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Feedback stabilization: from geometry to control

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Abstract: Youla parametrization of stabilizing controllers is a fundamental result of control theory: starting from a special, double coprime, factorization of the plant provides a formula for the stabilizing controllers as a function of the elements of the set of stable systems. In this case the set of parameters is universal, i.e., does not depend on the plant but only the dimension of the signal spaces. Based on the geometric techniques introduced in our previous work this paper provides an alternative, geometry based parametrization. In contrast to the Youla case, this parametrization is coordinate free: it is based only on the knowledge of the plant and a single stabilizing controller. While the parameter set itself is not universal, its elements can be generated by a universal algorithm. Moreover, it is shown that on the parameters of the strongly stabilizing controllers a simple group structure can be defined. Besides its theoretical and educative value the presentation also provides a possible tool for the algorithmic development.

Keywords: controller parametrisation; controller blending; stability guarantee

1. INTRODUCTION AND MOTIVATION

The branches of mathematics that are useful in dealing with engineering problems are analysis, algebra, and geometry. Although engineers favour graphic representations, geometry seems to have been applied to a limited extent and elementary geometrical treatment is often considered difficult to understand. Thus, in order to put geometry and geometrical thought in a position to become a reliable engineering tool, a certain mechanism is needed that translates geometrical facts into a more accessible form for everyday algorithms.

Klein proposed group theory as a mean of formulating and understanding geometrical constructions. In Szabó et al. [2014] the authors emphasise Klein's approach to geometry and demonstrate that a natural framework to formulate various control problems is the world that contains as points equivalence classes determined by stabilizable plants and whose natural motions are the Möbius transforms. The observation that any geometric property of a configuration, which is invariant under an euclidean or hyperbolic motion, may be reliably investigated after the data has been moved into a convenient position in the model, facilitates considerably the solution of the problems.

The link between algebra and geometry goes back to the introduction of real coordinates in the Euclidean plane by Descartes. Coordinates, in general, are the most essential tools for the applied disciplines that deal with geometry. The axiomatic approach to the Euclidean plane is seldom used because a truly rigorous development is very demanding while the Cartesian product of the reals provides an easy-to-use model. Descartes has managed to solve a lot of ancient problems by algebrizing geometry, and thus by finding a way to express geometrical facts in terms of other entities, in this case, numbers. Note that being a one-to-one mapping, this "naming" preserves information, so that we can study the corresponding group operations simply by looking at these operations' effect on the coordinates ("names"), even though the group elements themselves might be any kind of weird creatures. Descartes justifies algebra by interpreting it in geometry, but this is not the only choice: Hilbert will go the other way, using algebra to produce models of his geometric axioms. Actually this interplay between geometry, its group theoretical manifestation, algebra and control theory is what we are interested in our investigations.

In contrast to traditional geometric control theory, see, e.g., Wonham [1985], Basile and Marro [1992] for the linear and Isidori [1989], Jurdjevic [1997], Agrachev and Sachkov [2004] for the nonlinear theory, which is centered on a local view, this approach provides a global view. While the former uses tools from differential geometry, Lie algebra, algebraic geometry, and treats system concepts like controllability, as geometric properties of the state space or its subspaces the latter focuses on an input-output – coordinate free – framework where different transformation groups which leave a given global property invariant play a fundamental role.

In the first case the invariants are the so-called invariant or controlled invariant subspaces, and the suitable change of coordinates and system transforms (diffeomorphisms), see, e.g., the Kalman decomposition, reveal these properties. In contrast, our interest is in the transformation groups that leave a given global property, e.g., stability or $\mathcal{H}_\infty$ norm, invariant. One of the most important consequences of the approach is that through the analogous of the classical geometric constructions it not only might give hints for efficient algorithms but the underlaying algebraic structure, i.e., the given group operation, also provides tools for controller manipulations that preserves the property at hand, called controller blending.

Control theory should study also stability of feedback systems in which the open-loop operator is unstable or at least oscillatory. Such maps are clearly not contained in Banach spaces and some mathematical description is necessary if feedback stability is to be interpreted from open loop system descriptions. This is achieved by ruling out from the model class those unbounded operators that might "explode" and establishing the stability problem in an extended space which contains well-behaved as well as asymptotically unbounded functions, see Feintuch [1998]. The generalized extended space contains all functions which are integrable or summable over finite intervals. A disadvantage of the method is that the resulting space is a Banach space while we would prefer to work in a Hilbert space context for signals, and the set of stable operators for plants.

