8-2016

Martingales, Singular Integrals, and Fourier Multipliers

Michael A. Perlmutter
Purdue University

Graduate School Form
30 Updated 12/26/2015

PURDUE UNIVERSITY
GRADUATE SCHOOL
Thesis/Dissertation Acceptance

This is to certify that the thesis/dissertation prepared

By Michael Perlmutter

Entitled
MARTINGALES, SINGULAR INTEGRALS, AND FOURIER MULTIPLIERS

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Dr. Rodrigo Banuelos
Chair

Dr. Fabrice Baudoin

Dr. Jonathon Peterson

Dr. Aaron Nung Kwan Yip

To the best of my knowledge and as understood by the student in the Thesis/Dissertation Agreement, Publication Delay, and Certification Disclaimer (Graduate School Form 32), this thesis/dissertation adheres to the provisions of Purdue University's "Policy of Integrity in Research" and the use of copyright material.

Approved by Major Professor(s): Rodrigo Banuelos

Approved by: Dr. David Goldberg 6/21/2015

Head of the Departmental Graduate Program Date

MARTINGALES, SINGULAR INTEGRALS, AND FOURIER MULTIPLIERS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Michael A. Perlmutter

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2016

Purdue University

West Lafayette, Indiana

To my mother Elizabeth Ann Perlmutter and to my late father Franklin Lewis Perlmutter.

ACKNOWLEDGMENTS

I would like to acknowledge the invaluable advice of my advisor Professor Rodrigo Bañuelos and to also thank Professors Fabrice Baudoin, Jonathon Peterson, and Aaron Yip for agreeing to serve on my committee. I would also like to thank my family for supporting me throughout my entire life.

TABLE OF CONTENTS

ABSTRACT

Perlmutter, Michael A. PhD, Purdue University, August 2016. Martingales, Singular Integrals, and Fourier Multipliers. Major Professor: Rodrigo Bañuelos.

Many probabilistic constructions have been created to study the L^p-boundedness, $1 < p < \infty$, of singular integrals and Fourier multipliers. We will use a combination of analytic and probabilistic methods to study analytic properties of these constructions and obtain results which cannot be obtained using probability alone.

In particular, we will show that a large class of operators, including many that are obtained as the projection of martingale transforms with respect to the background radiation process of Gundy and Varapolous or with respect to space-time Brownian motion, satisfy the assumptions of Calderón-Zygmund theory and therefore boundedly map L^1 to weak-L^1.

We will also use a method of rotations to study the L^p boundedness, $1 < p < \infty$, of Fourier multipliers which are obtained as the projections of martingale transforms with respect to symmetric α-stable processes, $0 < \alpha < 2$. Our proof does not use the fact that $0 < \alpha < 2$ and therefore allows us to obtain a larger class of multipliers, indexed by a parameter, $0 < r < \infty$, which are bounded on L^p. As in the case of the multipliers which arise as the projection of martingale transforms, these new multipliers also have potential applications to the study of the Beurling-Ahlfors transform and are related to the celebrated conjecture of T. Iwaniec concerning its exact L^p norm.

1. Introduction

1.1 Overview

Martingale inequality methods provide a powerful tool to study the L^p boundedness, $1 < p < \infty$, of the basic Calderón-Zygmund singular integral operators and other Fourier multipliers on $\mathbb{R}^n$. An advantage of these techniques is that they give very good information on the size of these L^p bounds and, in particular, provide constants that are independent of the dimension. These same arguments can be used to extend results from $\mathbb{R}^n$ to manifolds and to the Ornstein-Uhleneck case. For some applications of these methods we refer the reader to [3], [7], [11], [8], [6], [16], [25], [35], [24], and the many references provided there. However, as powerful as these techniques are, weak-type martingale inequalities cannot be directly transferred to singular integral operators. For example, while Burkholder's celebrated L^p inequalities, $1 < p < \infty$, for martingale transforms [18], with his famous bound "$(p^* - 1)$", gives the same L^p bound for many singular integral operators, his weak-type martingale transform bound "2" provides no information for the weak-type inequalities of those operators. This is due to the fact that the probabilistic representation of such operators involves the use of conditional expectation which does not preserve weak-type inequalities. When viewed as analytic objects, many of the operators which are obtained as the projections of martingale transforms have natural generalizations which cannot be studied by purely probabilistic methods. Therefore, in chapters 2 and 3 we will use a combination of analytic and probabilistic techniques to study these operators.

The main result of chapter 2 is that a very general class of operators, including many of the operators considered in [11] and [8], which arise as the projections of martingale transforms, are in fact Calderón-Zygmund operators. This class includes operators which are not, in general, of convolution type. Once we know that these are

Calderón-Zygmund operators, they then satisfy all the properties of such operators, including their weak-type boundedness. This does not, of course, answer an important question that has been of interest to many people for many years, originally raised by Stein in [40] in the case of the Riesz transforms: do these operators have weak-type bounds independent of the dimension? An affirmative answer would in turn raise an even more important question: do weak-type inequalities hold for Riesz transforms on Wiener space? After nearly 35 years and the efforts of many, these questions remain completely open. For a more precise formulation of these questions, see [4].

The purpose of chapter 3 is to study the L^p boundedness of a class of Fourier multipliers which are closely related to multipliers obtained as the projection of martingale transform of α-stable processes, $0 < \alpha < 2$, in [9] and [5]. Using analytic methods, we are able to obtain a family of operators indexed by r, $0 < r < \infty$, that are bounded on $L^p(\mathbb{R}^n)$. When $0 < r < 2$, these operators coincide with the operators from [9] and [5]. However, for $r \geq 2$ these are a new family of operators whose L^p boundedness have not been previously studied (except in the trivial case that $p = 2$). These problems are motivated by a celebrated 1982 conjecture of Tadeusz Iwaniec [29] which asserts that the L^p norm of the Beurling-Ahlfors operator is the same as the L^p norm of martingale transforms, namely $(p^* - 1)$. Although great progress has been made on this conjecture, it too remains open.

1.2 Calderón-Zygmund Operators

Following standard terminology (see for example [26, p.175]), we will say that an operator T acting on the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ is a Calderón-Zygmund (CZ) operator if it admits a bounded extension to $L^2(\mathbb{R}^n)$ and is of the form

$$Tf(x) = \lim_{\epsilon \searrow 0} \int_{|x-\tilde{x}|>\epsilon} K(x,\tilde{x}) f(\tilde{x}) d\tilde{x} \tag{1.1}$$

where the kernel K is defined on the set $\{x \neq \tilde{x}\}$ and satisfies the following conditions

$$|K(x,\tilde{x})| \leq \frac{\kappa}{|x-\tilde{x}|^n} \tag{1.2}$$

$$|\nabla_x K(x,\tilde{x})| \leq \frac{\kappa}{|x-\tilde{x}|^{n+1}} \tag{1.3}$$

$$|\nabla_{\tilde{x}} K(x,\tilde{x})| \leq \frac{\kappa}{|x-\tilde{x}|^{n+1}}, \tag{1.4}$$

for some universal constant κ. Integrals as in (1.1) are referred to as *principal value integrals*. For the rest of this thesis, we will assume that all integrals, where the integrand has an isolated singularity, are to be interpreted as principal value integrals. If there exists a function $\bar{K}$, defined on $\mathbb{R}^n \setminus \{0\}$, so that $\bar{K}(x-\tilde{x}) = K(x,\tilde{x})$ for all $x \neq \tilde{x}$, then we say that T is of convolution type. The Hilbert, Riesz, and Beurling-Ahlfors transforms discussed below are basic examples of CZ operators of convolution type which give rise to interesting Fourier multipliers. It is well known (see for example [26, p.183]) that CZ operators are strong-type (p,p) for $1 < p < \infty$ and are weak-type $(1,1)$. More precisely, there exists universal constants $C_{p,n,\kappa}$, depending only on p, n, and κ, such that

$$\|Tf\|_p \leq C_{p,n,\kappa}\|f\|_p, \quad 1 < p < \infty \tag{1.5}$$

and

$$|\{x : |Tf(x)| > \lambda\}| \leq \frac{C_{1,n,\kappa}}{\lambda}\|f\|_1, \tag{1.6}$$

where here and below we use $|A|$ to denote the Lebesgue measure of a set A.

