{dist}(B, OX_3) \leq 1$. Consider the following cases:

(a) $A, B \in OX_2X_3$. Then $M_2 \cap OX_2X_3 \neq \emptyset$ and $M_3 \cap OX_2X_3 \neq \emptyset$. Hence M satisfies the Kouchnirenko condition for I in this case.
(b) $A \in OX_2X_3$ and $B \notin OX_2X_3$. Since $A \in OX_2X_3$ and $\operatorname{dist}(A, OX_2) \leq 1$ then $M_2 \cap OX_2X_3 \neq \emptyset$. Since $B \notin OX_2X_3$ and $\operatorname{dist}(B, OX_3) \leq 1$then $B \in OX_1X_3$ and B is at distance 1 to OX_3. Therefore $M_1 \cap OX_2X_3 \neq \emptyset$. Summing up M satisfies the Kouchnirenko condition for I in this case. (We consider analogously the case $A \notin OX_2X_3$ and $B \in OX_2X_3$.)
(c) $A \notin OX_2X_3$ and $B \notin OX_2X_3$. Then $A, B \in OX_1X_3$ and are at distance 1 to OX_3. Hence $M_1 \cap OX_2X_3 \neq \emptyset$. Since $\Gamma_+(f) \cap OX_2X_3 \neq \emptyset$ then there exists $C \in \operatorname{supp} f \cap OX_2X_3$. Therefore $M_j \cap OX_2X_3 \neq \emptyset$ for some $j \in \{2, 3\}$. Summing up M satisfies the Kouchnirenko condition for I in this case.

■

There are some equivalent combinatorial conditions to the Kouchnirenko condition. Hertling and Kurbel collected such conditions for quasihomogeneous polynomial in [HK12, Lemma 2.1] but this lemma is also true without the assumption of quasihomogeneity. Now we give a refined version of their lemma.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $|x| = |x_1| + \ldots + |x_n|$.

Lemma 3.8. *Let $M \subset \mathbb{N}^n$ and $|m| \geq 2$, $m \in M$. Then the following conditions are equaivalent.*

(K) *M satisfies the Kouchnirenko condition.*
(K') *M satisfies the Kouchnirenko condition for every $I \subset \{1, 2, \ldots, n\}$ such that $|I| \leq \frac{n+1}{2}$.*

(C1) *For every nonempty set* $I \subset \{1, 2, \ldots, n\}$ *we have* $M \cap OX_I \neq \emptyset$ *or there exists* $K \subset \{1, 2, \ldots, n\} \setminus I$ *with* $|K| = |I|$ *such that* $M_k \cap OX_I \neq \emptyset$ *for every* $k \in K$.

(C1') *As (C1), but only* I *with* $|I| \leq \frac{n+1}{2}$.

(C2) *For every* $I, J \subset \{1, 2, \ldots, n\}$ *with* $|I| < |J|$ *there exists* $k \in \{1, 2, \ldots, n\} \setminus I$ *such that* $M_k \cap OX_J \neq \emptyset$.

The proof is the same as the proof of [HK12, Lemma 2.1].

Now we give [Ko77, Thm. 1] in a slightly refined version.

Theorem 3.9. *Let* $M \subset \mathbb{N}^n$ *and* $|m| \geq 2$ *for every* $m \in M$. *Then the following conditions are equivalent.*

(ISe) *There exists an isolated singularity* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *such that* $\operatorname{supp} f \subset M$.

(ISg) *A singularity* f, $\operatorname{supp} f \subset M$ *with generic coefficients is an isolated singularity.*

(K) M *satisfies the Kouchnirenko condition.*

Remark 3.10. f_M is a singularity if and only if $|m| \geq 2$ for every $m \in M$.

Remark 3.11. (This remark is a slightly refined part of [HK12, Remarks 2.3]) Several people discovered parts of Theorem 3.9. We will not prove this theorem here, but comment on its history and references.

