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PURDUE UNIVERSITY
GRADUATE SCHOOL
Thesis/Dissertation Acceptance

This is to certify that the thesis/dissertation prepared

By Ru Zhang

Entitled
Selective Influences, Mental Architectures, and Contextuality

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Ehtibar N. Dzhafarov (Chair)	Sebastien Helie
Richard Schweickert	
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Robert W. Proctor	

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Approved by: Christopher R. Agnew 7/1/2016
Head of the Departmental Graduate Program Date

SELECTIVE INFLUENCES, MENTAL ARCHITECTURES, AND CONTEXTUALITY

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Ru Zhang

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2016

Purdue University

West Lafayette, Indiana

To my parents.

ACKNOWLEDGMENTS

I would like express my deepest gratitude to my advisor Ehtibar N. Dzhafarov for his support, encouragement, and guidance on my research and dissertation. I would like to thank my committee members Dr. Richard Schweickert, Dr. Zygmunt Pizlo, Dr. Robert W. Proctor, and Dr. Sebastien Helie for their reviews and comments on my dissertation.

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ABSTRACT

Zhang, Ru Ph.D., Purdue University, August 2016. Selective Influences, Mental Architectures, and Contextuality. Major Professor: Ehtibar N. Dzhafarov.

Given a system with say two external factors α and β and two random outputs A and B in response to the external factors. α forms the context of A and β forms the context of B. When the marginal distribution of A is not affected by the change of β and the marginal distribution of B is not affected by the change of α, we say marginal selectivity present in the system. Can we say there is no context effect then? Our answer is "not yet for interdependent A and B". If in addition, one can find a hidden variable R, so that A can be written as a function of α and R, and B can be written as a function of β and R, selective influences are established (Dzhafarov, 2003) and one speaks of "no context effect".

Perceptual separability understands if different stimulus attributes are perceived, evaluated, and responded in a separable fashion. Selective influences provide a new definition of perceptual separability. To realize the approach, we developed psychophysical matching experiments in which the responses A and B were extracted from an observer's choice of a stimulus that was adjusted to match the fixed stimulus with attributes α and β. We used α and β (also A and B) as simple geometric properties of dots or lines. α and β are considered perceptually separable if selective influences of α and β on A and B are established.

A mental architecture is a hypothetical network of underlying cognitive processes when a subject is performing a task. It is usually assumed that the durations of processes involved in the network are selectively influenced by different experimental factors. Usually the overall duration is observable but the duration components are not. One way to characterize different types of mental architectures, e.g. the parallel

vs. the serial is to compute the interaction contrast of the distribution functions of the overall durations (Townsend & Nozawa, 1995). Note that for any given value of R, the duration components and the overall duration become deterministic quantities (Zhang & Dzhafarov, 2015). Consequently, one can easily compute the interaction contrast as the probabilistic problem is reduced to simple numerical combinatorics. Our work provides a simpler method than the previously used ones to investigate theories of mental architectures.

In the behavioral systems, it is very likely that marginal selectivity is absent in a system. According to the contextuality-by-default theory (Dzhafarov & Kujala, 2014a, 2014b, 2014c; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2015, 2016; Kujala, Dzhafarov, & Larsson, 2015), if the covariance between A and B can be entirely attributed to α and β, and a hidden variable R, the system is not contextual. Otherwise it is contextual. Note that when marginal selectivity is present in a system, the framework of contextuality-by-default reduces to the framework of selective influences. Contextuality is tested for cyclic systems of ranks $N = 4, 6, 8$ using the psychophysical matching data.

INTRODUCTION

Let us consider a system that contains external factors and outputs that depend on these external factors. In behavioral sciences, one can use physical luminance and physical size of an object as the two external factors (denoted as α and β) and the perception of the luminance and the perception of the size as the two random outputs (denoted as A and B). The perception of luminance, of course, depends on the physical luminance. But the perception of luminance can also depend on the size of the object (that forms a context for the perception of luminance), and it generally covaries with the perception of the size. Similarly, the perception of the size depends on the size of the object, but it can also depend on the luminance of the object (that forms a context for the perception of size), and it generally covaries with the perception of the luminance. In this dissertation, three theoretical frameworks that relate to contextual effects will be discussed. The framework of selective influences is the basis for the other two. Selective influences define the "no context effect" for a system in the presence of marginal selectivity. The technique of interaction contrast requires selective influences present in order to characterize the mental architectures. Noncontextuality is the generalized version of selective influences, including the cases when marginal selectivity is breached.

In the (α, β, A, B) system, β forms the context of A, and α forms the context of B. If manipulating β does not change the marginal distribution of A and manipulating α does not change the marginal distribution of B, then we say that *marginal selectivity* is present in the (α, β, A, B) system. Now the question arises: Is marginal selectivity equivalent to "no context effect"? Our answer is: only if A and B are stochastically independent for all possible α and β. However, A and B are usually stochastically interdependent for some if not all values of α and β, and then using marginal selectivity to define "no context effect" is not satisfactory. The interdependence between A and B may be attributed to some hidden variable, whose distribution does not depend on α and β. If one can find a hidden variable, denoted as R, so that A can be written as a function of α and R, and B can be written as a function of β and R, then *selective influences* are established, and one speak of "no context effect." The definition of selective influences was formulated for a finite set of random variables by Dzhafarov (2003) and further characterized by Dzhafarov and Gluhovsky (2006). The *cosphericity test* (Kujala & Dzhafarov, 2008) was developed to test selectiveness, and it is a sufficient and necessary condition for selective influences if confining the system to a 2×2 factorial design and the two output variables are bivariate normally distributed. Selective influences can also be defined by the *joint distribution criterion* (Dzhafarov & Kujala, 2010). The *Linear Feasibility Test* (Dzhafarov & Kujala, 2012b) is a direct use of the joint distribution criterion, and it is a powerful tool to establish selective influences for finite number of external factors and outputs. The *Bell-CHSH-Fine inequality test* (Dzhafarov & Kujala, 2012a) is a special case of the Linear Feasibility Test, and it can be used in a 2×2 factorial system in which the two output variables are binary, or discretized to be binary.

Mental architectures are the arrangements of mental processes underlying a subject's performance. Suppose there are two underlying mental processes that process the information on the external factors (α, β), respectively. Let (T^α, T^β) be the durations of the two processes. The overall duration T can be considered a function of the duration components T^α and T^β. Three fundamental architectures are of the greatest traditional interest: $T = T^\alpha + T^\beta$, $T = \min(T^\alpha, T^\beta)$, and $T = \max(T^\alpha, T^\beta)$ (Townsend, Yang, & Burns, 2011). They are named *serial*, *minimum parallel*, and *maximum parallel*, respectively. In behavioral sciences, usually the overall duration can be measured by recording the response time (RT) but the durations of the underlying processes are not observable. Fortunately, one can characterize the architecture by analyzing the pattern of a linear combination (interaction contrast) of the distributions of RT in a factorial experiment (Townsend & Nozawa, 1995). Selective influences play a role as the "pre-assumption" for this technique. It assumes that (T^{α}, T^{β}) are selectively influenced by (α, β), respectively. With this assumption, as one manipulates the external factors, the duration components influenced by those factors vary and consequently the overall duration is changed as well. With these factorial manipulations, each mental architecture has a distributional pattern of RT that differentiates it from other architectures. The applicability of the distributional approach has been extended to general architectures (Dzhafarov, Schweickert, & Sung, 2004; Schweickert, Giorgini, & Dzhafarov, 2000), which contain the fundamental two-process architectures as subsystems, and to the architectures composed by multiple serial, minimum parallel, or maximum parallel processes (Yang, Fific, & Townsend, 2014).

It is very likely that marginal selectivity is absent, especially in the systems of behavioral sciences. That is, β affects the distribution of A, and α affect the distribution of B. There is nothing wrong to name the violation of marginal selectivity "contextual". However, the amount of violation of marginal selectivity may or may not account for all the possible context effects. In the *contextuality-by-default* theory (Dzhafarov & Kujala, 2014a, 2014b, 2014c; Dzhafarov et al., 2016; Dzhafarov et al., 2015; Kujala & Dzhafarov, 2015, 2016; Kujala et al., 2015), when marginal selectivity is breached, if the covariance between A and B can be entirely attributed to α, β, and a hidden variable, the system is not contextual. Otherwise it is contextual. Note that when marginal selectivity is present in a system, the framework of contextuality-by-default reduces to the framework of selective influences.

In this dissertation, I will introduce the theories of selective influences, mental architectures, and contextuality-by-default. I will focus on my contribution to the three topics: (1) the theoretical development of mental architectures including simple serial-parallel mental architectures of size 2 and size n and two and more processes in arbitrary serial-parallel mental architectures; (2) the empirical application of the three frameworks: Psychophysical experiments were conducted to understand if selective influences or noncontextuality exists in human behavior. I also used the psychophysical experiments to investigate which mental architecure was used when the unfixed geometric stimuli were moved or modified to match the target ones.

SELECTIVE INFLUENCES AND APPLICATIONS

A Brief Theoretical Review of Selective Influences

The mathematical framework of selective influences was primarily developed by Dzhafarov and Kujala (Dzhafarov, 2003; Dzhafarov & Kujala, 2010, 2012a, 2012b; Kujala & Dzhafarov, 2008). In this section, I will briefly review the definitions, theorems, and tests of selective influences.

Given a system of size n, let us denote the external factors $(\lambda^1, \ldots, \lambda^n)$. Their values belong to nonempty sets $(\Lambda^1, \ldots, \Lambda^n)$, respectively, where $\Lambda^k = \{\lambda^k_1, \ldots, \lambda^k_{m_k}\}$, $k \in \{1, \ldots, n\}$. A nonempty vector $\phi = \left(\lambda^1_{i_1}, \ldots, \lambda^n_{i_n}\right)$ is called a treatment when $\lambda^1_{i_1} \in \Lambda^1, \ldots, \lambda^n_{i_n} \in \Lambda^n$. $\left(X^1_\phi, \ldots, X^n_\phi\right)$ denotes the random variables jointly distributed for a given ϕ.

Definitions

There are three equivalent definitions of selective influences.

Definition 1. A vector of random variables $(X^1, \ldots, X^n)$ is selectively influenced by $(\lambda^1, \ldots, \lambda^n)$:

$$(X^1, \ldots, X^n) \looparrowleft (\lambda^1, \ldots, \lambda^n), \tag{1}$$

if for any treatment ϕ,

$$(X^1_\phi, \ldots, X^n_\phi) \sim (f_1(\lambda^1_{i_1}, S^1, \Theta), \ldots, f_n(\lambda^n_{i_n}, S^n, \Theta)), \tag{2}$$

where Θ is a common source of randomness for $(X^1, \ldots, X^n)$, $(S^1, \ldots, S^n)$ are specific sources of randomness for $(X^1, \ldots, X^n)$, respectively, and $(f_1, \ldots, f_n)$ are some measurable functions. $(\Theta, S^1, \ldots, S^n)$ have a joint distribution that does not depend on $(\lambda^1, \ldots, \lambda^n)$.

Definition 2. A vector of random variables $(X^1, \ldots, X^n)$ is selectively influenced by $(\lambda^1, \ldots, \lambda^n)$:

$$(X^1, \ldots, X^n) \looparrowleft (\lambda^1, \ldots, \lambda^n),$$

if for any treatment ϕ,

$$(X^1_\phi, \ldots, X^n_\phi) \sim (g_1(\lambda^1_{i_1}, R), \ldots, g_n(\lambda^n_{i_n}, R)), \tag{3}$$

where R is some random variable that is independent of $(\lambda^1, \ldots, \lambda^n)$, and $(g_1, \ldots, g_n)$ are some measurable functions.

Definition 3. A vector of random variables $(X^1, \ldots, X^n)$ is selectively influenced by $(\lambda^1, \ldots, \lambda^n)$ if and only if there exists a vector of jointly distributed random variables

$$H = \left(\overbrace{H_{\lambda^1_1}, \ldots, H_{\lambda^1_{m_1}}}^{\text{for } \lambda^1}, \ldots, \overbrace{H_{\lambda^n_1}, \ldots, H_{\lambda^n_{m_n}}}^{\text{for } \lambda^n} \right), \tag{4}$$

one random variable for each factor point, such that for any treatment ϕ,

$$\left(H_{\lambda^1_{i_1}}, \ldots, H_{\lambda^n_{i_n}} \right) \sim (X^1_\phi, \ldots, X^n_\phi). \tag{5}$$

There are two tests that can establish or falsify selective influences. I will discuss them in detail.

Cosphericity Test

The cosphericity test can be used to establish selective influences in a system containing two factors (λ^1, λ^2). Each factor has two levels: $\lambda^1 \in \{\lambda_1^1, \lambda_2^1\}$, $\lambda^2 \in \{\lambda_1^2, \lambda_2^2\}$. Let random variables $(X^1_{i_1i_2}, X^2_{i_1i_2})$, depending on the allowable treatment $\phi = \left(\lambda^1_{i_1}, \lambda^2_{i_2}\right)$, $i_1, i_2 \in \{1, 2\}$, be bivariate normally distributed:

$$N_2\left(\begin{bmatrix}\mu_{i_1}\\ \mu_{i_2}\end{bmatrix}, \begin{bmatrix}\sigma^2_{i_1} & \sigma_{i_1}\sigma_{i_2}\rho_{i_1i_2}\\ \sigma_{i_1}\sigma_{i_2}\rho_{i_1i_2} & \sigma^2_{i_2}\end{bmatrix}\right), \tag{6}$$

where μ_{i_1} and $\sigma^2_{i_1}$ are the mean and variance of $X^1_{i_1i_2}$, μ_{i_2} and $\sigma^2_{i_2}$ are the mean and variance of $X^2_{i_1i_2}$, and ρ_{ij} is the correlation of $X^1_{i_1i_2}$ and $X^2_{i_1i_2}$. The parameters $\mu_{i_1}, \sigma_{i_1}, \mu_{i_2}, \sigma_{i_2}$, and $\rho_{i_1i_2}$ generally depend on λ^1 and λ^2. If marginal selectivity is present, i.e., μ_{i_1} and σ_{i_1} are independent of $\lambda^2_{i_2}$, and μ_{i_2} and σ_{i_2} are independent of $\lambda^1_{i_1}$, then (X^1, X^2) are selectively influenced by (λ^1, λ^2) on $\{\lambda^1_1, \lambda^1_2\} \times \{\lambda^2_1, \lambda^2_2\}$ if and only if

$$|\rho_{11}\rho_{21} - \rho_{12}\rho_{22}| \leq \sqrt{(1-\rho^2_{11})(1-\rho^2_{21})} + \sqrt{(1-\rho^2_{12})(1-\rho^2_{22})}. \tag{7}$$

If the distributions of $(X^1_{i_1i_2}, X^2_{i_1i_2})$ are not bivariate normal, then the inequality above is a necessary but not sufficient condition for selective influences.

Linear Feasibility Test

The Linear Feasibility test (LFT) is a direct application of Definition 3. It can be used in a system with arbitrarily finite number of input variables and output variables, provided each variable in this system has arbitrarily finite number of values. Let us assume that the random variable $X^k, k \in \{1, \ldots, n\}$, has l_k possible values:

$\{x_1^k, \ldots, x_{l_k}^k\}$. Let $x_{\xi_{ki_k}}^k \in \{x_1^k, \ldots, x_{l_k}^k\}, i_k \in \{1, \ldots, m_k\}$. Let us write the joint probability for the vector H as

$$\Pr\left[\begin{array}{c} H_{\lambda_1^1} = x_{\xi_{11}}^1, \ldots, H_{\lambda_{m_1}^1} = x_{\xi_{1m_1}}^1 \\ , \ldots, \\ H_{\lambda_1^n} = x_{\xi_{n1}}^n, \ldots, H_{\lambda_{m_n}^n} = x_{\xi_{nm_n}}^n \end{array}\right] = Q\left(\overbrace{\xi_{11}, \ldots, \xi_{1m_1}}^{\text{for } X^1}, \ldots, \overbrace{\xi_{n1}, \ldots, \xi_{nm_n}}^{\text{for } X^n}\right).$$

Theorem 4. *Selective influences of $(\lambda^1, \ldots, \lambda^n)$ on $(X^1, \ldots, X^n)$ are established if and only if the $l_1^{m_1} \times, \ldots, \times l_n^{m_n}$ Q-probabilities are nonnegative,*

$$Q\left(\overbrace{\xi_{11}, \ldots, \xi_{1m_1}}^{\text{for } X^1}, \ldots, \overbrace{\xi_{n1}, \ldots, \xi_{nm_n}}^{\text{for } X^n}\right) \geq 0, \tag{8}$$

and these Q-probabilities are restrained by $l_1 \times, \ldots, \times l_n \times m_1 \times, \ldots, \times m_n$ equations:

$$\sum Q\left(\overbrace{\xi_{11}, \ldots, \xi_{1m_1}}^{\text{for } X^1}, \ldots, \overbrace{\xi_{n1}, \ldots, \xi_{nm_n}}^{\text{for } X^n}\right) = \Pr[(X^1 = x_{\xi_{1i_1}}^1, \ldots, X^n = x_{\xi_{ni_n}}^n) \mid \phi = \left(\lambda_{i_1}^1, \ldots, \lambda_{i_n}^n\right)], \tag{9}$$

where $\sum$ sums over all possible values of $x_{\xi_{11}}^1, \ldots, x_{\xi_{1m_1}}^1, \ldots, x_{\xi_{n1}}^n, \ldots, x_{\xi_{nm_n}}^n$ except $x_{\xi_{1i_1}}^1, \ldots, x_{\xi_{ni_n}}^n$, which are fixed.

Note that (9) implies marginal selectivity. If marginal selectivity is violated, nonnegative solutions for (9) do not exist.

Table 1 gives an example of joint probabilities $\Pr[(X^1 = x_{\xi_{1i_1}}^1, \ldots, X^n = x_{\xi_{ni_n}}^n) \mid \phi = \left(\lambda_{i_1}^1, \ldots, \lambda_{i_n}^n\right)]$, where $n = 2$, $i_1, i_2 \in \{1, 2\}$, and $l_1 = l_2 = 2$. The numbers outside the grids are marginal probabilities.

Table 1

An Example of Joint Probabilities of $(X^1_{i_1i_2}, X^2_{i_1i_2})$ *Given Treatments* $\phi = \left(\lambda^1_{i_1}, \lambda^2_{i_2}\right)$, $i_1, i_2 \in \{1, 2\}$

$(\lambda^1_1, \lambda^2_1)$	$X^2_{11} = x^2_1$	$X^2_{11} = x^2_2$	
$X^1_{11} = x^1_1$	.2	.2	.4
$X^1_{11} = x^1_2$	.1	.5	.6
	.3	.7	

$(\lambda^1_1, \lambda^2_2)$	$X^2_{12} = x^2_1$	$X^2_{12} = x^2_2$	
$X^1_{12} = x^1_1$	.3	.1	.4
$X^1_{12} = x^1_2$	.4	.2	.6
	.7	.3	

$(\lambda^1_2, \lambda^2_1)$	$X^2_{21} = x^2_1$	$X^2_{21} = x^2_2$	
$X^1_{21} = x^1_1$	.1	.5	.6
$X^1_{21} = x^1_2$	.2	.2	.4
	.3	.7	

$(\lambda^1_2, \lambda^2_2)$	$X^2_{22} = x^2_1$	$X^2_{22} = x^2_2$	
$X^1_{22} = x^1_1$	.4	.2	.6
$X^1_{22} = x^1_2$	.3	.1	.4
	.7	.3	

This example satisfies marginal selectivity since the distributions of X^1_{11}, X^1_{12}, X^1_{21}, and X^1_{22} meet the conditions below:

$$
\begin{aligned}
&P(X^1_{11}) = P(X^1_{12}), P(X^1_{21}) = P(X^1_{22}), \\
&P(X^2_{11}) = P(X^2_{21}), P(X^2_{12}) = P(X^2_{22}).
\end{aligned}
\tag{10}
$$

Substituting the joint probabilities in Table 1 into (9),

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
Q(1,1,1,1) \\ Q(1,1,1,2) \\ Q(1,1,2,1) \\ Q(1,1,2,2) \\ Q(1,2,1,1) \\ Q(1,2,1,2) \\ Q(1,2,2,1) \\ Q(1,2,2,2) \\ Q(2,1,1,1) \\ Q(2,1,1,2) \\ Q(2,1,2,1) \\ Q(2,1,2,2) \\ Q(2,2,1,1) \\ Q(2,2,1,2) \\ Q(2,2,2,1) \\ Q(2,2,2,2)
\end{pmatrix}
=
\begin{pmatrix}
.2 \\ .2 \\ .1 \\ .5 \\ .3 \\ .1 \\ .4 \\ .2 \\ .1 \\ .5 \\ .2 \\ .2 \\ .4 \\ .2 \\ .2 \\ .2
\end{pmatrix},
$$

the nonnegative solution

$$
\begin{aligned}
&(Q(1,1,1,1),\ Q(1,1,1,2),\ \ldots,\ Q(2,2,2,2))^T \\
&= (0,0,0,0,.1,.1,.2,0,0,.1,.4,.1,0,0,0,0)^T
\end{aligned}
$$

establishes selective influences in this example.

Bell-CHSH-Fine inequalities. The Bell-CHSH-Fine inequalities are a special case of the LFT. They are equivalent when dealing with a 2×2 factorial design and each of the two output variables has two possible distinct values. The inequalities were first proposed to investigate the problem of quantum entanglement (Bell, 1964; Clauser, Horne, Shimony, & Holt, 1969; Fine, 1982a, 1982b). If the inequality test is passed, the entanglement phenomenon can be described by a local hidden variable theory (Bohm & Aharonov, 1957; Einstein, Podolski, & Rosen, 1935). In our terminology, if the test is passed, it indicates the outcome of the measurement of particle 1 (X^1) is selectively influenced by the measurement of particle 1 (λ^1) and the outcome of the measurement settings for particle 2 (X^2) is selectively influenced by the measurement settings for particle 2 (λ^2).

In a typical quantum entanglement experiment, the spins of two entangled particles are measured simultaneously at different physical locations. Particle 1 is measured along one of two possible axes: $\{\lambda_1^1, \lambda_2^1\}$, and simultaneously particle 2 is measured along one of two possible axes: $\{\lambda_1^2, \lambda_2^2\}$. If the particles are spin-1/2 ones (e.g., electrons), the outcome of measurement of each particle has two possible values: {spin up, spin down}. Let us denote the two possible outcomes as $\{x_1^1, x_2^1\}$ for particle 1 and $\{x_1^2, x_2^2\}$ for particle 2. Then this system can be represented by Table 2. The symbols in the grids are the joint probabilities of particular outcomes. For instance, η_{11} is the joint probability of $(X_{11}^1 = x_1^1, X_{11}^2 = x_1^2)$ given the treatment $(\lambda_1^1, \lambda_1^2)$. The numbers outside the grids are the marginal probabilities. Marginal selectivity is usually automatically secured in quantum physics, in our description by a space-like separation between the particles (i.e., by the fact that the two measurements are simultaneous).

The Bell-CHSH-Fine inequalities are

$$
\begin{aligned}
&-1 \leq -\eta_{11} + \eta_{12} + \eta_{21} + \eta_{22} - b - d \leq 0, \\
&-1 \leq \eta_{11} - \eta_{12} + \eta_{21} + \eta_{22} - b - c \leq 0, \\
&-1 \leq \eta_{11} + \eta_{12} - \eta_{21} + \eta_{22} - a - d \leq 0, \\
&-1 \leq \eta_{11} + \eta_{12} + \eta_{21} - \eta_{22} - a - c \leq 0. \qquad (11)
\end{aligned}
$$

Table 2

A Representation of the Outcomes of Measurements of Two Entangled Particles

$(\lambda_1^1, \lambda_1^2)$	$X_{11}^2 = \uparrow$	$X_{11}^2 = \downarrow$	
$X_{11}^1 = \uparrow$	η_{11}	$a - \eta_{11}$	a
$X_{11}^1 = \downarrow$	$c - \eta_{11}$	$1 - c - a + \eta_{11}$	$1 - a$
	c	$1 - c$	

$(\lambda_1^1, \lambda_2^2)$	$X_{12}^2 = \uparrow$	$X_{12}^2 = \downarrow$	
$X_{12}^1 = \uparrow$	η_{12}	$a - \eta_{12}$	a
$X_{12}^1 = \downarrow$	$d - \eta_{12}$	$1 - a - d + \eta_{12}$	$1 - a$
	d	$1 - d$	

$(\lambda_2^1, \lambda_1^2)$	$X_{21}^2 = \uparrow$	$X_{21}^2 = \downarrow$	
$X_{21}^1 = \uparrow$	η_{21}	$b - \eta_{21}$	b
$X_{21}^1 = \downarrow$	$c - \eta_{21}$	$1 - b - c + \eta_{21}$	$1 - b$
	c	$1 - c$	

$(\lambda_2^1, \lambda_2^2)$	$X_{22}^2 = \uparrow$	$X_{22}^2 = \downarrow$	
$X_{22}^1 = \uparrow$	η_{22}	$b - \eta_{22}$	b
$X_{22}^1 = \downarrow$	$d - \eta_{22}$	$1 - b - d + \eta_{22}$	$1 - b$
	d	$1 - d$	

For our purpose, the Bell-CHSH-Fine inequalities are used to test selective influences. By substituting the values in Table 1 into (11), one has

$$-1 \leq -.2 + .3 + .1 + .4 - .6 - .7 \leq 0,$$

$$-1 \leq .2 - .3 + .1 + .4 - .6 - .3 \leq 0,$$

$$-1 \leq .2 + .3 - .1 + .4 - .4 - .7 \leq 0,$$

$$-1 \leq .2 + .3 + .1 - .4 - .4 - .3 \leq 0.$$

Therefore selective influences are confirmed in this example.

Applying Selective Influences to Perceptual Separability

Stimuli usually contain multiple attributes. These attributes may be perceived, evaluated, or responded to in a separable fashion. Let us consider again an object that varies in luminance α and size β. Intuitively, α and β are considered *perceptually separable* if the attribute α is perceived without regard to the attribute β and the attribute β is perceived without regard to the attribute α. On the other hand, it may be that different attributes are "*integral.*"

Garner (1974) pointed out that if the stimulus attributes are integral, they are not perceived as attributes at all. Attributes exist for the researcher but the immediate perceptual experience of the participant is a seamless gestalt. This implies that if (α, β) are perceived integrally, α "looks differently" for β_1 than for β_2, or/and β "looks differently" for α_1 than for α_2.

Understanding how different stimulus attributes are perceived is of fundamental importance in the study of perception and cognition. Several approaches were utilized to make the distinction between separable and integral stimuli operationally and mathematically rigorous, e.g., the *General Recognition Theory* (Ashby & Townsend, 1986), the framework of *multidimensional scaling* (Shepard, 1987), and the *generalized Fechnerian Scaling* (Dzhafarov, 2002, 2004).

The General Recognition Theory (GRT) is applied to the so-called feature-complete factorial design, in which stimuli consist of the factorial combination of each level on each stimulus attribute of interest. To give an example, let us consider a stimulus $(\alpha_{i_1}, \beta_{i_2})$ constructed from a physical attribute α at level i_1 and another attribute β at level i_2, where $i_1 \in \{1, 2, \ldots, I_1\}$ and $i_2 \in \{1, 2, \ldots, I_2\}$. Let us denote by A the perceptual dimension associated with α and by B the perceptual dimension associated with β. Let $P(A_{i_1 i_2})$ and $P(B_{i_1 i_2})$ be the marginal distributions of perceptual effects on the perceptual dimensions A and B, respectively, given the stimulus $(\alpha_{i_1}, \beta_{i_2})$. In the theory of GRT, perceptual separability occurs if and only if the marginal perceptual effect of one attribute is the same across all levels of the other attribute:

$$P(A_{i_1 1}) = P(A_{i_1 2}) = \ldots = P(A_{i_1 I_2})$$

and

$$P(B_{1i_2}) = P(B_{2i_2}) = \ldots = P(B_{I_1 i_2}).$$

Note that it is logically possible for perceptual separability in the theory of GRT to hold for one level of one attribute while failing for another level of the same attribute. So far numerous publications have successfully applied GRT to understand various types of perceptual phenomena, including visual perception (Blaha, Silbert, & Townsend, 2011; Thomas, 2001; Wenger & Ingvalson, 2002), auditory perception (Silbert, 2012; Silbert, Townsend, & Lentz, 2009), and haptic perception (Giordano et al., 2012; Louw, Kappers, & Koenderink, 2002; Oberle & Amazeen, 2003).