Since unbounded operators on a given space do not form an algebra – nor even a linear space, because each one is defined on its own domain – the association of the operator with a linear space, its graph subspace, turns to be fruitful. This leads us to the study of the generalized projective geometries that copy the constructions of the projective plane into a more complex mathematical setting while maintaining the original relations between the main entities and the original ideas. In doing this our main tools are algebraic: group theory, see Szabó and Bokor [2015], and the framework of the so called Jordan pairs will help us to obtain the proper interpretations and to achieve new results, see Szabó and Bokor [2016].

The main concern of this work is to highlight the deep relation that exists between the seemingly different fields of geometry, algebra and control. While the Kleinian view make the link between geometry and group theory, through different representations and homomorphism the abstract group theoretical facts obtain an algebraic (linear algebraic) formulation that opens the way to engineering applications. We would like to stress that it is a very fruitful strategy to try to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction; finally the solution of the original control problem can be formulated in an algorithmic way by transposing the geometric ideas into the proper algebraic terms.

The main contribution of this paper relative to the previous efforts is the following: it is shown that, in contrast to the classical Youla approach, there is a parametrisation of the entire controller set which can be described entirely in a coordinate free way, i.e., just by using the knowledge of the plant P and of the given stabilizing controller K_0. The corresponding parameter set is given in geometric terms, i.e., by providing an associated algebraic (semigroup, group) structure. It turns out that the geometry of stable controllers is surprisingly simple.

Section 2 gives the basic notions related to feedback stability and recalls the fundamental result of the Youla parametrization. Section 3 recalls some previous results of the authors: a natural blending method is introduced that acts directly on the controllers and keeps stability of the loop. The formulae on the corresponding operations in the parameter space are new results. Section 4 provides a geometric based parametrization of the stabilizing controllers by showing how the geometric view can be applied to reveal the coordinate free nature of the parametrization. In Section 5 we conclude the paper by illustrating how the abstract geometric framework interferes with the practical problems. Finally some conclusions and further research topics are formulated.

2. BASIC SETTINGS

A central concept of control theory is that of the feedback and the stability of the feedback loop. For practical reasons our basic objects, the systems, i.e., plants and controllers, are causal. Stability is actually a continuity property of a certain map, more precisely a property of boundedness and causality of the corresponding map. Boundedness here involves some topology. In what follows we consider linear systems, i.e., the signals are elements of some normed linear spaces and an operator means a linear map that acts between signals. Thus, boundedness of the systems is regarded as boundedness in the induced operator norm.

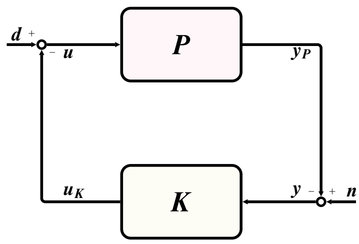

Fig. 1. Feedback connection

To fix the ideas let us consider the feedback-connection depicted on Figure 1. It is convenient to consider the signals

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, p = \begin{pmatrix} u \\ y_P \end{pmatrix}, k = \begin{pmatrix} u_K \\ y \end{pmatrix}, z = \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{H},$$

where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and we suppose that the signals are elements of the Hilbert space $\mathcal{H}_1, \mathcal{H}_2$ (e.g., $\mathcal{H}_i = \mathcal{L}^{n_i}[0,\infty)$) endowed by a resolution structure which determines the causality concept on these spaces. In this model the plant P and the controller K are linear causal maps. For more details on this general setting, see Feintuch [1998].

The feedback connection is called well-posed if for every $w \in \mathcal{H}$ there is a unique p and k such that $w = p + k$ (causal invertibility) and the pair (P, K) is called stable if the map $w \to z$ is a bounded causal map, i.e., the pair (P, K) is called well-posed if the inverse

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} S_u & S_c \\ S_p & S_y \end{pmatrix} = \begin{pmatrix} (I-KP)^{-1} & -K(I-PK)^{-1} \\ -P(I-KP)^{-1} & (I-PK)^{-1} \end{pmatrix} \tag{1}$$

exists (causal invertibility), and it is called stable if all the block elements are stable.

