Analytic and Algebraic Geometry

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ON SMOOTH HYPERSURFACES CONTAINING A GIVEN SUBVARIETY

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Abstract. We reprove some results about affine complete intersections.

1. Introduction.

Let k be an algebraically closed field. Let X^n be a smooth affine variety (of dimension n). Let us recall that a variety $H \subset X$ is a hypersurface if the ideal $I(H) \subset k[X]$ is generated by a single polynomial. Let $Y^r \subset X^n$ be a smooth subvariety. It was proved in [2] (see also [3]), that if $n \geq 2r+1$ then there is a smooth complete intersection $Z^{2r} \subset X^n$ such that $Y^r \subset Z^{2r}$. In general this result can not be improved- see Example 2.2. We also show how to use results from [6] to improve the result above in some special cases. In particular we show:

Theorem 1.1. *(Greco, Valabrega) Let X^n be a smooth variety and let Y^r be a smooth subvariety of X. Assume that the r^{th} Chow group $CH^r(Y^r)$ vanishes. If $n \geq 2r$, then there is a smooth complete intersection $Z^{2r-1} \subset X$ such that $Y^r \subset Z^{2r-1}$.*

and

Theorem 1.2. *(Murthy) Let $Y^r \subset \mathbb{A}^n$ be a smooth subvariety. If $n \geq 2r$ then there is a smooth hypersurface $H \subset \mathbb{A}^n$ such that $Y \subset H$.*

In particular a smooth surface $S \subset \mathbb{A}^4$ is contained in a smooth hypersurface $H \subset \mathbb{A}^4$. Let us note that this is not true in the projective case: it is well known that a smooth surface $S \subset \mathbb{P}^4$ is not contained in any smooth hypersurface $H \subset \mathbb{P}^4$,

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unless it is a complete intersection. Our approach is slightly different than the original ones.

2. Main Result.

We start with:

Theorem 2.1. *Let $Y \subset X$ be smooth affine varieties. Then there is a smooth hypersurface $V(f) \subset X$ which contains Y if and only if the normal bundle of Y contains a one dimensional trivial summand i.e.,*

$$\mathbf{N}_{X/Y} = \mathbf{T} \oplus \mathbf{E}^1,$$

where $\mathbf{E}^1$ denotes a trivial line bundle.

Proof. Assume that there is a smooth hypersurface $H = V(f) \subset X$ which contains Y. We have

$$TY \subset TH \subset TX,$$

in particular

$$\mathbf{N}_{X/Y} = \mathbf{N}_{H/Z} \oplus \mathbf{N}_{X/H}|_Y.$$

However, the normal bundle of the smooth hypersurface $H = V(f)$ is trivial (in fact the class of f is a generator of the conormal bundle of H).

Conversely, assume that

$$\mathbf{N}_{X/Y} = \mathbf{T} \oplus \mathbf{E}^1.$$

Hence also

$$\mathbf{N}^*_{X/Y} = \mathbf{T}^* \oplus \mathbf{E}^1.$$

This means that the conormal bundle $\mathbf{N}^*_{X/Y}$ has a nowhere vanishing section $\mathbf{s} \in \Gamma(Y, \mathbf{N}^*_{X/Y})$. But $\Gamma(Y, \mathbf{N}^*_{X/Y}) = I(Y)/I(Y)^2$, where $I(Y) \subset k[X]$ denotes the ideal of the subvariety Y. Hence $\mathbf{s}$ determines a polynomial $s \in I(X)$ such that the class of s is $\mathbf{s}$. Take a point $a \in Y$ and local coordinates $(u_1, ..., u_n)$ at a such that Y is described by local equations $u_1, ..., u_t$ ($t = \text{codim} Y$) near a. Since $\mathbf{u}_1, ..., \mathbf{u}_t$ freely generate the bundle $\mathbf{N}^*_{X/Y}$ near the point a, we have

$$\mathbf{s} = \sum_{i=1}^{t} \alpha_i \mathbf{u}_i,$$

where $\alpha_i \in k[U_a]$ (U_a denotes some open neighborhood of a in Y). Since the section $\mathbf{s}$ nowhere vanishes, there exists at least one i_0 such that $\alpha_{i_0} \neq 0$. Let us compute the derivative $d_y s$ of the polynomial s at the point $y \in Y$. We have