Shepard (1987) made an attempt to study perceptual separability within the framework of multidimensional scaling (MDS). According to his theory, stimuli p and q can be represented as points $(p_1, p_2, \ldots, p_N)$ and $(q_1, q_2, \ldots, q_N)$ in the N-dimensional perceptual space coordinates, respectively. The points in this perceptual space are separated by the distance d_{pq}, negative-exponentially related to some measure of the perceived similarity between the stimuli. This inter-stimulus distance forms a Minkowskian metric:

$$d_{pq} = \left(|p_1 - q_1|^w + |p_2 - q_2|^w + \ldots + |p_N - q_N|^w\right)^{1/w}, w \geq 1.$$

Shepard suggested that the exponent w equals 1 (city-block metric) if the N dimensions of the perceptual space are separable, and it equals 2 (Euclidean metric) if they are integral. Despite being widely used, MDS has been criticized for several issues that may produce misleading conclusions from the data. First, whether $w = 1$ or not can depend on the choice of dimensions of the perceptual space, rather than the inherent property of the stimulus. Second, city-block metric may be misidentified as

Euclidean metric because of various reasons such as the presence of noise (Shepard, 1986), low discriminability (Nosofsky, 1985; Tversky & Gati, 1982), and instability (Eisler & Knöppel, 1970).

Dzhafarov's (2002, 2004) approach to perceptual separability of stimulus attributes is based on the theory of Multidimensional Fechnerian Scaling (MDFS). He proposed that the attributes α and β are perceptually separable if the two conditions are satisfied: (a) the probability for a stimulus $(\alpha_{i_1}, \beta_{i_1})$ to be discriminated from a nearby stimulus $(\alpha_{i_2}, \beta_{i_2})$ can be computed from the probabilities that $(\alpha_{i_1}, \beta_{i_1})$ is discriminated from $(\alpha_{i_2}, \beta_{i_1})$ and from $(\alpha_{i_1}, \beta_{i_2})$; (b) the difference between the probability for the stimulus $(\alpha_{i_1}, \beta_{i_1})$ to be discriminated from a nearby stimulus $(\alpha_{i_2}, \beta_{i_1})$ and the probability for the stimulus $(\alpha_{i_1}, \beta_{i_1})$ to be discriminated from itself does not depend on the value of β_{i_1}; and analogously for the stimulus $(\alpha_{i_1}, \beta_{i_1})$ and the nearby stimulus $(\alpha_{i_1}, \beta_{i_2})$.

The framework of selective influences is a new approach to define perceptual separability. In this approach, perceptual separability is defined as a term relating certain responses A and B to certain stimulus attributes α and β: We have α and β perceptually separable with respect to responses A and B if $(A, B) \looparrowleft (\alpha, \beta)$. To realize the approach of selective influences, we have developed an experimental procedure in which the responses (A, B) are extracted from an observer's choice of a stimulus that is adjusted to match a fixed stimulus with attributes (α, β). The choices of (α, β) and (A, B) are flexible and unknown to the participants. Below we report the results of the experiments using α and β (also A and B) as simple geometric properties of dots or lines.

Experiments

Participants. All the participants were students at Purdue University. Three unpaid volunteers (P1, P2, & P3) attended Experiments 1(a) and 2(a). Two paid participants (P4 & P5) and one unpaid participant (P3) attended Experiments 1(b), 2(b), 2(c), 3(a), and 3(b). The author of this proposal, labeled as P3, participated in all the experiments. All participants were aged around 25 and had normal or corrected to normal vision.

Stimuli and procedure. In each experiment, dots and closed curves were presented on a flat-panel monitor. These geometric stimuli were grayish-white on a black background, of a comfortably low fixed luminance. The diameter of the dots and the width of the curves was 5 pixels (px). The participants viewed the stimuli in darkness using a chin rest with a forehead support from the distance of 90 cm, making 1 screen pixel approximately 62 sec arc. In each trial the participants were asked to match a given stimulus by adjusting a variable stimulus as accurately as possible by rotating a trackball using their dominant hand. The program allowed the participants to view the instantaneous movement of the dots and the change of the curves on the screen. Once a response was made to the participants' satisfaction, they clicked a button on the trackball device to terminate this trial, and a new stimulus appeared .5 second later. Each experiment included several sessions. We ran one session per day. Each session consisted of about 200 trials with a 10-min break in the middle; each session was preceded by a practice series of 10 trials (which were not recorded).

Experiment 1(a). Each trial began with presenting two circles with a dot in the first quadrant of each circle (exemplified in Figure 1(a)). The radius of each circle was 160 px. The circles' centers were located respectively at (-125 px, 200 px) and (125 px, -150 px), relative to the center of the screen. The dot in the bottom right circle was movable. It appeared randomly in the first quadrant. The dot in the left upper circle was fixed. Its location was randomly chosen from six possibilities. The six possibilities, if using the center of its circle as the origin, can be represented equivalently using the rectangular coordinates: {(24 px, 48 px), (32 px, 32 px), (32 px, 64 px), (48 px, 24 px), (64 px, 32 px), (64 px, 64 px)} or the polar coordinates: {(53.67 px, 63.43 deg), (45.25 px, 45 deg), (71.55 px, 63.43 deg), (53.67 px, 26.57 deg), (71.55 px, 26.56 deg), (90.51 px, 45 deg)}. Hence the experiment contained a rectangular subdesign {32 px, 64 px}×{32 px, 64 px} and a polar subdesign {53.67 px, 71.55 px}×{63.43 deg, 26.57 deg}.

The participants were asked to move the movable dot until its location matched that of the fixed one. Once a response was made, the program recorded the locations of the given dot and the reproduced dot in both rectangular coordinates and polar coordinates using the center of each circle as the origin. There were 1200 trials overall with approximately 200 trials per treatment.

Experiment 1(b). Experiment 1(b) was identical to Experiment 1(a) except that in Experiment 1(b) the horizontal coordinate and vertical coordinate of each immovable dot were random integers drawn from the rectangular [20 px, 80 px) × [20 px, 80 px). This experiment included 1800 trials overall. If the dots were represented in the polar coordinates, a subdesign in which the dots' radial coordinates varied within the interval [40 px, 90 px) and angular coordinates varied within the interval [30 deg, 60 deg), was included in this experiment. The polar subdesign contained about 900 trials.

Experiment 2(a). The stimuli presented in each trial are exemplified in Figure 1(b). In each trial, concentric circles together with their center appeared on the left part of the screen. The radii of circle 1 and circle 2 were randomly chosen from the sets {16 px, 56 px, 64 px} and {48 px, 72 px, 80 px}, respectively. Therefore a 3×3 factorial design was formed. On the right part of the screen there was an immovable dot, located at (250 px, 0 px) relative to the center of the concentric circles.

The participants aimed to reproduce the concentric circles. The program automatically made the right dot as the center of the reproduced circles. The two circles

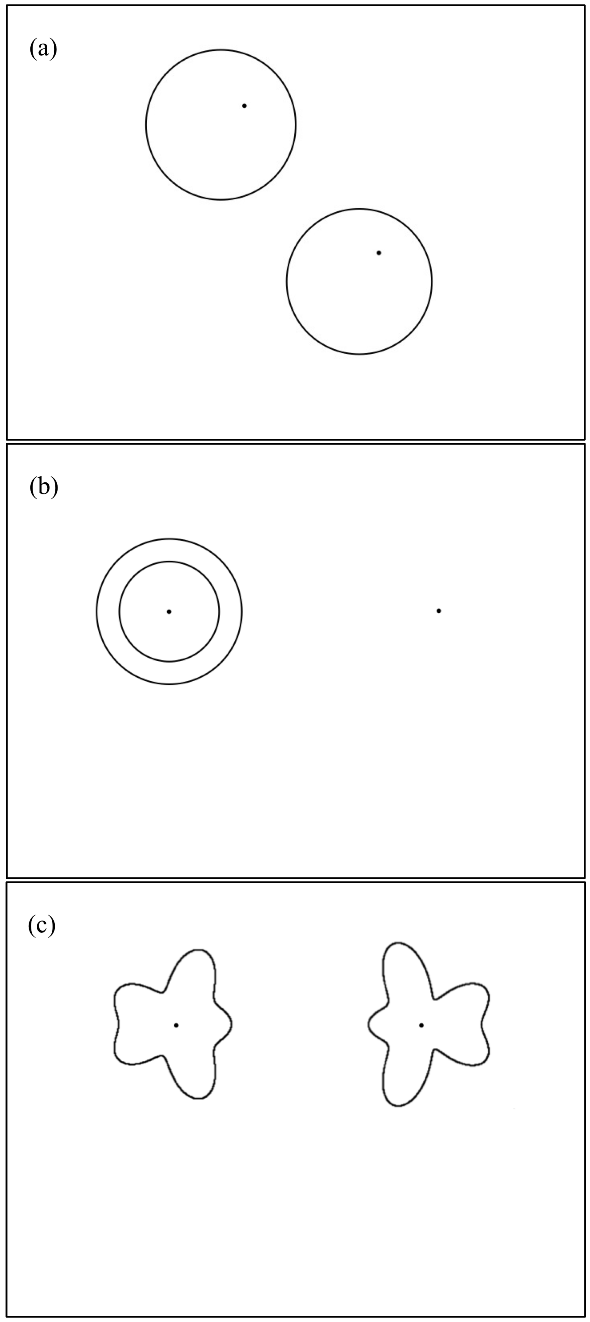

Figure 1. (a) Examples of stimuli used in Experiment 1. (b) Examples of stimuli used in Experiment 2. (c) Examples of stimuli used in Experiment 3.

were drawn successively and their sizes were controlled by rotating the trackball. Once the first matching circle was produced, the participants clicked a button on the trackball to stabilize this circle and then the program automatically enabled the user to draw the other. After the second response was made, the trial was terminated by clicking the same button on the trackball. The participants had the freedom to draw the inner circle first then the outer circle or vice versa. The program recorded the radii of the given and reproduced concentric circles in each trial. There were 1800 trials overall, approximately 200 trials per treatment.

Experiment 2(b). Experiment 2(b) was identical to Experiment 2(a) except that in each trial the radii of the given circle 1 and circle 2 were randomly chosen from four possibilities: $\{12\text{ px}, 24\text{ px}\} \times \{18\text{ px}, 30\text{ px}\}$. Besides, there were 1600 trials overall, about 400 trials per treatment.

Experiment 2(c). Experiment 2(c) was identical to Experiment 2(a) except that in each trial the radius of the given circle 1 was an integer randomly chosen from the interval [18 px, 48 px) and the radius of the given circle 2 was an integer randomly chosen from the interval [56 px, 86 px). In addition, the duration from the appearance of the stimuli to the terminating trial button click in each trial was also recorded. There were 1800 trials overall.

Experiment 3(a). Two floral shapes together with their centers are exemplified in Figure 1(c). Two such configurations were present simultaneously in each trial. One was on the left part of the screen and the other was on the right. The floral shape was generated using this function:

$$\begin{aligned} x &= cos(.02\pi\Delta)[70 + \alpha cos(.06\pi\Delta) + \beta cos(.1\pi\Delta)], \\ y &= sin(.02\pi\Delta)[70 + \alpha cos(.06\pi\Delta) + \beta cos(.1\pi\Delta)]. \end{aligned} \tag{12}$$

In each trial, amplitude 1 (α) and amplitude 2 (β) of the left floral shape were randomly chosen from the sets {-18 px, 10 px, 14 px} and {-16 px, -12 px, 20 px}, respectively. Δ were integers from 0 to 99. For each value of Δ, a point represented by the rectangular coordinates (x, y) was drawn to the screen and the floral shape was composed of 100 such points. The left shape was fixed. The shape on the right was modifiable by rotating the trackball, whose amplitudes were initialized by randomly selecting two numbers from the interval [-35 px, 35 px).

The participants were asked to reproduce the left shape by modifying the right shape. After each trial the program recorded the amplitudes of the left shape and the amplitudes of the reproduced shape. Since the computer can only record the horizontal move and the vertical move of the trackball, a transformation function that converts the trackball move to the amplitude move was imposed:

$$\begin{aligned} A_{new} &= A_{current} + \frac{\text{sign}(\triangle x)}{100}(70 - A_{current} - |B_{current}|), \\ B_{new} &= B_{current} + \frac{\text{sign}(\triangle y)}{100}(70 - |A_{current}| - B_{current}). \end{aligned} \tag{13}$$

Here $A_{current}$ and $B_{current}$ are amplitude 1 and amplitude 2 of the being reproduced shape. $\triangle x$ is the horizontal move of the trackball and $\triangle y$ is the vertical move of the trackball. $\triangle x$ and $\triangle y$ can be 1 px, 0 px, or -1 px. The sign function returns the sign of $\triangle x$ or $\triangle y$. Once $\triangle x$ (or $\triangle y$) $=\pm 1$, the amplitude, labeled as A_{new} (or B_{new}), is updated accordingly. Once the participant was satisfied with the shape that he/she produced, he/she terminated the trial and the program recorded the instant A_{new} and B_{new} as the final values of amplitudes of the reproduced shape.

The program recorded the amplitudes of the given and reproduced shapes in each trial. There were 1800 trials overall, about 200 trials per treatment.

Experiment 3(b). Experiment 3(b) was identical to Experiment 3(a) except that the two amplitudes of the left shape were randomly chosen from the interval [-30 px, 30 px). In addition, the duration from the appearance of the stimuli to the terminating trial button click in each trial was also recorded.

Speculations

There were three different types of tasks: dot position reproduction (Experiments 1(a) & 1(b)), concentric circle reproduction (Experiments 2(a), 2(b), & 2(c)), and floral shape reproduction (Experiments 3(a) & 3(b)). Table 3 presents what the external factors (α, β) and the random outputs (A, B) stand for in the three types of tasks.

In the concentric circle reproduction task, the two circles in each trial were reproduced successively. The first reproduced circle reflected the perception of corresponding given circle in the context of the given concentric circles. It is well documented as Delboeuf illusion that the size of a circle looks different when presented alone and when it is surrounded by another circle or a smaller circle is drawn inside it. In addition, the apparent size of that circle varies when the distance between the concentric circles varies (Pressey, 1977). Therefore the size of the first reproduced circle in each trial, say the inner circle, would be influenced by the size of the given outer circle. Due to the Delboeuf illusion, marginal selectivity, consequently selective influences, was expected to be violated in Experiments 2(a), 2(b), and 2(c).

The following speculation can be offered for the dot position reproduction task and the floral shape reproduction task. In Experiments 1(a) and 3(a), the participants may gradually realize there were only several distinct stimuli presented repeatedly. With just a few stimuli presented in more than 1,000 trials in each experiment could class their percepts into several categories. It can be expected that they gradually formed an automatic manner to respond to each of these categories. Therefore each

Table 3

The External Factors (α, β) and the Random Outputs (A, B) for the Three Types of Tasks

Task	α	β	A	B
Dot position reproduction (rectangular coordinates)	Horizontal coordinate of the given dot	Vertical coordinate of the given dot	Horizontal coordinate of the reproduced dot	Vertical coordinate of the reproduced dot
Dot position reproduction (polar coordinates)	Radial coordinate of the given dot	Angular coordinate of the given dot	Radial coordinate of the reproduced dot	Angular coordinate of the reproduced dot
Concentric circle reproduction	Radius of the given circle 1	Radius of the given circle 2	Radius of the reproduced circle 1	Radius of the reproduced circle 2
Floral shape reproduction	Amplitude 1 of the given shape	Amplitude 2 of the given shape	Amplitude 1 of the reproduced shape	Amplitude 2 of the reproduced shape

stimulus was perceived as a whole rather than by its attributes, resulting in violations of selective influences. By contrast, the participants were presented with more than one thousand distinct stimuli in Experiments 1(b) and 3(b). They had to deliberately observe the details of each stimulus before making response. Therefore, in

Experiments 1(b) and 3(b), the stimulus attributes were more likely to be perceived separably.

In addition, we were also interested in violation of selective influences when the presence of marginal selectivity is artificially imposed. By appropriately chosen transformations of the data sets for (A, B), marginal selectivity can be secured in the three tasks (I will discuss the transformations in detail in the next section.). By inspecting (7), if three of the four correlations have the same sign with their absolute values close to one and the other correlation value has the absolute value close to zero, the cosphericity test, as well as LFT (therefore Bell-CHSH-Fine inequalities), are all likely to be violated. Experiment 2(b) had four treatments: (16 px, 24 px), (16 px, 40 px), (32 px, 24 px), and (32 px, 40 px). Among the four concentric circles, one had two distant circles and the other three were composed of two extremely close circles. It was expected that the correlations of (A, B) corresponding to the three close concentric circles would be close to one and the other correlation value would be close to zero, resulting in failure of the tests of selective influences.

Results

Experiments with discrete factor points. Table 4 lists all possible treatments for the experimental designs with discrete factor points. The data are presented in Figure 2 (the rectangular subdesign of Experiment 1(a)), Figure 3 (the polar subdesign of Experiment 1(a)), Figure 4 (Experiment 2(a)), Figure 5 (Experiment 2(b)), and Figure 6 (Experiment 3(a)). The data points obviously falling far outside the cluster of the other data points were considered outliers and several outliers were removed in each experiment. Each panel contains approximately 200 data points. The mean and standard deviation of $(A_{i_1 i_2}, B_{i_1 i_2})$ for each treatment $(\alpha_{i_1}, \beta_{i_2})$, where $1 \leq i_1 \leq$ the number of levels of α, and $1 \leq i_2 \leq$ the number of levels of β, are also included in Figures 2-6. We then used these data to test selective influences. If the tests are passed, then α and β are considered perceptually separable.

Table 4

Possible Treatments in Experiments 1(a), 2(a), 2(b), and 3(a)

Experiment	Possible treatments
Rectangular subdesign of Experiment 1(a)	{32 px, 64 px}×{32 px, 64 px}
Polar subdesign of Experiment 1(a)	{53.67 px, 71.55 px}×{63.43 deg, 26.57 deg}
Experiment 2(a)	{16 px, 56 px, 64 px}×{48 px, 72 px, 80 px}
Experiment 2(b)	{12 px, 24 px}×{18 px, 30 px}
Experiment 3(a)	{-18 px, 10 px, 14 px}×{ -16 px, -12 px, 20 px}

Testing the original data. Two external factors (α, β) and two random outputs (A, B) were involved in the experiments. Marginal selectivity is satisfied if marginal distribution of A is independent of β and the marginal distribution of B is independent of α. We compared the distributions of $A_{i_1 i_2}$ across all levels of i_2 and compared the distributions of $B_{i_1 i_2}$ across all levels of i_1. If all the comparisons demonstrate nonsignificant differences ($p \geq .05$), then it is considered that marginal selectivity is obtained. The K-S test for 2-independent samples was used for paired comparisons. ANOVA was applied for multiple comparisons. The number in each cell represents the p value obtained from the K-S test (Table 5 & Table 7) or ANOVA (Table 6 & Table 8) for each comparison. The statistical results rejected marginal selectivity for each participant in the two 2×2 subdesigns of Experiment 1(a), the

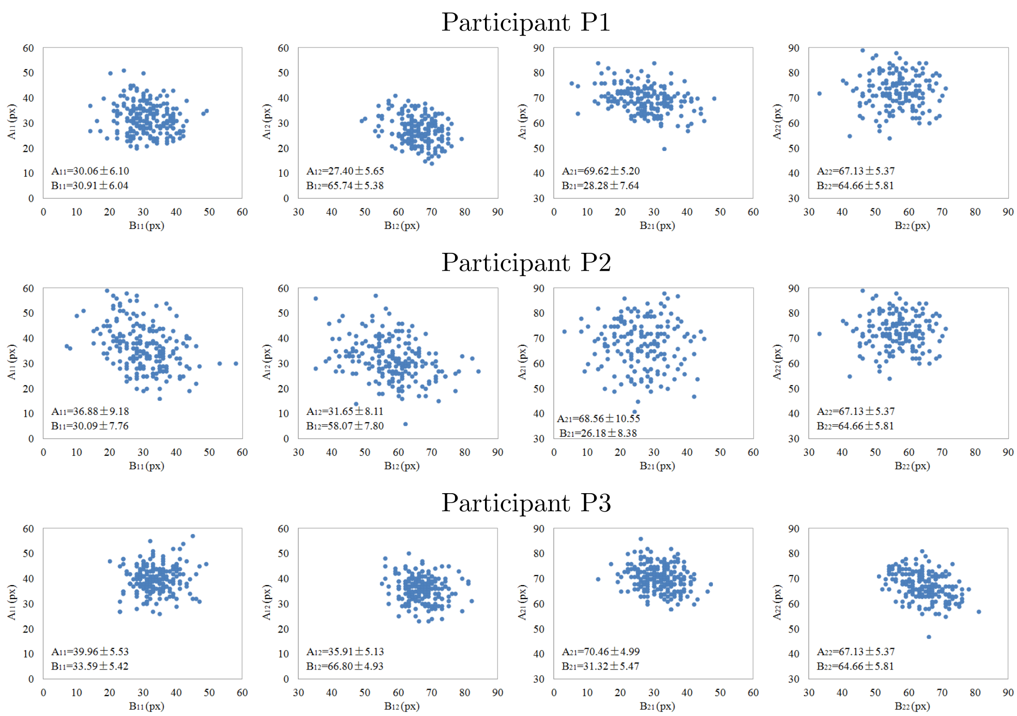

Figure 2. Rectangular coordinate responses to the rectangular subdesign in Experiment 1(a).

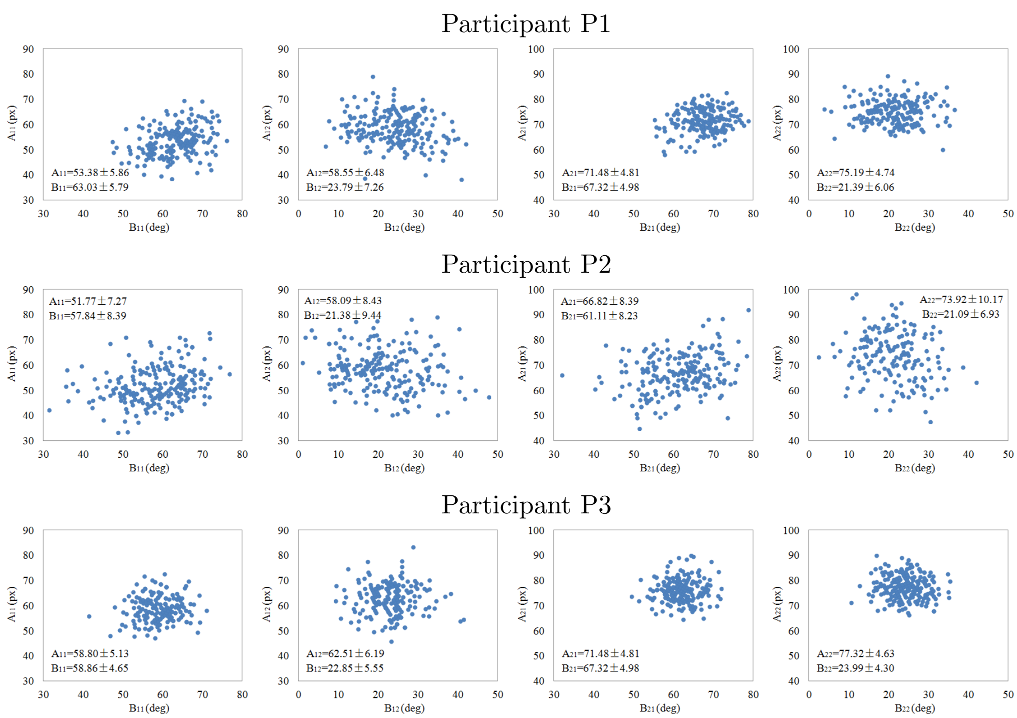

Figure 3. Polar coordinate responses to the polar subdesign in Experiment 1(a).

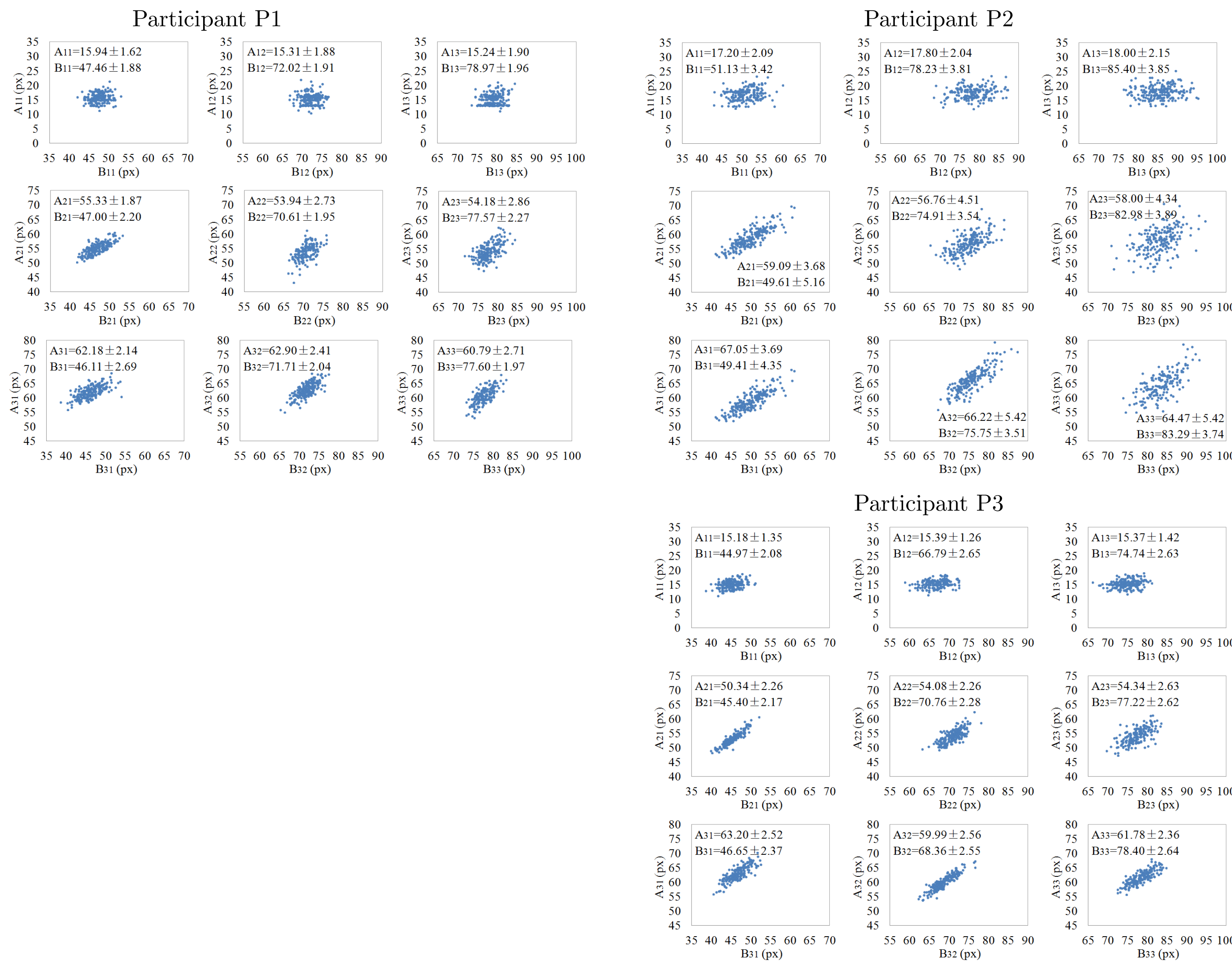

Figure 4. Radii of the reproduced circles in Experiment 2(a).

