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On the correspondence of hyperbolic geometry and system analysis

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Abstract: Different aspects of the relation between hyperbolic geometry and linear system theory are discussed in this paper. The underlying connection is presented by an intuitive example that points out the basic motivations. It is shown that the convergence factor of Laguerre series expansion is equal to the hyperbolic distance, under certain conditions. Preliminary results are also reported, connecting the H_∞ norm and ν-gap metric with the hyperbolic distance. Furthermore, the equivalence of (i) the H_∞ norm of the difference of two first order LTI system, (ii) the ν-gap of these systems and (iii) the hyperbolic distance is also proved, under specified assumptions.

Keywords: Linear systems, Hyperbolic geometry, Orthogonal basis functions, ν-gap metric, H_∞ norm,

1. INTRODUCTION

Hyperbolic geometry view on dynamical systems can offer a unique insight and reveal connections between certain properties of linear systems (Beardon and Minda, 2000; Anderson, 2005). This paper elaborates certain aspects of this connection. The main motivation is the successful utilization of the hyperbolic geometry in system identification and analysis. However, the underlying correspondence is rarely discussed from the engineering point of view.

The most successful application area of the hyperbolic approach is system identification, where various methodologies are developed in the literature. The main advantage is the appropriate model structure, offered by the hyperbolic approach. In these frameworks, the identification problem is generally translated as a search for a set of basis functions that provides series expansion of the model with fast convergence. The use of orthogonal basis functions for identification of stable systems in Hardy space H^2 has a great advantage that if the basis is properly chosen then the speed of convergence of the series expansion can be substantially increased (see Heuberger et al. (2006)). Therefore, only a few coefficients have to be estimated. The speed of convergence is characterized by the decay ratio that is the reciprocal of the convergence factor. So the quality of the chosen basis can be represented quantitatively by the convergence factor. In the early works, Linear Time Invariant (LTI) systems are considered Heuberger et al. (1995), while later extensions for Linear Parameter Varying (LPV) models have appeared Tóth et al. (2009).

The problem of Linear Time Invariant (LTI) system identification is discussed in Soumelidis et al. (2009) from a hyperbolic geometric point of view. The identification method is based on a hyperbolic wavelet construction that parametrizes the location of poles by operations similar to translation and dilatation. These are the basic mother wavelet transformations in wavelet theory. Furthermore, a hyperbolic wavelet transformation is proposed on the conceptual base of the Blaschke function, operating as a translation operator.

Another identification method, based on the intersection of hyperbolic circles is proposed in Soumelidis et al. (2012). The system is represented in Laguerre basis and the convergence of the Laguerre series expansion is connected to the hyperbolic radius of a hyperbolic circle. The identification method is developed in the Hardy space $H^2_\perp$, which is the set of all functions that is analytic inside the open unit disk $\mathbb{D}$ and has a finite norm. However, the control engineering convention is that a stable discrete LTI systems belong to the Hardy space H^2, with poles inside the unit circle.

The 2-dimensional Poincaré disc model is utilized in Tóth (2010) in order to aid system identification of Linear Parameter Varying (LPV) systems. The work connects Kolmogorov n-width optimal orthogonal basis (see Oliveira e Silva (1996)) functions with objects in hyperbolic geometry. This approach is an important application of hyperbolic geometry.

The main contribution of the paper is the establishment of the connection between ν-gap metric, H_∞ norm and hyperbolic distance, under specific conditions. The result can give an opportunity for new methods for calculating bounds on H_∞ norm or ν-gap metric with low numerical complexity. This is especially important in the case of large-scale systems, known to be ill-conditioned.

⋆ Sponsor and financial support acknowledgment goes here. Paper titles should be written in uppercase and lowercase letters, not all uppercase.

The remainder of this paper is organized as follows. Section 2 shortly introduces the most important features of hyperbolic geometry. An example presented in Section 3 which intuitively point out the correspondence between hyperbolic geometry and system behavior. Section 4 presents a detailed derivation of the convergence factor of Laguerre series expansion as hyperbolic distance. As a new result relation between $\nu - gap$ metric, H_∞ norm and hyperbolic distance is identified and presented in Section 5 followed by concluding remarks in Section 6.