(i) The implication $(ISe) \Rightarrow (K)$ is a consequence of [Ko76, Thm. I] and [Ko76, Remarque 1.13 (ii)], but the Kouchnirenko did not carry out the explanation of [Ko76, Remarque 1.13 (ii)] in detail. He gave a short proof of the refined version $(ISe) \Leftrightarrow (K')$ in [Ko77, Thm. 1]. This reference [Ko77] seems to have been cited up to now only in [Sh79], it seems to have been almost completely ignored.

(ii) Around the same time as Kouchnirenko, Orlik and Randell proved $(ISe) \Leftrightarrow (C2)$ in the preprint [OR76, Thm. 2.12], but the published paper [OR77] does not contain this result. It seems that they have not published this result.

(iii) O.P. Shcherbak stated a result for maps [Sh79, Thm. 1] from which one can extract $(ISe) \Leftrightarrow (C1)$, but he did not provide a proof. This was done by Wall [Wa96, Chap. 5], who also stated explicitly $(ISe) \Leftrightarrow (ISg) \Leftrightarrow (C1)$ for maps in [Wa96, Thm. 5-1] and quasihomogeneous version of $(ISe) \Leftrightarrow (ISg) \Leftrightarrow (C1)$ for maps in [Wa96, Thm. 5-3]. The hypersurface case was done by Wall explicitly in [Wa96, (5-7)].
(For details see Section 4.)

(iv) A short proof valid only in quasihomogeneous case of $(ISg) \Leftrightarrow (C1)$ is given by Kreuzer and Skarke [KS92, proof of Thm. 1]. Although it requires some work to see that the condition stated in [KS92, Thm. 1] is equivalent to $(C1)$.

As a direct consequence of Theorem 3.9 we have the following corollary.

Corollary 3.12. *The support of an isolated singularity f satisfies the Kouchnirenko condition.*

Proof. Put $M = \operatorname{supp} f$. Suppose to the contrary, there exists $I \subset \{1, \dots, n\}$ such that there are exactly $p < |I|$ nonempty sets $M_{j_1} \cap OX_I, \dots, M_{j_p} \cap OX_I$ among the sets $M_i \cap OX_i$, $i = 1, 2, \dots, n$. Therefore $M_k \cap OX_I = \emptyset$ for $k \in \{1, 2, \dots n\} \setminus \{j_1, \dots, j_p\}$. For such k we obviously get

$$(1) \qquad \frac{\partial f}{\partial z_k} = \sum_{i \notin I} z_i h_i \quad \text{and hence} \quad \{z \in \mathbb{C}^n : z_i = 0, i \notin I\} \subset \left\{ \frac{\partial f}{\partial z_k} = 0 \right\},$$

for some $h_i \in \mathcal{O}^n$. Substitute $z_i = 0$ for $i \notin I$ to the system of equations:

$$\frac{\partial f}{\partial z_{j_1}} = \cdots = \frac{\partial f}{\partial z_{j_p}} = 0.$$

We get a system of p equations with $|I|$ variables. Therefore by (1) and Corollary 8 in [G, p. 81] we get

$$\dim\{\nabla f = 0\} \geq |I| - p > 0,$$

which contradicts the assumption that zero of ∇f is isolated. ■

Remark 3.13. Saito proved that a support of an isolated singularity f satisfies condition (C1), which by Lemma 3.8 is equivalent to the Kouchnirenko condition (see Lemma 1.5 in [Sa71]). It can also be extracted from Remark 3 in [Sh79].

As a direct consequence of the above corollary and Property 3.4(i) we give the following property.

Property 3.14. *Every isolated singularity f is nearly convenient.*

4. Quasihomogeneous case

Quasihomogeneous singularities are a special class of singularities. Obviously to determine when they are isolated we may check whether they satisfy the Kouchnirenko condition. However, we would like to give combinatorial conditions in terms of their weights instead. By Milnor-Orlik formula [MO70] for quasihomogeneous isolated singularities the Milnor number $\mu(f)$ is equal to $\prod_{i=1}^n [(d/w_i) - 1]$. Hence a first necessary condition is that $\prod_{i=1}^n [(d/w_i) - 1]$ is a positive integer number. It is not a sufficient condition which the example below shows.