2.1 Youla parametrization

A fundamental result concerning feedback stabilization is the description of the set of the stabilizing controllers. A standard assumption is that among the stable factorizations there exists a special one, called double coprime factorization, i.e., $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and there are causal bounded systems $U, V, \tilde{U}$ and $\tilde{V}$, with invertible V and $\tilde{V}$, such that

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = \tilde{\Sigma}_P \Sigma_P = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad (2)$$

an assumption which is often made when setting the stabilization problem, Vidyasagar [1985], Feintuch [1998]. The existence of a double coprime factorization implies feedback stabilizability, actually $K_0 = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ is a stabilizing controller. In most of the usual model classes actually there is an equivalence.

For a fixed plant P let us denote by $\mathbb{W}_P$ the set of well-posed controllers, while $\mathbb{G}_P \subset \mathbb{W}_P$ denotes the set of stabilizing controllers.

Given a double coprime factorization the set of the stabilizing controllers is provided through the well-known Youla parametrization, Kucera [1975], Youla, Jabr and Bongiorno [1976]:

$$\mathbb{G}_P = \{K = \mathfrak{M}_{\Sigma_P}(Q) \mid Q \in \mathbb{Q}, (V + NQ)^{-1} \text{ exists}\},$$

where $\mathbb{Q} = \{Q \,|\, Q \text{ stable }\}$ and

$$\mathfrak{M}_{\Sigma_P}(Q) = (U + MQ)(V + NQ)^{-1}. \qquad (3)$$

Here $\mathfrak{M}_T(Z)$ is the Möbius transformation corresponding to the symbol T defined by

$$\mathfrak{M}_T(Z) = (B + AZ)(D + CZ)^{-1}, \text{ with } T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

on the domain $\mathrm{dom}_{\mathfrak{M}_T} = \{Z \,|\, (D + CZ)^{-1} \text{ exists}\}$. Note that

$$Q_K = \mathfrak{M}_{\tilde{\Sigma}_P}(K) = (\tilde{V}K - \tilde{U})(\tilde{M} - \tilde{N}K)^{-1}, \qquad (4)$$

and thus $Q = 0_K$ corresponds to $K_0 = UV^{-1}$.

Since the dimensions of the controller and plant are different, it is convenient to distinguish the zero controller and zero plant by an index, i.e., 0_K and 0_P, respectively.

Observe that the domain of (4) is exactly $\mathbb{W}_P$; thus we can introduce the corresponding extended parameter set $\mathbb{Q}_P^{wp} = \{Q_K = \mathfrak{M}_{\tilde{\Sigma}_P}(K) \,|\, K \in \mathbb{W}_P\}$. Note, that Q_0, i.e., $\mathfrak{M}_{\tilde{\Sigma}_P}(0_K) = -\tilde{U}\tilde{M}^{-1} = -M^{-1}U$, is not in $\mathbb{Q}$, in general. The content of the Youla parametrization is that K is stabilizing exactly when $Q_K \in \mathbb{Q}$.

3. GROUP OF CONTROLLERS

In order to design efficient algorithms that operate on the set of controllers that fulfil a given property, e.g., stability or a prescribed norm bound, it is important to have an operation that preserves that property, i.e., a suitable blending method. Available approaches use the Youla parameters in order to define this operation for stability in a trivial way. As these approaches ignore the well-posedness problem by assuming strictly proper plants, they do not provide a general answer to the problem.

In the particular case when $P = 0_P$ we have $\mathbb{G}_P = \mathbb{Q}$, i.e., mere addition preserves well-posedness and stability. Moreover, the set of these controllers forms the usual additive group $(\mathbb{Q}, +)$ with neutral element 0_K and inverse element $Q \to -Q$. In the general case, however, addition of controllers neither ensure well-posedness nor stability.

3.1 Indirect blending

The most straightforward approach to obtain a stability preserving operation is to find a suitable parametrization of the stabilizing controllers, where the parameter space possesses a blending operation. As an example for this indirect (Youla based) blending is provided by the Youla parametrization. However, this mere addition on the Youla parameter level does not lead, in general, to a "simple" operation on the level of controllers:

$$K = \mathfrak{M}_{\Sigma_P}((\mathfrak{M}_{\tilde{\Sigma}_P}(K_1) + \mathfrak{M}_{\tilde{\Sigma}_P}(K_2))). \qquad (5)$$

The unit element of this operation is the controller K_0 which defines Σ_P, see Figure 2. Its implementation involves three nontrivial transformations.