$$s = \sum_{i=1}^{t} \alpha_i u_i \mod I(Y)^2,$$

hence there are polynomials $f_j, h_j \in I(Y),\ j = 1, ..., m$, such that

$$s = \sum_{i=1}^{t} \alpha_i u_i + \sum_{j=1}^{m} f_j h_j.$$

Now we easily see that

$$d_a s = \sum_{i=1}^{t} \alpha_i d_a u_i.$$

Since $d_a u_i, i = 1, ..., n$, are linearly independent and not all α_i vanish at y we have $d_y s \neq 0$. Hence the hypersurface $V(s)$ is smooth along Y. Let $I(Y) = (g_1, ..., g_r)$. Consider the linear system on X given by the polynomials $(s, g_1^2, ..., g_r^2)$. The base locus of this system is exactly the subvariety Y. We can extend the set $\{g_1^2, ..., g_r^2\}$ adding some polynomials $\{g_j^2 \alpha_i,\ j = 1, ..., s, i = 0, 1, ..., k\}$ in such a way that a new system $(s, g_1^2, ..., g_r^2, g_j^2 \alpha_i)$ is unramified outside Y. Indeed, let $x \in X \setminus Y$. There is a polynomial $g_x \in I(Y)$, such that $g_x(x) \neq 0$. Let $\alpha_1, ..., \alpha_{2k+1}$ ($k = \dim X$) be polynomials which gives an embedding of X into k^{2n+1}. In some neighbourhood U_x of X we still have $g_x \neq 0$. Since $X \setminus Y$ is quasi-compact we can cover $X \setminus Y$ by a finite set $U_{x_i}, i \in I$ of such neighbourhoods. Associate with every such U_x the set $S_x := \{g_x^2, g_x^2 \alpha_1, ..., g_x^2 \alpha_{2k+1}\}$. It is easy to see, that the system given by polynomials $\{s, g_1^2, ..., g_r^2\} \cup \bigcup_{i \in I} S_{x_i}$ is unramified on $X \setminus Y$.

Hence by the Bertini Theorem (see [4], Corollary 12 and [5], Theorem 3.1) the hypersurface $V(s + \sum_{i=1}^r \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s)$ for generic $\beta_i,\ \beta_{j,s}$ is smooth outside Y. But for $y \in Y$,

$$d_y(s + \sum_{i=1}^{r} \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s) = d_y s.$$

This implies that the hypersurface $V(s + \sum_{i=1}^r \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s)$ is also smooth along Y. Hence we can take $f = s + \sum_{i=1}^r \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s$. □

Let X^{2n} be a smooth variety and Y^n be a smooth subvariety of X^{2n}. We show that in general does not exist a smooth hypersurface $H \subset X^{2n}$, such that $Y^n \subset H$. Indeed we have:

Example 2.2. Let $H_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > n + 2$. Let $Y \subset H$ be an affine open subset. By [7] we have $CH^n(Y) \neq 0$. Take a non-zero $z \in CH^n(Y)$. By Riemann-Roch without denominators and Serre Splitting Theorem (Theorem 2.3 below), there exists an algebraic vector bundle $\mathbf{F}$ on Y of rank n such that $c_n(\mathbf{F}) = (n-1)!z$. Since $CH^n(Y)$ has no $(n-1)!$ torsion (see e.g. [6]) we have $c_n(\mathbf{F}) \neq 0$. Now let X denote the total space of this vector bundle. Then $Y \subset X$ (as the zero-section) and $\mathbf{N}_{X/Y} \cong \mathbf{F}$. Since the top Chern class of $\mathbf{F}$ does not vanish, the bundle $\mathbf{F}$ does not have a one dimensional trivial summand. In particular Y is not contained in any smooth hypersurface in X (see Theorem 2.1).

In the sequel we need the following (see [1], p.177, Th. 7.1.8 and [5], Corollary 3.4):

Theorem 2.3. *(Serre Splitting Theorem) Let X be a smooth affine variety and let $\mathbf{F}$ be an algebraic vector bundle on X. If rank $\mathbf{F} > \dim \mathbf{X}$, then $\mathbf{F}$ has a one dimensional trivial summand i.e.,*

$$\mathbf{F} = \mathbf{T} \oplus \mathbf{E}^1.$$

Now we are in a position to prove:

Theorem 2.4. *Let X^n be a smooth variety and let Y^r be a smooth subvariety of X. If $n \geq 2r+1$ then there is a smooth complete intersection $Z^{2r} \subset X^n$ such that $Y^r \subset Z^{2r}$. Assume additionally that the r^{th} Chow group $CH^r(Y^r)$ vanishes. If $n \geq 2r$, then there is a smooth complete intersection $Z^{2r-1} \subset X$ such that $Y^r \subset Z^{2r-1}$.*

Proof. Assume first that $s = n - 2r > 0$. Since $\dim Y^r <$ rank $\mathbf{N}_{X/Y}$, Theorem 2.3 shows that $\mathbf{N}_{X/Y} = \mathbf{T} \oplus \mathbf{E}^1$, where $\mathbf{E}^1$ denotes a trivial line bundle. By Theorem 2.1 there exists a smooth hypersurface $H = V(f)$ (where f is a reduced polynomial) such that $Y \subset H$. Now we can apply the mathematical induction. This completes the proof of the first part of Theorem 2.4.