We note that CZ operators do not, in general, map $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, nor do they map $L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. They do, however, map the Hardy space $H^1(\mathbb{R}^n)$, an important subset of $L^1(\mathbb{R}^n)$, into $L^1(\mathbb{R}^n)$ and map $L^\infty(\mathbb{R}^n)$ into the set of functions with bounded mean oscillation. This topic will be discussed further in section 2.2.

1.2.1 The Hilbert Transform

The Hilbert transform is the prototypical example of a Calderón-Zygmund operator of convolution type. For a rapidly decreasing function, $f : \mathbb{R} \to \mathbb{R}$, we define

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

In other words, the Hilbert transform is the operator given by convolving a function with the singular kernel $\frac{1}{\pi x}$. It is important to note that this integral must be interpreted as a principal value integral since otherwise it may not converge. (Take, for instance, $x = 0$ and $f(y) = e^{-y^2}$.) An interesting property of the Hilbert transform is that if we let $u(x, y)$ and $v(x, y)$ be the extensions of $f(x)$ and $Hf(x)$ to the upper half-space by convolution against the Poisson kernel $p_y(x)$, then $u(x, y) + iv(x, y)$ is holomorphic on the upper half-space.

1.2.2 The Riesz Transforms

The natural generalizations of the Hilbert transform to higher dimensions are known as the Riesz transforms. For $f \in L^p(\mathbb{R}^n)$, we define the Riesz transform in direction j, $1 \leq j \leq n$, by

$$R_j f(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} \frac{x_j - \tilde{x}_j}{|x - \tilde{x}|^{n+1}} f(\tilde{x}) d\tilde{x}.$$

When $n = 1$, this reduces to the Hilbert transform. R_j is a Fourier multiplier with

$$\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|} \widehat{f}(\xi).$$

In [40], Stein showed that we may take the constant, $C_{p,n}$, to be independent of n in (1.5) for the Riesz transforms. Whether or not the constant in (1.6) can be taken independent of n is unknown with the best known result being that the constant is at worst $O(\sqrt{n})$ as $n \to \infty$ [31]. Gundy and Varopoulos showed in [27] that the Riesz transforms could be interpreted probabilistically as projections of martingale transforms, and from this it again follows that the constant may be taken to be independent of dimension. See [4] for more on this topic. These techniques were further explored by Bañuelos and Wang in [8] to prove the sharp inequalities

$$\|R_j f\|_p \leq C_p \|f\|_p \quad \text{and} \quad \|((R_j f)^2 + f^2)^{1/2}\|_p \leq \sqrt{C_p^2 + 1}\|f\|_p,$$

where

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}, \quad \text{and} \quad C_p = \cot\left(\frac{\pi}{2p^*}\right).$$

The first inequality had been proved earlier by Iwaniec and Martin in [30] using the method of rotations.

1.2.3 The Beurling-Ahlfors Transfrom

For $f \in L^p(\mathbb{C})$, we define the Beurling-Ahlfors operator by

$$Bf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dw.$$

B is a Fourier multiplier with

$$\widehat{Bf}(\xi) = \frac{\xi_1^2 - \xi_2^2 - 2i\xi_1\xi_2}{|\xi|^2}\widehat{f}(\xi).$$

Therefore, we can decompose the Beurling-Ahlfors transform into a linear combination of second order Riesz transforms,

$$B = R_2^2 - R_1^2 + 2iR_1R_2. \tag{1.7}$$

Because of its many connections to quasiconformal mappings and other topics in complex analysis (see for example [2]) there has been a lot of interest for many years in finding its operator norm on $L^p(\mathbb{C})$, $1 < p < \infty$, which we denote $\|B\|_p$. In [33], Lehto showed that $\|B\|_p \geq (p^* - 1)$. A long standing conjecture of Iwaniec [29] is that $\|B\|_p = (p^* - 1)$. Despite the efforts of many researchers, Iwaniec's conjecture remains open. There are, however, many partial results, and the techniques developed in these efforts have lead to many other interesting questions and applications. In particular, there are a number of probabilistic constructions which provide upper bounds for $\|B\|_p$.

In [8], Bañuelos and Wang showed that $\|B\|_p \leq 4(p^* - 1)$. This constant was reduced to $2(p^* - 1)$ by Nazarov and Volberg in [35] using a Littlewood-Paley inequality proved using Bellman functions techniques. The Bellman function in [35] is itself constructed from Burkholder martingale inequalities. In [11] the martingale techniques from [8] were applied to space-time Brownian motion to reproduce the bound $2(p^* - 1)$. The methods of [11] were refined in [10] to reduce this constant to $1.575(p^* - 1)$, which is the best known bound as of now valid for all $1 < p < \infty$. We do point out that for $1000 < p < \infty$, this bound was improved to $1.4(p^* - 1)$ in [16].

1.3 Multiplier Theorems

Two important tools for studying the $L^p(\mathbb{R}^n)$ boundedness of Fourier multipliers, which we will use in chapter 3, are the Marcinkiewicz mutliplier theorem and the Hörmander-Mikhlin multiplier theorem which we state below for convenience. For proofs of these results see [26] or [39].

Theorem 1.3.1 *(Marcinkiewicz). Let $m \in L^\infty(\mathbb{R}^n)$ with $\|m\|_\infty \leq K$ for some $0 < K < \infty$. Supposed that $m(\xi)$ is n-times continuously differentiable on the subset of $\mathbb{R}^n$ where none of the ξ_i are zero. For $j \in \mathbb{Z}$, let I_j denote the dyadic interval $(-2^{j+1}, -2^j] \cup [2^j, 2^{j+1})$. Suppose that for all $1 \leq k \leq n$, for all subsets $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ of order k, and for all integers $l_{i_1}, \ldots l_{i_k}$, we have that*

$$\int_{I_{l_{i_1}}} \cdots \int_{I_{l_{i_k}}} |\partial_{i_1} \ldots \partial_{i_k} m(\xi)| d\xi_{i_k} \ldots d\xi_{i_1} \leq K < \infty \tag{1.8}$$

whenever $\xi_j \neq 0$ for all $j \notin \{i_1, \ldots, i_k\}$. Then $m(\xi)$ is a bounded Fourier multiplier on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and

$$\|T_m f\|_p \leq C_n K (p^* - 1)^{6n} \|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}^n),$$

where C_n is a constant depending only on n.