Note that an obstruction might appear if the sum of the Youla parameters are not in the domain of $\mathfrak{M}_{\Sigma_P}$, e.g., for non strictly proper plants where some of the non strictly proper parameters are out-ruled.

We can formulate this process as a group homomorphism between the usual addition of parameters $\mathbb{Q}$ and the group of automorphisms $Q \mapsto \tau_Q$ associated tpspace formed by simple translations, i.e.,

$$\tau_Q = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}, \quad \tau_{Q_1}\tau_{Q_2} = \tau_{Q_1+Q_2}.$$

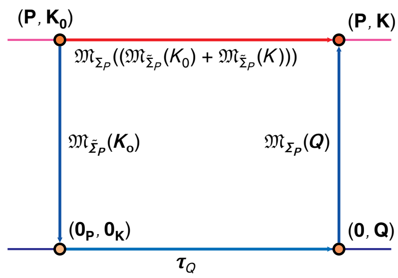

Fig. 2. Youla based blending

3.2 Direct blending

The observation that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - PK_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - PK_2 \end{pmatrix}. \qquad (6)$$

leads to operation

$$K = K_1(I - PK_2) + K_2 = K_1 \boxdot_P K_2, \qquad (7)$$

under which well-posed controllers form a group $(\mathbb{W}_P, \boxdot_P)$. The unit of this group is the zero controller $K = 0_K$ and the corresponding inverse elements are given by

$$K^{\boxminus_P} = -K(I - PK)^{-1}. \qquad (8)$$

Note that

$$I - PK^{\boxminus_P} = (I - PK)^{-1}. \qquad (9)$$

Clearly not all elements of $\mathbb{W}_P$ are stabilizing, e.g., 0_K is not stabilizing for an unstable plant.

Theorem 3.1. $(\mathbb{G}_P, \boxdot_P)$ with the operation (blending) defined in (7) is a semigroup.

Note, that

$$(I - PK)^{-1} = (I - PK_2)^{-1}(I - PK_1)^{-1}. \tag{10}$$

By using the notation

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - PK \end{pmatrix} = R_P T_K^{(P)}$$

we have the group homomorphism $T_{K_1}^{(P)} T_{K_2}^{(P)} = T_{K_1 \boxdot_P K_2}^{(P)}$ and $K = \mathfrak{M}_{R_P T_K^{(P)} R_P^{\ 1}}(0_K)$.

On the level of Youla parameters the corresponding operation is more complex:

$$\begin{aligned} Q_{K_2} \odot_P Q_{K_1} &= \tilde{V}U + \tilde{V}MQ_{K_1} + Q_{K_2}\tilde{M}V + Q_{K_2}\tilde{N}MQ_{K_1} = \\ &= (Q_{K_2} - Q_0)\tilde{M}(V + NQ_{K_1}) + Q_{K_1} = \\ &= Q_{K_2} + (\tilde{V} + Q_{K_2}\tilde{N})M(Q_{K_1} - Q_0), \qquad (11) \\ Q_{K^{\boxminus_P}} &= Q_0 - M^{\ \ 1} K \tilde{M}^{\ \ 1} = \\ &= Q_0 - (Q_K - Q_0)(I + V^{\ \ 1} N Q_K)^{\ \ 1} V^{\ \ 1} \tilde{M}^{\ \ 1}. \qquad (12) \end{aligned}$$

Note that $(\mathbb{G}_P, \boxdot_P)$ and $(\mathbb{Q}, \odot_P)$ are related by only a semigroup homomorphism, while $(\mathbb{W}_P, \boxdot_P)$ and $(\mathbb{Q}_P^{wp}, \odot_P)$ are related, however, through a group homomorphism.

3.3 Strong stability

The semigroup $(\mathbb{G}_P, \boxdot_P)$ does not have a unit, in general. However, if there is a stabilizing controller K_0 such that

$$K_0^{\boxminus_P} = -K_0(I - PK_0)^{-1}$$

is also a stabilizing controller, i.e., K_0 is stable, then $(\mathbb{G}_P, \boxtimes_P)$ with

$$K_1 \boxtimes_P K_2 = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2$$

is a semigroup with a unit (K_0). This may happen only if the plant is strongly stabilizable.