2. THE HYPERBOLIC DISTANCE

In the followings, a short summary is presented about the most important features of the hyperbolic geometry (for further information see Beardon and Minda (2000); Anderson (2005)). The main motivation is that stable LTI systems can be naturally represented in the hyperbolic setting over the unit disc.

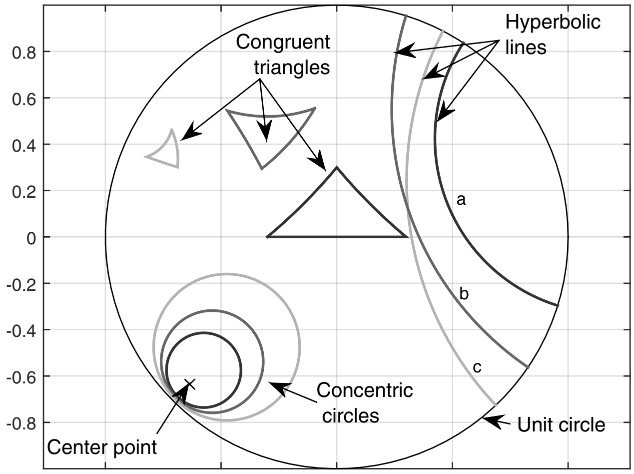

Fig. 1. Basic geometric objects on Poincaré disk model of hyperbolic geometry

In order to do so, Euclid's parallel postulate is substituted by the axiom that states for every line h, and a point P not on h, there are infinitely many lines through P which do not cross h (see Figure 1). Among the many models with this property, the 2-dimensional Poincaré disk model is widely used in the control engineering community Tóth (2010). Lines are represented by Euclidean circles that are orthogonal to the unit circle, while the hyperbolic line is the part of a circle which lies strictly inside the unit circle (see Fig. 1). The line a is parallel to line b and line c and it is easy to see that an infinite number of lines can be drawn that is parallel to line a and goes through the intersection of lines b and c.

The 2-dimensional Poincaré disk model is defined on the complex unit disk with the following distance metric:

$$d_h(\gamma_1, \gamma_2) = 2 \tanh^{-1} \frac{|\gamma_1 - \gamma_2|}{|1 - \gamma_2 \overline{\gamma}_1|}, \tag{1}$$

where $\gamma_1, \gamma_2 \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\gamma}$ is the complex conjugate of γ. It is obvious that:

$$\lim_{\gamma_1 \to \partial \mathbb{D}} d_h(\gamma_1, \gamma_2) \to \infty, \tag{2}$$

i.e. the distance approaches infinity as one of the points approaches the unit circle $\partial \mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$. In other words: the complex unit disk $\mathbb{D}$ represents the infinite hyperbolic 2-dimensional space in this model. Hyperbolic circles are the set of all points that are at a given hyperbolic distance from a given center point (see Fig. 1). In Poincaré disk model the hyperbolic circles can be represented by Euclidean circles that means there is an Euclidean circle for every hyperbolic circle so that each circle has the same set of points.

It is also important to note that one can define a so called pseudo-hyperbolic distance as:

$$d_{hp}(\gamma_1, \gamma_2) = \frac{|\gamma_1 - \gamma_2|}{|1 - \overline{\gamma}_1 \gamma_2|}, \tag{3}$$

The Poincaré disk model equipped with the pseudo-hyperbolic distance has the same geometric properties except the the pseudo-hyperbolic distance is never additive along geodesics (i.e. hyperbolic lines).

3. INTUITIVE INTRODUCTION

The advantage of this metric is demonstrated by a simple example. Consider a nominal second order, strictly proper SISO discrete transfer function $G_N(z)$ with the complex eigenvalue pair $0.9e^{\pm i\frac{\pi}{8}}$ inside the unit disk. Take the perturbed systems $G_1(z)$ and $G_2(z)$ with the poles $0.99e^{\pm i\frac{\pi}{8}}$ and $0.81e^{\pm i\frac{\pi}{8}}$ respectively. Set the static gain of each system equal to 1 and compare the time-domain behaviors.The result is plotted in Fig. 2. In this example the euclidean distance between the corresponding poles of the nominal and $G_1(z)$, $G_2(z)$ is equal it is 0.09. The time domain behavior is greatly differs from each other so it is clear that the euclidean distance does not captures the dynamic behavior of these systems. The hyperbolic distance of the corresponding poles respect to $G_N(z)$ and $G_1(z)$ is 2.3489 and respect to $G_N(z)$ and $G_2(z)$ is 0.6904. The hyperbolic distances between the poles suggests that the hyperbolic metric is more suitable for comparing dynamic behavior based only on pole locations.