Theorem 1.3.2 *(Hörmander-Mikhlin). Let* $n_0 = \lfloor \frac{n}{2} \rfloor + 1$*, and let* $m(\xi)$ *be* n_0*-times differentiable on* $\mathbb{R}^n \setminus \{0\}$*. Suppose there exists* $0 < K < \infty$ *such that* $\|m\|_\infty \leq K$ *and that also*

$$\sup_{R>0} R^{-n+2|\beta|} \int_{R<|\xi|<2R} |\partial^\beta m(\xi)|^2 d\xi < K^2 \tag{1.9}$$

for all multi-indices such that $|\beta| \leq n_0$*. Then* $m(\xi)$ *is a bounded Fourier multiplier on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$ *and there exists* C_n *depending only on* n *such that*

$$\|T_m f\|_p \leq C_n K(p^* - 1)\|f\|_p.$$

1.4 Lévy Processes

A Lévy process on $\mathbb{R}^n$ is an $\mathbb{R}^n$-valued stochastic process, $(X_t)_{t\geq 0}$, which almost surely starts at the origin, has stationary, independent increments, and satisfies the stochastic continuity condition $\lim_{t\searrow 0} \mathbb{P}(|X|_t > \epsilon) = 0$ for all $\epsilon > 0$. The famous Lévy-Khintchine formula states that there exists a point $b \in \mathbb{R}^n$, a non-negative symmetric $n \times n$ matrix B, and a measure ν such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^n} \min\{|z|^2, 1\} d\nu(z) < \infty,$$

such that the characteristic function of X_t is given by $\mathbb{E}(e^{i\xi\cdot X_t}) = e^{t\rho(\xi)}$ where

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2} B\xi \cdot \xi + \int_{\mathbb{R}^n} \left[e^{i\xi\cdot z} - 1 - i(\xi \cdot z)\mathbb{I}_{(|z|<1)} \right] \nu(dz).$$

(b, B, ν) is referred to as the Lévy triple of X_t. The triple $(b, 0, 0)$ corresponds to a drift process $X_t = bt$; $(0, B, 0)$ corresponds to a centered Gaussian process with whose covariance is given by $[X_s^i, X_t^j] = B_{i,j} \min\{s, t\}$; and $(0, 0, \nu)$ corresponds to a "pure-jump" process. If X_t and Y_t are independent Lévy processes with triples (b_X, B_X, ν_X) and (b_Y, B_Y, ν_Y), then $X_t + Y_t$ is a Lévy process with the triple $(b_X + b_Y, B_X + B_Y, \nu_X + \nu_Y)$. Therefore, the Lévy-Khinchtine formula says that any Lévy process can be decomposed into the sum of three independent Lévy processes, a drift process, a centered Gaussian process, and a pure-jump process. For further background on Lévy processes see [13], [14], and [38].

1.4.1 α-stable Processes

For $0 < \alpha \le 2$, the symmetric α-stable process is a Lévy processes, $(X_t)_{t\ge 0}$ with $\rho(\xi) = -|\xi|^\alpha$. In the case that $\alpha = 2$, $(b, B, \nu) = (0, I, 0)$, $(X_t)_{t\ge 0}$ is Brownian motion (running at twice the usual speed), and density of X_t is given by the Gaussian heat kernel

$$\frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}. \tag{1.10}$$

For $0 < \alpha \le 2$, we have that $(b, B, \nu) = (0, 0, d\nu(z) = C_{n,\alpha} \frac{1}{|z|^{n+\alpha}} dz)$. If $\alpha = 1$, then $(X_t)_{t\ge 0}$ is the Cauchy process and the density of X_t is given by the Poisson kernel

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}. \tag{1.11}$$

Except for in the cases $\alpha = 1$ and $\alpha = 2$, we do not have a simple analytic expression for the density of X_t as in (1.10) and (1.11). However, there are a number of integral representations which are available for any α.

1.5 Martingale Transforms

The study of martingale transforms and their boundedness on L^p dates back to D. L. Burkholder's 1966 paper [17]. Since that time, martingale transforms have been extensively studied for both their theoretical importance in probability theory and their applications to finance. As alluded to in subsection 1.1, they have also been widely used to study the boundedness of singular integrals and Fourier multipliers.

1.5.1 Discrete Martingales

If $(f_n)_{n\ge 0}$ is a discrete-time martingale defined on a probability space $(\Omega, \mathbb{F}_\infty, \mathbb{P})$, with filtration $\mathbb{F} = (\mathbb{F}_n)_{n\ge 0}$, then we may define a difference sequence,

$$d_k = f_k - f_{k-1}$$

for $k \ge 1$, and $d_0 = f_0$ so that

$$f_n = \sum_{k=0}^{n} d_k.$$

If v_k is a predictable sequence of random variables, in the sense that $v_0 = 0$ and $v_k \in \mathbb{F}_{k-1}$ for $k \geq 1$, such that $|v_k| \leq 1$ a.s. for all k, then the martingale transform of f by v is defined by

$$(v * f)_n = \sum_{k=0}^{n} v_k d_k.$$

It is straight forward to check that $(v * f)_n$ is a martingale. The primary result of [17] was that the mapping $f \to v * f$ is bounded on L^p, $1 < p < \infty$, and weak-type $(1, 1)$. That is, there exists constants C_p and C_1 such that

$$\left(\sup_n \mathbb{E}|(v * f)_n|^p\right)^{1/p} \leq C_p \left(\sup_n \mathbb{E}|f_n|^p)\right)^{1/p} \tag{1.12}$$

and

$$\mathbb{P}\{\sup_n |(v * f)_n| > \lambda\} \leq \frac{C_1}{\lambda}\|f\|_1, \quad \text{for all } \lambda > 0. \tag{1.13}$$

Eighteen years later in [18], Burkholder was able to show that, for all $1 < p < \infty$, the best possible value of C_p in (1.12) is $p^* - 1$ and that the best possible constant in (1.13) is 2.

In [19], Burkholder introduced a condition called differential subordination, which allows for a much simpler proof of the fact that (1.12) holds with constant $p^* - 1$. Furthermore, this construction allows us to extend this result to martingale transforms defined with respect to stochastic integrals. Let $(f_n)_{n \geq 0}$ and $(g_n)_{n \geq 0}$ be martingales taking values in a separable Hilbert space and let d_k and e_k be their difference sequences (so that $f_n = \sum_{k=0}^n d_k$ and $g_n = \sum_{k=0}^n e_k$). We say that g_n is differentially subordinate to f_n if $|e_k| \leq |d_k|$ a.s. for all k. Burkholder showed that if g_n is differentially subordinate to f_n, then

$$\left(\sup_n \mathbb{E}|g_n|^p\right)^{1/p} \leq (p^* - 1) \left(\sup_n \mathbb{E}|f_n|^p)\right)^{1/p}. \tag{1.14}$$

Note that $p^* - 1$ is the same as the constant that appears in Iwaniec's conjecture regarding the Beurling-Ahlfors transform.

1.5.2 Stochastic Integrals and Continuous-Time Martingales

Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be martingales on a probability space $(\Omega, \mathbb{F}_\infty, \mathbb{P})$, with a common filtration, $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ that take values in a separable Hilbert space. Assume that $\mathbb{F}$ is right continuous and $\mathbb{F}_0$ contains all events of probability zero. Let $[X]_t$ and $[Y]_t$ denote quadratic variations of X_t and Y_t respectively. Y_t is said to be differentially subordinate to X_t if $|Y_0| \leq |X_0|$ and the process $[X]_t - [Y]_t$ is non-decreasing. Note that the quadratic variation of a discrete-time martingale $f_n = \sum_{k=0}^n d_k$ is given by $[f]_n = \sum_{k=0}^n |d_k|^2$. Therefore, this condition is the natural generalization of the differential subordination condition for discrete martingales. In [8], Bañuelos and Wang, showed that if X_t and Y_t have continuous sample paths, and Y_t is differentially subordinate to X_t, then

$$\left(\sup_t \mathbb{E}|Y_t|^p\right)^{1/p} \leq (p^* - 1)\left(\sup_t \mathbb{E}|X_t|^p\right)^{1/p}. \tag{1.15}$$

In the case that X_t and Y_t are orthogonal, in the sense that their quadratic covariation is zero, the constant $p^* - 1$ can be improved to $\cot\left(\frac{\pi}{2p^*}\right)$. We note that in [41], Wang showed (1.15) holds even if the assumption of continuous sample paths is removed. This fact will be important when we consider martingale transforms with respect to general Lévy processes.