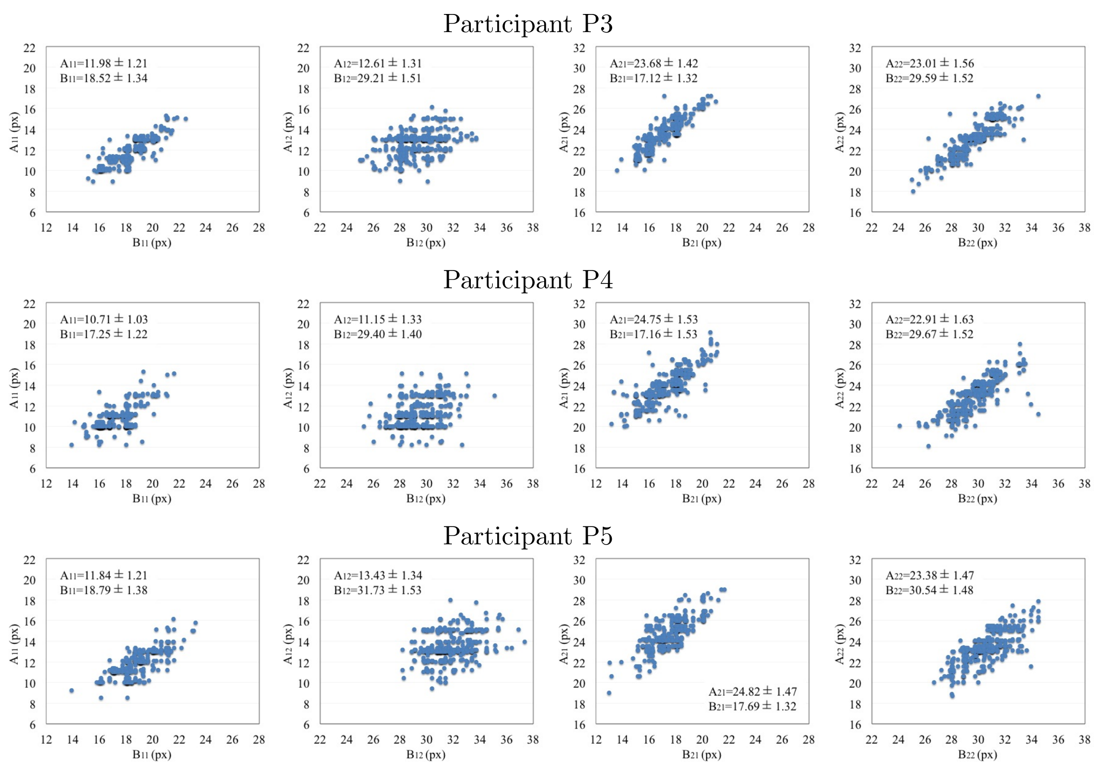

Figure 5. Radii of the reproduced circles in Experiment 2(b).

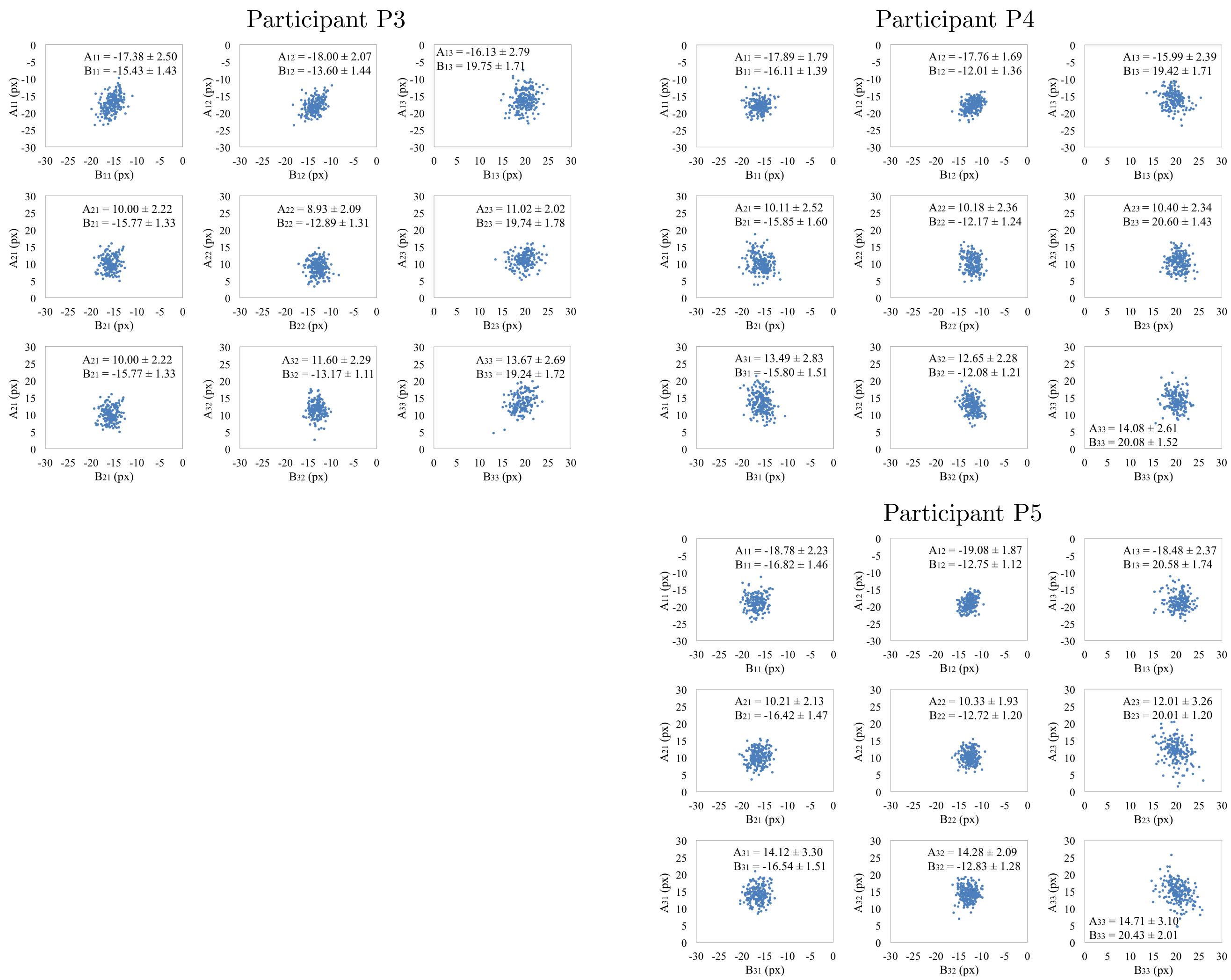

Figure 6. Amplitudes of the reproduced floral shapes in Experiment 3(a).

3×3 factorial design of Experiment 2(a), Experiment 2(b), and the 3×3 factorial design of Experiment 3(a).

Table 5

K-S Tests of Marginal Selectivity for Experiment 1(a)

		Participant		
		P1	P2	P3
Rect-angular subdesign	A_{11}, A_{12}	.000	.000	.000
	A_{21}, A_{22}	.000	.000	.000
	B_{11}, B_{21}	.000	.000	.008
	B_{12}, B_{22}	.000	.141	.001
Polar subdesign	A_{11}, A_{12}	.000	.000	.000
	A_{21}, A_{22}	.000	.000	.006
	B_{11}, B_{21}	.000	.003	.000
	B_{12}, B_{22}	.001	.123	.018

In addition, marginal selectivity was not present in all the 3×2, 2×3, and 2×2 subdesigns of Experiment 2(a) and Experiment 3(a) except three 2×2 subdesigns: {-18 px, 10 px}×{-16 px, -12 px}, {-18 px, 14 px}×{-16 px, -12 px}, and {10 px, 14 px}×{-16 px, -12 px} for participant P5 in Experiment 3(a).

Testing the transformed data. It was pointless to perform the cosphericity test, LFT, and Bell-CHSH-Fine inequalities on the original data in the two 2×2 subdesigns of Experiment 1(a), the 3×3 factorial designs of Experiment 2(a), Experiment 2(b), and the 3×3 factorial design of Experiment 3(a) due to the absence of marginal selectivity. However, for mathematical and practical interests, one can transform the original data to their percentile ranks, followed by an inverse Z-transformation as

Table 6

ANOVA Tests of Marginal Selectivity for Experiment 2(a)

	Participant P1	Participant P2	Participant P3
A_{11}, A_{12}, A_{13}	.000	.000	.218
A_{21}, A_{22}, A_{23}	.000	.000	.000
A_{31}, A_{32}, A_{33}	.000	.000	.000
B_{11}, B_{21}, B_{31}	.000	.000	.000
B_{12}, B_{22}, B_{32}	.000	.000	.000
B_{13}, B_{23}, B_{33}	.000	.000	.000

Table 7

K-S Tests of Marginal Selectivity for Experiment 2(b)

	Participant P3	Participant P4	Participant P5
A_{11}, A_{12}	.000	.000	.000
A_{21}, A_{22}	.000	.000	.000
B_{11}, B_{21}	.000	.000	.000
B_{12}, B_{22}	.034	.000	.000

illustrated in Figure 7. Let us name this two-step transformation Type N transformation. After the Type N transformation the data are bivariate normally distributed and marginal selectivity is automatically secured (standard normal distributions for all marginals).

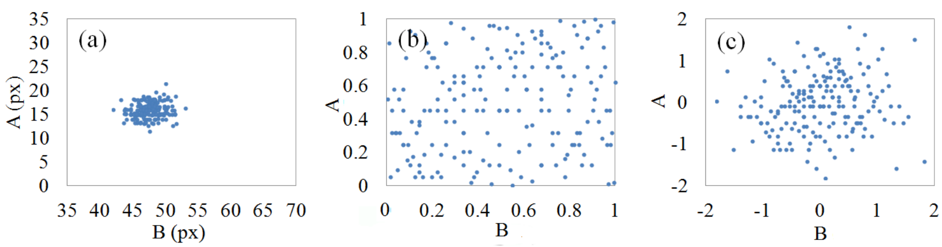

Figure 7. An illustration of transforming from (a) the original data to (b) the percentile ranks to (c) the inverse Z-scores.

Table 8

ANOVA Tests of Marginal Selectivity for Experiment 3(a)

	Participant P3	Participant P4	Participant P5
A_{11}, A_{12}, A_{13}	.000	.000	.029
A_{21}, A_{22}, A_{23}	.000	.450	.000
A_{31}, A_{32}, A_{33}	.000	.000	.053
B_{11}, B_{21}, B_{31}	.000	.076	.022
B_{12}, B_{22}, B_{32}	.000	.433	.581
B_{13}, B_{23}, B_{33}	.005	.000	.012

Now the cosphericity test (7) becomes a sufficient and necessary condition for selective influences in each 2×2 design. The correlations between the transformed responses for Experiments 1(a), 2(a), 2(b), and 3(a) are given in Tables 9-12. As mentioned earlier, the correlation values in Table 11 (Experiment 2(b)) were expected more likely to fail the test than the others. However, it was found the test was passed in the two 2×2 factorial subdesigns of Experiment 1, all the 2×2 subsets of Experiment 2(a), all the 2×2 subsets of Experiment 3(a), and also Experiment 2(b). Therefore selective influences were established for the Type N transformed data in all the 2×2 sets in the experiments with discrete factor points. Here I show how the correlations in the second row of Table 9 passed the cosphericity test as an example:

$$\begin{aligned}&|(-.105)(-.318) - (-.264)(-.466)|\\ &\qquad\le \sqrt{[1-(-.105)^2][1-(-.318)^2]} + \sqrt{[1-(-.264)^2][1-(-.466)^2]}.\end{aligned}$$

Table 9

Correlations of Transformed Responses in Experiment 1(a)

Rectangular subdesign				
Participant P1	$\rho_{11} = -.105$	$\rho_{12} = -.264$	$\rho_{21} = -.318$	$\rho_{22} = -.466$
Participant P2	$\rho_{11} = -.355$	$\rho_{12} = -.320$	$\rho_{21} = -.046$	$\rho_{22} = .018$
Participant P3	$\rho_{11} = .107$	$\rho_{12} = -.165$	$\rho_{21} = -.206$	$\rho_{22} = -.390$
Polar subdesign				
Participant P1	$\rho_{11} = .297$	$\rho_{12} = -.236$	$\rho_{21} = .150$	$\rho_{22} = -.020$
Participant P2	$\rho_{11} = .263$	$\rho_{12} = -.213$	$\rho_{21} = .280$	$\rho_{22} = -.190$
Participant P3	$\rho_{11} = .100$	$\rho_{12} = .013$	$\rho_{21} = .041$	$\rho_{22} = -.079$

The Linear Feasibility Test is used in a system that contains arbitrary finite number of inputs and outputs, each variable having arbitrary finite number of values. Therefore the output variables collected from Experiments 1(a), 2(a), 2(b), and 3(a) have to be discretized in order to apply this test. There are infinitely many ways to perform the discretization. However, if marginal selectivity is violated after just one of the discretizations, the test fails definitely. To avoid it, $A_{i_1 i_2}$ and $B_{i_1 i_2}$ can be discretized by some percentile ranks. For example, we can dichotomize the data set of $A_{i_1 i_2}$ by its median and discretize the data set of $B_{i_1 i_2}$ by the first quartile, the median, and the third quartile. Now there are two possible discrete values for $A_{i_1 i_2}$: {below the median, above the median} and four possible discrete values for $B_{i_1 i_2}$: {below the first quartile, above the first quartile and below the median, above the median and below the third quartile, above the third quartile}. Then the marginal distribution of A is independent of β and the marginal distribution of B is

Table 10

Correlations of Transformed Responses in Experiment 2(a)

Participant P1		
$\rho_{11} = .146$	$\rho_{12} = -.020$	$\rho_{13} = .162$
$\rho_{21} = .717$	$\rho_{22} = .558$	$\rho_{23} = .559$
$\rho_{31} = .655$	$\rho_{32} = .715$	$\rho_{33} = .694$
Participant P2		
$\rho_{11} = .253$	$\rho_{12} = .216$	$\rho_{13} = .091$
$\rho_{21} = .771$	$\rho_{22} = .643$	$\rho_{23} = .479$
$\rho_{31} = .663$	$\rho_{32} = .797$	$\rho_{33} = .653$
Participant P3		
$\rho_{11} = .317$	$\rho_{12} = .233$	$\rho_{13} = .234$
$\rho_{21} = .914$	$\rho_{22} = .716$	$\rho_{23} = .625$
$\rho_{31} = .810$	$\rho_{32} = .878$	$\rho_{33} = .800$

Table 11

Correlations of Transformed Responses in Experiment 2(b)

Participant P3	$\rho_{11} = .807$	$\rho_{12} = .341$	$\rho_{21} = .845$	$\rho_{22} = .852$
Participant P4	$\rho_{11} = .623$	$\rho_{12} = .365$	$\rho_{21} = .789$	$\rho_{22} = .826$
Participant P5	$\rho_{11} = .730$	$\rho_{12} = .331$	$\rho_{21} = .731$	$\rho_{22} = .726$

independent of α. Therefore, with this percentile-rank-discretization, marginal selectivity is guaranteed.

Table 12

Correlations of Transformed Responses in Experiment 3(a)

Participant P3		
$\rho_{11} = .359$	$\rho_{12} = .363$	$\rho_{13} = .120$
$\rho_{21} = .009$	$\rho_{22} = -.015$	$\rho_{23} = .217$
$\rho_{31} = .164$	$\rho_{32} = -.084$	$\rho_{33} = .190$
Participant P4		
$\rho_{11} = .106$	$\rho_{12} = .333$	$\rho_{13} =$ -.147
$\rho_{21} =$ -.086	$\rho_{22} =$ -.105	$\rho_{23} = .009$
$\rho_{31} =$ -.057	$\rho_{32} =$ -.259	$\rho_{33} = .035$
Participant P5		
$\rho_{11} = -.001$	$\rho_{12} = .041$	$\rho_{13} = .019$
$\rho_{21} = .072$	$\rho_{22} = -.063$	$\rho_{23} = -.312$
$\rho_{31} = .070$	$\rho_{32} = -.027$	$\rho_{33} = -.303$

The Bell-CHSH-Fine inequalities require a 2×2 factorial design and each output variable should have two possible distinct values. Marginal selectivity also needs to be satisfied beforehand. One can dichotomize the output variables by percentile ranks, so that the test is applicable.

Let us name discretizing data according to percentile ranks Type D transformation. Tables 13-16 present joint probabilities of $(A_{i_1i_2}, B_{i_1i_2})$ for each participant in Experiments 1(a), 2(a), 2(b), and 3(a) discretized by the medians. Marginal selectivity is automatically secured. Each number in the tables represents the joint probability $\Pr(A_{i_1i_2} \leq M_{A_{i_1i_2}},\ B_{i_1i_2} \leq M_{B_{i_1i_2}})$, where $M_{A_{i_1i_2}}$ denotes the median of $A_{i_1i_2}$ and

$M_{B_{i_1 i_2}}$ denotes the median of $B_{i_1 i_2}$. Knowing the value of $\Pr(A_{i_1 i_2} \leq M_{A_{i_1 i_2}},\ B_{i_1 i_2} \leq M_{B_{i_1 i_2}})$, the other three joint probabilities can be computed easily:

$$\begin{aligned}
\Pr(A_{i_1 i_2} \leq M_{A_{i_1 i_2}},\ B_{i_1 i_2} > M_{B_{i_1 i_2}}) &= .5 - \Pr(A_{i_1 i_2} \leq M_{A_{i_1 i_2}},\ B_{i_1 i_2} \leq M_{B_{i_1 i_2}}), \\
\Pr(A_{i_1 i_2} > M_{A_{i_1 i_2}},\ B_{i_1 i_2} \leq M_{B_{i_1 i_2}}) &= .5 - \Pr(A_{i_1 i_2} \leq M_{A_{i_1 i_2}},\ B_{i_1 i_2} \leq M_{B_{i_1 i_2}}), \\
\Pr(A_{i_1 i_2} > M_{A_{i_1 i_2}},\ B_{i_1 i_2} > M_{B_{i_1 i_2}}) &= \Pr(A_{i_1 i_2} \leq M_{A_{i_1 i_2}},\ B_{i_1 i_2} \leq M_{B_{i_1 i_2}}).
\end{aligned} \tag{14}$$

Table 13

Joint Probabilities of $(A_{i_1 i_2}, B_{i_1 i_2}), i_1, i_2 \in \{1, 2\}$*, Discretized by the Medians, Experiment 1(a)*

	Rectangular subdesign			Polar subdesign		
	P1	P2	P3	P1	P2	P3
$\Pr(A_{11} \leq M_{A_{11}},\ B_{11} \leq M_{B_{11}})$	.269	.209	.292	.208	.218	.240
$\Pr(A_{12} \leq M_{A_{12}},\ B_{12} \leq M_{B_{12}})$	.243	.199	.246	.333	.300	.267
$\Pr(A_{21} \leq M_{A_{21}},\ B_{21} \leq M_{B_{21}})$	.226	.231	.237	.240	.202	.237
$\Pr(A_{22} \leq M_{A_{22}},\ B_{22} \leq M_{B_{22}})$	.177	.291	.191	.259	.296	.256

The Bell-CHSH-Fine inequalities cannot be used in the 3×3 sets of Experiment 2(a) (Table 14) and Experiment 3(a) (Table 16), but they can be used in their nine 2×2 subsets. After substituting the joint probabilities in Table 13 and Table 15, and the nine 2×2 subsets of Table 14 and Table 16 into (11), it was found that the test

Table 14

Joint Probabilities of $(A_{i_1i_2}, B_{i_1i_2}), i_1, i_2 \in \{1,2,3\}$*, Discretized by the Medians, Experiment 2(a)*

	Participant		
	P1	P2	P3
$\Pr(A_{11} \leq M_{A_{11}},\ B_{11} \leq M_{B_{11}})$	.285	.275	.276
$\Pr(A_{12} \leq M_{A_{12}},\ B_{12} \leq M_{B_{12}})$	.273	.283	.290
$\Pr(A_{13} \leq M_{A_{13}},\ B_{13} \leq M_{B_{13}})$	.288	.284	.284
$\Pr(A_{21} \leq M_{A_{21}},\ B_{21} \leq M_{B_{21}})$	.376	.427	.461
$\Pr(A_{22} \leq M_{A_{22}},\ B_{22} \leq M_{B_{22}})$	.372	.387	.375
$\Pr(A_{23} \leq M_{A_{23}},\ B_{23} \leq M_{B_{23}})$	.375	.338	.384
$\Pr(A_{31} \leq M_{A_{31}},\ B_{31} \leq M_{B_{31}})$	.359	.377	.400
$\Pr(A_{32} \leq M_{A_{32}},\ B_{32} \leq M_{B_{32}})$	.374	.417	.439
$\Pr(A_{33} \leq M_{A_{33}},\ B_{33} \leq M_{B_{33}})$	.363	.365	.409

was passed in all the cases. Here I show how the numbers in a 2×2 set in the second column of Table 13 passed the test as an example:

$$-1 \leq -.269 + .243 + .226 + .177 - .5 - .5 \leq 0,$$

$$-1 \leq .269 - .243 + .226 + .177 - .5 - .5 \leq 0,$$

$$-1 \leq .269 + .243 - .226 + .177 - .5 - .5 \leq 0,$$

$$-1 \leq .269 + .243 + .226 - .177 - .5 - .5 \leq 0.$$

We also tested the Bell-CHSH-Fine inequalities using the data dichotomized by other percentile ranks. In each 2×2 set, there are four percentile values that can be

Table 15

Joint Probabilities of $(A_{i_1 i_2}, B_{i_1 i_2}), i_1, i_2 \in \{1, 2\}$*, Discretized by the Medians, Experiment 2(b)*

	Participant		
	P3	P4	P5
$\Pr(A_{11} \leq M_{A_{11}},\ B_{11} \leq M_{B_{11}})$	.429	.378	.453
$\Pr(A_{12} \leq M_{A_{12}},\ B_{12} \leq M_{B_{12}})$	.338	.357	.354
$\Pr(A_{21} \leq M_{A_{21}},\ B_{21} \leq M_{B_{12}})$	.436	.431	.402
$\Pr(A_{22} \leq M_{A_{22}},\ B_{22} \leq M_{B_{22}})$	.408	.427	.389

chosen. They are a, b, c, and d in (11). Each value was varied from the 5th percentile to the 95th percentile with increments of 5 percentile points. Therefore we ran the Bell-CHSH-Fine inequality test 19^4 times for each 2×2 set. It turned out that there was no violation in all the 2×2 sets except participant P3 in Experiment 2(b). 390 violations out of 19^4 trials were detected and the largest excess of boundaries of the inequalities was .056. In order to evaluate whether the violations were true or just statistical fluctuations, the data of this participant were divided into two groups. The first four experimental sections formed group 1 and the second four sections formed group 2. Therefore each of the four treatments in each group contained about 200 data points. Then the Bell-CHSH-Fine inequality test was run 19^4 times in each group. We were interested in whether the two groups shared the same values of (a, b, c, d), where the violations occurred, among the overall 19^4 trials. 46 such quadruples were found and the largest excess of the boundaries was smaller than .05. The violations seemed to occur at random positions for participant P3 and the extent of the violations did

Table 16

Joint Probabilities of $(A_{i_1 i_2}, B_{i_1 i_2}), i_1, i_2 \in \{1, 2, 3\}$*, Discretized by the Medians, Experiment 3(a)*

	Participant		
	P3	P4	P5
$\Pr(A_{11} \leq M_{A_{11}},\ B_{11} \leq M_{B_{11}})$	.333	.252	.258
$\Pr(A_{12} \leq M_{A_{12}},\ B_{12} \leq M_{B_{12}})$	.292	.322	.301
$\Pr(A_{13} \leq M_{A_{13}},\ B_{13} \leq M_{B_{13}})$	.283	.224	.271
$\Pr(A_{21} \leq M_{A_{21}},\ B_{21} \leq M_{B_{21}})$	.245	.196	.256
$\Pr(A_{22} \leq M_{A_{22}},\ B_{22} \leq M_{B_{22}})$	.217	.224	.255
$\Pr(A_{23} \leq M_{A_{23}},\ B_{23} \leq M_{B_{23}})$	.289	.244	.185
$\Pr(A_{31} \leq M_{A_{31}},\ B_{31} \leq M_{B_{31}})$	.290	.219	.245
$\Pr(A_{32} \leq M_{A_{32}},\ B_{32} \leq M_{B_{32}})$	.262	.198	.236
$\Pr(A_{33} \leq M_{A_{33}},\ B_{33} \leq M_{B_{33}})$	.314	.204	.198

not seem to be significant either. Hence, it was considered that the Bell-CHSH-Fine inequalities did not fail in Experiment 2(b).

The LFT, of course, produced the same conclusion as the Bell-CHSH-Fine inequalities when (A, B) were dichotomized. As discussed earlier, the LFT can be used to test selective influences when each of the output variables has arbitrarily finite number of discrete values. We performed the LFT using multiple percentile-rank-discretized data in all the 2×2 sets, 2×3 sets, 3×2 sets, and 3×3 sets of Experiments 1(a), 2(a), 2(b), and 3(a), nonnegative solutions always existed for all the multiple discretized data we tried except Experiment 2(b). However, there was no evidence that those violations in Experiment 2(b) were significant. Therefore, selective influences were considered established for all those conditions.

Tables 17-20 present the solutions to the LFT using the joint probabilities given in Tables 13-16.

Table 17

Solutions to the LFT for the Median-Discretized Data in Experiment 1(a)

Participant	$(Q_{1111}, Q_{1112}, \ldots, Q_{2222})^T$
Rectangular subdesign	
P1	$(0, .075, 0, .005, .018, .177, .225, 0, 0, .151, .177, .093, .080, 0, 0, 0)^T$
P2	$(0, 0, .050, .010, 0, .209, .149, .082, .032, .200, .209, 0, .060, 0, 0, 0)^T$
P3	$(.029, 0, 0, .062, .072, .191, .145, 0, 0, .208, .162, .038, 0, 0, .091, 0)^T$
Polar subdesign	
P1	$(.041, 0, .052, 0, .001, .167, .240, 0, 0, .200, .167, .041, 0, .093, 0, 0)^T$
P2	$(.016, 0, .080, 0, .002, .2, .202, 0, 0, .186, .200, .017, 0, .096, 0, 0)^T$
P3	$(.000, 0, .023, 0, .007, .233, .237, 0, 0, .237, .233, .007, 0, .023, 0, 0)^T$

Experiments with continuous factor points. Experiments 1(b), 2(c), and 3(b) have external factors that vary within specific intervals. We computed $(\alpha - A)$ and $(\beta - B)$ for each trial and took large deviations indicators of outliers. There were less than one percent outliers in each experiment and they were removed from further analysis. In order to perform the tests of selective influences, one has to convert the continuous factor points to discrete factor points. For example, one can create two levels for each factor according to the middle point of its confined interval. Then each experiment has four distinct treatments, denoted as $(\alpha_{i_1}, \beta_{i_2}), i_1, i_2 \in \{1, 2\}$. Tables

Table 18

Solutions to the LFT for the Median-Discretized Data in Experiment 2(a)

Participant	$(Q_{111111}, Q_{111112}, \ldots, Q_{222222})^T$
P1	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$ $0, 0, 0, .018, 0, .006, .021, .267, .188, 0, 0, 0, 0, 0, 0, 0, 0,$ $.047, 0, 0, .066, 0, .058, .061, 0, .030, 0, 0, .061, 0,$ $.076,.065, 0, 0, 0, 0, .010, 0, .015, .009, 0)^T$
P2	$(.274, 0, 0, 0, 0, .004, 0, .003, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$ $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .004, .009, .205, .053, 0,$ $0, 0, 0, 0, 0, 0, 0, 0, 0, .099, 0, .056, .010, 0, .003, 0,$ $0, .046, 0, .082, .034, 0, .023, 0, 0, 0, 0, 0, .093, 0)^T$
P3	$(.276, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$ $0, 0, 0, 0, 0, 0, 0, .014, .009, .201, .098, 0, 0, 0, 0, 0, 0, 0, 0,$ $0, 0, 0, .087, 0, .001, .008, .030, 0, 0, 0, .026, 0, .$ $.065, .035, 0, 0, 0, 0, .013, 0, .046, .075, .016)^T$

Table 19

Solutions to the LFT for the Median-Discretized Data in Experiment 2(b)

Participant	$(Q_{1111}, Q_{1112}, \ldots, Q_{2222})^T$
P3	$(.338, .027, 0, 0, 0, .064, 0, .071, .006, .065, .064, 0, 0, 0, .092, .273)^T$
P4	$(.356, 0, 0, 0, 0, .022, 0, .122, .001, .073, .069, 0, 0, .048, .073, .24)^T$
P5	$(.354, .001, 0, 0, 0, .098, 0, .047, 0, .047, .035, .063, 0, 0, .111, .244)^T$

Table 20

Solutions to the LFT for the Median-Discretized Data in Experiment 3(a)

Participant	$(Q_{111111}, Q_{111112}, \ldots, Q_{222222})^T$
P3	$(.135, 0, .049, 0, 0, 0, .009, .057, 0, 0, 0, 0, 0, 0,$ $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .073, 0, 0, .077, 0, .083,$ $.017, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .061, 0, .082, .096,$ $.012, .052, 0, 0, .054, 0, .075, 0.069, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$
P4	$(.123, 0, 0, 0, .002, .036, 0, .128, .021, 0, 0, 0,$ $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .067, 0, 0, .040, 0, .072,$ $.010, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .051, 0, .041, .097,$ $0, 0, 0, 0, .095, 0, .036, .079, 0, .101, 0, 0, 0, 0, 0, 0, 0)^T$
P5	$(.116, .052, 0, 0, 0, .016, 0, .104, .008, 0, 0, 0,$ $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .067, 0, 0, .0150, 0, .041,$ $.080, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .080, 0, .062, .061,$ $0, 0, 0, 0, .077, 0, .052, .083, 0, .085, 0, 0, 0, 0, 0, 0, 0)^T$

21-23 present the mean and standard deviation of $(A_{i_1i_2}, B_{i_1i_2})$ for each treatment $(\alpha_{i_1}, \beta_{i_2})$ in Experiments 1(b), 2(c), and 3(b), respectively. α and β are considered perceptually separable if the data for the discretized factors pass the tests of selective influences.