Example 4.1. Let $f(z_1, z_2, z_3) = z_1^5 + z_2^4 + z_1^2 z_3^2$. It is a quasihomogeneous polynomial with weight system $(4, 5, 6, 20)$ and

$$\left(\frac{20}{4} - 1\right)\left(\frac{20}{5} - 1\right)\left(\frac{20}{6} - 1\right) = 28 \in \mathbb{N}.$$

On the other hand f is not nearly convenient. Hence by Property 3.14 the singularity f is not an isolated singularity. ■

A good tool to examine whether singularities are isolated is the Poincaré function. For quasihomogeneous polynomial with weight system $(w_1, \ldots, w_n, d)$, $w_i < d, i = 1, 2, \ldots, n$, the Poincaré function is a rational function

$$\rho_{w,d}(t) = \prod_{i=1}^{n} \frac{(t^d - t^{w_i})}{(t^{w_i} - 1)}.$$

It is well known that if there exists a quasihomogeneous isolated singularity with weight system $(w_1, \ldots, w_n, d)$ then $\rho_{w,d}(t) \in \mathbb{N}[t]$ (see [AGV] or [Bou, Chap. V, sec. 5.1). Hence we have a second necessary condition for quasihomogeneous singularities to be isolated. It turns out that for dimensions $n = 2, 3$, it is also a sufficient condition.

Theorem 4.2. [Sa87, Thm. 3] *Let $(w_1, \ldots, w_n, d), w_i < d, i = 1, 2, \ldots, n$ be a weight system and $n \leq 3$. Then $\rho_{w,d}(t) \in \mathbb{Z}[t]$ if and only if there exists an isolated quasihomogeneous singularity with weight system $(w_1, \ldots, w_n, d)$.*

Remark 4.3. The above theorem is also stated in [Ar74, remark after Cor. 4.13] and [AGV, 2nd remark in 12.3].

The condition $\rho_{w,d}(t) \in \mathbb{Z}[t]$ is equivalent to a simple numerical condition.

Lemma 4.4. ([HK12], Lemma 2.4) *Let $(w_1, \ldots, w_n, d), w_i < d, i = 1, 2, \ldots, n$ be a weight system. The following conditions are equivalent:*

(P) $\rho_{w,d}(t) \in \mathbb{Z}[t]$,

(GCD) *for every $J \subset \{1, \ldots, n\}$ the* $\gcd\{w_j : j \in J\}$ *divides at least $|J|$ of the numbers $d - w_k, k = 1, \ldots, n$.*

Example 4.5. For the quasihomogeneous singularity $f(z_1, z_2, z_3) = z_1^5 + z_2^4 + z_1^2 z_3^2$ with weight system $(4, 5, 6, 20)$ from Example 4.1 the condition (GCD) is not satisfied. Indeed, take $J = \{3\}$, then $w_3 = 6$ does not divide any of numbers: $d - w_1 = 15, d - w_2 = 16, d - w_3 = 14$. Hence by the above lemma $\rho_{w,d}(t) \notin \mathbb{Z}[t]$ and by Theorem 4.2 there is no isolated quasihomogeneous singularity with such weight system.

On the other hand for quasihomogeneous singularity $f(z_1, z_2, z_3) = z_1^5 + z_2^4 + z_1 z_3^2$ with weight system $(4, 5, 8, 20)$ we easily check the condition (GCD) is satisfied. Therefore by Theorem 4.2 and Theorem 3.9 a quasihomogeneous singularity with weight system $(4, 5, 8, 20)$ with generic coefficients is an isolated singularity. ■

For $n \geq 4$ the condition $\rho_{w,d}(t) \in \mathbb{Z}[t]$ is not a sufficient condition in Theorem 4.2. See the following example which comes from [AGV, 12.3] and was given by Ivlev.