A particularly important class of examples are martingales of the form

$$X_t = \int_0^t H_s \cdot dB_s$$

where B_t is n-dimensional Brownian motion and H_s is a $\mathbb{R}^n$-valued predictable process. If A_s is a predictable, matrix-valued process such that for all $s > 0$, and all $v \in \mathbb{R}^n$, $|A_s v| \leq |v|$, then

$$(A * X)_t = \int_0^t A_s H_s \cdot dB_s$$

is called the martingale transform of X by A. Similarly to the discrete martingale transforms in the previous subsection, $(A * X)_t$ is a martingale that is differentially subordinate to X_t. We remark that if for all $s > 0$ and all $v \in \mathbb{R}^n$, $Av \cdot v = 0$, then X_t and $(A * X)_t$ are orthogonal.

1.6 Martingales Transforms and Harmonic Analysis

As alluded to in section 1.1, there are several constructions which use martingale transforms to study the L^p boundedness of the classical Calderón-Zygmund singular integrals mentioned in section 1.2 and other Fourier multipliers. In many of these constructions, the method is based on the same fundamental idea. For a function f in $L^p(\mathbb{R}^n)$, we construct a martingale $M(f)_t$ such that $\sup_t \|M(f)_t\|_p = \|f\|_p$. Then we apply a martingale transform to get a new martingale, $N(f)_t$, such that $\sup_t \|N(f)_t\|_p \leq C_p \sup_t \|M(f)_t\|_p$. Finally, we project $N(f)_t$ onto $L^p(\mathbb{R}^n)$ using conditional expectation to get a new function which we denote by $Sf(x)$. Conditional expectation is a contraction on $L^p(\mathbb{R}^n)$ so $\|Sf\|_p \leq \sup_t \|N(f)_t\|_p$. Combining these three inequalities yields $\|Sf\|_p \leq C_p\|f\|_p$. If appropriate choices are made at each step, this operator will coincide with an operator of classical interest in analysis such as the Riesz Transforms or Beurling-Ahlfors transform.

1.6.1 The Background Radiation Process

We first consider the construction developed by Gundy and Varopoulos in [27] and used by Bañuelos and Wang in [8]. The first step is to construct a martingale corresponding to each function $f \in L^p$. Let

$$p_y(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \frac{y}{(|x|^2 + y^2)^{(n+1)/2}} \tag{1.16}$$

be the Poisson kernel for the upper half-space, $\mathbb{R}^{n+1}_+$, and for $f(x) \in C_0^\infty(\mathbb{R}^n)$, let $(p_y * f)(x) = u_f(x, y)$ be the Poisson extension of f. (Note that by (1.11) p_y is the density of the Cauchy process at time y.) Background radiation is a "time-reversed Brownian motion," $(B_t)_{t\leq 0}$, taking values in $\mathbb{R}^{n+1}_+$ such that $B_{-\infty}$ has distribution given by the Lebesgue measure on $\mathbb{R}^n \times \{\infty\}$, and B_0 is distributed by the Lebesgue measure on $\mathbb{R}^n \times \{0\}$. We write $B_t = (X_t, Y_t)$ with X_t taking values in $\mathbb{R}^n$ and $Y_t > 0$. The standard rules of stochastic calculus, in particular Itô's formula, hold for the background radiation process. Therefore, $u_f(X_t, Y_t)$ is a martingale and

$$f(X_0) = u_f(B_0) = \int_{-\infty}^{0} \nabla u_f(X_s, Y_s) \cdot dB_s,$$

where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_y)$. If $A(x, y)$ is an $(n+1) \times (n+1)$ matrix-valued function such that

$$\|A\| = \| \sup_{|v| \leq 1} (|A(x, y)v|) \|_{L^\infty(\mathbb{R}^n \times [0,\infty))} < \infty,$$

we define the martingale transform of f by A as

$$(A * f) = \int_{-\infty}^{0} A(X_s, Y_s) \nabla u_f(X_s, Y_s) \cdot dB_s. \tag{1.17}$$

The random variable $A * f$ is not a function of the endpoint, X_0. This motivates us to define a projection operator by averaging the integral in (1.17) over all paths ending at x, that is,

$$T_A f(x) = \mathbb{E}\left(\int_{-\infty}^{0} A(X_s, Y_s) \nabla u_f(X_s, Y_s) \cdot dB_s | X_0 = x \right).$$

It is known (see [4]) that $\mathbb{E}(|(f(B_0)|^p) = \int_{\mathbb{R}^n} |f(x)|^p dx$, which implies

$$\sup_{t \geq 0} \|u_f(B_t)\|_p = \|f\|_p$$

since $|u_f(B_t)|^p$ is a submartingale. In other words, lifting $f \in L^p(\mathbb{R}^n)$ to the space of martingales does not change its norm. Combining this with the fact that conditional expectation is a contraction in $L^p(\mathbb{R}^n)$, we see that the operator norm of T_A is the same as the operator norm of the martingale transform $X \to A * X$. Thus, we have

$$\|T_A f(x)\|_p \leq (p^* - 1)\|A\|\|f\|_p.$$

It is known (see for example [18]) that martingale transforms are weak-type $(1, 1)$ and in fact we have the sharp inequality

$$P\{|A * X| > \lambda\} \leq \frac{2\|A\|}{\lambda} \|X\|_1.$$

Unfortunately, this does not give us information about the weak-type behavior of T_A because weak-type inequalities are not preserved under conditional expectation. However, we can represent T_A analytically by finding a kernel $K_A(x,\tilde{x})$ such that

$$T_A f(x) = \int_{\mathbb{R}^n} K_A(x,\tilde{x}) f(\tilde{x}) d\tilde{x}.$$

Let $f, g \in C_0^\infty(\mathbb{R}^n)$ and note that

$$g(B_0) = \int_{-\infty}^{0} \nabla u_g(B_s) \cdot dB_s$$

by Itô's formula. Therefore, using basic facts about the covariation of stochastic integrals and the occupation time formula for the background radiation process, (see [23, p.31 and 57] and [27])

$$\begin{aligned}
\int_{\mathbb{R}^n} T_A f(x) g(x) dx &= \int_{\mathbb{R}^n} \mathbb{E}\left(\int_{-\infty}^{0} A(X_s,Y_s) \nabla u_f(X_s,Y_s) \cdot dB_s | X_0 = x \right) g(x) dx \\
&= \mathbb{E}\left(\int_{-\infty}^{0} A(X_s,Y_s) \nabla u_f(X_s,Y_s) \cdot dB_s g(B_0) \right) \\
&= \mathbb{E}\left(\int_{-\infty}^{0} A(X_s,Y_s) \nabla u_f(X_s,Y_s) \cdot dB_s \int_{-\infty}^{0} \nabla u_g(B_s) \cdot dB_s \right) \\
&= \mathbb{E}\left(\int_{-\infty}^{0} A(X_s,Y_s) \nabla u_f(X_s,Y_s) \cdot \nabla u_g(B_s) ds \right) \\
&= \int_0^\infty \int_{\mathbb{R}^n} 2y A(x,y) \nabla u_f(x,y) \cdot \nabla u_g(x,y) dx dy. \qquad (1.18)
\end{aligned}$$

Using the fact that $\nabla u_f(x,y) = ((\nabla p_y) * f)(x)$ and applying Fubini's theorem, we see that we have

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} 2y A(\bar{x},y) \nabla p_y(\bar{x}-\tilde{x}) \cdot \nabla p_y(\bar{x}-x) d\bar{x} dy.$$

This representation will be used in chapter 2 show that T_A is, under mild assumptions, a Calderón-Zygmund operator.