The hyperbolic distance has another important feature that is the distance is not defined i.e meaningless if one compares stable poles with unstable poles which is coherent with the expectations. This feature does not hold for euclidean distance. On the other way measuring the hyperbolic distance between unstable poles is possible since one can transform them with the transformation $\hat{p} = 1/\overline{p}$ where $\hat{p}$ is the transformed pole and bar means complex conjugate.

The presented simple example shows that the hyperbolic distance can be better in comparing LTI systems based on their pole location then Euclidean distance but does not point out any suggestion why one should use hyperbolic distance from the bunch of other possible distances. The following two theorems (Anderson (2005)) show the motivation of using hyperbolic distance.

Theorem 1. Any holomorphic homeomorphism $f : \mathbb{D} \to \mathbb{D}$ is an isometry of the hyperbolic metric.

Theorem 2. Any holomorphic homeomorphism f of $\mathbb{D}$ to itself is a Möbius transformation

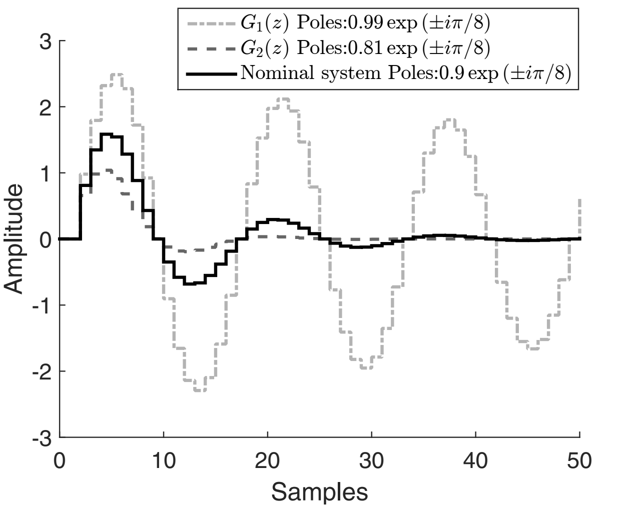

Fig. 2. Impulse responses of $G_1(z)$, $G_2(z)$ and $G_N(z)$ systems

$$f(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0 \tag{4}$$

Where $\mathbb{D}$ is the open unit disc.

Möbius transformation itself is a transfer function of a first order LTI system. This intrinsic relation gives the motivation of the analysis of hyperbolic geometry in correlation with system and control theory. The problem is that the mentioned theorems do not give useful results from system and control point of view, so this require further investigation, some aspects of this problem is discussed in this paper. In the sequel the derivation of the convergence factor of Laguerre series expansion as hyperbolic distance, and the relation of H_∞ norm and $\nu - gap$ metric with hyperbolic distance is shown.

4. CONVERGENCE FACTOR OF LAGUERRE SERIES EXPANSION AS HYPERBOLIC DISTANCE

This section shows that the convergence factor of the Laguerre series expansion of a first order discrete LTI system is exactly the pseudo-hyperbolic distance between the Laguerre parameter and the corresponding pole of the first order system. Similar derivations can be found in the literature: in Soumelidis et al. (2012) equivalence in the Hardy space $H^2_\perp$ is proven, while in Heuberger et al. (2006) only real poles are considered. The present derivation is carried out in the H^2 Hardy space, important from engineering point of view. Furthermore it is not limited to real poles.

First the formal definition of the H^2 Hardy space is discussed.

4.1 H^2 Hardy space

Let $H^2(\mathbb{D})$ be the set of all functions that is analytic outside the unit circle plate $\mathbb{D}$ and has a finite norm with respect to the following norm definition.