If we denote by $\mathbb{S}_P$ the set of strongly stabilising controllers, then if this set is not empty, then

Theorem 3.2. $(\mathbb{S}_P, \boxtimes_P)$ with the operation (blending) defined as

$$\begin{aligned} K = K_1 \boxtimes_P K_2 &= K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2 = \\ &= K_2 + (K_1 - K_0)(I - PK_0)^{-1}(I - PK_2) \end{aligned} \tag{13}$$

is the group of strongly stable controllers, where $K_0 \in \mathbb{S}_P$ is arbitrary. The corresponding inverse is given by

$$K^{\boxtimes_P^{\ 1}} = K_0 - (K - K_0)(I - PK)^{-1}(I - PK_0). \tag{14}$$

Opposed to the possible expectations, we not only have simple expressions for these operations in the Youla parameter space, but the formulae also resemble (7) and (8):

$$Q_K = Q_{K_2} \otimes Q_{K_1} = Q_{K_2} + Q_{K_1} + Q_{K_2}V^{-1}NQ_{K_1}, \tag{15}$$

$$Q_K^{\otimes\ 1} = -Q(I + V^{-1}NQ)^{-1}. \tag{16}$$

It is important to note that while (15) keeps the strong stabilizability, as a property, invariant it does not guarantee that the property is fulfilled. This means that the formula also makes sense for parameters that does not correspond to stable controllers.

4. A GEOMETRY BASED CONTROLLER PARAMETRIZATION

In what follows we fix a stabilizing controller, say K_0, and in the formulae we associate, according to (1), the corresponding sensitivities to this controller. Considering

$$\hat{\Sigma}_{P,K_0} = \begin{pmatrix} UV^{-1} & M - UV^{-1}N \\ V^{-1} & -V^{-1}N \end{pmatrix}$$

we obtain the lower LFT representation of the Youla parametrization, i.e.,

$$K = \mathfrak{M}_{\Sigma_{P,K_0}}(Q) = \mathfrak{F}_l(\hat{\Sigma}_{P,K_0}, Q), \tag{17}$$

see, e.g., Zhou and Doyle [1999]. Rearranging the terms one has

$$K = \mathfrak{F}_l(\Psi_{K_0,P}, R), \quad \text{with} \quad \Psi_{K_0,P} = \begin{pmatrix} K_0 & I \\ I & S_p \end{pmatrix} \tag{18}$$

and

$$R \in \mathbb{R}_{K_0}^Y = \{\, \tilde{V}^{-1}QV^{-1} \,|\, Q \in \mathbb{Q} \,\}. \tag{19}$$

This fact was already observed for a while, e.g., Mirkin [2016] or Bendtsen et al. [2005], where it was used as a starting point for a Youla parametrization based gain scheduling scheme of rational LTI systems. We have recalled this result with the intention to demonstrate how our previous ideas on the geometry of stabilizing controllers can be applied in order to find significantly new information on an already known configuration.

4.1 A coordinate free parametrization

In order to relate a Möbius transform to an LFT we prefer to use the formalism presented in Szabó et al. [2014]. Thus, recall that $\hat{\Sigma}_P$ is the Potapov-Ginsburg transform of Σ_P and formulae like (18) can be easily obtained by using the group property of the Möbius transform. Accordingly, we have that

$$K = \mathfrak{M}_{\Gamma_{P,K_0}}(R) = \mathfrak{F}_l(\Psi_{P,K_0}, R), \tag{20}$$

$$R = \mathfrak{M}_{\Gamma_{P,K_0}^{\ 1}}(K) = \mathfrak{F}_l(\Phi_{P,K_0}, K), \tag{21}$$

where

$$\Gamma_{P,K_0} = \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix}, \quad \Psi_{P,K_0} = \hat{\Gamma}_{P,K_0}, \tag{22}$$

$$\Gamma_{P,K_0}^{-1} = \begin{pmatrix} I & -K_0 \\ S_p & S_y \end{pmatrix}, \; \Phi_{P,K_0} = \begin{pmatrix} -K_0S_y^{-1} & S_u^{-1} \\ S_y^{-1} & P \end{pmatrix}. \tag{23}$$