If we define $A_j = (a^j_{l,m})$ by

$$a^j_{l,m} = \left\{ \begin{array}{ll} 1 & l = n+1,\ m = j \\ -1 & l = j,\ m = n+1 \\ 0 & \text{otherwise} \end{array} \right\},$$

then plugging into (1.18) and Fourier transforming shows that $T_{A_j} = R_j$. Since A_j satisfies the orthogonality condition $A_j v \cdot v = 0$ for all $v \in \mathbb{R}^n$, it follows that

$$\|R_j\|_p \leq \cot\left(\frac{\pi}{2p^*}\right)\|f\|_p, \quad 1 < p < \infty.$$

We can also define A in such a way that $T_A = R_{i,j}$ and $\|A\| = 1$. This implies,

$$\|R_i R_j\|_p \leq (p^* - 1)\|f\|_p, \quad 1 < p < \infty.$$

By (1.7), this also implies that $\|B\|_p \leq 4(p^* - 1)$. If A is any matrix with constant coefficients, T_A will be a linear combination of the identity and first and second order Riesz transforms. Moreover, if $A(x,y) = A(y)$ is independent of x and $\|A\| < \infty$, then T_A is a Fourier multiplier. For more examples of multipliers corresponding to various choices of A, see [8] and [4].

1.6.2 Space-Time Brownian Motion

The approach of [11] is similar to the construction discussed in the previous subsection, but uses space-time Brownian motion and the heat kernel for the half Laplacian,

$$h_t(x) = \frac{1}{(2\pi t)^{n/2}} e^{-|x|^2/2t}, \tag{1.19}$$

instead of background radiation and the Poisson kernel. (We remark that h_t is the density of a standard Brownian motion at time t. Observe that this is, up to a simple time change, $t = 2s$, the density of the stable process given in (1.10).) Fix $T > 0$, and let $Z_t = (B_t, T - t)$ for $0 \leq t \leq T$ where B_t is Brownian motion on $\mathbb{R}^n$ with initial distribution given by the Lebesgue measure. Letting $u_f(x,t)$ denote the extension of f to the upper half-space by convolution with h_t, Itô's formula shows that $u_f(Z_t)$ is a martingale and

$$u_f(Z_t) = \int_0^s \nabla_x u_f(B_s, T - s) \cdot dB_s.$$

For an $n \times n$ matrix-valued function, $A(x,t)$, such that $\|A\| < \infty$, we define a martingale transform and a projection operator by

$$A * f = \int_0^T A(B_s, T - s) \nabla_x u_f(B_s, T - s) \cdot dB_s$$

and

$$S_A^T f(x) = \mathbb{E}(A * f | Z_T = (x, 0)).$$

It is shown in [11] that $\lim_{T\to\infty} S_A^T = S_A$ exists in $L^2(\mathbb{R}^n)$. Moreover,

$$\|S_A f(x)\|_p \le (p^* - 1)\|A\|\|f\|_p. \tag{1.20}$$

If $A^{(i,j)}$ is defined by

$$a_{l,m}^{(i,j)} = \left\{ \begin{array}{ll} -1 & l = i,\ m = j \\ 0 & \text{otherwise} \end{array} \right\},$$

then S_A is the second order Riesz transform, $R_i R_j$. By (1.7), this easily leads us to the conclusion that $\|B\|_p \le 2(p^* - 1)$. As with the operators arising from background radiation, if $A(x, y) = A(y)$ is independent of x, then S_A is a Fourier multiplier. Furthermore, we may again find a kernel so that

$$S_A f(x) = \int_{\mathbb{R}^n} \mathcal{K}_A(x, \tilde{x}) f(\tilde{x}) d\tilde{x},$$

where

$$\mathcal{K}_A(x, \tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} A(\bar{x}, t) \nabla_x h_t(\bar{x} - \tilde{x}) \nabla_x h_t(\bar{x} - x) d\bar{x} dt.$$

1.6.3 Martingale Transforms with respect to General Lévy Processes

In [9] and [5], the construction discussed in the previous subsection was generalized by replacing Brownian motion with more general Lévy processes. This results in a large class of Fourier multipliers, with formulas given in terms of the characteristics of the Lévy process, which are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Let ν be a Lévy measure on $\mathbb{R}^n$, φ a complex-valued function on $\mathbb{R}^n$ with $\|\varphi\|_\infty \le 1$, μ a finite Borel measure on $\mathbb{S}^{n-1}$, and ψ a complex-valued function on $\mathbb{S}^{n-1}$ with $\|\psi\|_\infty \le 1$. Define $m_{\mu,\nu}(\xi)$ by

$$m_{\mu,\nu}(\xi) = \frac{\int_{\mathbb{R}^n} (\cos(\xi \cdot z) - 1)\varphi(z)\nu(dz) + A\xi \cdot \xi}{\int_{\mathbb{R}^n} (\cos(\xi \cdot z) - 1)\nu(dz) + B\xi \cdot \xi} \tag{1.21}$$

where

$$A = \left(\int_{\mathbb{S}^{n-1}} \theta_i \theta_j \psi(\theta) d\mu(\theta) \right)_{1 \le i,j \le n} \quad \text{and} \quad B = \left(\int_{\mathbb{S}^{n-1}} \theta_i \theta_j d\mu(\theta) \right)_{1 \le i,j \le n}.$$

Note that $(\cos(\xi \cdot z) - 1) = \Re(e^{i\xi \cdot z} - 1 - i(\xi \cdot z)\mathbb{I}_{(|z|<1)})$. Therefore, $m_{\mu,\nu}$ may be interpreted as a "modulation" of the real part of the Lévy exponent of some process, X_t, divided by the "unmodulated" real part of the Lévy exponent of X_t. The primary result of [5] is to show that $m_{\mu,\nu}$ a bounded multiplier on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and

$$\|T_{m_{\mu,\nu}} f\|_p \le (p^* - 1)\|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

We will now give a brief summary of how this multiplier is obtained in the case where $\mu = 0$ and ν is symmetric and finite, which corresponds to X_t being a compound Poisson process. (The general case can then be proved by symmetrization and approximation arguments. See [5] for details.) Similarly to [11], we fix $T > 0$, let $(Z_t)_{0 \le t \le T} = (X_t, T - t)_{0 \le t \le T}$, and let $V_f(x, t) = P_t f(x) = \mathbb{E}^T(f(X_t + x))$. It is shown in [9] that $V_f(Z_t)$ is a martingale, with $\sup_t \|V_f(Z_t)\|_p = \|f\|_p$ for all $1 < p < \infty$, and by the generalized Itô's formula (see for example [36])

$$V_f(Z_t) - V_f(Z_0) = \int_0^{t+} \int_{\mathbb{R}^n} [V_f(Z_{s-} + z) - V_f(Z_{s-})] \tilde{N}(ds, dz),$$

where $Z_{s-} = \lim_{u \nearrow s} Z_u$, and $\tilde{N}$ is the so-called compensator, defined for each fixed $t > 0$ on Borel sets of $\mathbb{R}^n$ by

$$\tilde{N}(t, A) = N(t, A) - t\nu(A)$$

where N is the Poisson random measure that describes the jumps of X_t, i.e.