Let $f \in H^2(\mathbb{D})$ and let $M_2(f,r)$ be a following function:

$$M_2(f,r) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(re^{\mathrm{i}\,\omega}\right) \right|^2 d\omega \right\}^{\frac{1}{2}}, \tag{5}$$

where r and ω are the magnitude and argument of the complex number with i being the imaginary unit. For any $f \in H^2(\mathbb{D})$ the 2-norm is defined as:

$$\|f\|_2 = \lim_{r \to 1} M_2(f,r). \tag{6}$$

The $L^2(\partial\,\mathbb{D})$ space is the space of functions g on the unit circle $\partial\,\mathbb{D}$ for which the following norm

$$\|g\| = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| g\left(e^{\mathrm{i}\,\omega}\right) \right|^2 d\omega \right\}^{\frac{1}{2}}$$

is bounded.

In the followings useful theorems are summarized regarding $H^2(\mathbb{D})$ (see Rudin (1987)).

- If $f \in H^2(\mathbb{D})$ then f has radial limits $f^*(e^{i\omega})$ at almost all points of $\partial\,\mathbb{D}$.
- $f^* \in L^2(\partial\,\mathbb{D})$.
- The mapping $f \to f^*$ is an isometry of $H^2(\mathbb{D})$ onto the subspace of $L^2(\partial\,\mathbb{D})$.
- Let $f, g \in H^2(\mathbb{D})$ and the inner product in $H^2(\mathbb{D})$ is defined by
$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*\left(e^{i\omega}\right) \overline{g^*\left(e^{i\omega}\right)} d\omega, \tag{7}$$
then the $H^2(\mathbb{D})$ space is a Hilbert space equipped with the above described inner product.

In this paper every $F(z) \in H^2(\mathbb{D})$ under investigation is a strictly proper rational transfer function that do not have zeros on the unit circle. In this case the followings are true (see Heuberger et al. (2006)):

- The radial limits $f^*(e^{i\omega})$ of $f \in H^2(\mathbb{D})$ are equal to $f(e^{i\omega})$.
- The inner product among these functions can be expressed as:
$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\omega}\right) \overline{g\left(e^{i\omega}\right)} d\omega \quad and$$
$$\langle f, g \rangle = \frac{1}{2\pi\,\mathrm{i}} \oint_{\mathbb{T}} f(z) \overline{g\left(\frac{1}{\bar{z}}\right)} \frac{dz}{z} \tag{8}$$

Note that the there is an orthogonal complement of H^2 in L^2 and it is denoted by $H^2_\perp$. In short $H^2_\perp$ is the set of all functions that is analytic inside the unit circle plate $\mathbb{D}$ and has a finite norm. From engineering point of view the H^2 space is more important since the convention is that a discrete stable LTI systems have their poles inside the unit circle.

4.2 Convergence factor

The unit pulse response $g(t)$ of a stable casual LTI system can be expressed in an orthonormal series expansion as:

$$g(t) = \sum_{k=1}^{\infty} c_k f_k(t), \tag{9}$$

where $f_k(t)$ is the k-th element of an orthonormal basis and c_k is the k-th coefficient. In practical applications all the element of the series expansion can not be used so $g(t)$ is approximated by the first n elements of the series expansion. Let $g(t;n)$ be the the approximation of $g(t)$ of order n. The error of this approximation is

$$\epsilon_g(t;n) = g(t;n) - g(t). \tag{10}$$

The convergence factor describes the rate of convergence for the series expansion. Under exponential approximation error one can write,

$$\|\epsilon_g(t;n+k)\| \approx \rho^k \|\epsilon_g(t;n)\| \tag{11}$$

where ρ is the convergence factor ($0 \leq \rho < 1$).