Marginal selectivity needs to be tested firstly. We compared the distributions of A_{i_11} with A_{i_12} and compared the distributions of B_{1i_2} with B_{2i_2}. The K-S test for 2-independent samples was used to make the four paired comparisons for each participant in each experiment. If one of the four paired comparisons was significant ($p < .05$), it was considered as an absence of marginal selectivity. Tables 24-26 present

Table 21

Means and Standard Deviations of $(A_{i_1i_2}, B_{i_1i_2})$,$i_1, i_2 \in \{1, 2\}$ *in Experiment 1(b)*

Rectangular design									
i_1	i_2	α_{i_1} (px)	β_{i_2} (px)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (px)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (px)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (px)
				Participant P3		Participant P4		Participant P5	
1	1	[20, 50)	[20, 50)	44.09±11.90	37.58±11.15	38.46±10.74	34.14±12.42	37.00±10.25	31.99±10.52
1	2	[20, 50)	[50, 80)	39.69±10.24	68.99±9.12	34.21±10.40	64.89±10.99	37.49±11.34	60.92±10.72
2	1	[50, 80)	[20, 50)	71.89±8.23	37.37±11.47	63.39±10.62	38.50±12.10	64.87±9.87	30.78±10.82
2	2	[50, 80)	[50, 80)	67.32±9.09	67.62±9.60	59.47±10.26	67.96±12.42	65.85±11.12	60.33±11.07
Polar subdesign									
i_1	i_2	α_{i_1} (px)	β_{i_2} (deg)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (deg)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (deg)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (deg)
				Participant P3		Participant P4		Participant P5	
1	1	[40, 65)	[30, 45)	63.86±8.70	32.99±7.20	56.76±9.53	36.36±10.41	53.44±8.26	33.27±8.92
1	2	[40, 65)	[45, 60)	61.30±9.61	50.25±6.51	55.53±9.49	49.57±10.05	54.44±7.94	48.83±8.45
2	1	[65, 90)	[30, 45)	84.34±7.00	35.95±6.34	77.97±9.14	40.70±8.19	76.55±7.83	34.86±8.36
2	2	[65, 90)	[45, 60)	84.43±7.50	52.56±5.80	77.91±9.07	54.21±8.12	76.48±9.09	50.16±7.72

Table 22

Means and Standard Deviations of $(A_{i_1i_2}, B_{i_1i_2})$,$i_1, i_2 \in \{1, 2\}$ *in Experiment 2(c)*

i_1	i_2	α_{i_1} (px)	β_{i_2} (px)	$A_{i_1i_2}$ (px)	$B_{i_1i_2}$ (px)
Participant P3					
1	1	[18, 33)	[56, 71)	24.15±4.17	58.47±4.66
1	2	[18, 33)	[71, 86)	24.41±4.23	73.31±4.64
2	1	[33, 48)	[56, 71)	38.86±4.44	60.28±4.45
2	2	[33, 48)	[71, 86)	38.45±4.71	74.03±4.71
Participant P4					
1	1	[18, 33)	[56, 71)	25.29±4.54	62.76±4.96
1	2	[18, 33)	[71, 86)	24.98±4.58	77.81±4.89
2	1	[33, 48)	[56, 71)	39.34±4.45	61.59±4.83
2	2	[33, 48)	[71, 86)	39.47±4.61	77.32±5.46
Participant P5					
1	1	[18, 33)	[56, 71)	25.81±4.66	64.49±4.75
1	2	[18, 33)	[71, 86)	25.19±4.15	78.25±4.74
2	1	[33, 48)	[56, 71)	39.81±4.35	63.49±4.87
2	2	[33, 48)	[71, 86)	38.81±4.80	77.44±4.87

the p values for the comparisons of responses to the corresponding given factor point across the levels of the other factor. For Experiment 1(b) participant P5 passed the tests of marginal selectivity in both rectangular design and polar subdesign. The other participants failed the tests in both designs. Experiment 2(c) failed the tests for all the participants. Experiment 3(b) passed the tests for all the participants.

Table 23

Means and Standard Deviations of $(A_{i_1 i_2}, B_{i_1 i_2})$,$i_1, i_2 \in \{1, 2\}$ *in Experiment 3(b)*

i_1	i_2	α_{i_1} (px)	β_{i_2} (px)	$A_{i_1 i_2}$ (px)	$B_{i_1 i_2}$ (px)
Participant P3					
1	1	$[-30, 0)$	$[-30, 0)$	-14.55±8.53	-15.03±8.10
1	2	$[-30, 0)$	$[0, 30)$	-14.33±8.78	15.21±8.44
2	1	$[0, 30)$	$[-30, 0)$	14.41±8.43	-14.94±8.59
2	2	$[0, 30)$	$[0, 30)$	14.91±8.06	15.48±8.71
Participant P4					
1	1	$[-30, 0)$	$[-30, 0)$	-14.68±9.50	-15.30±9.07
1	2	$[-30, 0)$	$[0, 30)$	-14.69±8.99	15.00±9.60
2	1	$[0, 30)$	$[-30, 0)$	14.39±8.99	-15.74±9.24
2	2	$[0, 30)$	$[0, 30)$	15.69±8.94	15.86±8.93
Participant P5					
1	1	$[-30, 0)$	$[-30, 0)$	-15.94±8.78	-15.52±8.72
1	2	$[-30, 0)$	$[0, 30)$	-14.92±9.29	14.11±8.22
2	1	$[0, 30)$	$[-30, 0)$	14.66±9.01	-15.26±8.48
2	2	$[0, 30)$	$[0, 30)$	14.39±9.39	13.81±8.72

The cosphericity test and LFT (of course Bell-CHSH-Fine inequalities) confirmed selective influences present in the data collected from participant P5 in Experiment 1(b) and all the participants in Experiment 3(b). Here I present the results of these tests for Experiment 3(b).

Correlations $\rho_{i_1 i_2}$ in Experiment 3(b) are given in Table 27. The cosphericity test was passed for all the participants.

Table 24

Tests of Marginal Selectivity for Experiment 1(b)

		Participant		
		P3	P4	P5
Rect-angular design	A_{11}, A_{12}	.000	.000	.332
	A_{21}, A_{22}	.000	.000	.122
	B_{11}, B_{21}	.758	.000	.329
	B_{12}, B_{22}	.289	.002	.621
Polar design	A_{11}, A_{12}	.475	.283	.616
	A_{21}, A_{22}	.854	.393	.122
	B_{11}, B_{21}	.000	.000	.394
	B_{12}, B_{22}	.003	.000	.306

Table 25

Tests of Marginal Selectivity for Experiment 2(c)

	Participant P3	Participant P4	Participant P5
A_{11}, A_{12}	.352	.696	.031
A_{21}, A_{22}	.427	.198	.011
B_{11}, B_{21}	.000	.003	.009
B_{12}, B_{22}	.010	.122	.030

The cosphericity test was only a necessary condition for selective influences in Experiment 3(b) as each $(A_{i_1 i_2}, B_{i_1 i_2})$ was not bivariate normally distributed.. Since marginal selectivity was satisfied, I then used A_{i_1} to represent $A_{i_1 1}$ and $A_{i_1 2}$, and B_{i_2}

Table 26

Tests of Marginal Selectivity for Experiment 3(b)

	Participant P3	Participant P4	Participant P5
A_{11}, A_{12}	.950	.728	.106
A_{21}, A_{22}	.610	.230	.338
B_{11}, B_{21}	.838	.187	.876
B_{12}, B_{22}	.496	.069	.295

Table 27

Correlations of (A, B) *in Experiment 3(b)*

Participant P3	$\rho_{11} = -.139$	$\rho_{12} = -.073$	$\rho_{21} = .153$	$\rho_{22} = -.104$
Participant P4	$\rho_{11} = .012$	$\rho_{12} = .099$	$\rho_{21} = .070$	$\rho_{22} = .045$
Participant P5	$\rho_{11} = .002$	$\rho_{12} = .010$	$\rho_{21} = .095$	$\rho_{22} = .019$

to represent B_{1i_2} and B_{2i_2}. In order to apply Bell-CHSH-Fine inequalities, the output variables have to be dichotomized and marginal selectivity has to be conserved after that. One can discretize A_1, A_2, B_1, and B_2 according to particular values in px. To give an example, I created two levels for A_1 and A_2: {smaller than or equal to -15 px, larger than -15 px}, labeled as $\{a_1, a_2\}$, and two levels for B_1 and B_2: {smaller than or equal to 15 px, larger than 15 px}, labeled as $\{b_1, b_2\}$. The joint probabilities and marginal probabilities are presented in Table 28.

Table 28

Joint Distributions of the Discretized $(A_{i_1i_2},\ B_{i_1i_2}), i_1, i_2 \in \{1, 2\}$ *in Experiment 3(b)*

Participant P3							
(α_1, β_1)	$B_{11}{=}b_1$	$B_{11}{=}b_2$		(α_1, β_2)	$B_{12}{=}b_1$	$B_{12}{=}b_2$	
$A_{11}{=}a_1$	.526	0	.526	$A_{12}{=}a_1$	.222	.294	.516
$A_{11}{=}a_2$	.474	0	.474	$A_{12}{=}a_2$	.237	.247	.484
	1	0			.459	.541	
(α_2, β_1)	$B_{21}{=}b_1$	$B_{21}{=}b_2$		(α_2, β_2)	$B_{22}{=}b_1$	$B_{22}{=}b_2$	
$A_{21}{=}a_1$	0	0	0	$A_{22}{=}a_1$	0	0	0
$A_{21}{=}a_2$	1	0	1	$A_{22}{=}a_2$	.459	.541	1
	1	0			.459	.541	

Participant P4							
(α_1, β_1)	$B_{11}{=}b_1$	$B_{11}{=}b_2$		(α_1, β_2)	$B_{12}{=}b_1$	$B_{12}{=}b_2$	
$A_{11}{=}a_1$	.509	0	.509	$A_{12}{=}a_1$	.254	.272	.526
$A_{11}{=}a_2$	.491	0	.491	$A_{12}{=}a_2$	.226	.249	.475
	1	0			.480	.521	
(α_2, β_1)	$B_{21}{=}b_1$	$B_{21}{=}b_2$		(α_2, β_2)	$B_{22}{=}b_1$	$B_{22}{=}b_2$	
$A_{21}{=}a_1$	0	0	0	$A_{22}{=}a_1$	0	0	0
$A_{21}{=}a_2$	1	0	1	$A_{22}{=}a_2$	.460	.540	1
	1	0			.460	.540	

(table continues)

Participant P5							
(α_1, β_1)	$B_{11}{=}b_1$	$B_{11}{=}b_2$		(α_1, β_2)	$B_{12}{=}b_1$	$B_{12}{=}b_2$	
$A_{11}{=}a_1$	.577	0	.577	$A_{12}{=}a_1$	.277	.246	.523
$A_{11}{=}a_2$	.423	0	.423	$A_{12}{=}a_2$	.246	.232	.478
	1	0			.523	.478	
(α_2, β_1)	$B_{21}{=}b_1$	$B_{21}{=}b_2$		(α_2, β_2)	$B_{22}{=}b_1$	$B_{22}{=}b_2$	
$A_{21}{=}a_1$	0	0	0	$A_{22}{=}a_1$	0	0	0
$A_{21}{=}a_2$	1	0	1	$A_{22}{=}a_2$	.552	.448	1
	1	0			.552	.448	

In this Table, it is not always exact that

$$P(A_{11}) = P(A_{12}), P(A_{21}) = P(A_{22}),$$
$$P(B_{11}) = P(B_{21}), P(B_{12}) = P(B_{22}).$$

However equalities were still considered satisfied since marginal selectivity was previously established. In order to conduct the Bell-CHSH-Fine inequalities, let (a, b, c, d) in (11) be

$$([P(A_{11} = a_1) + P(A_{12} = a_1)]/2, [P(A_{21} = a_1) + P(A_{22} = a_1)]/2,$$
$$[P(B_{11} = b_1) + P(B_{21} = b_1)]/2, [P(B_{12} = b_1) + P(B_{22}) = b_1]/2),$$

for each participant in this example. After substituting those values and joint probabilities in Table 28 into (11), Bell-CHSH-Fine inequalities were found not violated for all the participants.

We also tested the Bell-CHSH-Fine inequalities using A_1, A_2, B_1, and B_2 dichotomized by other values in px. Four integers were generated from -40 px to 40 px with an increment of 1 px per trial. So 81^4 distinct quadruples were created. We then used each of them to discretize the (A_1, A_2, B_1, B_2) and computed the corresponding joint probabilities and marginal probabilities (a, b, c, d). The test was passed for all the 81^4 systems. Therefore, selective influences on (A, B) by the midpoint-discretized (α, β) were certain in Experiment 3(b). Consequently, the two given amplitudes in Experiment 3(b) were considered perceptually separable for all the three participants.

The LFT, of course, produced the same conclusion as the Bell-CHSH-Fine inequalities when (A, B) in Experiment 3(b) were dichotomized. Table 29 presents the nonnegative solutions to the joint probabilities given in Table 28. The LFT is also applicable for multiply discretized (A, B). The test was passed for all the discretizations that we tried.

Table 29

Solutions to the LFT for Experiment 3(b)

Participant	$(Q_{1111}, Q_{1112}, \ldots, Q_{2222})^T$
P3	$(0, 0, 0, 0, .225, .296, 0, 0, 0, 0, 0, 0, .234, .245, 0, 0)^T$
P4	$(0, 0, 0, 0, .244, .273, 0, 0, 0, 0, 0, 0, .226, .257, 0, 0)^T$
P5	$(0, 0, 0, 0, .291, .259, 0, 0, 0, 0, 0, 0, .246, .204, 0, 0)^T$

Conclusions

Selective influences were demonstrated verifiable and falsifiable by performing the cosphericity test, LFT, and Bell-CHSH-Fine inequalities using the empirical data.

The failure to detect marginal selectivity in Experiments 2(a), 2(b) and 2(c) confirmed that perceptual separability absent in the Delboeuf illusion phenomenon. The participants' performance in the dot position reproduction task and the floral shape reproduction task was not exact within the expectation. The results from the dot position reproduction task indicated that selective influences were absent in Experiment 1(a) but detected for participant P5 in Experiment 1(b) for both rectangular design and polar subdesign. For the floral reproduction task, selective influences were indeed violated for all the participants (except some 2×2 subdesigns for participant P5) in Experiment 3(a) but supported in Experiment 3(b) for all the participants. It was found, consistently in the three types of tasks, selective influences were more likely to exist in the designs with continuous factor points than the designs with several discrete factor points, supporting the speculation that responding to each stimulus deliberately rather than automatically has a higher chance to result in processing stimulus attributes separably. Nevertheless, it may not be complete to attribute the presence of selective influences in Experiment 3(b) to this reason only. We speculated that different levels of awareness of the two factors were involved in the three tasks. In the concentric circle reproduction task, the participants were at the highest level of awareness of the two factors. It was apparent that the two factors were the sizes of circle one and circle two. In the dot position reproduction task, the participants were at the middle level of awareness. They were asked to move the dot to the "correct" location. Whether the two factors were represented in the rectangular coordinates or the polar coordinates, they should realize that if they moved the trackball to the right or up, the dot was further from the center of the circle. So they sensed that there were two factors involved to some extent. In the floral shape reproduction task, it was believed that the participants had no way to consciously know there were two factors. In the dot position reproduction task and the concentric circle reproduction task, selective influences were rarely present, probably because once people realized two distinct factors were involved, the responses to one factor tended to be influenced by the other factor, resulting in the failure of marginal selectivity. However, selective influences for the amplitudes in the floral shape reproduction task stood a better chance to be detected because the participants had no clue if there were distinct factors and how many of them were used. This speculation seems counterintuitive but was supported by the results from the experiments.

It was theoretically proved that the chance to violate selective influences is relatively low in the presence of marginal selectivity (Dzhafarov & Kujala, 2011): If the marginal probabilities are constrained to .5 in a 2×2 factorial design and the values of $\Pr(A_{i_1 i_2} \leq M_{A_{i_1 i_2}},\ B_{i_1 i_2} \leq M_{B_{i_1 i_2}})$, $i_1,\ i_2 \in \{1, 2\}$ are randomly picked up from four independent uniform distributions from 0 to .5, the chance to sustain selective influences is .67. Therefore it was not a surprise that in Experiments 1(a), 2(a), 2(b), and 3(a), no violation of selective influences was found with the artificially imposed marginal selectivity. In Experiment 2(b), three correlations of Type N transformed (A, B) were as high as about .8 and the other correlation was about .3. But the cosphericity test did not fail. Besides the LFT for Type D transformed (A, B) were passed except for a few particular discretizations of the data set. These violations were attributed to statistical fluctuations. We are not sure if there exists such an empirical paradigm in which selective influences are violated but marginal selectivity is present. If the answer is no, it may imply that in humans behavior selective influences are essentially synonymous to marginal selectivity.

MENTAL ARCHITECTURES AND APPLICATIONS

A Historical Review of Mental Architectures

As mentioned earlier, a mental architecture is a hypothetical network of processes carried in the mind when the task is being performed. One way of understanding the arrangement of the processes is to investigate the distribution functions of the overall processing time in different experimental conditions and compute a linear combination of them. It is assumed that the durations of the processes in the network are selectively influenced by different external factors. Let us consider only two processes X^α and X^β, selectively responding to external factors α and β, respectively. Let us denote durations of processes X^α and X^β as T^α and T^β, respectively. There are infinitely many possible architectures even if only two processes are considered. Three elementary schemes (Figure 8) are of the greatest traditional interest:

(a) Minimum parallel ($T = \min(T^\alpha, T^\beta)$),

(b) Maximum parallel ($T = \max(T^\alpha, T^\beta)$),

(c) Serial ($T = T^\alpha + T^\beta$).

The study of serial and parallel processing of selective influenced components can be traced back to Sternberg (1969). His Additive Factor Method was used in a 2×2 factorial design. Let $T_{i_1 i_2}$ denote the time to complete the task given the treatment $\phi = (\alpha_{i_1}, \beta_{i_2}), i_1, i_2 \in \{1, 2\}$. He suggested that if the processes are serial and the process durations are pairwise independent, then the *mean interaction contrast* ($\overline{C}$) presented below equals zero:

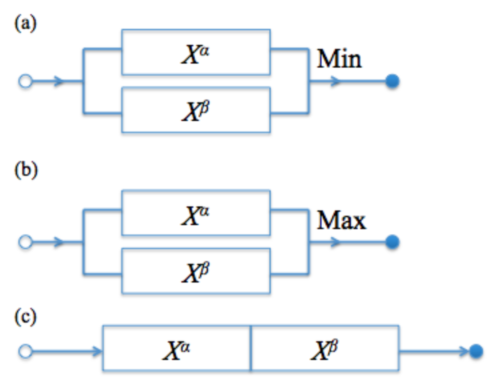

Figure 8. Three elementary schemes: (a) minimum parallel, (b) maximum parallel, and (c) serial.

$$\overline{C} = \overline{T}_{11} - \overline{T}_{12} - \overline{T}_{21} + \overline{T}_{22},$$

where $\overline{T}_{i_1 i_2}$ is the mean value of $T_{i_1 i_2}$.

The use of $\overline{C}$ was later extended to architectures other than serial processing (Schweickert, 1978, 1982; Schweickert & Townsend, 1989; Townsend & Schweickert, 1989). However, $\overline{C}$ is only a rough summary of response times. The complete information of the response time is carried in its distribution function. Townsend and Nozawa (1995) constructed the *interaction contrast* of survivor functions for response time to provide an insight of the underlying mental architecture operating in a given psychological task. It can be equivalently expressed as the linear combination of the distribution functions:

$$C\left(t\right) = \Pr\left(T_{11} \leq t\right) - \Pr\left(T_{12} \leq t\right) - \Pr\left(T_{21} \leq t\right) + \Pr\left(T_{22} \leq t\right). \tag{15}$$

Let us denote $T_{i_1}^{\alpha}$ the duration to process the factor α_{i_1} and $T_{i_2}^{\beta}$ the duration to process the factor β_{i_2}. By imposing the assumption of *stochastic dominance* to the distribution functions of durations,

$$\Pr(T_1^{\alpha} \leq t) \geq \Pr(T_2^{\alpha} \leq t), \Pr(T_1^{\beta} \leq t) \geq \Pr(T_2^{\beta} \leq t), \tag{16}$$

they found that $\overline{C}$ is positive and $C(t)$ always nonpositive for the minimum parallel model, and $\overline{C}$ is negative and $C(t)$ always nonnegative for the maximum parallel model. For the serial model, $C(t)$ is positive for small times t and later becomes negative, while $\overline{C}$ is zero.

The assumption of selective influences is critical for the technique of interaction contrast. Townsend and Thomas (1994) proved that if selectivity does not hold for the interdependent T^α and T^β, the characteristic patterns associated with the minimum parallel, maximum parallel, and serial models are distorted: $C(t)$ can be negative, zero, and positive in each of the three models. Therefore, without presence of selective influences, one architecture is indistinguishable from the others.

In this dissertation, I will use $X \wedge Y$ or $\wedge(X, Y)$ to denote $\min(X, Y)$, and $X \vee Y$ or $\vee(X, Y)$ to denote $\max(X, Y)$ for convenience. The pairwise serial and parallel processes are the fundamental units of an *serial-parallel (SP) mental architecture*. An SP mental architecture is a network defined by the three arguments.

Definition 5. (1) A single process is an SP mental architecture. (2) If X and Y are SP mental architectures that do not share components, then $X \wedge Y$, $X \vee Y$, and $X + Y$ are SP mental architectures. (3) There are no other SP mental architectures than those construable by rules 1 and 2.

Definition 6. An SP mental architecture is *homogeneous* if it does not contain both $\wedge$ and $\vee$ in one network. Those constructed of plus and min are $\text{SP}_\wedge$ mental architectures. Those constructed of plus and max are $\text{SP}_\vee$ mental architectures.

Definition 7. An SP mental architecture is *simple* if it contains only one particular operation.

Figure 9(a) is an example of an SP mental architecture but it is not homogeneous. Figure 9(b) is a homogeneous SP mental architecture. Figure 9(c) is a simple SP mental architecture.

Most of the results previously obtained for mental networks are confined to homogeneous SP mental architectures. Schweickert et al. (2000) studied the three pairwise operations-minimum parallel, maximum parallel, and serial, in homogeneous SP mental architectures assuming T^α and T^β are stochastically independent. They demonstrated that the three operations are distinguishable as they have distinct patterns

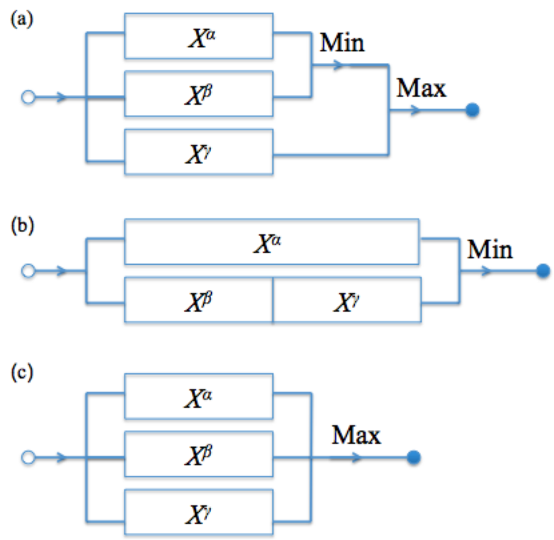

Figure 9. Examples of (a) an SP mental architecture, (b) a homogeneous SP mental architecture, and (c) a simple SP mental architecture.

of $C(t)$. Dzhafarov et al. (2004) generalized the results to interdependent T^{α} and T^{β} and found the patterns preserve in the interdependent cases. By setting the common randomness Θ in Definition 1 to a particular value θ, interdependent T^{α} and T^{β} become conditionally independent. Then the corresponding interaction contrast $C(t)$ is identical to that developed by Schweickert et al. (2000). The interaction contrast for the interdependent processes can be obtained by integrating the $C(t)$ conditional on $\Theta = \theta$ over the measure space of Θ.

Despite the success of earlier work in classifying mental architectures, those approaches are limited by several "auxiliary" assumptions (e.g., existence and certain properties of probability density functions). Zhang and Dzhafarov (2015) reduced the interaction contrast (15) to a linear combination of deterministic numbers by conditioning all random variables involved on a particular value r of the hidden variable R in Definition 2. This method requires fewer assumptions and reduces the probabilistic problem to simple numerical combinatorics. In addition, the consideration is not constrained to homogenous SP mental architectures: it can be extended to general SP mental architectures. Below I present the theoretical work that we have done and the results of some empirical studies guided by the the theory of mental architectures.

Note that SP mental architectures do not span the entire range of possible configurations of mental architectures. There is significant theoretical work on more general architectures, in which the operations are only assumed to be commutative and associative (Cortese & Dzhafarov, 1996; Dzhafarov & Cortese, 1996; Dzhafarov & Schweickert, 1995). They are outside the scope of the discussion here.

Theoretical Achievement of SP Mental Architectures

Simple SP Mental Architectures of Size 2

Let us consider a system of only two processes X^{α} and X^{β}, responding to the external factors α and β, respectively. Let us denote the durations of the two processes T^{α} and T^{β}, respectively. Suppose T^{α} and T^{β} are selectively influenced by α and β: $(T^{\alpha}, T^{\beta}) \looparrowleft (\alpha, \beta)$. Let each factor have two levels: $\alpha \in \{\alpha_1, \alpha_2\}$ and $\beta \in \{\beta_1, \beta_2\}$. $T_{i_1 i_2}$, $T^{\alpha}_{i_1 i_2}$, and $T^{\beta}_{i_1 i_2}$ denote, respectively, the overall duration, duration of a response to α, and duration of a response to β, given the treatment $(\alpha_{i_1}, \beta_{i_2}), i_1, i_2 \in \{1, 2\}$. According to the assumption of selective influences, $T^{\alpha}_{i_1 i_2}$ is the same for all values of i_2 and $T^{\beta}_{i_1 i_2}$ is the same for all values of i_1. I therefore write $T^{\alpha}_{i_1 i_2} = T^{\alpha}_{i_1} = f_{\alpha}(i_1, R)$ and $T^{\beta}_{i_1 i_2} = T^{\beta}_{i_2} = f_{\beta}(i_2, R)$ for convenience. $T_{i_1 i_2}$, $T^{\alpha}_{i_1}$, and $T^{\beta}_{i_2}$ for each treatment become deterministic when R defined in Definition 2 is fixed to some value r. These deterministic quantities are denoted as $T_{i_1 i_2 r}$, $T^{\alpha}_{i_1 r}$, and $T^{\beta}_{i_2 r}$. The distribution function $\Pr(T_{i_1 i_2} \leq t)$ conditioned on $R = r$ is reduced to a (shifted) *Heaviside step function*:

$$H_{i_1 i_2 r}(t) = \begin{cases} 0, \; if \; t < T_{i_1 i_2 r} \\ 1, \; if \; t \geq T_{i_1 i_2 r} \end{cases}.$$

The distribution function for the unconditional $T_{i_1 i_2}$ is

$$H_{i_1 i_2}(t) = \int_{\mathcal{R}} H_{i_1 i_2 r}(t) \, d\mu_r,$$

where $\mathcal{R}$ is the set of all possible values of R, and μ_r is its probability measure. When conditioned on $R = r$, the interaction contrast $C(t)$ (15) is reduced to:

$$C_r(t) = H_{11r}(t) - H_{12r}(t) - H_{21r}(t) + H_{22r}(t).$$

Hence the unconditional $C(t)$ can be expressed as

$$\begin{aligned} C(t) &= \int_{\mathcal{R}} C_r\left(t\right) d\mu_r \\ &= \int_{\mathcal{R}} \left[H_{11r}\left(t\right) - H_{12r}\left(t\right) - H_{21r}\left(t\right) + H_{22r}\left(t\right)\right] d\mu_r. \\ &= H_{11}\left(t\right) - H_{12}\left(t\right) - H_{21}\left(t\right) + H_{22}\left(t\right) \end{aligned}$$

We have to make one auxiliary assumption: the *prolongation assumption*, which is the deterministic version of the stochastic dominance assumption (16). For any choice $R = r$,

$$T_{1r}^{\alpha} \leq T_{2r}^{\alpha},\, T_{1r}^{\beta} \leq T_{2r}^{\beta}. \tag{17}$$

The graphical representation of $C_r(t)$ with the prolongation assumption is displayed below (with the possibility that some of the points on the time axis may coincide). It is easy to see that if $T_{12r} \wedge T_{21r} = T_{11r}$, then $C_r(t) \leq 0$; if $T_{12r} \vee T_{21r} = T_{22r}$, then $C_r(t) \geq 0$.