$$N(t, A) = |\{s : 0 \le s \le t, X_s - X_{s-} \in A\}|.$$

Therefore if $\varphi : \mathbb{R}^n \to \mathbb{C}$ with $\|\varphi\|_\infty \le 1$, we can define the martingale transform of $V_f(Z_t)$ by φ as

$$\varphi * V_f(Z_t) = \int_0^{t+} \int_{\mathbb{R}^n} [V_f(Z_{s-} + z) - V_f(Z_{s-})] \varphi(z) \tilde{N}(ds, dz).$$

The quadratic variations of $V_f(Z_t)$ and $\varphi * V_f(Z_t)$ are given by

$$[V_f(Z)]_t = \int_0^{t+} \int_{\mathbb{R}^n} |V_f(Z_{s-} + z) - V_f(Z_{s-})|^2 N(ds, dz)$$

and

$$[\varphi * V_f(Z)]_t = \int_0^{t+} \int_{\mathbb{R}^n} |V_f(Z_{s-} + z) - V_f(Z_{s-})|^2 |\varphi(z)|^2 N(ds, dz).$$

Therefore, $\varphi * V_f(Z_t)$ is differentially subordinate to $V_f(Z_t)$ and

$$\sup_t \|\varphi * V_f(Z_t)\|_p \leq (p^* - 1)\|f\|_p.$$

A projection operator can be defined by

$$S_\varphi^T f(x) = \mathbb{E}^T(\varphi * V_f(Z_T)|Z_T = (x, 0))$$

and we again have that

$$\|S_\varphi^T f(x)\|_p \leq (p^* - 1)\|f\|_p.$$

It is shown that as $T \to \infty$, a limiting operator, S_φ, exists and satisfies the bound

$$\|S_\varphi f(x)\|_p \leq (p^* - 1)\|f\|_p.$$

Moreover, S_φ is a Fourier multiplier and $\widehat{S_\varphi f}(\xi) = m_{\mu,\nu}(\xi)\widehat{f}(\xi)$.

A particularly interesting class of operators occurs when we take X_t to be the symmetric α-stable process with $0 < \alpha < 2$ and assume that φ is homogeneous of order zero. In polar coordinates, we may write $d\nu(z) = C_{n,\alpha} r^{-1-\alpha} dr d\sigma(\theta)$ where $C_{n,\alpha}$ is a constant chosen so that

$$\rho(\xi) = \int_{\mathbb{R}^n} (\cos(\xi \cdot z) - 1) d\nu(z) = -|\xi|^\alpha.$$

In this case, the numerator of (1.21) is given by

$$\begin{aligned} C_{n,\alpha} \int_{\mathbb{R}^n} (\cos(\xi \cdot z) - 1)\varphi(z) d\nu(z) &= C_{n,\alpha} \int_{\mathbb{S}^{n-1}} \varphi(\theta) \int_0^\infty \cos(r\xi \cdot \theta) r^{-1-\alpha} dr d\sigma(\theta) \\ &= C_{n,\alpha} \int_{\mathbb{S}^{n-1}} \varphi(\theta)|\xi \cdot \theta|^\alpha \int_0^\infty \cos(s) s^{-1-\alpha} ds d\sigma(\theta) \\ &= C'_{n,\alpha} \int_{\mathbb{S}^{n-1}} \varphi(\theta)|\xi \cdot \theta|^\alpha d\sigma(\theta). \end{aligned}$$

Therefore, the corresponding multiplier is given by

$$m_\alpha(\xi) = \frac{\int_{\mathbb{S}^{n-1}} |\xi \cdot \theta|^\alpha \varphi(\theta) d\sigma(\theta)}{\int_{\mathbb{S}^{n-1}} |\xi \cdot \theta|^\alpha d\sigma(\theta)}.$$

If we set $n = 2$ and choose $\varphi(\theta) = e^{-2i \arg \theta}$, then it is shown in [5] that $m_\alpha(\xi) = \frac{\alpha}{\alpha+2} \frac{\bar{\xi}}{\xi}$. Therefore, for all $0 < \alpha < 2$ and all $f \in L^p(\mathbb{R}^n)$

$$\|Bf\|_p \leq \frac{\alpha+2}{\alpha}(p^* - 1)\|f\|_p.$$

Letting $\alpha \nearrow 2$, we recover the bound $\|B\|_p \leq 2(p^* - 1)$.

The condition $0 < \alpha < 2$ is natural from a probabilistic prospective. Otherwise, the measure $d\nu(z) = \frac{C_{n,\alpha}}{|z|^{n+\alpha}}$ is not a Lévy measure on $\mathbb{R}^n$. However, for any $r > 0$, the multiplier

$$m_r(\xi) = \frac{\int_{\mathbb{S}^{n-1}} |\xi \cdot \theta|^r \varphi(\theta) d\sigma(\theta)}{\int_{\mathbb{S}^{n-1}} |\xi \cdot \theta|^r d\sigma(\theta)} \tag{1.22}$$

satisfies $\|m_r\|_\infty \leq 1$. Therefore, T_{m_r} is a bounded operator on $L^2(\mathbb{R}^n)$. Furthermore, for any $r > 0$, if we choose $\varphi(\theta) = e^{-2i \arg \theta}$, the formula $T_{m_r} f(x) = \frac{r}{r+2} Bf(x)$ is valid for all $f \in C_0^\infty(\mathbb{R}^n)$. Therefore, if we could prove conjecture (1), stated below, then letting $r \to \infty$ it would follow that $\|B\|_p \leq p^* - 1$, and therefore the celebrated conjecture of Iwaniec would be proved. This motivated the following conjecture of Bañuelos which first appeared in [4].

Conjecture 1 *Let $n \geq 2$, $0 < r < \infty$, $\varphi \in L^\infty(\mathbb{S}^{n-1})$, $\|\varphi\|_\infty \leq 1$, and let m_r be defined as in (1.22). Then the corresponding operator, T_{m_r}, is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and*

$$\|T_{m_r} f\|_p \leq (p^* - 1)\|f\|_p, \quad \textit{for all } f \in L^p(\mathbb{R}^n).$$

This is a very strong conjecture since it includes Iwaniec's conjecture, which has remained unproved for over thirty years, as a special case.

1.7 Calderón-Zygmund Operators Arising from Martingale Transforms, Statement of Results

The constructions used in [8] and [11] give very good constants for the $L^p(\mathbb{R}^n)$ boundedness, $1 < p < \infty$, of the operators which are constructed there as the projection of martingale transforms. However, using purely probabilistic methods, we are not able to get any information about the behavior of these operators on $L^1(\mathbb{R}^n)$. The purpose of chapter 2 is to show that these operators are Calderón-Zygmund operators and therefore are weak-type $(1,1)$. Specifically, we prove the following theorem. In the case that $\alpha = 1$ or 2, these operators are the conditional expectations of martingale transforms which were used in [8] and [11] respectively. (See subsections 1.6.1 and 1.6.2.)