4.3 Derivation of the convergence factor of the Laguerre series expansion

The Laguerre basis on $H^2(\mathbb{D})$ is defined by the following basis functions:

$$\Phi_n(z) = \frac{\sqrt{1-|a|^2}}{z-a}\left(\frac{1-\bar{a}z}{z-a}\right)^n \quad n = 0,1,2\ldots \qquad a \in \mathbb{D};\ a \neq 0 \tag{12}$$

Let $F(z) \in H^2(\mathbb{D})$ a strictly proper rational function that is analytic outside the unit circle and has no pole on the unit circle. Then compute the n-th Laguerre coefficient l_n of $F(z)$ as:

$$\begin{aligned} l_n = \langle \Phi_n(z), F(z)\rangle &= \frac{1}{2\pi\,\mathrm{i}}\oint_{\mathbb{T}} \Phi_n(z)\overline{F\left(\frac{1}{\bar{z}}\right)}\frac{dz}{z} = \\ &= \frac{\sqrt{1-|a|^2}}{2\pi\,\mathrm{i}}\oint_{|z-a|=\epsilon} \frac{(1-\bar{a}z)^n\,\overline{F\left(\frac{1}{\bar{z}}\right)}\frac{1}{z}}{(z-a)^{n+1}}dz + \\ &+ \frac{1}{2\pi\,\mathrm{i}}\oint_{|z|=\epsilon} \frac{\frac{(1-\bar{a}z)^n}{(z-a)^{n+1}}\overline{F\left(\frac{1}{\bar{z}}\right)}}{z}dz \end{aligned} \tag{13}$$

where $0 < \epsilon < |a|$.

Applying the Cauchy integral formulas to (13) the Laguerre coefficients are rewritten as:

$$\begin{aligned} l_n = &\left[\frac{\sqrt{1-|a|^2}}{n!}\frac{d^n}{dz^n}\left(\frac{1}{z}(1-\bar{a}z)^n\,\overline{F\left(\frac{1}{\bar{z}}\right)}\right)\right]_{z=a} + \\ &+ \left[\sqrt{1-|a|^2}\frac{(1-\bar{a}z)^n}{(z-a)^{n+1}}\overline{F\left(\frac{1}{\bar{z}}\right)}\right]_{z=0} \end{aligned} \tag{14}$$

At this point we restrict ourselves for stable, first-order transfer functions, in the following form:

$$F(z) = \frac{A}{z-b} \tag{15}$$

$$\overline{F\left(\frac{1}{\bar{z}}\right)} = \frac{Az}{1-\bar{b}z} \tag{16}$$

where $b \in \mathbb{D}$. Substituting (15) and (16) into (14) the Laguerre coefficients for the given form of $F(z)$ is computed through :

$$\begin{aligned} l_n = &\left[\frac{A\sqrt{1-|a|^2}}{n!}\frac{d^n}{dz^n}\left(\frac{(1-\bar{a}z)^n}{1-\bar{b}z}\right)\right]_{z=a} + \\ &+ \left[\sqrt{1-|a|^2}\frac{(1-\bar{a}z)^n}{(z-a)^{n+1}}\frac{Az}{1-\bar{b}z}\right]_{z=0} \end{aligned} \tag{17}$$

It is obvious that the second therm of (17) is zero, therefore the Laguerre coefficients of (15) are:

$$l_n = \left[\frac{A\sqrt{1-|a|^2}}{n!}\frac{d^n}{dz^n}\left(\frac{(1-\bar{a}z)^n}{1-\bar{b}z}\right)\right]_{z=a}. \tag{18}$$

In order to calculate l_n the n-th derivative of

$$\frac{(1-\bar{a}z)^n}{1-\bar{b}z} = (1-\bar{a}z)^n\left(1-\bar{b}z\right)^{-1} \tag{19}$$

has to be computed, for which the generalized Leibniz rule can be applied. Let u and v be two n-times differentiable functions, then:

$$(uv)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)} \tag{20}$$

Applying (20) for (19), we get:

$$\begin{aligned} u^{(n-k)} &= \left((1-\bar{a}z)^n\right)^{(n-k)} = \\ &= (-1)^{n-k}(n-k)!\binom{n}{n-k}(1-\bar{a}z)^k(\bar{a})^{n-k} \\ v^{(k)} &= \left(\left(1-\bar{b}z\right)^{-1}\right)^{(k)} = k!\left(1-\bar{b}z\right)^{-1-k}\left(\bar{b}\right)^k \end{aligned} \tag{21}$$