We also define two *cumulative interaction contrasts* conditioned on $R = r$:

$$C_r\left(0, t\right) = \int_0^t C_r(t) dt, \tag{18}$$

$$C_r\left(t, \infty\right) = \int_t^{\infty} C_r\left(t\right) dt = \lim_{u \to \infty} \int_t^u C_r\left(t\right) dt. \tag{19}$$

The corresponding unconditional cumulative interaction contrasts are

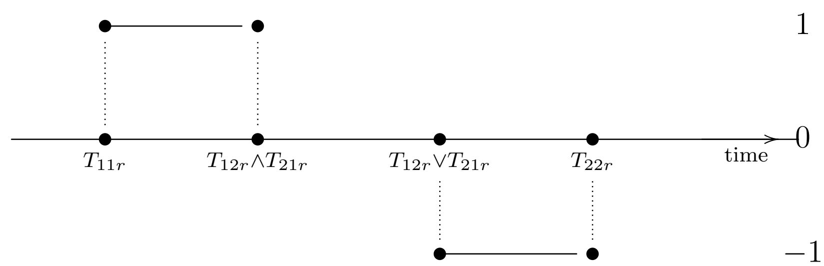

Figure 10. The graphical representation of $C_r(t)$.

$$
\begin{aligned}
C\left(0,t\right) &= \int_{\mathcal{R}} C_r\left(0,t\right) d\mu_r \\
&= \int_{\mathcal{R}} \left(\int_0^t C_r\left(t\right) dt\right) d\mu_r \\
&= \int_0^t \left(\int_{\mathcal{R}} C_r\left(t\right) d\mu_r\right) dt \\
&= \int_0^t C\left(t\right) dt,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
C\left(t,\infty\right) &= \int_{\mathcal{R}} C_r\left(t,\infty\right) d\mu_r \\
&= \int_{\mathcal{R}} \left(\int_t^\infty C_r\left(t\right) dt\right) d\mu_r \\
&= \int_t^\infty \left(\int_{\mathcal{R}} C_r\left(t\right) d\mu_r\right) dt \\
&= \int_t^\infty C\left(t\right) dt.
\end{aligned} \tag{21}
$$

Theorem 8 below states that the three two-process simple mental architectures have distinct patterns for the conditional interaction contrast or for the conditional cumulative interaction contrast.

Theorem 8. *(i) For $T = T^\alpha \wedge T^\beta$, $C_r\left(t\right) \leq 0$ for any r, t; (ii) for $T = T^\alpha \vee T^\beta$, $C_r\left(t\right) \geq 0$ for any r, t; (iii) for $T = T^\alpha + T^\beta$, $C_r(0,t) \geq 0$ and $C_r(t,\infty) \leq 0$ for any r, t; moreover, $\lim_{t\to\infty} C_r(0,t) = \lim_{t\to 0} C_r(t,\infty) = 0$.*

Proof. (i) If $T = T^\alpha \wedge T^\beta$, we have, for any r,

$$
\begin{aligned}
T_{11r} &= T^\alpha_{1r} \wedge T^\beta_{1r}, \\
T_{12r} &= T^\alpha_{1r} \wedge T^\beta_{2r}, \\
T_{21r} &= T^\alpha_{2r} \wedge T^\beta_{1r}.
\end{aligned}
$$

With the prolongation assumption (17), it follows that

$$T_{12r} \wedge T_{21r} = T_{1r}^{\alpha} \wedge T_{2r}^{\beta} \wedge T_{2r}^{\alpha} \wedge T_{1r}^{\beta} = T_{1r}^{\alpha} \wedge T_{1r}^{\beta} = T_{11r}.$$

By observing Figure 10, $C_r(t) \leq 0$ is apparent.

(ii) The proof is analogous if $T = T^{\alpha} \vee T^{\beta}$.

(iii) If $T = T^{\alpha} + T^{\beta}$,

$$T_{11r} = T_{1r}^{\alpha} + T_{1r}^{\beta},$$

$$T_{12r} = T_{1r}^{\alpha} + T_{2r}^{\beta},$$

$$T_{21r} = T_{2r}^{\alpha} + T_{1r}^{\beta},$$

$$T_{22r} = T_{2r}^{\alpha} + T_{2r}^{\beta}.$$

Hence,

$$T_{12r} \wedge T_{21r} - T_{11r} = \wedge(T_{1r}^{\alpha} + T_{2r}^{\beta}, T_{2r}^{\alpha} + T_{1r}^{\beta}) - (T_{1r}^{\alpha} + T_{1r}^{\beta}),$$

$$T_{22r} - T_{12r} \vee T_{21r} = (T_{2r}^{\alpha} + T_{2r}^{\beta}) - \vee(T_{1r}^{\alpha} + T_{2r}^{\beta}, T_{2r}^{\alpha} + T_{1r}^{\beta}).$$

Without loss of generality, let $T_{1r}^{\alpha} + T_{2r}^{\beta} \leq T_{2r}^{\alpha} + T_{1r}^{\beta}$. Then

$$T_{12r} \wedge T_{21r} - T_{11r} = (T_{1r}^{\alpha} + T_{2r}^{\beta}) - (T_{1r}^{\alpha} + T_{1r}^{\beta}) = T_{2r}^{\beta} - T_{1r}^{\beta},$$

$$T_{22r} - T_{12r} \vee T_{21r} = (T_{2r}^{\alpha} + T_{2r}^{\beta}) - (T_{2r}^{\alpha} + T_{1r}^{\beta}) = T_{2r}^{\beta} - T_{1r}^{\beta}.$$

So we have

$$T_{12r} \wedge T_{21r} - T_{11r} = T_{22r} - T_{12r} \vee T_{21r}.$$

By observing Figure 10, the statement follows. $\square$

Corollary 9 below follows from Theorem 8 immediately. The unconditional $C(t)$ is the result of integrating $C_r(t)$ over the measure space of R. The integral preserves the sign of $C_r(t)$, therefore the pattern of $C(t)$ is preserved in the unconditional condition.

Corollary 9. *(i) For* $T = T^\alpha \wedge T^\beta$, $C(t) \leq 0$ *for any* t*; (ii) for* $T = T^\alpha \vee T^\beta$, $C(t) \geq 0$ *for any* t*; (iii) for* $T = T^\alpha + T^\beta$, $C(0,t) \geq 0$ *and* $C(t,\infty) \leq 0$ *for any* t*; moreover,* $\lim_{t\to\infty} C(0,t) = \lim_{t\to 0} C(t,\infty) = 0$.

Two Processes in an Arbitrary SP Mental Architecture

The technique of interaction contrast is still applicable to characterize two processes arranged in parallel or in sequence in an SP network. Let us, as before, write T^α the duration of the response to α and T^β the duration of the response to β. The overall duration T of this SP mental architecture can be considered a function of T^α, T^β and other components of SP: $T = \mathrm{SP}(T^\alpha, T^\beta, \ldots)$. We assume that T^α, T^β and all other components are selectively influenced by α, β, and empty set, respectively: $(T^\alpha, T^\beta, \ldots) \looparrowleft (\alpha, \beta, \emptyset)$. Using the same notation as in the previous section, $T_{i_1 i_2}$, $T^\alpha_{i_1}$, and $T^\beta_{i_2}$ denote the overall duration of the entire SP mental architecture, the duration in response to α, and the duration in response to β, given the treatment $(\alpha_{i_1}, \beta_{i_2}), i_1, i_2 \in \{1,2\}$. The prolongation assumption (17) is imposed on the system.

Definition 10. Two durations T^α, T^β in an SP mental architecture are *minimum parallel* if there is a subnetwork of the form $\mathrm{SP}^1(T^\alpha, \ldots) \wedge \mathrm{SP}^2\left(T^\beta, \ldots\right)$; they are *maximum parallel* if there is a subnetwork of the form $\mathrm{SP}^1(T^\alpha, \ldots) \vee \mathrm{SP}^2\left(T^\beta, \ldots\right)$; they are *serial* (or *sequential*) if there is a subnetwork $\mathrm{SP}^1(T^\alpha, \ldots) + \mathrm{SP}^2\left(T^\beta, \ldots\right)$.

Lemma 11. *If* T^α *and* T^β *are arranged in a minimum parallel way in an SP mental architecture, then* $\mathrm{SP}(T^\alpha, T^\beta, \ldots)$ *can be represented as* $\mathrm{SP}^1(T^\alpha, \ldots) \wedge \mathrm{SP}^2\left(T^\beta, \ldots\right)$*; if they are arranged in maximum parallel, then* $\mathrm{SP}(T^\alpha, T^\beta, \ldots)$ *can be represented as* $\mathrm{SP}^1(T^\alpha, \ldots) \vee \mathrm{SP}^2\left(T^\beta, \ldots\right)$.

Proof. Let $\diamondsuit$ denote either $\wedge$ or $\vee$. According to Definition 5 and Definition 10, if T^α, T^β are minimum parallel, then $\mathrm{SP}(T^\alpha, T^\beta, \ldots)$ can be presented either as (i) $\mathrm{SP}^1(T^\alpha, \ldots) \wedge \mathrm{SP}^2(T^\beta, \ldots)$ or (ii) $\left(\mathrm{SP}^1(T^\alpha, \ldots) \wedge \mathrm{SP}^2(T^\beta, \ldots) + T'\right) \diamondsuit T''$ or (iii) $\left(\mathrm{SP}^1(T^\alpha, \ldots) \wedge SP^2(T^\beta, \ldots) \diamondsuit T'\right) + T''$, where T^β does not enter in SP^1, T^α does not enter in SP^2, and T' and T'' are durations of certain subnetworks. Now we only need to observe that (ii) and (iii) can be written in the form of $\mathrm{SP}^1\left(T^\alpha, \ldots\right) \wedge \mathrm{SP}^2\left(T^\beta, \ldots\right)$: (ii) $= \left(\mathrm{SP}^1(T^\alpha, \ldots) + T'\right) \wedge \left(\left(\mathrm{SP}^2(T^\beta, \ldots) + T'\right) \diamondsuit T''\right)$ and (iii) $= \left(\mathrm{SP}^1(A, \ldots) + T''\right) \wedge \left(\mathrm{SP}^2(B, \ldots) \diamondsuit T' + T''\right)$. The proof for the maximum parallel case is analogous. $\square$

Note that if T^α and T^β are arranged in a sequence in an SP mental architecture, $\mathrm{SP}(T^\alpha, T^\beta, \ldots)$ cannot be represented as $\mathrm{SP}^1\left(T^\alpha, \ldots\right) + \mathrm{SP}^2\left(T^\beta, \ldots\right)$ in many cases. Figure 9(b) is an example for it.

Lemma 12. *If T^α and T^β are arranged in parallel or in sequence in an SP mental architecture, then $\left(\mathrm{SP}^1\left(T^\alpha, \ldots\right), \mathrm{SP}^2\left(T^\beta, \ldots\right)\right) \looparrowleft (\alpha, \beta)$, and for any fixed $R = r$, the following version of the prolongation assumption holds:* $\mathrm{SP}^1\left(T^\alpha_{1r}, \ldots\right) \leq \mathrm{SP}^1\left(T^\alpha_{2r}, \ldots\right)$, $\mathrm{SP}^2\left(T^\beta_{1r}, \ldots\right) \leq \mathrm{SP}^2\left(T^\beta_{2r}, \ldots\right)$.

Proof. According to Definition 10, $\left(\mathrm{SP}^1\left(T^\alpha, \ldots\right), \mathrm{SP}^2\left(T^\beta, \ldots\right)\right) \looparrowleft (\alpha, \beta)$ is obvious. Fixing $R = r$, by the prolongation assumption (17) and (nonstrict) monotonicity of SP mental architectures, $\mathrm{SP}^1\left(T^\alpha_{1r}, \ldots\right) \leq \mathrm{SP}^1\left(T^\alpha_{2r}, \ldots\right)$ and $\mathrm{SP}^2\left(T^\beta_{1r}, \ldots\right) \leq \mathrm{SP}^2\left(T^\beta_{2r}, \ldots\right)$ is apparent. $\square$

Lemma 13. *In any SP mental architecture, for any r,*

$$T_{11r} \leq T_{12r} \wedge T_{21r} \leq T_{12r} \vee T_{21r} \leq T_{22r}.$$

Proof. Follows from Lemma 12 and the property of the (nonstrict) monotonicity of an SP mental architecture. $\square$

According to Lemma 12, it is not hard to see that Figure 10 is also the graphical representation of the conditional interaction contrast $C_r(t)$ for two processes in an arbitrary SP mental architecture.

Lemma 14. *In any SP mental architecture, for any* r*,*

$$C_r(t) = \begin{cases} 1-0-0+0>0, & \text{if } T_{11r} \le t < T_{12r} \wedge T_{21r} \\ 1-1-1+0<0, & \text{if } T_{12r} \vee T_{21r} \le t < T_{22r} \\ 0, & \text{otherwise} \end{cases}.$$

Proof. By direct computation. □

Lemma 15. *In any SP mental architecture, for any* r, t*,* $C_r(t) \le 0$ *if and only if* $T_{11r} = T_{12r} \wedge T_{21r}$*;* $C_r(t) \ge 0$ *if and only if* $T_{12r} \vee T_{21r} = T_{22r}$*.*

Proof. Immediately follows from Lemma 13. □

Lemma 16. *In any SP mental architecture, for any* r, t*,*

(i) $C_r(0,t) \ge 0$ *if and only if* $-T_{11r} + T_{12r} + T_{21r} - T_{22r} \ge 0$, and

(ii) $C_r(t,\infty) \le 0$ *if and only if* $-T_{11r} + T_{12r} + T_{21r} - T_{22r} \le 0$.

Proof. By observing Figure 10, it is immediate that $C_r(0,t) \ge 0$ for any t if and only if $T_{12r} \wedge T_{21r} - T_{11r} \ge T_{22r} - T_{12r} \vee T_{21r} \Longrightarrow -T_{11r} + T_{12r} + T_{21r} - T_{22r} \ge 0$. Therefore statement (i) is proved. The proof for (ii) is analogous. □

Theorem 17. *(i) If* T^α *and* T^β *in an SP mental architecture are minimum parallel, then* $C_r(t) \le 0$ *for any* r, t*; (ii) if* T^α *and* T^β *in an SP mental architecture are maximum parallel, then* $C_r(t) \ge 0$ *for any* r, t*; (iii) if* T^α *and* T^β *in an SP mental architecture are serial, then either* $C_r(0,t) \ge 0$ *for any* r, t*, or* $C_r(t,\infty) \le 0$ *for any* r, t*.*

Proof. (i) If T^{α}, T^{β} in an SP mental architecture are minimum parallel, then according to Lemma 11, the overall duration is $T = \mathrm{SP}^1\left(T^{\alpha}, \ldots\right) \wedge \mathrm{SP}^2\left(T^{\beta}, \ldots\right)$. In addition, according to Lemma 12, $\mathrm{SP}^1\left(T^{\alpha}_{1r}, \ldots\right) \leq \mathrm{SP}^2\left(T^{\alpha}_{2r}, \ldots\right)$, $\mathrm{SP}^2\left(T^{\beta}_{1r}, \ldots\right) \leq \mathrm{SP}^1\left(T^{\beta}_{2r}, \ldots\right)$, $C_r\left(0, t\right) \leq 0$ can be obtained by replacing T^{α}_r and T^{β}_r in the proof of Theorem 8 with $\mathrm{SP}^1\left(T^{\alpha}_r, \ldots\right)$ and $\mathrm{SP}^2\left(T^{\beta}_r, \ldots\right)$, respectively.

(ii) The proof for the maximum parallel case is analogous.

(iii) According to Definition 10, if T^{α}, T^{β} are serial in an SP mental architecture, there is a subnetwork of the form $\mathrm{SP}^1\left(T^{\alpha}, \ldots\right) + \mathrm{SP}^2\left(T^{\beta}, \ldots\right)$. Let us denote the overall duration for this subnetwork T^s. We have

$$\begin{aligned} -T^s_{11r} + T^s_{12r} + T^s_{21r} - T^s_{22r} = &-\left(\mathrm{SP}^1\left(T^{\alpha}_{1r}, \ldots\right) + \mathrm{SP}^2\left(T^{\beta}_{1r}, \ldots\right)\right) \\ &+ \left(\mathrm{SP}^1\left(T^{\alpha}_{1r}, \ldots\right) + \mathrm{SP}^2\left(T^{\beta}_{2r}, \ldots\right)\right) + \left(\mathrm{SP}^1\left(T^{\alpha}_{2r}, \ldots\right) + \mathrm{SP}^2\left(T^{\beta}_{1r}, \ldots\right)\right) \\ &- \left(\mathrm{SP}^1\left(T^{\alpha}_{2r}, \ldots\right) + \mathrm{SP}^2\left(T^{\beta}_{2r}, \ldots\right)\right) = 0. \end{aligned}$$

The entire SP mental architecture $\mathrm{SP}\left(T^{\alpha}, T^{\beta}, \ldots\right)$ can be represented as either

$$\left(\mathrm{SP}^1(T^{\alpha}, \ldots) + \mathrm{SP}^2(T^{\beta}, \ldots)\right) \wedge T' + T'' \tag{22}$$

or

$$\left(\mathrm{SP}^1(T^{\alpha}, \ldots) + \mathrm{SP}^2(T^{\beta}, \ldots)\right) \vee T' + T''. \tag{23}$$

The overall duration for the entire SP mental architecture is $T = T^s \wedge T' + T''$ for case (22). We have for this case

$$-T_{11r} + T_{12r} + T_{21r} - T_{22r} = -T^s_{11r} \wedge T' + T^s_{12r} \wedge T' + T^s_{21r} \wedge T' - T^s_{22r} \wedge T'.$$

Without loss of generality, assuming $T^s_{12r} \leq T^s_{21r}$, the above expression equals

$$\begin{cases} 0 & \text{if } T' < T^s_{11r} \\ -T^s_{11r} + T' & \text{if } T^s_{11r} \leq T' < T^s_{12r} \\ -T^s_{11r} + T^s_{12r} & \text{if } T^s_{12r} \leq T' < T^s_{21r} \\ -T^s_{11r} + T^s_{12r} + T^s_{21r} - T' & \text{if } T^s_{21r} \leq T' < T^s_{22r} \\ -T^s_{11r} + T^s_{12r} + T^s_{21r} - T^s_{22r} & \text{if } T' \geq T^s_{22r}. \end{cases}$$

The nonnegativity of the first three expressions is obvious, the fifth one is zero, and the fourth expression is larger than the fifth because $T' < T^s_{22r}$. Hence $-T_{11r} + T_{12r} + T_{21r} - T_{22r} \geq 0$. By Lemma 16, $C_r(0, t) \geq 0$ for any t for case (22). The proof for $C_r(t, \infty) \leq 0$ for case (23) is analogous. □

Corollary 18. *(i) If T^α and T^β in an SP mental architecture are minimum parallel, then $C(t) \leq 0$ for any t; (ii) if T^α and T^β in an SP mental architecture are maximum parallel, then $C(t) \geq 0$ for any t; (iii) if T^α and T^β in an SP mental architecture are serial, then either $C(0, t) \geq 0$ for any t, or $C(t, \infty) \leq 0$ for any t.*

If the serial T^α, T^β are in an homogeneous SP mental architecture, the statement of theorem can be made more specific.

Theorem 19. *If T^α and T^β are serial in an* $\mathrm{SP}_\wedge$ *mental architecture, then $C(0, t) \geq 0$ for any t; if T^α and T^β are serial in an* $\mathrm{SP}_\vee$ *mental architecture, then $C(t, \infty) \leq 0$ for any t.*

Simple SP Mental Architectures of Size n

Now let us consider simple SP mental architectures of n processes $X^1, \ldots, X^n$, whose durations are $T^1, \ldots, T^n$. The overall duration of this architecture T can be considered a function of $T^1, \ldots, T^n$. Suppose $T^1, \ldots, T^n$ are selectively influenced by the external factors $\lambda^1, \ldots, \lambda^n$, respectively: $(T^1, \ldots, T^n) \looparrowleft (\lambda^1, \ldots, \lambda^n)$. Suppose in addition each factor has two levels: $\lambda^k \in \{\lambda_1^k, \lambda_2^k\}, k \in \{1, \ldots, n\}$. Denote $T_{i_1 \ldots i_n}$ and $T_{i_1 \ldots i_n}^k$ the overall duration for the entire mental architecture and the duration for process X^k, respectively, given the treatment $(\lambda_{i_1}^1, \ldots, \lambda_{i_n}^n), i_1, \ldots, i_n \in \{1, 2\}$. According to the assumption of selective influences, $T_{i_1 \ldots i_n}^k$ is independent of factors other than $\lambda_{i_k}^k$. We therefore can write $T_{i_1 \ldots i_n}^k = T_{i_k}^k$.

Yang et al. (2014) generalized the 2nd order interaction contrast (15) to the n-th order interaction contrast. It can be expressed as

$$C^{(n)}(t) = \sum_{i_1, \ldots, i_n} (-1)^{n + \sum_{k=1}^n i_k} \Pr(T_{i_1 \ldots i_n} \leq t). \tag{24}$$

They used the idea similar to that proposed by Dzhafarov et al. (2004): By setting the common randomness Θ in Definition 1 to a particular value θ, $(T^1, \ldots, T^n)$ become independent and the patterns of $C^{(n)}(t)$ conditioned on $\Theta = \theta$ for the three simple SP mental architectures $\wedge(T^1, \ldots, T^n)$, $\vee(T^1, \ldots, T^n)$, and $T^1 + \ldots + T^n$ were proved distinct. The patterns still hold in the unconditional case.

However, as mentioned earlier, this approach is limited by certain auxiliary assumptions. Similarly to what was discussed in the previous sections, Zhang and Dzhafarov (2015) fixed the random entity $R = r$, then $T_{i_1}^1, \ldots, T_{i_n}^n$ and $T_{i_1 \ldots i_n}$ reduce to numbers, written as $T_{i_1 r}^1, \ldots, T_{i_n r}^n$ and $T_{i_1 \ldots i_n r}$. Consequently the distribution function $\Pr(T_{i_1 \ldots i_n} \leq t)$ is a (shifted) Heaviside step function:

$$H_{i_1 \ldots i_n r}(t) = \begin{cases} 0, \; if \; t < T_{i_1 \ldots i_n r} \\ 1, \; if \; t \geq T_{i_1 \ldots i_n r} \end{cases}.$$

The distribution function for the unconditional $T_{i_1 \dots i_n}$ is

$$H_{i_1 \dots i_n}(t) = \int_{\mathcal{R}} H_{i_1 \dots i_n r}(t)\, d\mu_r.$$

The n-th order $C^{(n)}(t)$ conditioned on $R = r$ can be written as:

$$C_r^{(n)}(t) = \sum_{i_1, \dots, i_n} (-1)^{n + \sum_{k=1}^{n} i_k} H_{i_1 \dots i_n r}(t).$$

Thus,

$$C_r^{(1)}(t) = \sum_{i_1} (-1)^{1+i_1} H_{i_1 r}(t) = H_{1r}(t) - H_{2r}(t),$$

$$\begin{aligned} C_r^{(2)}(t) &= \sum_{i_1, i_2} (-1)^{2+i_1+i_2} H_{i_1 i_2 r}(t) \\ &= H_{11r}(t) - H_{12r}(t) - H_{21r}(t) + H_{22r}(t), \end{aligned}$$

$$\begin{aligned} C_r^{(3)}(t) &= \sum_{i_1, i_2, i_3} (-1)^{3+i_1+i_2+i_3} H_{i_1 i_2 i_3 r}(t) \\ &= H_{111r}(t) - H_{112r}(t) - H_{121r}(t) - H_{211r}(t) \\ &+ H_{122r}(t) + H_{212r}(t) + H_{221r}(t) - H_{222r}(t), \end{aligned}$$

etc. Hence $C^{(n)}(t)$ (24) can be written as

$$\begin{aligned} C^{(n)}(t) &= \int_{\mathcal{R}} C_r^{(n)}(t)\, d\mu_r \\ &= \int_{\mathcal{R}} \sum_{i_1, \dots, i_n} (-1)^{n + \sum_{k=1}^{n} i_k} H_{i_1 \dots i_n r}(t)\, d\mu_r \\ &= \sum_{i_1, \dots, i_n} (-1)^{n + \sum_{k=1}^{n} i_k} H_{i_1 \dots i_n}(t) \end{aligned}$$

We define the n-th order cumulative contrast conditioned on $R = r$:

$$C_r^{[n]}(0,t) = \int_0^t \left(\int_0^{t_1} \ldots \int_0^{t_{n-2}} C_r^{(n)}(t_{n-1})\, dt_{n-1} \ldots dt_2 \right) dt_1,$$

$$C_r^{[n]}(t,\infty) = \int_t^\infty \left(\int_{t_1}^\infty \ldots \int_{t_{n-2}}^\infty C_r^{(n)}(t_{n-1})\, dt_{n-1} \ldots dt_2 \right) dt_1.$$

Thus,

$$C_r^{[1]}(0,t) = C_r^{[1]}(t,\infty) = H_{1r}(t) - H_{2r}(t),$$

$$C_r^{[2]}(0,t) = \int_0^t C_r^{(2)}(t_1)\, dt_1,$$

$$C_r^{[2]}(t,\infty) = \int_t^\infty C_r^{(2)}(t_1)\, dt_1,$$

$$C_r^{[3]}(0,t) = \int_0^t \int_0^{t_1} C_r^{(3)}(t_2)\, dt_2 dt_1,$$

$$C_r^{[3]}(t,\infty) = \int_t^\infty \int_{t_1}^\infty C_r^{(3)}(t_2)\, dt_2 dt_1,$$

etc. Hence the n-th order unconditional cumulative interaction contrast is written as

$$C^{[n]}(0,t) = \int_0^t C_r^{[n]}(0,t)\, d\mu_r,$$

$$C^{[n]}(t,\infty) = \int_t^\infty C_r^{[n]}(t,\infty)\, d\mu_r.$$

We also denote

$$C_{i_w r}^{(n-1)}(t) = \sum_{i_1,\dots,i_{w-1},i_{w+1},\dots,i_n} (-1)^{n-1-i_w+\sum_{k=1}^{n} i_k} H_{i_1\dots i_{w-1} i_w i_{w+1}\dots i_n r}(t), \tag{25}$$

where $w \in \{1, \dots, n\}$ and i_w is fixed at 1 or 2. We then can write

$$C_r^{[1]}(0,t) = C_r^{[1]}(t,\infty) = H_{1r}(t) - H_{2r}(t), \tag{26}$$

$$\begin{aligned} C_r^{[2]}(0,t) &= \int_0^t C_r^{(2)}(t_1)\,dt_1 \\ &= \int_0^t (H_{11r}(t_1) - H_{12r}(t_1) - H_{21r}(t_1) + H_{22r}(t_1))\,dt_1 \\ &= \int_0^t \left[C_{i_w=1,r}^{[1]}(t_1) - C_{i_w=2,r}^{[1]}(t_1)\right] dt_1, \end{aligned}$$

$$\begin{aligned} C_r^{[2]}(t,\infty) &= \int_t^\infty C_r^{(2)}(t_1)\,dt_1 \\ &= \int_t^\infty (H_{11r}(t_1) - H_{12r}(t_1) - H_{21r}(t_1) + H_{22r}(t_1))\,dt_1 \\ &= \int_t^\infty \left[C_{i_w=1,r}^{[1]}(t_1) - C_{i_w=2,r}^{[1]}(t_1)\right] dt_1, \end{aligned}$$

$$\begin{aligned} C_r^{[3]}(0,t) &= \int_0^t \int_0^{t_1} C_r^{(3)}(t_2)\,dt_2 dt_1 \\ &= \int_0^t \int_0^{t_1} \left[C_{i_w=1,r}^{(2)}(t_2) - C_{i_w=2,r}^{(2)}(t_2)\right] dt_2 dt_1 \\ &= \int_0^t \left[\int_0^{t_1} C_{i_w=1,r}^{(2)}(t_2)\,dt_2 - \int_0^{t_1} C_{i_w=2,r}^{(2)}(t_2)\,dt_2\right] dt_1 \\ &= \int_0^t \left[C_{i_w=1,r}^{[2]}(0,t_1) - C_{i_w=2,r}^{[2]}(0,t_1)\right] dt_1, \end{aligned}$$

$$\begin{aligned}
C_r^{[3]}(t,\infty) &= \int_t^\infty \int_{t_1}^\infty C_r^{(3)}(t_2)\, dt_2 dt_1 \\
&= \int_t^\infty \int_{t_1}^\infty \left[C_{i_w=1,r}^{(2)}(t_2) - C_{i_w=2,r}^{(2)}(t_2) \right] dt_2 dt_1 \\
&= \int_t^\infty \left[\int_{t_1}^\infty C_{i_w=1,r}^{(2)}(t_2)\, dt_2 - \int_{t_1}^\infty C_{i_w=2,r}^{(2)}(t_2)\, dt_2 \right] dt_1 \\
&= \int_t^\infty \left[C_{i_w=1,r}^{[2]}(t_1,\infty) - C_{i_w=2,r}^{[2]}(t_1,\infty) \right] dt_1,
\end{aligned}$$

and generally, the n-th order cumulative interaction contrast conditioned on $R = r$ can be written as,

$$C_r^{[n]}(0,t) = \int_0^t C_{i_w=1,r}^{[n-1]}(0,t)\, dt - \int_0^t C_{i_w=2,r}^{[n-1]}(0,t)\, dt, \tag{27}$$

$$C_r^{[n]}(t,\infty) = \int_t^\infty C_{i_w=1,r}^{[n-1]}(t,\infty)\, dt - \int_t^\infty C_{i_w=2,r}^{[n-1]}(t,\infty)\, dt. \tag{28}$$

The conditional prolongation assumption is again made: For any choice $R = r$,

$$T_{1r}^k \leq T_{2r}^k. \tag{29}$$

Theorem 20. *(i) If $T = T^1 \wedge \ldots \wedge T^n$, $C_r^{(n)}(t) \leq 0$ if n is even and $C_r^{(n)}(t) \geq 0$ if n is odd; (ii) if $T = T^1 \vee \ldots \vee T^n$, $C_r^{(n)}(t) \geq 0$; (iii) if $T = T^1 + \ldots + T^n$, $C_r^{[n]}(0,t) \geq 0$, and $C_r^{[n]}(t,\infty) \leq 0$ if n is even and $C_r^{[n]}(t,\infty) \geq 0$ if n is odd; additionally,* $\lim_{t\to\infty} C_r^{[n]}(0,t) = \lim_{t\to 0} C_r^{[n]}(t,\infty) = 0$.