Theorem 1.7.1 *Let $0 < \alpha \le 2$. Let $(X_t)_{t>0}$ be a symmetric α-stable process on $\mathbb{R}^n$ and let φ denote the density of X_1. For $y \ge 0$, let $\varphi_y(x) = \frac{1}{y^n}\varphi(\frac{x}{y})$. Let $A(x,y) = (a^{i,j}(x,y))$ be an $(n+1)\times(n+1)$ matrix-valued function with*

$$\|A\| = \|\sup_{|v|\le 1}(|A(x,y)v|)\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} < \infty. \tag{1.23}$$

Further assume that $a^{i,j}(x,y) = a^{i,j}(y)$ is independent of x whenever i or $j = n+1$. Consider the kernel

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} 2yA(\bar{x},y)\nabla\varphi_y(\bar{x}-\tilde{x})\nabla\varphi_y(\bar{x}-x)d\bar{x}dy, \tag{1.24}$$

where $\nabla = (\partial_{x_1},\ldots,\partial_{x_n},\partial_y)$. Then the operator

$$T_A f(x) = \int_{\mathbb{R}^n} K(x,\tilde{x})f(\tilde{x})d\tilde{x}$$

is a CZ operator.

Remark 1 *If we make the additional assumption that $a^{i,j}(y) = 0$ whenever i or $j = n+1$, we may also write our kernel in terms of the density of X_t, which we denote ψ_t. It is well known (see e.g. [13]) that ψ_t obeys the scaling relation $\psi_t(x) = \frac{1}{t^{n/\alpha}}\psi(\frac{x}{t^{1/\alpha}})$ which implies $\varphi_{t^{1/\alpha}} = \psi_t$. Therefore, after a simple change of variables we see that*

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} \frac{2}{\alpha}t^{\frac{2}{\alpha}-1}A(\bar{x},t^{1/\alpha})\nabla\psi_t(\bar{x}-\tilde{x})\nabla\psi_t(\bar{x}-x)d\bar{x}dt. \tag{1.25}$$

The reason why we need the assumption that $a^{i,j}(y) = 0$ whenever i or $j = n+1$ is because these entries correspond to "vertical" derivatives with respect to the dilation parameter t, and the change of variables $y = t^{1/\alpha}$ does not commute with the taking of vertical derivatives.

1.8 A Method of Rotations for Lévy Multipliers, Statement of Results

The main results of chapter 3 are two theorems which are partial solutions to Conjecture 1. The probabilistic methods used in [9] and [5] do not apply when $r \geq 2$. Instead, we will study T_{m_r} by analytic methods which make use of the Marcinkiewicz mutliplier theorem and the Hörmander-Mikhlin multiplier theorem (see section 1.3).

Theorem 1.8.1 *Let $n \geq 2$, $0 < r < \infty$, $\varphi \in L^\infty(\mathbb{S}^{n-1})$, $\|\varphi\|_\infty \leq 1$, and let m_r be defined as in (1.22). Then the corresponding operator, T_{m_r}, is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and*

$$\|T_{m_r} f\|_p \leq C_n (p^* - 1)^{6n} \frac{\Gamma(\frac{r+n}{2})}{\Gamma(\frac{r+1}{2})} \|f\|_p, \quad \text{for all } f \in L^p(\mathbb{R}^n),$$

where C_n is a constant which depends only on n.

Remark 2 *Sterling's formula implies that if $a > 0$*

$$\frac{\Gamma(x+a)}{\Gamma(x)} = O(x^a) \quad \text{as } x \to \infty.$$

Therefore,

$$\frac{\Gamma(\frac{r+n}{2})}{\Gamma(\frac{r+1}{2})} = O(r^{(n-1)/2}) \quad \text{as } r \to \infty.$$

In the case that r is sufficiently large, we can use the Hörmander-Mikhlin multiplier theorem to obtain estimates on the L^p bounds of T_{m_r} that are linear in p as $p \to \infty$.

Theorem 1.8.2 *Let $n \geq 2$ and define $n_0 = \lfloor \frac{n}{2} \rfloor + 1$. Let $n_0 \leq r < \infty$, $\varphi \in L^\infty(\mathbb{S}^{n-1})$, $\|\varphi\|_\infty \leq 1$, and let m_r be defined as in (1.22). Then the corresponding operator, T_{m_r}, is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and*

$$\|T_{m_r} f\|_p \leq C_n \max\{r^{n_0}, 1\} (p^* - 1) \|f\|_p, \quad \text{for all } f \in L^p(\mathbb{R}^n),$$

where C_n is a constant depending only on n. Furthermore, T_{m_r} is weak-type $(1,1)$ and

$$|\{T_{m_r}f(x) > \lambda\}| \leq C_n \max\{r^{n_0}, 1\}\frac{\|f\|_1}{\lambda}.$$

Remark 3 *Comparing the estimates in theorem 1.8.1 and theorem 1.8.2, we see that each has some advantages over the other. The constants obtained in theorem 1.8.1 have slower growth as $r \to \infty$ than those obtained in theorem 1.8.2 and have the advantage of being valid for all $r > 0$. On the other hand, theorem 1.8.2 gives estimates which are linear in p as $p \to \infty$ and includes weak-type (1,1) estimates which theorem 1.8.1 does not. This is because the proof of theorem 1.8.1 involves the method of rotations and the Marcinkiewicz multiplier theorem, neither of which give weak-type inequalities. We also remark that it is unknown if the operators which are obtained in [9] and [5] satisfy weak-type (1,1) inequalities. While it is true that martingale transforms do satisfy weak-type (1,1) estimates, these estimates are not preserved under conditional expectation, as we already mentioned several times before.*

2. Calderón-Zygmund Operators Arising from Martingale Transforms

2.1 The Proof of Theorem 1.7.1

Proof We need to verify that T_A is bounded on $L^2(\mathbb{R}^n)$ and that K_A satisfies the estimates (1.2), (1.3), and (1.4). From the definition of T_A, we observe that (1.3) and (1.4) are equivalent.

Lemma 1 *T_A is bounded on $L^2(\mathbb{R}^n)$. In particular, there exists a constant $C_{n,\alpha}$, depending only on n and α, such that for all $f \in C_0^\infty(\mathbb{R}^n)$*

$$\|T_A f\|_2 \leq C_{n,\alpha}\|A\|\|f\|_2. \tag{2.1}$$

Proof Let $f, g \in C_0^\infty(\mathbb{R}^n)$. We will show that

$$\left|\int_{\mathbb{R}^n} T_A f(x)g(x)dx\right| \leq C_{n,\alpha}\|A\|\|f\|_2\|g\|_2.$$

Letting u_f and u_g denote $\varphi_y * f$ and $\varphi_y * g$ respectively,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_A f(x)g(x)dx\right| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_A(x,\tilde{x})f(\tilde{x})g(x)d\tilde{x}dx\right| \\
&= \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n} 2yA(\bar{x},y)\nabla\varphi_y(\bar{x}-\tilde{x})\cdot\nabla\varphi_y(\bar{x}-x)f(\tilde{x})g(x)d\bar{x}dyd\tilde{x}dx\right| \\
&= \left|\int_0^\infty\int_{\mathbb{R}^n} 2yA(\bar{x},y)\int_{\mathbb{R}^n}\nabla\varphi_y(\bar{x}-\tilde{x})f(\tilde{x})d\tilde{x}\cdot\int_{\mathbb{R}^n}\nabla\varphi_y(\bar{x}-x)g(x)dxd\bar{x}dy\right| \\
&= \left|\int_0^\infty\int_{\mathbb{R}^n} 2yA(\bar{x},y)\nabla u_f(\bar{x},y)\cdot\nabla u_g(\bar{x},y)d\bar{x}dy\right| \\
&\leq 2\|A\|\int_0^\infty\int_{\mathbb{R}^n} y^{1/2}|\nabla u_f(x,y)|y^{1/2}|\nabla u_g(x,y)|dxdy.
\end{aligned}$$