Notice that:

$$(n-k)!\binom{n}{n-k} = \frac{n!}{k!},$$

so equation (20) takes the following form:

$$\begin{aligned} &(uv)^{(n)} = \\ &\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{n!}{k!}(1-\bar{a}z)^k(\bar{a})^{n-k}k!\left(1-\bar{b}z\right)^{-1-k}\left(\bar{b}\right)^k \end{aligned} \tag{22}$$

After some algebraic manipulation the binomial theorem can be applied to get:

$$(uv)^{(n)} = \frac{n!}{\left(1-\bar{b}z\right)^{n+1}}K(z) \tag{23}$$

where

$$\begin{aligned} K(z) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(\bar{a})^{n-k}\left(1-\bar{b}z\right)^{n-k}\left(\bar{b}\right)^k(1-\bar{a}z)^k = \\ &= \left(\bar{b}(1-\bar{a}z) - \bar{a}\left(1-\bar{b}z\right)\right)^n = \left(\bar{b}-\bar{a}\right)^n. \end{aligned} \tag{24}$$

Therefore, we arrive to the following formula:

$$(uv)^{(n)} = \frac{n!\left(\bar{b}-\bar{a}\right)^n}{\left(1-\bar{b}z\right)^{n+1}} \tag{25}$$

Consequently, the Laguerre coefficients (18) are:

$$\begin{aligned} l_n &= \left[\frac{A\sqrt{1-|a|^2}}{n!}\frac{n!\left(\bar{b}-\bar{a}\right)^n}{\left(1-\bar{b}z\right)^{n+1}}\right]_{z=a} = \\ &= \frac{A\sqrt{1-|a|^2}}{1-\bar{b}a}\frac{\left(\bar{b}-\bar{a}\right)^n}{\left(1-\bar{b}a\right)^n}. \end{aligned} \tag{26}$$

From (26) it is obvious that the convergence factor ρ of $F(z)$ is

$$\rho = \frac{\left|\bar{b}-\bar{a}\right|}{\left|1-\bar{b}a\right|} = \frac{|b-a|}{\left|1-\bar{b}a\right|} \tag{27}$$

which is equal to the pseudo-hyperbolic distance between b and a in (3). It is worth to mention that this result is true in a more general cases. If $F(z)$ has a partial fractional representation and every pole is distinct and stable then the contribution of each partial fraction to the series expansion has the form of (26). For large n the convergence factor of $F(z)$ is obviously equal to the convergence factor of the therm in partial fractional representation whose convergence factor is the largest.

5. RELATION OF H_∞ NORM AND $\nu - GAP$ METRIC WITH HYPERBOLIC DISTANCE

In this section two preliminary results are presented on the correspondence of H_∞ norm and ν-gap metric with hyperbolic distance.

Theorem 3. Let $P_1(s)$ and $P_2(s)$ are two first order continuous time LTI SISO systems and let $G_1(z)$ and $G_2(z)$ are the discrete zero-order hold equivalent of $P_1(s)$, $P_2(s)$ with an appropriate sampling time. Then, if the static gains are equal to one and the sampling time is approaching zero:

- the ν-gap metric of continuous systems and the pseudo-hyperbolic distance of the poles of the discrete systems are equivalent metrics
- the H_∞ norm of the difference of the continuous systems and the pseudo-hyperbolic distance of the poles of the discrete systems are equivalent

Proof. Let the system be a first order LTI system in the form:

$$P(s) = \frac{A}{s+b}. \tag{28}$$

The normalized right graph symbol of $P(s)$ is

$$G_r = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} \dfrac{-A}{s+\sqrt{A^2+b^2}} \\ \dfrac{-s-b}{s+\sqrt{A^2+b^2}} \end{bmatrix}. \tag{29}$$

In order to see that $N(s)$ and $D(s)$ in (29) are the co-prime factorization of (28) we write:

$$P(s) = ND^{-1} = \frac{-A}{s+\sqrt{A^2+b^2}} \frac{s+\sqrt{A^2+b^2}}{-s-b} = \frac{A}{s+b}, \tag{30}$$