Proof. (i) By induction on n, the statement for $T = T^1 \wedge \ldots \wedge T^n$ is true when $n = 1$ according to the prolongation assumption (29):

$$C_r^{(1)}(t) = H_{1r}(t) - H_{2r}(t) \geq 0.$$

Let the statement be true up to $C_r^{(n-1)}(t)$, $n-1 \geq 1$. Let

$$T_{1r}^1 \wedge T_{1r}^2 \wedge \ldots \wedge T_{1r}^n = T_{1r}^v,$$

where $1 \leq v \leq n$. We then have for any values of $i_1 ... i_{v-1}, i_{v+1} \ldots i_n$,

$$T_{i_1 \ldots i_{v-1} 1 i_{v+1} \ldots i_n r} = T_{1r}^1 \wedge \ldots \wedge T_{i_{v-1} r}^{v-1} \wedge T_{1r}^v \wedge T_{i_{v+1} r}^{v+1} \wedge \ldots \wedge T_{i_n r}^n = T_{1r}^v.$$

Consequently

$$H_{i_1 i_2 \ldots i_{v-1} 1 i_{v+1} \ldots i_n r}(t) = \begin{cases} 0, \ if \ t < T_{1r}^v \\ 1, \ if \ t \geq T_{1r}^v \end{cases}.$$

Therefore $C_{i_v=1,r}^{(n-1)}(t) = 0$, and

$$C_r^{(n)}(t) = C_{i_v=1,r}^{(n-1)}(t) - C_{i_v=2,r}^{(n-1)}(t) = -C_{i_v=2,r}^{(n-1)}(t) = \begin{cases} \leq 0, \ \text{if } n \text{ is even} \\ \geq 0, \ \text{if } n \text{ is odd} \end{cases}.$$

(ii) For $T = T^1 \vee \ldots \vee T^n$, by induction on n, the case $n = 1$ is true by the prolongation assumption:

$$C_r^{(1)}(t) = H_{1r}(t) - H_{2r}(t) \geq 0.$$

Let the statement be true up to $C_r^{(n-1)}(t)$, where $n-1 \geq 1$. Let

$$T_{2r}^1 \vee T_{2r}^2 \vee \ldots \vee T_{2r}^n = T_{2r}^m,$$

where $1 \leq m \leq n$. We then have

$$T_{i_1 i_2 \ldots i_{m-1} 2 i_{m+1} \ldots i_n r} = T_{2r}^m,$$

and

$$H_{i_1 \ldots i_{m-1} 2 i_{m+1} \ldots i_n r}(t) = \begin{cases} 0, \ if \ t < T_{2r}^m \\ 1, \ if \ t \geq T_{2r}^m \end{cases},$$

for any $i_1 \ldots i_{m-1}, i_{m+1} \ldots i_n$. Then $C_{i_m=2,r}^{(n-1)}(t) = 0$, and

$$C_r^{(n)}(t) = C_{i_m=1,r}^{(n-1)}(t) - C_{i_m=2,r}^{(n-1)}(t) = C_{i_m=1,r}^{(n-1)}(t) \geq 0.$$

(iii) For $T = T^1 + \ldots + T^n$, by induction on n, the case $n = 1$ is true by the prolongation assumption:

$$C_r^{[1]}(0,t) = C_r^{[1]}(t,\infty) = H_{1r}(t) - H_{2r}(t) \geq 0,$$

and

$$\lim_{t \to \infty} C_r^{[1]}(0,t) = \lim_{t \to 0} C_r^{[1]}(t,\infty) = 0.$$

Let the statement be true up to $n - 1 \geq 1$. We have

$$\begin{aligned} C_r^{[n]}(0,t) &= \int_0^t C_{i_w=1,r}^{[n-1]}(0,t)\,dt - \int_0^t C_{i_w=2,r}^{[n-1]}(0,t)\,dt \\ &= \int_0^{t-T_{1r}^w} C_r^{[n-1]}(0,t)\,dt - \int_0^{t-T_{2r}^w} C_r^{[n-1]}(0,t)\,dt \\ &= \int_{t-T_{2r}^w}^{t-T_{1r}^w} C_r^{[n-1]}(0,t)\,dt, \end{aligned} \tag{30}$$

which is ≥ 0 since $C_r^{[n-1]}(0,t) \geq 0$ and $t - T_{2r}^w \leq t - T_1^w$. Analogously,

$$\begin{aligned} C_t^{[n]}(t,\infty) &= \int_t^\infty C_{i_w=1,r}^{[n-1]}(t,\infty)\,dt - \int_t^\infty C_{i_w=2,r}^{[n-1]}(t,\infty)\,dt \\ &= \int_{t-T_{1r}^w}^\infty C_r^{[n-1]}(t,\infty)\,dt - \int_{t-T_{2r}^w}^\infty C_r^{[n-1]}(t,\infty)\,dt \\ &= -\int_{t-T_{2r}^w}^{t-T_{1r}^w} C_r^{[n-1]}(t,\infty)\,dt, \end{aligned} \tag{31}$$

which is ≤ 0 if n is even and ≥ 0 if n is odd. Applying the mean value theorem to the results of (30) and (31), we get, for some $t - T_{2r}^w < t', t'' < t - T_{1r}^w$

$$\lim_{t\to\infty} \int_{t-T_{2r}^w}^{t-T_{1r}^w} C_r^{[n-1]}(0,t)\,dt = \lim_{t\to\infty} C_r^{[n-1]}(0,t')\left(-T_{1r}^w + T_{2r}^w\right),$$

$$\lim_{t\to 0} \int_{t-T_{2r}^w}^{t-T_{1r}^w} C_r^{[n-1]}(t,\infty)\,dt = \lim_{t\to 0} C_r^{[n-1]}(t'',\infty)\left(-T_{1r}^w + T_{2r}^w\right).$$

$t \to \infty$ implies $t' \to \infty$ and $t \to 0$ implies $t'' \to 0$. Both expressions tend to zero since $C_r^{[n-1]}(0,\infty) = 0$ $\square$

Corollary 21. *(i) If* $T = T^1 \wedge \ldots \wedge T^n$, $C^{(n)}(t) \leq 0$ *if* n *is even and* $C^{(n)}(t) \geq 0$ *if* n *is odd; (ii) if* $T = T^1 \vee \ldots \vee T^n$, $C^{(n)}(t) \geq 0$*; (iii) if* $T = T^1 + \ldots + T^n$,

$C^{[n]}(0,t) \geq 0$, and $C^{[n]}(t,\infty) \leq 0$ if n is even and $C^{[n]}(t,\infty) \geq 0$ if n is odd; additionally, $\lim_{t\to\infty} C^{[n]}(0,t) = \lim_{t\to 0} C^{[n]}\ (t,\infty) = 0$.

Multiple Processes in an Arbitrary SP Network

Now let us consider multiple processes $X^1,\ldots,X^n$ in an arbitrary SP mental architecture. The overall duration of this SP architecture T can be considered a function of the durations of processes $T^1,\ldots,T^n$ and other duration components, written as $T = \mathrm{SP}(T^1,\ldots,T^n,\ldots)$. Suppose $T^1,\ldots,T^n$ and other components are selectively influenced by the external factors $\lambda^1,\ldots,\lambda^n$ and the empty set, respectively: $(T^1,\ldots,T^n,...) \looparrowleft (\lambda^1,\ldots,\lambda^n,\emptyset)$. Using the same notation as in the previous section, $T_{i_1\ldots i_n}$ and $T^k_{i_k}, k \in \{1,\ldots,n\}$ denote the overall duration and the duration of process X^k, respectively, given the treatment $(\lambda^1_{i_1},\ldots,\lambda^n_{i_n}), i_1,\ldots,i_n \in \{1,2\}$. The prolongation assumption (29) is imposed in the system as well.

Definition 22. $T^1,\ldots,T^n$ in an SP mental architecture are *minimum parallel* if there is a subnetwork of the form $\mathrm{SP}^1(T^1,\ldots) \wedge \ldots \wedge \mathrm{SP}^n(T^n,\ldots)$ or *maximum parallel* if there is a subnetwork of the form $\mathrm{SP}^1(T^1,\ldots) \vee \ldots \vee \mathrm{SP}^n(T^n,\ldots)$ or *serial* if there is a subnetwork of the form $\mathrm{SP}^1(T^1,\ldots) + \ldots + \mathrm{SP}^n(T^n,\ldots)$.

Lemma 23. *If $T^1,\ldots,T^n$ are all minimum parallel in an SP mental architecture, then this architecture can be represented as* $\mathrm{SP}(T^1,\ldots,T^n,\ldots) = \mathrm{SP}^1(T^1,\ldots) \wedge \ldots \wedge \mathrm{SP}^n(T^n,\ldots)$*; if they are arranged in a maximum parallel way, then* $\mathrm{SP}(T^1,\ldots,T^n,\ldots) = \mathrm{SP}^1(T^1,\ldots) \vee \ldots \vee \mathrm{SP}^n(T^n,\ldots)$.

Proof. For the minimum parallel case, similar to the proof in Lemma 11, we write the SP mental architecture as (i) $\mathrm{SP}^1(T^1,\ldots)\wedge\mathrm{SP}^2(T^2,\ldots,T^n,\ldots)$ or (ii) $(\mathrm{SP}^1(T^1,\ldots)\wedge\mathrm{SP}^2(T^2,\ldots,T^n,\ldots)+T')\Diamond T''$ or (iii) $(\mathrm{SP}^1(T^1,\ldots)\wedge\mathrm{SP}^2(T^2,\ldots,T^n,\ldots)\Diamond T')+T''$, where T' and T'' are durations of certain subnetworks. We observe that (ii) and (iii) can be rewritten as the form of (i): (ii) $= \big(\mathrm{SP}^1(T^1,\ldots)+T'\big)\wedge((\mathrm{SP}^2(T^2,\ldots,T^n,\ldots)+$

$T')\Diamond T'')$ and (iii) $= (\mathrm{SP}^1(T^1,\ldots) + T'') \wedge (\mathrm{SP}^2(T^2,\ldots,T^n,\ldots)\Diamond T' + T'')$. We then decompose $\mathrm{SP}^2(T^2,\ldots,T^n,\ldots)$ achieving $\mathrm{SP}^1(T^1,\ldots)\wedge\mathrm{SP}^2(T^2,\ldots)\wedge\mathrm{SP}^3(T^3,\ldots,T^n,$ $\ldots)$ and carry on this manner until the required $\mathrm{SP}^1\left(T^1,\ldots\right) \wedge \ldots \wedge \mathrm{SP}^n\left(T^n,\ldots\right)$ is obtained. The proof for the maximum parallel case is analogous. $\square$

Observe that if multiple durations $T^1,\ldots,T^n$ are arranged in a sequence in $\mathrm{SP}(T^1,$ $\ldots, T^n,\ldots)$, this SP mental architecture generally cannot be represented as $\mathrm{SP}^1(T^1,$ $\ldots) + \ldots + \mathrm{SP}^n\left(T^n,\ldots\right)$.

Lemma 24. *If $T^1,\ldots,T^n$ are arranged in parallel or in a sequence in an SP mental architecture, $\left(\mathrm{SP}^1\left(T^1,\ldots\right),\ldots,\mathrm{SP}^n\left(T^n,\ldots\right)\right) \looparrowleft \left(\lambda^1,\ldots,\lambda^n\right)$ and, for any fixed $R = r$, the following version of the prolongation assumption holds:* $\mathrm{SP}^k\left(T^k_{1r},\ldots\right) \leq \mathrm{SP}^k\left(T^k_{2r},\ldots\right), k \in \{1,\ldots,n\}$.

Proof. According to Definition 22, $\left(\mathrm{SP}^1\left(T^1,\ldots\right),\ldots,\mathrm{SP}^n\left(T^n,\ldots\right)\right) \looparrowleft \left(\lambda^1,\ldots,\lambda^n\right)$ is obvious. Fixing $R = r$, by the prolongation assumption (29) and (nonstrict) monotonicity of SP mental architectures, $\mathrm{SP}^k\left(T^k_{1r},\ldots\right) \leq \mathrm{SP}^k\left(T^k_{2r},\ldots\right)$, $k \in \{1,\ldots,$ $n\}$ is apparent. $\square$

The statement of Theorem 20 can be generalized to multiple parallel processes in an SP mental architecture but there is no extension for the multiple serial processes.

Theorem 25. *If $T^1,\ldots,T^n$ are minimum parallel in an* SP *mental architecture, then for any r,t, $C_r^{(n)}(t) \leq 0$ if n is even and $C_r^{(n)}(t) \geq 0$ if n is odd; if $T^1,\ldots,T^n$ are maximum parallel in an* SP *mental architecture, then for any r,t, $C_r^{(n)}(t) \geq 0$.*

Proof. Follows from Lemma 23, 24 and statements (i), (ii) of Theorem 20. $\square$

Corollary 26. *If $T^1,\ldots,T^n$ are minimum parallel in an* SP *mental architecture, then for any t, $C^{(n)}(t) \leq 0$ if n is even and $C^{(n)}(t) \geq 0$ if n is odd; if $T^1,\ldots,T^n$ are maximum parallel in an* SP *mental architecture, then for any r,t, $C^{(n)}(t) \geq 0$.*

Conclusions

We have demonstrated a new way to characterize different types of mental architectures. According to the assumption of selective influences, one can reduce the components in the network from random variables to deterministic values by conditioning the common source of randomness R on a fixed value. The interaction contrast of distribution functions is then equivalent to a linear combination of shifted Heaviside functions that involve only 0's and 1's at every time moment. This method simplifies the arithmetic compared to the traditional approach. By using this method we presented the proofs of the known results for two-process and multiple-process mental architectures. We also characterized two processes and multiple processes in the SP mental architectures. We expect that this method can be extended to investigate more complex networks.

Diagnosing Mental Architectures Implemented in Psychophysical Experiments

The interaction contrast has been widely used to investigate mental architectures implemented in various cognitive tasks, such as the simple detection task (Townsend & Nozawa, 1995), Stroop task (Eidels, Townsend, & Algom, 2010), Gestalt principles (Eidels, Townsend, & Pomerantz, 2008), visual search (Fific, Townsend, & Eidels, 2008; Sung, 2008), short term memory search (Townsend & Fific, 2004), face perception (Fific & Townsend, 2010; Wenger & Townsend, 2001), and even in the clinical domain (Johnson, Blaha, Houpt, & Townsend, 2010). As explained above, however, one has to impose several assumptions on the system when using the technique of interaction contrast to investigate mental architectures. Some of these assumptions are untestable when separately taken.

Assumptions

Assumption 1: selective influences. If one considers only two processes, it is assumed that T^α is selectively influenced by the factor α and T^β is selectively influenced by the factor β:

$$(T^\alpha, T^\beta) \looparrowleft (\alpha, \beta). \tag{32}$$

To establish selective influences of α and β on T^α and T^β, one has to know the distributions of T^α and T^β. In psychological research, usually the overall duration T can be measured but T^α and T^β are unobservable. So as a rule the assumption of selective influences separately taken cannot be tested.

Assumption 2: stochastic dominance. The assumption of stochastic dominance (16) states that the distribution of T^α at level one dominates that of level two and the distribution of T^β at level one dominates that of level two. It follows from the prolongation assumption (17) in our treatment. With the assumption of selective influences (32), stochastic dominance implies the four inequalities:

$$\begin{aligned}
\Pr(T_{11} \le t) &\ge \Pr(T_{12} \le t), \\
\Pr(T_{11} \le t) &\ge \Pr(T_{21} \le t), \\
\Pr(T_{12} \le t) &\ge \Pr(T_{22} \le t), \\
\Pr(T_{21} \le t) &\ge \Pr(T_{22} \le t).
\end{aligned} \tag{33}$$

The inequalities state that the distribution of T_{11} dominates that of T_{12} and T_{21}. The distributions of T_{22} is dominated by T_{12} and T_{21}. These four distributions are observable.

Note that the conjunction of (16) and (32) is a sufficient but not necessary condition for (33). If (33) is confirmed, it corroborates (16) and (32) but does not guaranteed them. If (33) is violated, then either (16) or (32), or both, are violated.

Assumption 3: only a single type of mental architecture used from trial to trial. Though this assumption is not explicitly stated in the literature, it is implicitly imposed on the investigated systems. This assumption could be invalid since a person may implement a maximum parallel arrangement in one experimental trial and switch to a serial one in another trial. In psychological research, it is usually impossible to track the mental architectures in each trial. So this assumption is again not testable when taken separately.

In addition to using these untestable assumptions, all the earlier studies on mental architectures focused on the tasks with short response times: the participants made a response within one second or so. In our study, it took the participants several seconds to make a response. Therefore the current study investigated mental architectures in a more complex situation, which broadens the application of mental architectures.

The study conducted in our lab involved two psychophysical tasks: the dot position reproduction task and the floral shape reproduction task (with minor modifications as compared to the experiments reported in Chapter 1). Here we are using the same notation as in Chapter 1: α and β denote the coordinates of the target dots or the amplitudes of the target shapes. A and B represent the eventual responses to α and β, that is, the coordinates of the reproduced dots or the amplitudes of of the reporduced shapes. We label T^{α} the time to produce A and T^{β} the time to produce B. Factorial subdesigns were extracted from the experiments, so that the interaction contrasts could be computed. We investigated the manner of the trackball movement (parallel or serial) when the geometric stimuli were being reproduced. This experimental paradigm allows us to test the assumptions about processes that are usually unobservable in other paradigms.

Experiments

Three paid volunteers (P6, P7, and P8) participated in Experiment 1(c) and Experiment 3(c). All the participants were students at Purdue University, aged around 30 with normal or corrected to normal vision.

Experiment 1(c). The stimuli and procedure were identical to Experiment 1(b) except that the movable dot on the bottom right located initially in the center of the circle and the program tracked the movement of the trackball by recording the rectangular coordinates and the polar coordinates of the moving dot every 10 ms in every trial.

Experiment 3(c). The stimuli and procedure were identical to Experiment 3(b) except that the program tracked the movement of the trackball by recording the amplitudes of the changing floral shape every 10 ms in every trial.

Results

By using the computations of interaction contrasts, one can investigate whether the processes are arranged in parallel or in a sequence in the experiments. In our experiment, in addition, one can directly learn the process arrangements by plotting and analyzing the trackball movement data. It was expected that the conclusions from these two lines of analysis would agree with each other. We also expected that the minimum parallel arrangement was not chosen by any participant in the tasks since in order to match a given stimuli both coordinates or both amplitudes had to be set at “correct” values, not just one of them.

Analysis of the typical trackball movements.

Experiment 1(c). Figure 11 plots the trackball movements in a typical trial for each participant in Experiment 1(c). The positions of the moving dot are plotted every 50 ms. The upper figure shows a trackball movement in rectangular coordinates, and the bottom one is the same movement represented by the polar coordinates. The red dot represents the position of the fixed target dot. The movement started from (0 px, 0 px) or (0 px, 0 deg) and proceeded toward the target position. The final position of the dot was very close to the target position. The movement formally confirms the obvious expectation that the minimum parallel arrangement could not be used by the participants, otherwise the final position of the dot would be close to the target with respect to one coordinate but far with respect to the other coordinate. By observing the trajectory, the horizontal coordinate and the vertical coordinate changed together, in parallel, in most steps in the upper figure. This fact suggests that the horizontal movements and the vertical movements were not arranged in a serial manner. When representing the movement using the polar coordinates, we observe the same parallel changes in the most steps, again excluding a serial arrangement. The trackball movement plots suggests that the maximum parallel arrangement was used by Participants P6, P7, and P8.

Experiment 3(c). Figure 12 plots the trackball movement in a typical trial of Experiment 3(c). The red dot represents the amplitudes of the fixed target shape. After a complex sequence of moves, eventually a shape close to the given shape was reproduced. This plot confirms the obvious expectation that the minimum parallel arrangement could not be used to accomplish the task. It also suggests that a serial arrangement was not used by Participants P6, P7, and P8, as in most steps the two coordinates changed together.

In addition, in both experiments the trackball movement data confirmed Assumption 3 that the participants maintained a stable manner to perform the tasks from trial to trial.

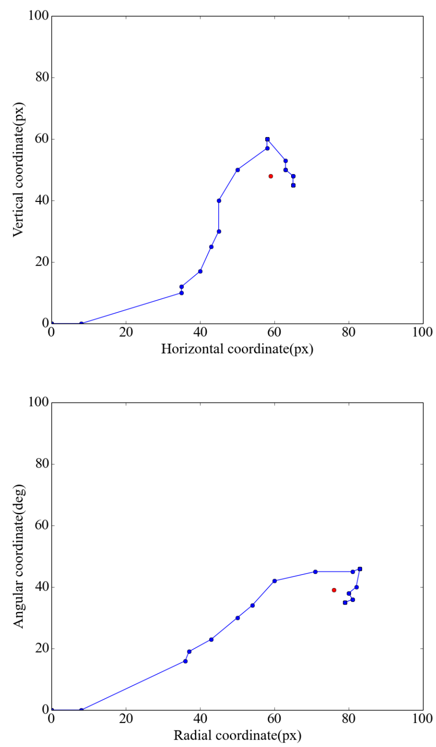

Figure 11. The trackball movement in a typical trial in Experiment 1(c), represented by the rectangular coordinates (upper) and the polar coordinates (bottom).

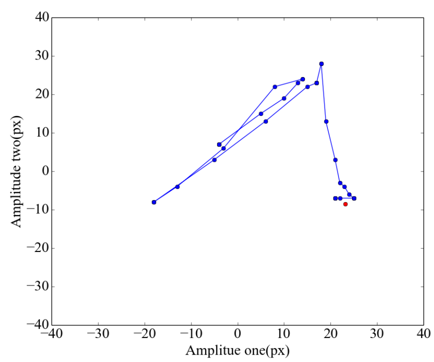

Figure 12. The trackball movement in a typical trial in Experiment 3(c).

Testing selective influences of α and β on T^α and T^β . Selective influences of α and β on T^α and T^β have ideally to be tested before the interaction contrasts are computed. In our experimental paradigm, the two processes were characterized by two properties. One was the physical parameters of the eventual responses, i.e., A and B, and the other was the durations for the processes, i.e., T^α and T^β. We speculated that $(A, B) \looparrowleft (\alpha, \beta)$ is a sufficient (and perhaps also necessary) condition for $(T^\alpha, T^\beta) \looparrowleft (\alpha, \beta)$. In other words, we find it unlikely that $(T^\alpha, T^\beta) \looparrowleft (\alpha, \beta)$ but nevertheless the final outcomes of the two processes, A and B are not selectively influenced by the same factors (and perhaps this is also true in the opposite direction). If this speculation is accepted, we can test $(T^\alpha, T^\beta) \looparrowleft (\alpha, \beta)$ by inspecting whether $(A, B) \looparrowleft (\alpha, \beta)$.

Experiment 1(c). Two 2×2 factorial subdesigns were extracted from Experiment 1(c). Each subdesign contained about 800 data points. One was the rectangular subdesign, in which α and β denote the horizontal coordinate and the vertical coordinate of the given immovable dot:

$$\alpha = \{\alpha_1, \alpha_2\} = \{[60\,\text{px}, 80\,\text{px}]\,, [20\,\text{px}, 40\,\text{px}]\}\,,$$
$$\beta = \{\beta_1, \beta_2\} = \{[60\,\text{px}, 80\,\text{px}]\,, [20\,\text{px}, 40\,\text{px}]\}\,.$$

The other was the polar subdesign, in which α and β denote the radial coordinate and the angular coordinate of the given immovable dot:

$$\alpha = \{\alpha_1, \alpha_2\} = \{[65\,\text{px}, 90\,\text{px}]\,, [40\,\text{px}, 65\,\text{px}]\}\,,$$
$$\beta = \{\beta_1, \beta_2\} = \{[45\,\text{deg}, 60\,\text{deg}]\,, [30\,\text{deg}, 45\,\text{deg}]\}\,.$$

For the rectangular subdesign, A and B denote the horizontal coordinate and the vertical coordinate of the reproduced dot. For the polar subdesign, A and B

denote the radial coordinate and the angular coordinate of the reproduced dot. We computed $(\alpha - A)$ and $(\beta - B)$ for each trial and took large deviations as indicators of outliers. There were less than one percent outliers in each experiment and they were removed. $A_{i_1 i_2}$ and $B_{i_1 i_2}$ denote the coordinates of the reproduced dot for treatment $(\alpha_{i_1}, \beta_{i_2}), i_1, i_2 \in \{1, 2\}$.

Marginal selectivity needs to be tested first. We compared the distributions of $A_{i_1 1}$ with $A_{i_1 2}$ and compared the distributions of $B_{1 i_2}$ with $B_{2 i_2}$. The K-S test for 2 independent samples was used to make the four paired comparisons for each participant. If one of the four paired comparisons was significant ($p < .05$), it was considered an absence of marginal selectivity. Table 30 presents the p values for the paired comparisons. Participant P6 passed the tests of marginal selectivity in both the rectangular subdesign and the polar subdesign ($p \geq .05$). The other participants failed the tests in both designs. The Linear Feasibility Test was then performed on Participant P6. The result supported selective influences of α and β on A and B for this participant. We consider this an indication that selective influences of α and β on T^{α} and T^{β} are established for this participant.