Observe that (21) is defined exactly on $\mathbb{W}_P$ and let the restriction on the stabilizing controllers be denoted by $\mathbb{R}_{K_0} = \{\mathfrak{F}_l(\Phi_{P,K_0}, K) \,|\, K \in \mathbb{G}_P\}$. Apparently, apart the structure of the set $\mathbb{R}_{K_0}^Y$ these formulae do not depend on any special factorization. Moreover, they can be also obtained directly, i.e., without any reference to some factorization of the plant or of the controller, starting from

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} (K - K_0)\,(0 \;\, I)$$

and applying two times the matrix inversion lemma to obtain first

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix}^{-1} - \begin{pmatrix} S_u \\ S_p \end{pmatrix} R\,(S_p \;\, S_y)\,,$$

with $R = (K - K_0)(I + S_p(K - K_0))^{-1}$ and then

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & K_0 \\ P & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} R(I - S_pR)^{-1}\,(0 \;\, I)\,. \tag{24}$$

Thus, it would be desirable to provide, if it exists, a coordinate free description of $\mathbb{R}_{K_0}$. Exactly this is the point where the geometric view and the coordinate free results of Section 3 can be applied.

As a starting point observe that

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix} = \tag{25}$$

$$= \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R)^{-1} \end{pmatrix}. \tag{26}$$

Analogous to (6) we have the factorization

$$\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} = \begin{pmatrix} S_u & 0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & S_u^{-1}K \\ 0 & I - PK \end{pmatrix}. \tag{27}$$

By using the notations

$$R_{(P,K_0)} = \begin{pmatrix} S_u & 0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix}$$

$$T_K^{(P,K_0)} = \begin{pmatrix} S_u & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & S_u^{-1}K \\ 0 & I - PK \end{pmatrix} \begin{pmatrix} S_u^{-1} & 0 \\ 0 & I \end{pmatrix}$$

we have

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = R_{(P,K_0)} T_K^{(P,K_0)}$$

and

$$T_{K_1}^{(P,K_0)} T_{K_2}^{(P,K_0)} = T_{K_1 \boxdot_P K_2}^{(P,K_0)},$$

moreover

$$K = \mathfrak{M}_{R_{(P,K_0)} T_K^{(P,K_0)} R_{(P,K_0)}^{\ 1}}(0_K) = \mathfrak{M}_{\Gamma_{P,K_0}}(R),$$

see (26) for the last equality. Thus, it is immediate that the operation (7) is a natural choice for this new configuration, too.

4.2 Geometric description of the parameters

Considering (11) and keeping in mind that $R = \tilde{V}^{-1} Q V^{-1}$ we have the blending rule on $\mathbb{R}_{K_0}^Y$:

$$R_2 \odot_{P,K_0} R_1 = K_0 + S_u R_1 + R_2 S_y - R_2 S_y S_p R_1. \tag{28}$$

For the stable controllers the parameter blending is more simple:

$$R_2 \otimes_{P,K_0} R_1 = R_2 + R_1 - R_2 S_p R_1, \tag{29}$$

$$R^{\otimes_{P,K_0}^{\ 1}} = -R(I - S_p R)^{-1}, \tag{30}$$

see (15) and (16).

Observe that $K_0 = \tilde{V}^{-1} \tilde{V} U V^{-1} \in \mathbb{R}_{K_0}^Y$ and that the corresponding controller is $K = K_0 \boxdot_P K_0 = [2K_0]_{\boxdot_P}$. Based on (28) it is easy to show that to the controller $K = [nK_0]_{\boxdot_P}$ corresponds the parameter $R = (I + \cdots + S_u^{n-1})K_0 \in \mathbb{R}_{K_0}^Y$. Thus, if K_0 is stable, then all these parameters are stable. However, the corresponding controllers are not necessarily stable.

Theorem 4.1. The algebraic structures defined by (28) and (29) holds also on $\mathbb{R}_{K_0}$, i.e., they can be introduced in a complete coordinate free way.