Furthermore, to see that (29) is normalized it has to satisfy the following Bezout identity:

$$N^*N + D^*D = I \tag{31}$$

where $N(s)^* = N(-s)^T$ and $D(s)^* = D(-s)^T$. Substitute (29) in (31):

$$\frac{-A}{-s+\sqrt{A^2+b^2}} \frac{-A}{s+\sqrt{A^2+b^2}} + \\ + \frac{s-b}{-s+\sqrt{A^2+b^2}} \frac{-s-b}{s+\sqrt{A^2+b^2}} = \\ = \frac{A^2}{A^2+b^2-s^2} + \frac{b^2-s^2}{A^2+b^2-s^2} = \frac{A^2+b^2-s^2}{A^2+b^2-s^2} = 1$$

so one can conclude that (29) is the normalized right graph symbol of $P(s)$. The same argument can be applied to the left normalized graph symbol, that is:

$$G_l = [-D(s) \;\; N(s)] = \begin{bmatrix} \dfrac{s+b}{s+\sqrt{A^2+b^2}} & \dfrac{-A}{s+\sqrt{A^2+b^2}} \end{bmatrix}. \tag{32}$$

Now we are at the position to compute the ν-gap between two stable systems $P_1(s)$ and $P_2(s)$, by using the corresponding co-prime factorizations. The definition of ν-gap metric (see Vinnicombe (1993)) is

$$\delta_\nu(P_1, P_2) = \begin{cases} \|G_{l2}G_{r1}\|_\infty & \text{if } det(G^*_{r2}G_{r1})(j\omega) \neq 0 \\ & \forall\omega \in (-\infty, \infty) \\ & \text{and winding number of} \\ & det(G^*_{r2}G_{r1}) = 0 \\ 1 & \text{otherwise} \end{cases} \tag{33}$$

Now, substitute the first order dynamics of $P_1(s)$ and $P_2(s)$ into (33) using (29) and (32)

$$\delta_\nu(P_1, P_2) = \\ \left\| \begin{bmatrix} \dfrac{s+b_2}{s+\sqrt{A_2^2+b_2^2}} & \dfrac{-A_2}{s+\sqrt{A_2^2+b_2^2}} \end{bmatrix} \begin{bmatrix} \dfrac{-A_1}{s+\sqrt{A_1^2+b_1^2}} \\ \dfrac{-s-b_1}{s+\sqrt{A_1^2+b_1^2}} \end{bmatrix} \right\|_\infty = \\ \left\| \frac{(A_2-A_1)s + (A_2b_1 - A_1b_2)}{s^2+(c_1+c_2)s+c_1c_2} \right\|_\infty \tag{34}$$

where $c_1 = \sqrt{A_1^2+b_1^2}$ and $c_2 = \sqrt{A_2^2+b_2^2}$.

Let $A_1 = b_1 = d_1$ and $A_2 = b_2 = d_2$ in order to set the static gain to one and substitute in (34):

$$\delta_\nu(P_1, P_2) = \left\| \frac{(A_2-A_1)s + (A_2b_1 - A_1b_2)}{s^2+(c_1+c_2)s+c_1c_2} \right\|_\infty = \\ \left\| \frac{(d_2-d_1)s}{s^2+\sqrt{2}(d_1+d_2)s+2d_1d_2} \right\|_\infty \tag{35}$$

From the definition of H_∞ norm it follows that:

$$\left\| \frac{(d_2-d_1)s}{s^2+\sqrt{2}(d_1+d_2)s+2d_1d_2} \right\|_\infty = \\ \max_\omega \left| \frac{(d_2-d_1)i\omega}{-\omega^2+\sqrt{2}(d_1+d_2)i\omega+2d_1d_2} \right| \tag{36}$$

In (36) the transfer function has two real poles, which can be shown by simply solving the quadratic formula to obtain:

$$p_{1,2} = \frac{-\sqrt{2}(d_1+d_2) \pm \sqrt{2(d_1+d_2)^2 - 8d_1d_2}}{2} = \\ \frac{-\sqrt{2}(d_1+d_2) \pm \sqrt{2(d_1-d_2)^2}}{2} = \\ \frac{-\sqrt{2}(d_1+d_2) \pm \sqrt{2}(d_1-d_2)}{2} = \begin{cases} -\sqrt{2}d_1 \\ -\sqrt{2}d_2 \end{cases}$$

I.e. the Bode magnitude plot has exactly one global maximum for positive ω and there is no local minimum or inflection point.