Experiments 3(b) and 3(c). A 2×2 subdesign was extracted from Experiment 3(c). Each treatment contained about 450 data points.

$$\alpha = \{\alpha_1, \alpha_2\} = \{[-30\,\mathrm{px}, 0\,\mathrm{px}], [0\,\mathrm{px}, 30\,\mathrm{px}]\},$$

$$\beta = \{\beta_1, \beta_2\} = \{[-30\,\mathrm{px}, 0\,\mathrm{px}], [0\,\mathrm{px}, 30\,\mathrm{px}]\}.$$

Table 31 presents the test results for marginal selectivity (the outliers were removed in the same way as in Experiment 1(c)). Only Participant P6 passed the test. The Linear Feasibility Test confirmed selective influences of α and β on A and B for this participant. The external factors of Experiment 3(b) was also discretized in the same

Table 30

p Values of the Two Sample K-S Tests for Marginal Selectivity, Experiment 1(c)

		Participant		
		P6	P7	P8
Rect-angular subdesign	A_{11}, A_{12}	.365	.001	.560
	A_{21}, A_{22}	.206	.000	.364
	B_{11}, B_{21}	.120	.000	.002
	B_{12}, B_{22}	.582	.061	.287
Polar subdesign	A_{11}, A_{12}	.570	.039	.641
	A_{21}, A_{22}	.331	.327	.388
	B_{11}, B_{21}	.393	.232	.002
	B_{12}, B_{22}	.204	.343	.004

way and all three participants (P3, P4, and P5) passed the test of selective influences on A and B (see Tables 26, 28, and 29). Therefore we consider selective influences of α and β on T^{α} and T^{β} established for Participants P3, P4, and P5 in Experiment 3(b) and for Participant P6 in Experiment 3(c).

Testing stochastic dominance.

Experiment 1(c). The assumption of stochastic dominance was tested using the inequalities (33) for participant P6. The trials with response time that obviously fell outside the cluster of the other data points were considered outliers and were removed from the test. Two one tail K-S tests were performed for each pair of variables. For instance, in order to test the first equation in (33), we required

$$\max\left(\Pr(T_{11} \leq t) - \Pr(T_{12} \leq t)\right) \geq 0, \tag{34}$$

Table 31

p Values of the Two Sample K-S Tests for Marginal Selectivity, Experiment 3(c)

Pair	Participant P6	Participant P7	Participant P8
A_{11}, A_{12}	.191	.003	.023
A_{21}, A_{22}	.442	.476	.032
B_{11}, B_{21}	.388	.142	.351
B_{12}, B_{22}	.927	.336	.522

and

$$\max\left(\Pr(T_{12} \le t) - \Pr(T_{11} \le t)\right) = 0. \tag{35}$$

Table 32 lists the p values of the one tail K-S tests for Participant P6. The upper number in each cell, for instance .013, is the p value for (34). The bottom number, for instance .995, is the p value for (35). This table indicates that the stochastic dominance assumption held for both designs for this person since the p values in each bottom line were not significant.

Experiments 3(b) and 3(c). Again, trials with outliers were removed from the test. Table 33 shows Participants P3, P4, and P5 in Experiment 3(b) and Participant P6 in Experiment 3(c) passed the stochastic dominance test (Participant P6 passed the test marginally as $p = .011$ for $\max(\Pr(T_{22} \le t) - \Pr(T_{12} \le t))$.

Plotting interaction contrasts.

Experiment 1(c). With the confirmation of the three assumptions, the interaction contrasts (15) were plotted (Figure 13).

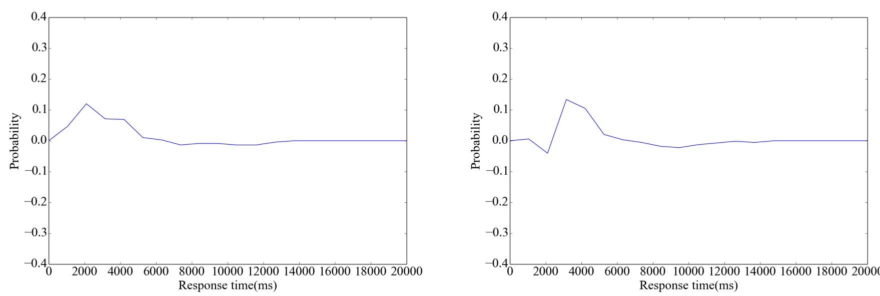

Figure 13. The empirical interaction contrast patterns for the rectangular subdesign (left) and polar subdesign (right) of Experiment 1(c), Participant P6.

Table 32

p Values of the One Tail K-S Tests for Stochastic Dominance in Experiment 1(c), Participant P6

Experimental design	T_{11}, T_{12}	T_{11}, T_{21}	T_{12}, T_{22}	T_{21}, T_{22}
Rectangular	.013	.065	.702	.040
subdesign	.995	.773	.467	.536
Polar	.123	.023	.580	.881
subdesign	.330	.771	.786	.454

Table 33

p Values of the One Tail K-S Tests for Stochastic Dominance in Experiment 3(b) and 3(c)

Participant	T_{11}, T_{12}	T_{11}, T_{21}	T_{12}, T_{22}	T_{21}, T_{22}
P3	.000	.027	.496	.005
	.990	.961	.843	.961
P4	.000	.000	.223	.290
	.997	1	.070	.354
P5	.000	.016	.056	.000
	.979	.991	.176	.841
P6	.000	.000	.721	.208
	.991	.959	.011	.465

The maximum parallel model for both the rectangular subdesign and polar subdesign was confirmed since $C(t) \geq 0$. This conclusion was consistent with that drawn from the trackball movement data.

Experiments 3(b) and 3(c). Figure 14 plots the interaction contrasts for P3, P4, and P5 in Experiment 3(b) and Participant P6 in Experiment 3(c). The patterns indicate that the participants reproduced amplitude one and amplitude two of the floral shape in the maximum parallel manner, except for Participant P5. The interaction contrast pattern of this participant was not consistent with any of the three mental architectures that we considered. The negative part was greater than the positive part indicating that he/she may use the coactive manner to make responses. The coactive arrangement (Townsend & Nozawa, 1995) is a type of parallel processing, which takes the sum of the two parallel processes and a response is made if the sum exceeds some criterion. The four figures confirmed that no one implemented the minimum parallel arrangement, as expected.

Testing stochastic dominance in the absence of $(A, B) \looparrowleft (\alpha, \beta)$. Participants P7 and P8 in Experiments 1(c) and 3(c) have violated $(A, B) \looparrowleft (\alpha, \beta)$. We then tested the stochastic dominance assumption for these two persons and plotted the interaction contrast if the test of stochastic dominance was passed. If the test of stochastic dominance is failed or the pattern of interaction contrast is misleading, it further validates the idea that $(T^\alpha, T^\beta) \looparrowleft (\alpha, \beta)$ can be tested by inspecting whether $(A, B) \looparrowleft (\alpha, \beta)$.

Experiment 1(c). The K-S test (Table 34) indicates that the ordering of response time in Experiment 1(c) passed the test of stochastic dominance for Participants P7 and P8 (P7 in polar subdesign passed the test marginally as $p = .014$ for $\max(\Pr(T_{22} \leq t) - \Pr(T_{21} \leq t))$.

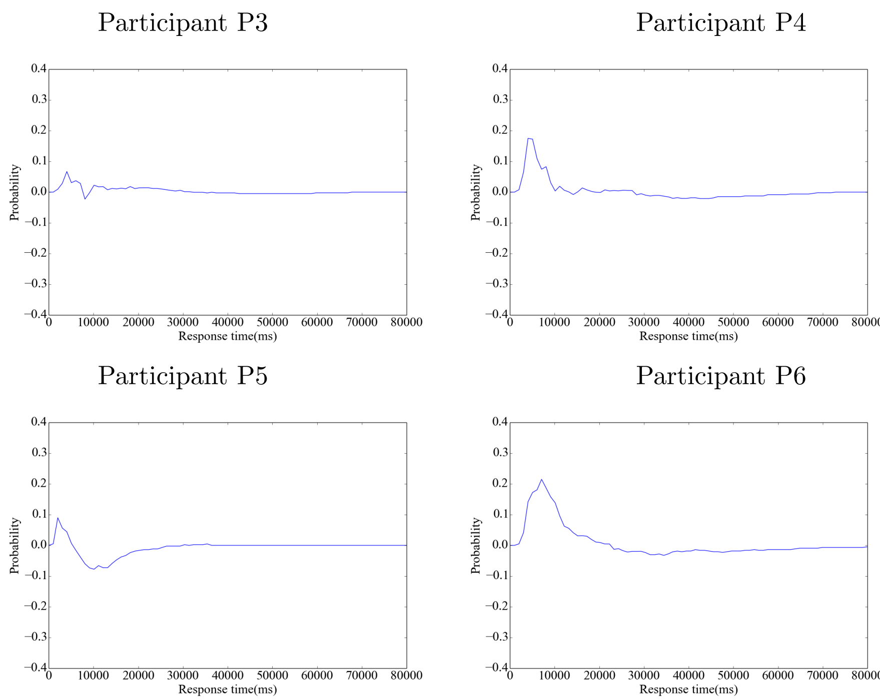

Figure 14. The empirical interaction contrast patterns for Participants P3, P4, and P5 in Experiment 3(b) and Participant P6 in Experiment 3(c).

Table 34

p Values of the One Tail K-S Tests for Stochastic Dominance for Participants P7 and P8, Experiment 1(c)

Participant		T_{11}, T_{12}	T_{11}, T_{21}	T_{12}, T_{22}	T_{21}, T_{22}
P7	Rectangular	.796	.549	0	0
	subdesign	.065	.345	.994	1
	Polar	.563	0	.003	1.000
	subdesign	.312	1.0	.982	.014
P8	Rectangular	0	.066	.168	.002
	subdesign	.988	1.000	.981	.884
	Polar	.593	.377	.046	.006
	subdesign	.505	.201	.942	.451

Some of the interaction contrast patterns in Figure 15 were misleading as $C(t) \leq 0$. Our interpretation is that the misleading patterns are caused by the violations of $(T^{\alpha}, T^{\beta}) \looparrowleft (\alpha, \beta)$.

Experiment 3(c). Participants P7 and P8 in Experiment 3(c) violated (33): Some of the p values in each bottom line of Table 35 were extremely low.

Conclusions

Mental architectures are hypothetical networks that are usually impossible to be observed directly. In the psychophysical experiments developed in our lab, we were able to directly see the process arrangements by tracking changes of the physical parameters of the reproduced stimuli. We also showed that the patterns of interaction contrast resulted in the same diagnosis of process arrangements as the direct

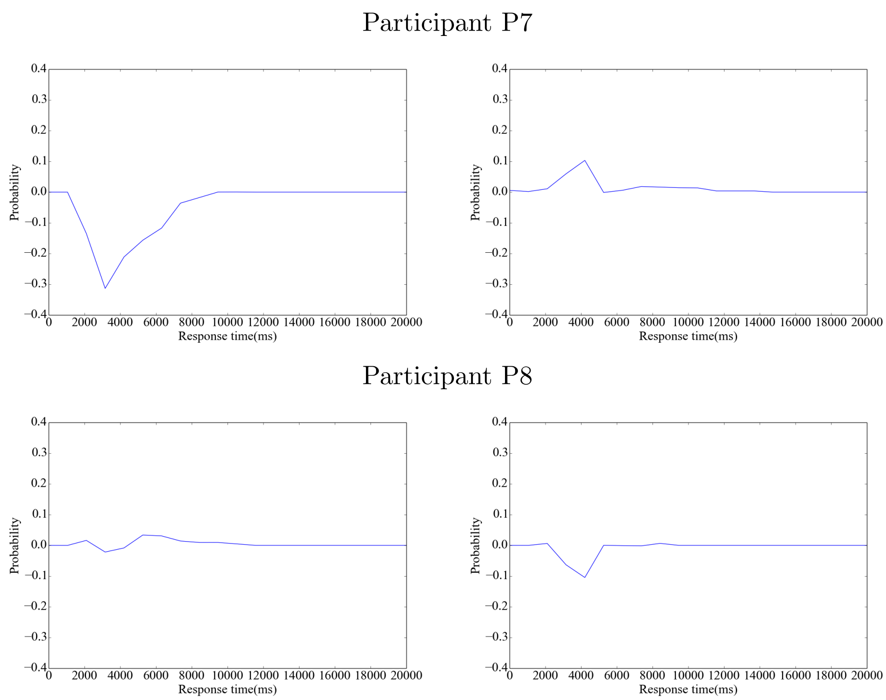

Figure 15. The empirical interaction contrast patterns for Participants P7 and P8 for the rectangular subdesign (left) and polar subdesign (right), Experiment 1(c).

Table 35

p Values of the One Tail K-S Tests for Stochastic Dominance for Participants P7 and P8, Experiment 3(c)

Participant	T_{11}, T_{12}	T_{11}, T_{21}	T_{12}, T_{22}	T_{21}, T_{22}
P7	.000	.000	.829	.558
	1	1	.001	.000
P8	.364	.962	.055	.000
	.071	0	.993	.968

observation of the trackball movements. This experimental paradigm provides support for the view that mental architectures are indeed real rather than imaginary.

Our work demonstrated that the framework of mental architectures can be applied in the tasks that consume longer reaction times than in the traditional experimental paradigms. This allowed us to observe greater complexity of performance than is usually assumed. For instance, in Experiment 1(c), the participants tended to move the dots in the maximum parallel manner in most steps, but it seems that in the final several steps they applied several micro adjustments to the positions of the moving dot in the serial manner. Strictly speaking, the participants implemented both maximum parallel and serial arrangements in a combined fashion in most trials. The exact pattern of the combined arrangement in theory is unclear and further investigation is needed. However, it was generally true that the participants used the maximum parallel arrangement most of the time. This observation was supported by the observed pattern of the interaction contrast. We were concerned in the beginning of our study that the participants may switch the manner of process arrangements from one trial to another. However, it was found that the participants generally maintained a stable way to perform the tasks. If people tend to stay with the same manner to make responses in cognitive tasks, then assumption 3 is justified.

I think that our new experimental design improves the reliability of the technique of interaction contrast. It provides a direct way to examine selective influences of the external factors on the duration components. This direct method was confirmed by observing that when $(A, B) \looparrowleft (\alpha, \beta)$ was established, the test of stochastic dominance (33) was passed and the patterns of interaction contrast behaved as expected; whereas when $(A, B) \looparrowleft (\alpha, \beta)$ was violated, (33) was violated (Table 35) too, or the patterns of interaction contrast could be misleading (Figure 15). Our analysis also demonstrated that taking (33) as a confirmation of $(T^{\alpha}, T^{\beta}) \looparrowleft (\alpha, \beta)$ is risky as a misleading interaction contrast patterns can be obtained even when (33) is satisfied. $(A, B) \looparrowleft (\alpha, \beta)$ seems to be a better indicator of $(T^{\alpha}, T^{\beta}) \looparrowleft (\alpha, \beta)$ than examining the inequalities (33) only.

This method may have broad applications. For instance, one can design similar paradigms to study the process architectures underlying the eye movement tasks or body movement tasks. Of course this method has its limitations: One cannot record physical parameters of mental responses in every experimental paradigm.

CONTEXTUALITY AND APPLICATIONS

A Brief Theoretical Review of Contextuality-by-Default

Recall the system of external factors $(\lambda^1, \ldots, \lambda^n)$ and the random outputs $(X^1, \ldots, X^n)$. Denote $\phi = \left(\lambda^1_{i_1}, \ldots, \lambda^n_{i_n}\right)$ a treatment. The entities in ϕ belong to nonempty sets $(\Lambda^1, \ldots, \Lambda^n)$, respectively, where $\Lambda^k = \{\lambda^k_1, \ldots, \lambda^k_{m_k}\}$, $k \in \{1, \ldots, n\}$. Given a treatment ϕ, the random outputs are written as $(X^1_\phi, \ldots, X^n_\phi)$. We denote the collection of treatments $\{(\lambda^1_1, \ldots, \lambda^n_1), \ldots, \left(\lambda^1_{m_1}, \ldots, \lambda^n_{m_n}\right)\}$ as a set Φ.

The Definition of Contextuality

The formal definition of contextuality is formed according to the idea of "*all-possible-couplings*" (Dzhafarov & Kujala, 2014a, 2014b, 2014c; Dzhafarov et al., 2016; Dzhafarov, et al., 2015; Kujala & Dzhafarov, 2015, 2016; Kujala, et al., 2015). Let us start with notations. Recall that X^k_ϕ denotes the measurement outcome of the external factor λ^k given treatment ϕ. Suppose each random variable X^k_ϕ is discrete. We call $\left\{X^k_\phi\right\}_{\lambda^k_{i_k} \in \phi}$ for every $\phi \in \Phi$ a *bunch*, and we say that the variables belong to the same bunch share a *context*. The joint distribution of the random variables within each bunch is observable. If $\lambda^k_{i_k} \in \phi$, $\lambda^l_{i_l} \in \phi'$, and $\phi \neq \phi'$, then the joint distribution of the corresponding random variables X^k_ϕ and $X^l_{\phi'}$ does not exist empirically. We say that they are stochastically unrelated.

Now we consider the union of all bunches:

$$\mathfrak{X} = \bigcup_{\phi \in \Phi} \mathfrak{X}_\phi = \bigcup_{\phi \in \Phi} \left\{X^k_\phi\right\}_{\lambda^k_{i_k} \in \phi}.$$

$\mathfrak{X}$ is a set rather than a multi-component random variable. The elements in $\mathfrak{X}$ are not jointly distributed except when they are within the same bunch.

Let us collect the treatments that contain the external factor point $\lambda_{i_k}^k$ and denote this set $\Phi_{i_k}^k$. Let us consider all random variables measuring this external factor point in different treatments:

$$\mathfrak{X}_{i_k}^k = \left\{X_\phi^k\right\}_{\phi \in \Phi_{i_k}^k}.$$

We call this set a *connection* for $\lambda_{i_k}^k$. For every external factor point and any two treatments ϕ, ϕ' containing that factor point, if $X_\phi^k \sim X_{\phi'}^k$ we call this system *consistently connected*. The term is synonymous with marginal selectivity within the framework of selective influences. If $X_\phi^k \nsim X_{\phi'}^k$, for some ϕ, ϕ', the system is called *inconsistently connected*.

In contextuality analysis we are interested in whether and how one could impose a joint distribution on $\mathfrak{X}$. This means to find a collection of jointly distributed random variables $\mathfrak{M} = \{\mathfrak{M}_\phi\}_{\phi \in \Phi} = \left\{M_\phi^k\right\}_{\lambda_{i_k}^k \in \phi \in \Phi}$, such that for every $\phi \in \Phi$,

$$\mathfrak{M}_\phi = \left\{M_\phi^k\right\}_{\lambda_{i_k}^k \in \phi} \sim \left\{X_\phi^k\right\}_{\lambda_{i_k}^k \in \phi} = \mathfrak{X}_\phi. \tag{36}$$

$\mathfrak{M}$ and $\mathfrak{M}_\phi$ are multi-component random variables, and M_ϕ^k is a single-component random variable. In probability theory $\mathfrak{M}$ is called a *coupling for* $\mathfrak{X}$. Note that one can always find a coupling $\mathfrak{M}$ for $\mathfrak{X}$, such that (36) is satisfied.

Any subset of the components of $\mathfrak{M}$ is its *marginal*. Every coupling $\mathfrak{M}$ for $\mathfrak{X}$ has a marginal $\mathfrak{M}_{i_k}^k = \left\{M_\phi^k\right\}_{\phi \in \Phi_{i_k}^k}$ that forms a coupling for the connection $\mathfrak{X}_{i_k}^k$. Since $\Phi_{i_k}^k = \left\{\left(\lambda_1^1, \ldots, \lambda_{i_k}^k, \ldots, \lambda_1^n\right), \ldots, \left(\lambda_{m_1}^1, \ldots, \lambda_{i_k}^k, \ldots, \lambda_{m_n}^n\right)\right\}$, the probability that all the components in $\mathfrak{M}_{i_k}^k$ share the same values can be written as

$$\text{couple}(\mathfrak{M}^k_{i_k}) = \Pr\left[M^k_{\left(\lambda^1_1,\ldots,\lambda^k_{i_k},\ldots,\lambda^n_1\right)} = \ldots = M^k_{\left(\lambda^1_{m_1},\ldots,\lambda^k_{i_k},\ldots,\lambda^n_{m_n}\right)}\right].$$

Also denote

$$\text{couple}(\mathfrak{M}) = \sum_k \sum_{i_k} \text{couple}(\mathfrak{M}^k_{i_k}).$$

Among all possible couplings for $\mathfrak{X}$, we are interested in the maximum of couple($\mathfrak{M}$):

$$\max(\mathfrak{M}) = \max_{\text{all possible couplings } \mathfrak{M}} \text{couple}(\mathfrak{M}).$$

Let us take the connection $\mathfrak{X}^k_{i_k}$ for $\lambda^k_{i_k}$ in isolation and consider a coupling for the connection $\mathfrak{G}^k_{i_k} = \left\{G^k_\phi\right\}_{\phi\in\Phi^k_{i_k}}$, such that

$$G^k_\phi \sim X^k_\phi. \tag{37}$$

$\mathfrak{G}^k_{i_k}$ is a multi-component random variables and G^k_ϕ is a single-component random variable. Also we define

$$\text{couple}(\mathfrak{G}^k_{i_k}) = \Pr\left[G^k_{\left(\lambda^1_1,\ldots,\lambda^k_{i_k},\ldots,\lambda^n_1\right)} = \ldots = G^k_{\left(\lambda^1_{m_1},\ldots,\lambda^k_{i_k},\ldots,\lambda^n_{m_n}\right)}\right],$$

$$\text{couple}(\mathfrak{G}) = \sum_k \sum_{i_k} \text{couple}(\mathfrak{G}^k_{i_k}),$$

$$\max(\mathfrak{G}) = \max_{\text{all possible couplings } \mathfrak{G}} \text{couple}(\mathfrak{G}).$$

Every subcoupling of $\mathfrak{M}$ corresponding to a connection is a coupling for this connection. So it is easy to see that $\max(\mathfrak{M}) \leq \max(\mathfrak{G})$.

Definition 27. If $\max(\mathfrak{M}) = \max(\mathfrak{G})$, the system $\mathfrak{X}$ is *maximally coupled.*

Definition 28. The system $\mathfrak{X}$ is *noncontextual* if it is maximally coupled. Otherwise, it is *contextual.*

If in a particular $\mathfrak{X}$ system

$$\max(\mathfrak{G}) = \sum_{k} \sum_{i_k} 1,$$

then Definition 28 reduces to Definition 3. That is, selective influences is a special case of noncontextuality.

An example. Here I present an example to help the readers to understand the definition of contextuality. Let us assume α has two levels α_1 and α_2 and β has two levels β_1 and β_2. There are four treatments

$$(\alpha_1, \beta_1), (\alpha_1, \beta_2), (\alpha_2, \beta_1), (\alpha_2, \beta_2). \tag{38}$$

Let us denote the responses to the four treatments

$$(A_{11}, B_{11}), (A_{12}, B_{12}), (A_{21}, B_{21}), (A_{22}, B_{22}).$$

Each pair of $(A_{i_1 i_2}, B_{i_1 i_2}), i_1, i_2 \in \{1, 2\}$ forms a bunch. In this 2×2 system, there are eight random variables,

$$\{A_{11}, B_{11}, A_{12}, B_{12}, A_{21}, B_{21}, A_{22}, B_{22}\}. \tag{39}$$

Some of these variables, for instance A_{11} and B_{11}, have observable joint distributions; others do not have such joint distributions because they do not coexist in the same context. Random variables, say A_{11} and A_{12}, are stochastically unrelated as they cannot be observed in the same context. There are four connections in this paradigm: $\{A_{11}, A_{12}\}$, $\{A_{21}, A_{22}\}$, $\{B_{11}, B_{21}\}$, and $\{B_{12}, B_{22}\}$.

One can impose a coupling on the entire system, which consists of eight jointly distributed variables

$$(A^*_{11}, B^*_{11}, A^*_{12}, B^*_{12}, A^*_{21}, B^*_{21}, A^*_{22}, B^*_{22}), \tag{40}$$

such that

$$\begin{aligned}
(A^*_{11}, B^*_{11}) &\sim (A_{11}, B_{11}),\\
(A^*_{12}, B^*_{12}) &\sim (A_{12}, B_{12}),\\
(A^*_{21}, B^*_{21}) &\sim (A_{21}, B_{21}),\\
(A^*_{22}, B^*_{22}) &\sim (A_{22}, B_{22}).
\end{aligned}$$

One can also impose couplings on the connections separately taken,

$$(A'_{11}, A'_{12}), (A'_{21}, A'_{22}), (B'_{11}, B'_{21}), (B'_{12}, B'_{22}), \tag{41}$$

such that

$$\begin{aligned}
A'_{i_1 i_2} &\sim A_{i_1 i_2},\\
B'_{i_1 i_2} &\sim B_{i_1 i_2}.
\end{aligned}$$

The choice of these couplings is not unique. Among all possible choices for couplings (40), there is one choice that maximizes

$$\Pr(A^*_{11} = A^*_{12}) + \Pr(A^*_{21} = A^*_{22}) + \Pr(B^*_{11} = B^*_{21}) + \Pr(B^*_{12} = B^*_{22}). \tag{42}$$

Similarly, among all the choices for couplings (41), there is also one choice that achieves the maximum of

$$\Pr(A'_{11} = A'_{12}) + \Pr(A'_{21} = A'_{22}) + \Pr(B'_{11} = B'_{21}) + \Pr(B'_{12} = B'_{22}). \tag{43}$$

Let us denote the maximum of (42) M^* and maximum of (43) M'. It is always true that $M' \geq M^*$. If $M' = M^*$, this system is noncontextual. Otherwise the system is contextual.

It is mathematically possible that $M' = 1 + 1 + 1 + 1 = 4$. It implies that the distributions of the responses to the same factor point in different treatments are identical: marginal selectivity (or consistent connectedness) is present in the system. If in addition $M^* = 4$, we say selective influences are satisfied in the system, and the system is noncontextual.

Testing Contextuality in a Cyclic System

Recall that for selective influences, the Linear Feasibility Test should be applied to a system that contains finite number of inputs and outputs, in which each input and output have multiple levels. Consider the case when each treatment contains exactly two deterministic factors and these treatments form a *cycle* in the sense that every entity enters exactly two treatments (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2016; Kujala et al., 2015):

$$\begin{array}{ccccc} \text{Treatment}\,1, & \text{Treatment}\,2, & \ldots, & \text{Treatment}\,N-1, & \text{Treatment}\,N \\ (q^1, q^2), & (q^2, q^3), & \ldots, & (q^{N-1}, q^N), & (q^N, q^1). \end{array} \tag{44}$$

Here N is said to be the *rank* of this cyclic system. We assume in addition that each random output is binary: $\{-1, +1\}$. Let us denote by Z_c^k the response to the external factor q^k given the treatment indexed by c, where $1 \le c, k \le N$ and $k = c \,\mathrm{or}\, c \oplus 1$. So the random outputs corresponding to (44) are written as

$$(Z_1^1, Z_1^2), \quad (Z_2^2, Z_2^3), \quad \ldots, \quad (Z_{N-1}^{N-1}, Z_{N-1}^{N}), \quad (Z_N^N, Z_N^1).$$

It was proved (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2016; Kujala et al., 2015) that this system is noncontextual if and only if the following inequality is satisfied:

$$\begin{aligned} \Delta C = \mathrm{s}_1 \left(\left\langle Z_1^1 Z_1^2 \right\rangle, \ldots, \left\langle Z_{N-1}^{N-1} Z_{N-1}^{N} \right\rangle, \left\langle Z_N^N Z_N^1 \right\rangle \right) - (N-2) \\ - \sum_{k=1}^{N} \left| \left\langle Z_{k \ominus 1}^k \right\rangle - \left\langle Z_k^k \right\rangle \right| \le 0, \end{aligned} \tag{45}$$

where $\langle \cdot \rangle$ denotes the expected value, and s_1 is the maximum of all linear combinations $\pm \langle Z_1^1 Z_1^2 \rangle \pm \ldots \pm \left\langle Z_{N-1}^{N-1} Z_{N-1}^{N} \right\rangle \pm \left\langle Z_N^N Z_N^1 \right\rangle$ with odd number of minuses.