Example 4.6. Let $f(z_1, z_2, z_3, z_4) = z_1^{265} + z_2^8 z_1 + z_3^4 z_2 + z_4^{11} z_1$. It is a quasihomogeneous singularity with weight system $(1, 33, 58, 24, 265)$. We easily check that f satisfies (GCD) condition and hence by Lemma 4.4 the Poincaré function

$\rho_{w,d}(t) \in \mathbb{Z}[t]$. On the other hand, $\operatorname{supp} f$ does not satisfy the Kouchnirenko condition for $I = \{2,4\}$ since only $OX_I \cap \operatorname{supp} f'_{z_1} \neq \emptyset$. Therefore, by Corollary 3.12, f cannot be an isolated singularity.

5. Nondegenarate class

In the previous sections we examined the characterization of isolated singularities in the case of generic coefficients. In this section we will consider the same problem for fixed coefficients in the class of nondegenerate singularities. Precisely, we take a nondegenerate singularity $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ and ask if there exist combinatorial conditions for the support of f, which imply (or are equivalent) to f being an isolated singularity. For dimensions $n = 2, 3$ we give such equivalent conditions.

Theorem 5.1. *Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) *f is an isolated singularity,*
(b) *f is nearly convenient.*

Remark 5.2. The definition of near convenience for $n = 2$ appeared for the first time in [Len96] and Theorem 5.1 was stated in this paper. See also [Len08].

Theorem 5.3. [BKO] *Let $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) *f is an isolated singularity,*
(b) *f is nearly convenient and $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$, $i, j \in \{1, 2, 3\}$, $i \neq j$.*

By Properties 3.6, 3.7 we can merge Theorems 5.1 and 5.3 in one following theorem.

Theorem 5.4. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \leq 3$, be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition,*
(b) *f is an isolated singularity.*

The proof of the above theorem is given after the proof of Theorem 5.6. It seems that for $n \geq 4$ Theorem 5.4 is also true. Therefore we may state the following conjecture.

Conjecture 5.5. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition,*
(b) *f is an isolated singularity.*

Now, we give some sufficient combinatorial conditions for nondegenerate singularity to be isolated.

Theorem 5.6. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, be a nondegenerate singularity such that*

(i) *f is nearly convenient,*
(ii) $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$, $i, j \in \{1, \ldots, n\}$, $i \neq j$.

Then f is an isolated singularity.

Remark 5.7. Observe that condition (ii) only is not necessary for an isolated singularity. Indeed, take $f(z_1, z_2, z_3, z_4) = z_1z_2 + z_3z_4$. Of course, f is an isolated singularity, but does not satisfy the condition (ii).

Since every convenient singularity satisfies the conditions (i) and (ii), as a direct consequence of the above theorem we have the following corollary.

Corollary 5.8. *Every convenient nondegenarate singularity is an isolated singularity.*

To prove Theorem 5.6 we give some lemmas and properties. Most of them can be found in [O13] and [BKO] but we repeat them for the convenience of the reader in slightly refined versions. For a series $\phi \in \mathbb{C}\{t\}$, $\phi \neq 0$, by $\operatorname{info}\phi$ (resp. $\operatorname{inco}\phi$) we mean the initial form of ϕ (resp. the coefficient of $\operatorname{info}\phi$). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a nonzero holomorphic function in an open neighborhood of $0 \in \mathbb{C}^n$ and let $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. Let $w = (w_1, \ldots, w_n) \in (\mathbb{N}_+)^n$. We define the number

$$\operatorname{ord}_w f = \inf\{\nu_1 w_1 + \ldots + \nu_n w_n : \nu = (\nu_1, \ldots, \nu_n) \in \operatorname{supp} f\}$$

and we call it the *order of f with respect to w*. The sum of such monomials $a_{\nu_1 \ldots \nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n}$ for which $\nu_1 w_1 + \ldots + \nu_n w_n = \operatorname{ord}_w f$ is called the *initial form of f with respect to w* and is denoted by $\operatorname{info}_w f$. Now we give two simple and useful properties. We omit their easy proofs.