The calculation of the maximum in (36) is a long and standard process, therefore only the basic steps are outlined here. Let the frequency function of (36) denoted by $M(\omega)$.

(1) Calculate the absolute value of $M(\omega)$.
  (a) Expanding $M(\omega)$ by the complex conjugate of its denominator.
  (b) Separate real and imaginary part.
  (c) Calculate $\sqrt{Re(M(\omega))^2 + Im(M(\omega))^2}$
(2) Since the square root is monotonic, $H(\omega) = |M(\omega)|^2$ can be used, without loss of generality.
(3) Calculate the ω derivative of $H(\omega)$.

(4) Since the transfer function of equation (36) has two real poles, it is sufficient to involve the equation $\frac{dH(\omega)}{d\omega} = 0$ for the correct result.

The final result is

$$\delta_\nu(P_1, P_2) = \frac{1}{\sqrt{2}} \left| \frac{d_2 - d_1}{d_1 + d_2} \right| = \left| \frac{1}{\sqrt{2}} \frac{b_2 - b_1}{b_1 + b_2} \right| \quad (37)$$

where we applied $d_1 = b_1$ and $d_2 = b_2$.

To see the correlation of (37) with the hyperbolic distance metric, we substitute the discrete poles of the system in the formula of the pseudo-hyperbolic distance in eq. (3). In addition, in order to connect the discrete representation with the continuous one, we investigate the limit of the distance, with sampling time T approaching zero. Hence:

$$\left| \lim_{T \to 0} \frac{e^{b_1 T} - e^{b_2 T}}{1 - e^{b_1 T} e^{b_2 T}} \right| = \left| \lim_{T \to 0} \frac{b_1 e^{b_1 T} - b_2 e^{b_2 T}}{-(b_1 + b_2) e^{(b_1 + b_2) T}} \right| = \left| \frac{b_2 - b_1}{b_1 + b_2} \right| \quad (38)$$

By comparing (37) and (38) it can be depicted that the only difference between the ν-gap metric and the pseudo hyperbolic distance is the scalar coefficient $\frac{1}{\sqrt{2}}$. This concludes the first part of the proof.

Similar argument can be applied to the correspondence of the H_∞ norm and pseudo hyperbolic distance. In order to see, we compute the H_∞ norm of the difference of the stable systems $P_1(s)$ and $P_2(s)$ as:

$$\left\| \frac{A_1}{s + b_1} - \frac{A_2}{s + b_2} \right\|_\infty = \left\| \frac{(A_2 - A_1)s + (A_2 b_1 - A_1 b_2)}{s^2 + (b_1 + b_2)s + b_1 b_2} \right\|_\infty \quad (39)$$

Again, let $A_1 = b_1 = d_1$ and $A_2 = b_2 = d_2$ then we get:

$$\left\| \frac{A_1}{s + b_1} - \frac{A_2}{s + b_2} \right\|_\infty = \left\| \frac{(d_2 - d_1)s}{s^2 + (d_1 + d_2)s + d_1 d_2} \right\|_\infty \quad (40)$$

Note that, the obtained formula has the same structure as in equation (35), hence the same train of thought can be followed. The final result shows:

$$\left\| \frac{A_1}{s + b_1} - \frac{A_2}{s + b_2} \right\|_\infty = \left| \frac{b_2 - b_1}{b_1 + b_2} \right| \quad (41)$$

Therefore, it can be concluded from (41) and (38) that the H_∞ norm of the difference of $P_1(s)$, $P_2(s)$ is equivalent with the pseudo hyperbolic distance as the sample time approaches zero.

6. CONCLUSIONS AND FUTURE WORK

Detailed derivation of the convergence factor of Laguerre series expansion is carried out in the Hardy space H^2, important from the engineering point of view. It has been shown that the convergence factor of Laguerre series expansion is equal to the hyperbolic distance.

The connection between H_∞ norm and ν-gap metric with hyperbolic distance is discussed. The generalization of this theorem may give an opportunity for potential new methods of calculation of bounds on H_∞ norm or ν-gap metric based on hyperbolic geometry. Since the calculation of hyperbolic distance is computationally not expensive the applicability of this methods can be extended to a larger dimensional system.

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