Due to lack of space, we do not continue to deduce all the formulae, e.g., inverse, shifted blending, etc., for the parameters. Instead we show, in what follows, that the operation (28) can be obtained directly, without the Youla parametrization. To do so, observe that

$$I - PK = (I - PK_0)(I - S_p R_1)^{-1},$$

thus

$$\begin{pmatrix} I & S_u^{\ 1} K \\ 0 & I - PK \end{pmatrix} = \begin{pmatrix} I & S_u^{\ 1}(K_0 + S_u R) \\ 0 & S_o^{\ 1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R)^{\ 1} \end{pmatrix}.$$

Then, according to (26) we have

$$\begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix} = \begin{pmatrix} S_u & K_0 \\ -S_p & I \end{pmatrix} \begin{pmatrix} I & R_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R_1)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} I & S_u^{-1}(K_0 + S_u R_2) \\ 0 & S_o^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (I - S_p R_2)^{-1} \end{pmatrix}. \tag{31}$$

Now, keeping in mind that

$$R = \mathfrak{M}_{\Gamma_{P,K_0}^{\ 1}}(K) = \mathfrak{M}_{\Gamma_{P,K_0}^{\ 1}} \begin{pmatrix} I & K \\ P & I \end{pmatrix}^{(0_K)} = \mathfrak{M}_{\Gamma_{P,K_0}^{\ 1}} \begin{pmatrix} S_u & K \\ -S_p & I \end{pmatrix}^{(0_K)},$$

the assertion follows after evaluating (31).

We have already seen that $\{0, K_0\} \subset \mathbb{R}_{K_0}$. Moreover, we have seen that $\mathbb{Q} \subset \mathbb{R}_{K_0}^Y$ by an identification of $Q \to \tilde{V} Q V$, i.e., $R \to Q$. It turns out that this inclusion is also a coordinate free property, i.e., the inclusion holds regardless the existence of any coprime factorization:

Theorem 4.2. The inclusion $\mathbb{Q} \subset \mathbb{R}_{K_0}$ holds.

Indeed, by taking a controller $K \in \mathcal{K}_{K_0}$, where

$$\mathcal{K}_{K_0} = \{ K = \mathfrak{F}_l(\Psi_{K_0,P}, Q) = K_0 + Q(I - S_p Q)^{\ 1} \mid Q \in \mathbb{Q}\}, \tag{32}$$

after some standard computations, that are left out for brevity, we obtain

$$(I - PK)^{\ 1} = (I - S_p Q)(I - PK_0)^{\ 1} \tag{33}$$

$$(I - KP)^{\ 1} = (I - K_0 P)^{\ 1}(I - Q S_p) \tag{34}$$

$$(I - PK)P^{\ 1} = -(I - S_p Q) S_p \tag{35}$$

$$K(I - PK)^{\ 1} = -S_c + (I - K_0 P)^{\ 1} Q (I - K_0 P)^{\ 1} - - (I - K_0 P)^{\ 1} Q + Q. \tag{36}$$

Thus $\mathcal{K}_{K_0} \subset \mathbb{G}_P$, as desired.

From a mathematical point of view, there is a small missing here. When a double coprime factorization exists, we should also prove that the set defined by (19), and the set defined by (24) are equal, i.e., $\mathbb{R}_{K_0}^Y = \mathbb{R}_{K_0}$. But this is equivalent to the fact that the Youla characterization of stabilizing controllers is exhaustive. This is a highly nontrivial issue and it is beyond the scope of this paper to address this topic in general. We should mention, however, that this property holds for discrete time systems, see, e.g., Feintuch [1998].

5. FROM GEOMETRY TO CONTROL

As it was already pointed out in the introduction of this paper we have fund very useful to formulate a control problem in an abstract setting, then translate it into an elementary geometric fact or construction. In the previous sections some examples were presented to illustrate this point. Now, it is time to demonstrate the way that starts from the abstract level and ends into a directly control relevant result.

The reader customised with system classes, like LTI, LPV (linear parameter varying), nonlinear, switching, etc. might find our presentation of the geometric ideas quite informal. We stress that this is a "feature" of the method. Recall that geometry – and also group theory – does not deal with the existence and the actual nature of the objects that are the primitives of the given geometry but rather captures the "rules" they obeys to. It gives the abstract structures that can be, for a given application, associated with actual objects, i.e., responds to the question "what can be done with these objects" rather than "how to synthesise the object having a given property (e.g., stability)".