Property 5.9. (see Property 2.1 in [O13]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $f(0) = 0$ and $\phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n$ be a parametrization such that $\phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \ldots, n$. Put $w = (\operatorname{ord}\phi_i)_{i=1}^n$. If $\operatorname{info}_w f \circ \operatorname{info}\phi \neq 0$, then*

$$\operatorname{info}(f \circ \phi) = \operatorname{info}_w f \circ \operatorname{info}\phi, \quad \operatorname{ord}(f \circ \phi) = \operatorname{ord}_w f.$$

Property 5.10. (see Property 2.2 in [O13]) *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $f(0) = 0$, $w \in (\mathbb{N} \setminus \{0\})^n$, $i \in \{1, \ldots, n\}$. Suppose that $\operatorname{info}_w f$ depends on z_i, then*

$$(\operatorname{info}_w f)'_{z_i} = \operatorname{info}_w f'_{z_i}.$$

The following lemma is used in the proof of Lemma 5.14, which in turn is the main tool in the proof of Theorem 5.6.

Lemma 5.11. (see Lemma 2.3 in [O13]) *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, be a singularity and $\phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n$ be a parameterization such that $\phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \ldots, n$. Put $w = (\operatorname{ord}\phi_i)_{i=1}^n$ and*

$$K = \{i \in \{1, \ldots, n\} : f'_{z_i} \circ \phi = 0\} \neq \emptyset.$$

Then for the face $S = \Delta(w, \Gamma_+(f)) \in \Gamma(f)$ we get that $(f_S)'_{z_i} \circ \operatorname{info}\phi = 0$ for $i \in K$.

Proof. Put $J = \{j \in K : S \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j = 0\}\}$. Then for every $i \in K \setminus J$ we can find a monomial in $\operatorname{info}_w f$ in which the variable z_i appears. Therefore by Property 5.10 we get $(\operatorname{info}_w f)'_{z_i} = \operatorname{info}_w f'_{z_i}$ for $i \in K \setminus J$. Therefore by Property 5.9 we get for $i \in K \setminus J$

$$0 = \operatorname{info}_w f'_{z_i} \circ \operatorname{info}\phi = (\operatorname{info}_w f)'_{z_i} \circ \operatorname{info}\phi = (f_S)'_{z_i} \circ \operatorname{info}\phi.$$

On the other hand $(f_S)'_{z_i} \circ \operatorname{info}\phi = 0$, for $i \in J$. ■

The following proposition is a direct consequence of the above lemma.

Proposition 5.12. (see Corollary 2.4 in [O13]) *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, be a singularity and $\phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n$ be a parametrization such that $\phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \dots, n$. If $(\nabla f) \circ \phi = 0$, then there exists a face $S \in \Gamma(f)$ such that $(\nabla f_S) \circ \operatorname{info}\phi = 0$. Thus f is degenerate on the face S.*

The following well-known property says that the Newton boundary of the restriction $f|_{\{z_{k+1} = \ldots = z_n = 0\}}$ is the restriction of the Newton boundary of f to the set $\{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_{k+1} = \ldots = x_n = 0\}$.

Property 5.13. *Let $f \in \mathcal{O}^n$, $n \geq 2$. Assume that $g(z_1, \dots, z_k) = f(z_1, \dots, z_k, 0, \dots, 0) \in \mathcal{O}^k$, $k < n$, is a nonzero germ. Then*

$$\Gamma(g) = \{S \in \Gamma(f) : S \subset \{x_{k+1} = \ldots = x_n = 0\}\}. \tag{2}$$

Proof. " $\subset$ ". Let $S \in \Gamma(g)$, then $S = \Delta(u, \Gamma_+(g))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^k$. Of course, $S \subset \Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\}$. Set

$$u' = (u_1, \dots, u_k, l(u, \Gamma_+(g)) + 1, \dots, l(u, \Gamma_+(g)) + 1) \in \mathbb{R}^n.$$

We show that $S = \Delta(u', \Gamma_+(f))$. By definition of u' we have that $l(u', \Gamma_+(f))$ can be attained only for $v \in \Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\}$. On the other hand it is easy to check that

$$\Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\} = \Gamma_+(g).$$

So we get $l(u', \Gamma_+(f)) = l(u, \Gamma_+(g))$ and $\Delta(u', \Gamma_+(f)) = \Delta(u, \Gamma_+(g))$. Summing up we obtain $S = \Delta(u', \Gamma_+(f))$, so $S \in \Gamma(f)$.