An example. The 2×2 system discussed earlier (38) can be written as a cyclic system:

$$\begin{matrix} \text{Treatment 1}, & \text{Treatment 2}, & \text{Treatment 3}, & \text{Treatment 4} \\ (\alpha_1, \beta_1), & (\beta_1, \alpha_2), & (\alpha_2, \beta_2), & (\beta_2, \alpha_1). \end{matrix} \tag{46}$$

Correspondingly, the random outputs are

$$(A_{11}, B_{11}), (B_{21}, A_{21}), (A_{22}, B_{22}), (B_{12}, A_{12}).$$

Consequently, ΔC is expressed as

$$\begin{aligned} \Delta C = \mathrm{s}_1 \left(\langle A_{11} B_{11} \rangle, \langle B_{21} A_{21} \rangle, \langle A_{22} B_{22} \rangle, \langle B_{12} A_{12} \rangle \right) - 2 \\ - \left| \langle A_{11} \rangle - \langle A_{12} \rangle \right| - \left| \langle B_{11} \rangle - \langle B_{21} \rangle \right| - \left| \langle A_{21} \rangle - \langle A_{22} \rangle \right| - \left| \langle B_{12} \rangle - \langle B_{22} \rangle \right|, \end{aligned} \tag{47}$$

where

$$\mathrm{s}_1\left(\gamma_1,\gamma_2,\gamma_3,\gamma_4\right)=\max(\left|\gamma_1+\gamma_2+\gamma_3-\gamma_4\right|,\left|\gamma_1+\gamma_2-\gamma_3+\gamma_4\right|,$$
$$\left|\gamma_1-\gamma_2+\gamma_3+\gamma_4\right|,\left|-\gamma_1+\gamma_2+\gamma_3+\gamma_4\right|).$$

Theoretical physicists studied contextuality without considering the violation of consistent connectedness. Given a sequence of measurements on the space-like separated entangled particles, the measurement set-up chosen in one particle is irrelevant for the measurement results on the other particle. In other words, marginal selectivity, or consistent connectedness, is expected to be preserved in quantum entanglement. In such cases, the term of $\sum_{k=1}^{N}\left|\left\langle Z_{k\ominus 1}^{k}\right\rangle-\left\langle Z_{k}^{k}\right\rangle\right|$ in (45) is zero. (45) then reduces to the so-called Leggett-Garg inequality when $N=3$ (Suppes & Zanotti, 1981), the Bell-CHSH-Fine inequality when $N=4$ (Clauser et al., 1969), and the so-called KCBS inequality when $N=5$ (Klyachko, Can, Binicioğlu, & Shumovsky, 2008). By testing these inequalities, experimental physicists demonstrated that the corresponding systems are contextual. However, marginal selectivity can be violated in these experiments due to signaling or measurement errors. They are usually dealt with by using some correction techniques, or even ignored. Instead of working around it, the framework of "contextuality-by-default" allows testing contextuality on top of inconsistent connectedness. By using the test (45), one can detect the "context-dependent" behaviors of the inconsistently connected quantum physics system (for $N=5$, see discussion in Kujala et al., 2015).

In contrast to quantum physics, Dzhafarov, Zhang, and Kujala (2015) showed that evidence of contextuality cannot be found in various social and behavioral data sets, from polls of public opinion to visual illusions to conjoint choices to word combinations to psychophysical matching. These studies were confined to systems of lower ranks

($N \leq 4$). Below I report results of the analyses for cyclc systems of ranks $N = 4, 6, 8$ using the psychophysical matching data.

Testing Contextuality on the Psychophysical Data

For the experimental details of the psychophysical matching task, see Experiments 1-3 in Chapter 1.

Testing Contextuality for Rank 4

In order to test contextuality using inequality (45), one has to form a cyclic system. In the "rectangular" subdesign of Experiment 1(a), the four treatments, represented in the rectangular coordinates were (α_1, β_1), (α_1, β_2), (α_2, β_1), and (α_2, β_2), where $\alpha_1 = 32\,\mathrm{px}$, $\alpha_2 = 64\,\mathrm{px}$, $\beta_1 = 32\,\mathrm{px}$, and $\beta_2 = 64\,\mathrm{px}$. A cyclic system is formed with (45) applicable in the form (46).

"Polar" subdesign of Experiment 1(a) and Experiment 2(b) also had 2×2 treatments that can also be represented in the cyclic manner of rank 4. Experiment 2(a) or Experiment 3(a) were 3×3 designs. 9 cyclic systems of rank 4 are contained in each of them. "Rectangular" design of Experiment 1(b), "Polar" subdesign of Experiment 1(b), Experiment 2(c), and Experiment 3(b) had external factors spanning certain intervals. In order to have a cyclic system of rank 4, each interval was dichotomized into two subintervals. For instance, two factor levels for the interval [20 px, 80 px) in the "rectangular" design of Experiment 1(b) was created according to the midpoint. Four treatments (α_1, β_1), (α_1, β_2), (α_2, β_1), and (α_2, β_2) were then formed, where $\alpha_1 = [20\,\mathrm{px}, 50\,\mathrm{px})$, $\alpha_2 = [50\,\mathrm{px}, 80\,\mathrm{px})$, $\beta_1 = [20\,\mathrm{px}, 50\,\mathrm{px})$, and $\beta_2 = [50\,\mathrm{px}, 80\,\mathrm{px})$. Of course other points can be chosen to dichotomize the intervals. In this dissertation, I only report the results from the midpoint-dichotomized treatments.

In addition, the random outputs should each be dichotomized. The two levels were defined according to a value a_{i_1} and a value $b_{i_2}, 1 \leq i_1, i_2 \leq 2$:

$$A_{1i_2} = \begin{cases} +1 & \text{if} \quad A_{1i_2} > a_1 \\ -1 & \text{if} \quad A_{1i_2} \leq a_1 \end{cases}, \quad A_{2i_2} = \begin{cases} +1 & \text{if} \quad A_{2i_2} > a_2 \\ -1 & \text{if} \quad A_{2i_2} \leq a_2 \end{cases},$$

$$B_{i_1 1} = \begin{cases} +1 & \text{if} \quad B_{i_1 1} > b_1 \\ -1 & \text{if} \quad B_{i_1 1} \leq b_1 \end{cases}, \quad B_{i_1 2} = \begin{cases} +1 & \text{if} \quad B_{i_1 2} > b_2 \\ -1 & \text{if} \quad B_{i_1 2} \leq b_2 \end{cases}.$$

The values of (a_1, a_2, b_1, b_2) can be chosen in various ways. We chose a value a_1 as any integer (in pixels) between max(minA_{11}, minA_{12}) and min(maxA_{11}, maxA_{12}), b_1 as any integer (in pixels or degrees) between max(minB_{11}, minB_{21}) and min(maxB_{11}, maxB_{21}), and analogously for a_2 and b_2. For each choice of the quadruple, we applied the test (47) to the distributions of the obtained A and B variables. 3024 to 11,663,568 tests were run for the systems we investigated. No positive $\triangle C$ was observed, indicating the absence of contextuality for rank 4 in all the experiments.

Here we present an example to illustrate how the test of (non)contextuality was conducted. For participant P3 in the "polar" subdesign of Experiment 1(a), one choice of the quadruple is $(a_1, a_2, b_1, b_2) =$ (72 px, 67 px, 60 deg, 23 deg). The distributions of the random outputs for the four treatments (indexed as (46)) are presented in Table 36 (Tr abbreviates Treatment).

Given

$$\begin{aligned} \langle XY \rangle = {} & (+1)(+1)\Pr(X = 1, Y = 1) + (+1)(-1)\Pr(X = 1, Y = -1) \\ & + (-1)(+1)\Pr(X = -1, Y = 1) + (-1)(-1)\Pr(X = -1, Y = -1) \end{aligned}$$

Table 36

Distributions of the Random Outputs for the Cyclic System of Rank 4, P3 in the "Polar" Subdesign of Experiment 1(a)

Tr 1	$B_{11} > b_1$	$B_{11} \leq b_1$	
$A_{11} > a_1$	.0056	0	.0056
$A_{11} \leq a_1$	.3944	.6	.9944
	.4	.6	

Tr 2	$B_{21} > b_1$	$B_{21} \leq b_1$	
$A_{21} > a_2$	.6403	.3399	.9802
$A_{21} \leq a_2$	.0099	.0099	.0198
	.6502	.3498	

Tr 3	$B_{22} > b_2$	$B_{22} \leq b_2$	
$A_{22} > a_2$	.5789	.4167	.9956
$A_{22} \leq a_2$	.0044	0	.0044
	.5833	.4167	

Tr 4	$B_{12} > b_2$	$B_{12} \leq b_2$	
$A_{12} > a_1$	.0273	.0219	.0492
$A_{12} \leq a_1$	.4699	.4809	.9508
	.4972	.5028	

and

$$\langle X \rangle = (+1)\Pr(X = 1) + (-1)\Pr(X = -1),$$

we have

$$\begin{aligned}
\Delta C &= \mathrm{s}_1 \left(\langle A_{11}B_{11} \rangle , \langle B_{21}A_{21} \rangle , \langle A_{22}B_{22} \rangle , \langle B_{12}A_{12} \rangle \right) - 2 \\
&\quad - \left| \langle A_{11} \rangle - \langle A_{12} \rangle \right| - \left| \langle B_{11} \rangle - \langle B_{21} \rangle \right| - \left| \langle A_{21} \rangle - \langle A_{22} \rangle \right| - \left| \langle B_{12} \rangle - \langle B_{22} \rangle \right| , \\
&= \mathrm{s}_1 \left(.2112, .3004, .1578, .0164 \right) - 2 - \left| (-.9016) - (-.9888) \right| - \left| (-.2) - .3004 \right| \\
&\quad - \left| .9604 - .9912 \right| - \left| .1666 - .0056 \right| \\
&= -2.1376
\end{aligned}$$

Testing Contextuality for Rank 6

Both Experiment 2(a) and Experiment 3(a) had 3×3 designs, $\{\alpha_1, \alpha_2, \alpha_3\} \times \{\beta_1, \beta_2, \beta_3\}$. Each of these designs included a cyclic system of rank 6:

$$\begin{array}{ccc} \text{Treatment 1}, & \text{Treatment 2}, & \text{Treatment 3}, \\ (\alpha_1, \beta_1), & (\beta_1, \alpha_2), & (\alpha_2, \beta_2), \\ \\ \text{Treatment 4}, & \text{Treatment 5}, & \text{Treatment 6}, \\ (\beta_2, \alpha_3), & (\alpha_3, \beta_3), & (\beta_3, \alpha_1). \end{array} \tag{48}$$

The corresponding outputs are

$$(A_{11}, B_{11}), \quad (B_{21}, A_{21}), \quad (A_{22}, B_{22}),$$

$$(B_{32}, A_{32}), \quad (A_{33}, B_{33}), \quad (B_{13}, A_{13}).$$

"Rectangular" design of Experiment 1(b), "polar" subdesign of Experiment 1(b), Experiment 2(c), and Experiment 3(b) are the systems with quasi-continuous factors. These factors were discretized into three levels. Two points should be chosen to make this discretization. There are infinitely many such choices. The data collected from the experiments with the quasi continuous factors were analyzed based on selecting the one-third point and the two-third point of each interval. For instance, a 3×3 design was formed in the "Rectangular" design of Experiment 1(b) according to this rule: $\alpha_1 = [20\,\text{px}, 40\,\text{px})$, $\alpha_2 = [40\,\text{px}, 60\,\text{px})$, $\alpha_3 = [60\,\text{px}, 80\,\text{px})$, $\beta_1 = [20\,\text{px}, 40\,\text{px})$, $\beta_2 = [40\,\text{px}, 60\,\text{px})$, and $\beta_3 = [60\,\text{px}, 80\,\text{px})$.

Again, each of the random outputs should be dichotomized. We chose a value a_{i_1} and a value $b_{i_2}, 1 \leq i_1, i_2 \leq 3$, and define

$$A_{1i_2} = \begin{cases} +1 & if \quad A_{1i_2} > a_1 \\ -1 & if \quad A_{1i_2} \leq a_1 \end{cases}, \quad A_{2i_2} = \begin{cases} +1 & if \quad A_{2i_2} > a_2 \\ -1 & if \quad A_{2i_2} \leq a_2 \end{cases},$$

$$A_{3i_2} = \begin{cases} +1 & if \quad A_{3i_2} > a_3 \\ -1 & if \quad A_{3i_2} \leq a_3 \end{cases}, \quad B_{i_1 1} = \begin{cases} +1 & if \quad B_{i_1 1} > b_1 \\ -1 & if \quad B_{i_1 1} \leq b_1 \end{cases},$$

$$B_{i_1 2} = \begin{cases} +1 & if \quad B_{i_1 2} > b_2 \\ -1 & if \quad B_{i_1 2} \leq b_2 \end{cases}, \quad B_{i_1 3} = \begin{cases} +1 & if \quad B_{i_1 3} > b_3 \\ -1 & if \quad B_{i_1 3} \leq b_3 \end{cases}.$$

For each rank 6 cyclic system, we chose a value a_1 as any integer between max(minA_{11}, minA_{13}) and min(maxA_{11}, maxA_{13}), we chose b_1 as any integer between max(minB_{11}, minB_{21}) and min(maxB_{11}, maxB_{21}), and analogously for a_2, a_3, b_2, and b_3. For each such choice of the sextuple $(a_1, a_2, a_3, b_1, b_2, b_3)$, we conducted the test (45). Using the obtained A and B variables, (45) can be equivalently represented as

$$\begin{aligned} \Delta C = {} & \mathrm{s}_1 \left(\langle A_{11} B_{11} \rangle, \langle B_{21} A_{21} \rangle, \langle A_{22} B_{22} \rangle, \langle B_{32} A_{32} \rangle, \langle A_{33} B_{33} \rangle, \langle B_{13} A_{13} \rangle \right) - 4 \\ & - \left| \langle A_{13} \rangle - \langle A_{11} \rangle \right| - \left| \langle B_{11} \rangle - \langle B_{21} \rangle \right| - \left| \langle A_{22} \rangle - \langle A_{21} \rangle \right| - \left| \langle B_{22} \rangle - \langle B_{32} \rangle \right| \\ & - \left| \langle A_{32} \rangle - \langle A_{33} \rangle \right| - \left| \langle B_{33} \rangle - \langle B_{13} \rangle \right|, \end{aligned} \tag{49}$$

Here we present an example to show how the test (49) was conducted. For participant P1 in Experiment 2(a), in which $\{\alpha_1, \alpha_2, \alpha_3\} \times \{\beta_1, \beta_2, \beta_3\} = \{16$ px, 56 px, 64 px$\} \times \{48$ px, 72 px, 80 px$\}$, one choice of the sextuple is $(a_1, a_2, a_3, b_1, b_2, b_3) =$ (16 px, 56 px, 64 px, 48 px, 72 px, 80 px). The distributions of the random outputs for the six treatments (indexed as (48)) are presented in Table 37:

Table 37

Distributions of the Random Outputs for the Cyclic System of Rank 6, P1 in Experiment 2(a)

Tr 1	$B_{11} > b_1$	$B_{11} \leq b_1$	
$A_{11} > a_1$	.2124	.2487	.4611
$A_{11} \leq a_1$	.1917	.3472	.5389
	.4041	.5959	

Tr 2	$B_{21} > b_1$	$B_{21} \leq b_1$	
$A_{21} > a_2$	.2353	.1041	.3394
$A_{21} \leq a_2$	.1538	.5068	.6606
	.3891	.6109	

Tr 3	$B_{22} > b_2$	$B_{22} \leq b_2$	
$A_{22} > a_2$	.1221	.0814	.2035
$A_{22} \leq a_2$	.1628	.6337	.7965
	.2849	.7151	

Tr 4	$B_{32} > b_2$	$B_{32} \leq b_2$	
$A_{32} > a_3$	.2703	.0586	.3288
$A_{32} \leq a_3$	.1982	.4730	.6712
	.4685	.5316	

Tr 5	$B_{33} > b_3$	$B_{33} \leq b_3$	
$A_{33} > a_3$	.0702	.0468	.1170
$A_{33} \leq a_3$	.0409	.8421	.8830
	.1111	.8889	

Tr 6	$B_{13} > b_3$	$B_{13} \leq b_3$	
$A_{13} > a_1$	.1321	.1981	.3302
$A_{13} \leq a_1$	.1651	.5047	.6698
	.2972	.7028	

Then we have

$$\begin{aligned}\Delta C &= \mathrm{s}_1\left(.1192, .4842, .5116, .4865, .8246, .2736\right) - 4 \\ &\quad - \left|-.0778 + .3396\right| - \left|-.1918 + .2218\right| - \left|-.3212 + .593\right| - \left|-.4302 + .0632\right| \\ &\quad - \left|-.3424 + .7660\right| - \left|-.7778 + .4056\right| \\ &= -3.2651.\end{aligned}$$

No positive $\triangle C$ was observed for the systems of rank 6 extracted from "rectangular" design of Experiment 1(b), "polar" subdesign of Experiment 1(b), Experiment 2(a), Experiment 2(c), Experiment 3(a), and Experiment 3(b). We concluded that there was no contextuality in all the investigated cyclic systems of rank 6.

Testing Contextuality for Rank 8

"Rectangular" design of Experiment 1(b), "polar" subdesign of Experiment 1(b), Experiment 2(c), and Experiment 3(b) have quasi-continuous factors. These factors were discretized into four discrete levels in order to form the rank 8 cyclic systems. Three points should be chosen for each factor to make this discretization. One choice could be the first quartile point, the second quartile (median) point, and the third quartile point of each interval. For instance, a 4×4 design was formed in Experiment 3(b): $\alpha_1 = [-30\,\text{px}, -15\,\text{px})$, $\alpha_2 = [-15\,\text{px}, 0\,\text{px})$, $\alpha_3 = [0\,\text{px}, 15\,\text{px})$, $\alpha_4 = [15\,\text{px}, 30\,\text{px})$, $\beta_1 = [-30\,\text{px}, -15\,\text{px})$, $\beta_2 = [-15\,\text{px}, 0\,\text{px})$, $\beta_3 = [0\,\text{px}, 15\,\text{px})$, and $\beta_4 = [15\,\text{px}, 30\,\text{px})$. The data were analyzed based on this particular type of discretization. Each cyclic system of rank 8 extracted from the experiment should be written as

$$\begin{array}{cccc}
\text{Treatment}\,1, & \text{Treatment}\,2, & \text{Treatment}\,3, & \text{Treatment}\,4,\\
(\alpha_1, \beta_1), & (\beta_1, \alpha_2), & (\alpha_2, \beta_2), & (\beta_2, \alpha_3),\\
\\
\text{Treatment}\,5, & \text{Treatment}\,6, & \text{Treatment}\,7, & \text{Treatment}\,8,\\
(\alpha_3, \beta_3), & (\beta_3, \alpha_4), & (\alpha_4, \beta_4), & (\beta_4, \alpha_1).
\end{array} \tag{50}$$

The corresponding outputs are

$$\begin{array}{cccc}
(A_{11}, B_{11}), & (B_{21}, A_{21}), & (A_{22}, B_{22}), & (B_{32}, A_{32}),\\
\\
(A_{33}, B_{33}), & (B_{43}, A_{43}), & (A_{44}, B_{44}), & (B_{14}, A_{14}).
\end{array}$$

We chose a value a_{i_1} and a value $b_{i_2}, 1 \leq i_1, i_2 \leq 4$ and define

$$A_{1i_2} = \begin{cases} +1 & if \quad A_{1i_2} > a_1 \\ -1 & if \quad A_{1i_2} \leq a_1 \end{cases}, \quad A_{2i_2} = \begin{cases} +1 & if \quad A_{2i_2} > a_2 \\ -1 & if \quad A_{2i_2} \leq a_2 \end{cases},$$

$$A_{3i_2} = \begin{cases} +1 & if \quad A_{3i_2} > a_3 \\ -1 & if \quad A_{3i_2} \leq a_3 \end{cases}, \quad A_{4i_2} = \begin{cases} +1 & if \quad A_{4i_2} > a_4 \\ -1 & if \quad A_{4i_2} \leq a_4 \end{cases},$$

$$B_{i_1 1} = \begin{cases} +1 & if \quad B_{i_1 1} > b_1 \\ -1 & if \quad B_{i_1 1} \leq b_1 \end{cases}, \quad B_{i_1 2} = \begin{cases} +1 & if \quad B_{i_1 2} > b_2 \\ -1 & if \quad B_{i_1 2} \leq b_2 \end{cases},$$

$$B_{i_1 3} = \begin{cases} +1 & if \quad B_{i_1 3} > b_3 \\ -1 & if \quad B_{i_1 3} \leq b_3 \end{cases}, \quad B_{i_1 4} = \begin{cases} +1 & if \quad B_{i_1 4} > b_4 \\ -1 & if \quad B_{i_1 4} \leq b_4 \end{cases}.$$

For each rank 8 cyclic system, we chose a value a_1 as any integer between max(minA_{11}, minA_{14}) and min(maxA_{11}, maxA_{14}), we chose b_1 as any integer between max(minB_{11}, minB_{21}) and min(maxB_{11}, maxB_{21}), and analogously for a_2, a_3, a_4, b_2, b_3, and b_4. For each choice we conducted the test (45). Using the obtained A and B variables, (45) can be equivalently represented as

$$\begin{aligned} \Delta C = {} & \mathrm{s}_1(\langle A_{11}B_{11}\rangle, \langle B_{21}A_{21}\rangle, \langle A_{22}B_{22}\rangle, \langle B_{32}A_{32}\rangle, \langle A_{33}B_{33}\rangle, \langle B_{43}A_{43}\rangle, \\ & \langle A_{44}B_{44}\rangle, \langle B_{14}A_{14}\rangle) - 6 - |\langle A_{11}\rangle - \langle A_{14}\rangle| - |\langle B_{11}\rangle - \langle B_{21}\rangle| - |\langle A_{22}\rangle - \langle A_{21}\rangle| \\ & - |\langle B_{22}\rangle - \langle B_{32}\rangle| - |\langle A_{33}\rangle - \langle A_{32}\rangle| - |\langle B_{43}\rangle - \langle B_{33}\rangle| - |\langle A_{44}\rangle - \langle A_{43}\rangle| \\ & - |\langle B_{14}\rangle - \langle B_{44}\rangle|. \end{aligned}$$

To give an example, for participant P4 in Experiment 3(b), one choice of the octuple is $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ = (-21 px, -6 px, 6 px, 21 px, -21 px, -9 px, 9 px, 21 px). The distributions of the random outputs for the eight treatments (indexed as (50)) are presented in Table 38:

Table 38

Distributions of the Random Outputs for the Cyclic System of Rank 8, P4 in Experiment 3(b)

Tr 1	$B_{11} > b_1$	$B_{11} \leq b_1$	
$A_{11} > a_1$	.1532	.2823	.4355
$A_{11} \leq a_1$	.1855	.3790	.5645
	.3387	.6613	

Tr 2	$B_{21} > b_1$	$B_{21} \leq b_1$	
$A_{21} > a_2$	.1619	.2667	.4286
$A_{21} \leq a_2$	.1905	.3810	.5715
	.3524	.6477	

Tr 3	$B_{22} > b_2$	$B_{22} \leq b_2$	
$A_{22} > a_2$	.2759	.2155	.4914
$A_{22} \leq a_2$	.2586	.2500	.5086
	.5345	.4655	

Tr 4	$B_{32} > b_2$	$B_{32} \leq b_2$	
$A_{32} > a_3$	.4130	.1739	.5869
$A_{32} \leq a_3$	.1957	.2174	.4131
	.6087	.3913	

Tr 5	$B_{33} > b_3$	$B_{33} \leq b_3$	
$A_{33} > a_3$	.2736	.3208	.5944
$A_{33} \leq a_3$	.1604	.2453	.4057
	.4340	.5661	

Tr 6	$B_{43} > b_3$	$B_{43} \leq b_3$	
$A_{43} > a_4$	.2460	.3095	.5555
$A_{43} \leq a_4$	.1667	.2778	.4445
	.4127	.5873	

Tr 7	$B_{44} > b_4$	$B_{44} \leq b_4$	
$A_{44} > a_4$	.3209	.3134	.6343
$A_{44} \leq a_4$	.1493	.2164	.3657
	.4702	.5298	

Tr 8	$B_{14} > b_4$	$B_{14} \leq b_4$	
$A_{14} > a_1$	.1619	.2571	.4190
$A_{14} \leq a_1$	.2381	.3429	.5810
	.4	.6	

We have

$$\begin{aligned}\Delta C &= \mathrm{s}_1\,(.0644,.0857,.0518,.2608,.0377,.0476,.0746,.0096) - 6 - .6902\\ &= -\,6.0772\end{aligned}$$

No positive $\triangle C$ was observed. When testing the "polar" subdesign of Experiment 1(b), each treatment contained only about 50 data points. Even with such small sample sizes, no positive $\triangle C$ was observed for a single case. We concluded that there was no contextuality in all the investigated cyclic systems of rank 8.

Conclusions

Contextuality-by-default is a mathematical framework that differentiates contextual systems and noncontextual systems. The empirical data suggest that the noncontextuality boundaries are generally breached in quantum physics. Experimental physicists showed that with the assumption of marginal selectivity, $\triangle C \geq 0$ for cyclic systems of ranks $N = 3, 4, 5$. Sometimes marginal selectivity is not satisfied in quantum physics. By admitting this fact and analyzing the quantum physics data using the contextuality test (45), one concludes that the quantum systems are still contextual, even when inconsistently connected.

Dzhafarov, Zhang, and Kujala (2015) reviewed several behavioral scenarios, and none of them provided any evidence for contextuality. By examining the psychophysical data collected in our lab, we did not find contextuality across cyclic systems of different ranks ($N = 4, 6, 8$). With marginal selectivity imposed on the same dataset (recall the discussion of selective influences in Chapter 1), we did not find contextuality either. Though it is not conclusive yet, we suspect that it may be generally true that human and social behaviors are not contextual.

SUMMARY

In the (α, β, A, B) system, if A depends on β and B depends on α, we say that marginal selectivity, or consistent connectedness, are not satisfied in the system. If A and B are stochastically interdependent, then one has to inquire about the origin of the interdependence. If A and B have the inherent interaction that cannot be attributed to α, β, or any hidden variable, then the system is contextual. Otherwise it is noncontextual.

Usually the behavioral systems are inconsistently connected. The psychophysical paradigms used in our lab all resulted in inconsistently connected systems. By imposing appropriate transformations, we obtained artificial consistently connected datasets. The contextual effects were then evaluated under the framework of selective influences. Once selective influences are established, we consider perceptual separability is confirmed as well. The Linear Feasibility Test (and therefore the Bell-CHSH-Fine inequalities) was not failed indicating a lack of contextual effects. Hence selective influences (and therefore perceptual separability) were established for the transformed datasets.

If the datasets were analyzed under the framework of contextuality-by-default without any transformations, the results of the contextuality test (45) also confirmed noncontextuality for cyclic systems of various ranks. The behavioral systems we have examined were shown to be different from the contextual quantum entanglement systems. Though it is still open to question, we suspect that human and social behaviors are noncontextual in general.

Mental architectures can be characterized according to the pattern of the interaction contrast, which is a linear combination of distributions of response times in a factorial experiment. Selective influences are assumed to hold in the investigated systems to ensure that the factorial manipulations influence the durations of the target processes only. Otherwise patterns of the interaction contrast for different types of mental architectures can mimic each other. By conditioning R in Definition 2 of selective influences on some value, we reduced the interaction contrast of distribution functions to simple arithmetic of 0's and 1's at every time moment.

The technique of interaction contrast was applied to empirical studies. We investigated how the subjects moved the trackball to match a target stimuli in psychophysical experiments. We tested the assumption of selective influences for the duration components by inspecting selective influences for the physical parameters of the reproduced stimuli. This seems to be a better way to examine selective influences of the experimental factors on the duration components than testing stochastic dominance alone. More importantly, we were able to investigate the process arrangements through directly observing the trackball movements. The analysis of the trackball movements led to the same conclusion as the computational results of the interaction contrast.

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2005-2010	Ohio University, Athens, OH M.S. in Experimental Psychology, 2011 M.S. in Physics, 2008
2001-2005	Special Class for the Gifted Young, University of Science and Technology of China, Hefei, Anhui, China B.S. in Physics, 2005

Research Experience

2016-present	Postdoctoral Researcher in Department of Psychological and Brain Sciences, Indiana University, Bloomington, IN
2010-2016	Research Assistant in Department of Psychological Sciences, Purdue University, West Lafayette, IN
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2005-2008	Research Assistant in Department of Physics and Astronomy, Ohio University, Athens, OH
2003-2005	Research Assistant in Center of Physical Sciences in Hefei National Laboratory for Physical Sciences at Microscale, Hefei, Anhui, China

Teaching Experience

2005-2008	Lab Instructor in Physics and Astronomy Department, Ohio University, Athens, OH

Awards and Honor

2016	Young scientist travel award, Indiana University Cognitive Science Program & National Science Foundation, Bloomington, IN
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2015	Student travel award, Society for Mathematical Psychology
2013-2014	Ben Winer memorial graduate fellowship, Department of Psychological Sciences, Purdue University, West Lafayette, IN
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- Hou, Ye; Zhang, Ru; Qian, Cheng; Liu, Jie (2009). Photoluminescence Quenching in Double-Walled Carbon Nanotubes. American Chemical Society Spring Meeting, Salt Lake City, UT.
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- Zhang, Ru; Chen, Liwei (2007). Fermi Energy of Different Single-Walled Carbon Nanotube Species Probed Through Ground State Redox Reactions. AVS Western PA Symposium for Nanoelectronic Devices and Materials, Pittsburgh, PA.