We illustrate this fact by the example of the Youla parametrization. A basic knowledge is to place the topic in the context of finite rank LTI systems, i.e., those associated with rational transfer functions $\mathcal{R}$, and to interpret the result only in this context. However, we should not confine ourselves to this class: it is clear that an LTI plant can be also stabilized by more "complex" controllers, e.g., nonlinear ones, see, e.g., the IQC approach of Szabó et al. [2013]. This is also clear from the geometry: nothing prevents the Youla parameter to be any stable plant (not necessarily linear) in order to generate the stabilizing controller. Moreover, the nature of the parameter (e.g., nonlinear) is inherited by the controller through the Möbius transform.

We stress that the geometric picture behind the Youla parametrization has been applied under the hood even in the cases when the classes at hand do not have a sound input-output description, e.g., the class of switching systems or even the LPV systems. For the difficulties around these systems when we want to cast them exclusively into in input-output framework see, e.g., Blanchini et al. [2009]. These difficulties does not prevent engineers to reduce the design of the switching controllers to switching between the corresponding values of the parameters, see, e.g., Niemann and Stoustrup [1999], Bendtsen et al. [2005], Trangbaek et al. [2008]. Moreover, the idea can be extended also for plants that are switching systems themselves, Hespanha and Morse [2002], Blanchini et al. [2009], or LPV plants, Xie and Eisaka [2004]. Observe that in all these examples the authors spend a considerable amount of effort to solve the existential problem, i.e., how to obtain K_0. In all these cases this problem is cast in a state space framework and the taxonomy of the methods revolves around the type of the Lyapunov function (quadratic vs. polyhedral norm, constant Lyapunov matrix vs. parameter varying) involved that is used as stability certifier.

The motivation behind the increased complexity of the controller is that some additional performance demand is imposed either for the closed loop or for the controller, which cannot be fulfilled in the LTI setting. Concerning closed loop performances, the advantage of the Youla based approaches is that the performance transfer function is affine in the design parameter.

As an example consider the strong stabilizability problem. It is a standard knowledge that in $\mathcal{R}$ the problem does not always have a solution. However, it is less known that if one considers time variant (LTV) controllers, the problem is always solvable, see Khargonekar et al. [1988]. Moreover, for the discrete time case the problem is solvable in the disc algebra $\mathcal{A}$ or even in $\mathcal{H}_\infty$, see Quadrat [2003, 2004].

To conclude this section we point out some additional properties of the parametrization presented in Section 4. As a consequence of (18) and (32), for every controller K_0 there is a stable perturbation ball Δ, contained in the image of the ball with radius $\frac{1}{\|S_p\|}$ under the map $x(1-x)^{-1}$, such that the pair $(P, K_0 + \delta)$ is stable for all $\delta \in \Delta$. In particular, if the controller K_0 is strongly stabilizing, then all the controllers from $K_0 + \Delta$ are strongly stabilizing. This fact reveals the role of the sensitivity S_p in relation to the robustness of the stabilizing property of K_0. Due to the symmetry, analogous role is played by S_c for P.

This knowledge, together with (28) can be exploited to generate a hole branch of strongly stabilizing controllers starting from an initial one, K_0, with this property. E.g., one has to choose arbitrarily a stable R with a sufficiently small norm (less than $\frac{1}{\|S_p\|}$) and then apply (28) iteratively.

6. CONCLUSIONS

In this paper we have shown that based on the direct blending operation the set $\mathbb{R}_{K_0}$ can be defined and characterized in a completely coordinate free way, without any reference to a coprime factorization. For practical purposes it is also interesting to know that $K_0 \in \mathbb{R}_{K_0}$, moreover $\mathbb{Q} \subset \mathbb{R}_{K_0}$ holds as a coordinate free property, too. We emphasize, that a fairly large set of stabilizing controllers can be constructed (parametrized) just starting from the knowledge of a single stabilizing controller, without any additional knowledge (e.g., factorization). This underlines an important property of the geometric (and also input-output) view: describes the structure of the given set – in our case those of the stabilizing controllers – but does not provide a direct method to find any of the actual objects at hand. To do so, we need to ensure (e.g., by a construction algorithm) the existence at least of a single element with the given property.

Up to this point only projective geometric structures were considered. In order to qualify a given controller K as a stabilizing one (validation problem) metric aspects should be also considered, i.e., euclidean, hyperbolic, etc. geometries. E.g., concerning the Youla parametrization $Q_K \in \mathbb{Q}$, or in the geometric parametrization $R_K \in \mathbb{R}_{K_0}$, should be decided. It is subject of further research how these geometries find their way to control theory and vice versa.

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