" $\supset$ ". Let $S \in \Gamma(f)$ and $S \subset \{x_{k+1} = \ldots = x_n = 0\}$. Then $S = \Delta(u, \Gamma_+(f))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^n$ and as we observed above $\Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\} = \Gamma_+(g)$. So $l(u, \Gamma_+(f)) = l(u', \Gamma_+(g))$, where $u' = (u_1, \dots, u_k)$. It follows that $\Delta(u', \Gamma_+(g)) = \Delta(u, \Gamma_+(f))$. Hence $S = \Delta(u', \Gamma_+(g))$, so $S \in \Gamma(g)$. That ends the proof. ■

Denote $OZ_iZ_j = \{z \in \mathbb{C}^n : z_k = 0, k \notin \{i, j\}\}$, $i \neq j$, $i, j = 1, 2, \dots n$. The following lemma is a stronger version of Proposition 5.12.

Lemma 5.14. (see Lemma 4.3 in [BKO]) *Let $f \in \mathcal{O}^n$, $n \geq 2$, be a singularity and $\nabla f \circ \phi = 0$ for some $\phi = (\phi_1, \dots, \phi_n) \in \mathbb{C}\{t\}^n$, $\phi(0) = 0$. Assume there exist $i \neq j$, such that $\phi_i \neq 0, \phi_j \neq 0$ and $f_{|OZ_iZ_j} \not\equiv 0$. Then there exists $S \in \Gamma(f)$ on which f is degenerate.*

Proof. For simplicity we may assume that $\phi_1, \dots, \phi_k \neq 0$, $\phi_{k+1} = \ldots = \phi_n = 0$ for some $k \geq 2$. We can represent f in the form

$$f(z_1, \dots, z_n) = g(z_1, \dots, z_k) + z_{k+1}h_{k+1}(z_1, \dots, z_n) + \ldots + z_n h_n(z_1, \dots, z_n)$$

By the assumption we get $g \neq 0$, $g(0) = 0$, $\nabla g(\phi_1, \dots, \phi_k) = 0$. By Proposition 5.12 there exists $S \in \Gamma(g)$, such that $(\operatorname{ord} \phi_i)_{i=1}^k$ is a primitive vector of S and $\nabla g_S \circ \operatorname{info} \phi = 0$. By Property 5.13 we get $S \in \Gamma(f)$. Of course $f_S = g_S$. Therefore we have

$$(f_S)'_{z_i}(\operatorname{info} \phi_1(t), \dots, \operatorname{info} \phi_k(t), t, \dots, t) \equiv 0,\ i = k+1, \dots, n$$

and since $(\nabla g_S) \circ \operatorname{info} \phi = 0$, then

$$(f_S)'_{z_i}(\operatorname{info} \phi_1(t), \dots, \operatorname{info} \phi_k(t), t, \dots, t) \equiv 0,\ i = 1, \dots k.$$