For the proof of the second part let us note that the bundle $\mathbf{F} = \mathbf{N}^*_{Z^{2r}/Y^r}$ has a one dimensional trivial summand as $c_r(\mathbf{F}) = 0$, by the Theorem of Murthy (see [6], Th. 3.8). Now we can finish by applying Theorem 2.1. □

Theorem 2.5. *Let X^n, Y^r be as above. If $n \geq 2r+1$ then there is a smooth hypersurface $H = V(f)$ such that $Y^r \subset H$. If the r^{th} Chow group $CH^r(X^n)$ vanishes, then it is enough to assume $n \geq 2r$.*

Proof. It is enough to consider only the last statement. Moreover, we can assume that $n = 2r$. Let $Y^r = \bigcup_{i=1}^s Y_i$ be the decomposition of Y into irreducible components. Of course $Y_i \cap Y_j = \emptyset$ for $i \neq j$. We show that the bundle $\mathbf{F} = \mathbf{N}_{X/Y}$ has a one dimensional trivial summand over every Y_i. Indeed, if dim $Y_i < r$ then it follows from the Serre Splitting Theorem. Assume that $\dim Y_i = r$. Let $\iota : Y_i \to X$ be the inclusion. By the self-intersection formula we have the following expression for the top Chern class of the normal bundle of Y_i:

$$c_r(\mathbf{F}|_{Y_i}) = \iota^* \circ \iota_*[Y_i],$$

where $[Y_i] \in CH^0[Y_i] = \mathbb{Z}$ is a generator. By our assumption we have $c_r(\mathbf{F}|_{Y_i}) = 0$. Now by the Theorem of Murthy, invoked above, the normal bundle $\mathbf{N}_{X/Y}$ splits over Y_i in a suitable way. Finally we can use Theorem 2.1. □

The last statement of Theorem 2.5 can be applied to $X = \mathbb{A}^n$, or more generally to $X =$ *open affine subset of* $\mathbb{A}^n$. In particular we have:

Corollary 2.6. *Let $Y^r \subset \mathbb{A}^n$ be a smooth subvariety. If $n \geq 2r$ then there is a smooth hypersurface $H \subset \mathbb{A}^n$ such that $Y \subset H$.*

Theorems above suggest that if all (positive) Chow groups of X and Y vanish, then it is easier to find a smooth hypersurface which contains a given smooth subvariety $Y \subset X$. However, we show that also in that case there are examples of smooth subvarieties $Y \subset X$ which are not contained in any smooth hypersurface of X. In our example X will be an open affine subset of $\mathbb{A}^9$ and Y be an affine open subset of $\mathbb{A}^7$. In particular Y and X have all positive Chow groups trivial.

Example 2.7. Consider the variety $\Gamma = \{(x,y) \in k^3 \times k^3 : \sum_{i=1}^3 x_i y_i = 1\}$. By the Raynaud Theorem (see [8] and [9]) the algebraic vector bundle given by the unimodular row (x_1, x_2, x_3) is not free. Let $\Lambda = \{(x,y) \in k^3 \times k^3 : \sum_{i=1}^3 x_i y_i = 0\}$ be an affine cone and let $Y' = \mathbb{A}^6 \setminus \Lambda$. Hence Y' is an affine open subset of $\mathbb{A}^6$. Moreover, the algebraic vector bundle $\mathbf{F}$ given by the unimodular row (x_1, x_2, x_3) is not trivial, because it is not trivial after restriction to Γ. Since every stably trivial line bundle is trivial and rank $\mathbf{F} = 2$, we see that the vector bundle $\mathbf{F}$ does not split.

Take $Y'' = Y' \times k,\ X = Y' \times k^3$ and consider the embedding

$$\phi : Y'' \ni ((x,y),t) \mapsto ((x,y), x_1 t, x_2 t, x_3 t) \in X.$$

Take $Y = \phi(Y'')$. By direct computations we see that the normal bundle $\mathbf{N}_{X/Y}$ restricted to the subvariety $Y' \times \{0\}$ is equal to

$$\mathbf{E}^3 / < (x_1, x_2, x_3) > \cong \mathbf{F}$$

(where $\mathbf{E}^s$ denotes the trivial bundle of rank s). Since the bundle $\mathbf{F}$ does not split, neither does $\mathbf{N}_{X/Y}$. In particular Y is not contained in any smooth hypersurface in X. Moreover, X is an open subset of $\mathbb{A}^9$ and Y is isomorphic to an open subset of $\mathbb{A}^7$.

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