Hence

$$(f_S)'_{z_i}(\operatorname{inco} \phi_1, \dots, \operatorname{inco} \phi_k, 1, \dots, 1) = 0, \quad i = 1, \dots, n,$$

thus f is degenerate on S. ■

Proof of Theorem 5.6 Suppose to the contrary, that f is not an isolated singularity. Then by the Curve Selection Lemma there exists a non-zero parametization $\phi = (\phi_1, \dots, \phi_n)$ such that $(\nabla f) \circ \phi = 0$. It is not possible for ϕ to have $n-1$ coordinates equal to zero. Indeed, if for example $\phi = (0, \dots, 0, \phi_n)$, $\phi_n \neq 0$, then by Property 3.14 we get that $f = a z_n^k z_i + \ldots$ for some $i \in \{1, \dots, n\}$, $a \neq 0$ and $k \geq 1$. Hence $f'_{z_i}(0, \dots, 0, \phi_n) \neq 0$, which contradicts the assumption $(\nabla f) \circ \phi = 0$. Therefore we may assume that $\phi_i \neq 0, \phi_j \neq 0$ for some $i \neq j$. Without loss of generality we may assume that $\phi_1 \neq 0, \phi_2 \neq 0$. Since $\Gamma_+(f) \cap OX_1X_2 \neq \emptyset$, by Lemma 5.14 we have that f is degenerate on some face $S \in \Gamma(f)$, which contradicts the assumption on f. ■

Now we can prove Theorem 5.4.

Proof of Theorem 5.4 If f is an isolated singularity then by Corollary 3.12 $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Now suppose that f satisfies the Kouchnirenko condition. Then by Properties 3.6, 3.7 and Theorem 5.6 we get that f is an isolated singularity. ■

Remark 5.15. Wall considered another type of nondegeneracy than the Kouchnirenko nondegeneracy. He got similar results to the ones obtained in this section, see Lemma 1.2 and Theorem 1.4 in [Wa98].

6. The Milnor and Newton numbers

By the main theorem of [Ko76] we always have $\mu(f) \geq \nu(f)$, with equality for nondegenerate isolated singularities. Hence, if $\mu(f)$ is finite, then $\nu(f)$ is also finite. The inverse implication is false, which shows the following simple example.

Example 6.1. *Let $f(z_1, \dots, z_n) = (z_1 + \dots + z_n)^2$. Obviously f is not an isolated singularity, but since f is convenient we have $\nu(f) < \infty$.*

It is well known by the local Nullstellensatz that $\mu(f)$ is finite if and only if f is an isolated singularity. On the other hand, Kouchnirenko writes in Remark 1.13 (ii) of his celebrated paper [Kou76] that the Newton number of a singularity f is finite if and only if $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Summing up, we can reformulate the results of the previous sections in terms of the Newton and Milnor numbers. By Theorem 3.9 we have the following corollary.

Corollary 6.2. *Let $M \subset \mathbb{N}^n$, $|m| \geq 2$ for every $m \in M$. Assume that $\nu(f_M) < \infty$. Then a singularity f, $\operatorname{supp} f \subset M$ with generic coefficients is an isolated singularity i.e. $\mu(f) < \infty$.*

We can also reformulate the results of Section 5. Observe that the singularity from Example 6.1 is degenerate. However the implication $\nu(f) < \infty \Rightarrow \mu(f) < \infty$ is true in the class of nondegenarate singularities in dimensions $n \leq 3$. Indeed, using Remarque 1.13 (ii) in [Ko76] we can reformulate Theorem 5.4, Corollary 5.8 and Conjecture 5.5 in terms of the Newton and Milnor numbers in the following way.

Proposition 6.3. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \leq 3$, be a nondegenerate singularity. Then*

$$\nu(f) < \infty \Leftrightarrow \mu(f) < \infty$$

Proposition 6.4. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate convenient singularity. Then*

$$\nu(f) < \infty \Leftrightarrow \mu(f) < \infty$$

Conjecture 6.5. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate singularity. Then*

$$\nu(f) < \infty \Leftrightarrow \mu(f) < \infty$$

Using Proposition 6.4 we may slightly weaken the assumptions of part (ii) of Theorem I in [Ko76] in the following way.

Corollary 6.6. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate convenient singularity. Then $\mu(f)$, $\nu(f)$ are finite and $\mu(f) = \nu(f)$.*

Remark 6.7. Wall obtained a result analogous to the above corollary in the class of singularities nondegenerate in his sense, see Theorem 1.6 in [Wa98].

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