Now by the Cauchy-Schwartz inequality and Holder's inequality,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} y^{1/2}|\nabla u_f(x,y)| y^{1/2}|\nabla u_g(x,y)|dxdy \\
&\leq \int_0^\infty \left(\int_{\mathbb{R}^n} y|\nabla u_f(x,y)|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^n} y|\nabla u_g(x,y)|^2 dx\right)^{1/2} dy \\
&\leq \left(\int_0^\infty \int_{\mathbb{R}^n} y|\nabla u_f(x,y)|^2 dxdy\right)^{1/2} \left(\int_0^\infty \int_{\mathbb{R}^n} y|\nabla u_g(x,y)|^2 dxdy\right)^{1/2}.
\end{aligned}$$

The proof will be complete once we show that

$$\left(\int_0^\infty y \int_{\mathbb{R}^n} |\nabla u_f(x,y)|^2 dxdy\right) \leq C_{n,\alpha}\|f\|_2^2.$$

Since φ is the density of X_1, which has characteristic function $\mathbb{E}(e^{iX_1\cdot\xi}) = e^{-|\xi|^\alpha}$, we have that $\widehat{\varphi(\xi)} = e^{-(2\pi|\xi|)^\alpha}$. Therefore, we may apply Plancherel's theorem, use the scaling relation for the Fourier transform, and substitute $t = y|\xi|$, to see that

$$\begin{aligned}
\int_0^\infty y \int_{\mathbb{R}^n} |\nabla_x u_f(x,y)|^2 dxdy &= \int_0^\infty y \int_{\mathbb{R}^n} 4\pi^2|\xi|^2|\widehat{\varphi_y}(\xi)|^2|\widehat{f}(\xi)|^2 d\xi dy \\
&= C\int_0^\infty y \int_{\mathbb{R}^n} |\xi|^2|\widehat{\varphi}(\xi y)|^2|\widehat{f}(\xi)|^2 d\xi dy \\
&= C\int_0^\infty t \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi' t)|^2|\widehat{f}(\xi)|^2 d\xi dt \\
&= C\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \int_0^\infty t e^{-2(2\pi t)^\alpha} dt d\xi \\
&\leq C_{n,\alpha}\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi = C_{n,\alpha}\|f\|_2^2.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\int_0^\infty y \int_{\mathbb{R}^n} |\partial_y u_f(x,y)|^2 dxdy &= \int_0^\infty y \int_{\mathbb{R}^n} |\partial_y \widehat{\varphi_y}(\xi)|^2|\widehat{f}(\xi)|^2 d\xi dy \\
&= C\int_0^\infty y \int_{\mathbb{R}^n} |\partial_y \widehat{\varphi}(\xi y)|^2|\widehat{f}(\xi)|^2 d\xi dy \\
&= C\int_0^\infty y \int_{\mathbb{R}^n} |\xi \cdot \nabla\widehat{\varphi}(\xi y)|^2|\widehat{f}(\xi)|^2 d\xi dy \\
&\leq C\int_0^\infty y \int_{\mathbb{R}^n} |\xi|^2|\nabla\widehat{\varphi}(\xi y)|^2|\widehat{f}(\xi)|^2 d\xi dy \\
&\leq C\int_{\mathbb{R}^n} |\xi|^4 \int_0^\infty y^3|\widehat{\varphi}(\xi y)|^2|\widehat{f}(\xi)|^2 dy d\xi \\
&= C\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \int_0^\infty t|\widehat{\varphi}(\xi' t)|^2 dt d\xi \leq C_{n,\alpha}\|f\|_2^2.
\end{aligned}$$

■

Now that we know T_A is bounded on $L^2(\mathbb{R}^n)$, we will show that it is, in fact, a CZ operator. It suffices to show that $K_A^{i,j}$ satisfies (1.2) and (1.3) for $1 \le i, j, \le n+1$ where

$$K_A^{i,j}(x, \tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} 2y a^{i,j}(\bar{x}, y) \partial_{x_i} \varphi_y(\bar{x} - \tilde{x}) \partial_{x_j} \varphi_y(\bar{x} - x) d\bar{x} dy. \tag{2.2}$$

The following lemma will be used to see that certain integrals converge.

Lemma 2 *There exists a constant $C_{n,\alpha}$, depending only on n and α, such that for all $x \in \mathbb{R}^n$, $1 \le i, j \le n$,*

$$|\varphi(x)| \le \frac{C_{n,\alpha}}{(1+|x|^2)^{(n+\alpha)/2}} \tag{2.3}$$

$$|\partial_{x_i} \varphi(x)| \le \frac{C_{n,\alpha}|x|}{(1+|x|^2)^{(n+2+\alpha)/2}} \le \frac{C_{n,\alpha}}{(1+|x|^2)^{(n+1+\alpha)/2}} \tag{2.4}$$

and

$$|\partial_{x_i} \partial_{x_j} \varphi(x)| \le \frac{C_{n,\alpha}}{(1+|x|^2)^{(n+2+\alpha)/2}}. \tag{2.5}$$

Proof Inverting the characteristic function of X_1 we see

$$\varphi(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-|\xi|^\alpha} d\xi. \tag{2.6}$$

From this we readily see that $\varphi \in C^\infty(\mathbb{R}^n)$, so in order to show (2.4) it suffices to show that there exists a constant $C_{n,\alpha}$ so that

$$|\partial_{x_i} \varphi(x)| \le C_{n,\alpha}|x| \tag{2.7}$$

and

$$|\partial_{x_i} \varphi(x)| \le \frac{C_{n,\alpha}}{|x|^{n+1+\alpha}}. \tag{2.8}$$

Using the fact that

$$\int_{\mathbb{R}^n} \xi_i e^{-|\xi|^\alpha} d\xi = 0,$$

we see that

$$\begin{aligned}|\partial_{x_i}\varphi(x)| &= \left|\int_{\mathbb{R}^n} \xi_i e^{-ix\cdot\xi} e^{-|\xi|^\alpha} d\xi\right| \\ &= \left|\int_{\mathbb{R}^n} \xi_i (e^{-ix\cdot\xi} - 1) e^{-|\xi|^\alpha} d\xi\right| \\ &\le \int_{\mathbb{R}^n} |\xi|\, |e^{-ix\cdot\xi} - 1| e^{-|\xi|^\alpha} d\xi \\ &\le 2\int_{\mathbb{R}^n} |\xi|^2 |x| e^{-|\xi|^\alpha} d\xi \le C_{n,\alpha}|x|,\end{aligned}$$

with the last inequality following because

$$|e^{ix\cdot\xi} - 1| \le |\cos(x\cdot\xi) - 1| + |\sin(x\cdot\xi)| \le 2|x\cdot\xi|.$$

Therefore (2.7) holds.

To show (2.8), we express X_t as a process subordinated to Brownian motion. A subordinator is an a.s. increasing one-dimensional Lévy process. It is well known (see [14] for details) that there exists a subordinator, T_t, such that

$$X_t = B_{T_t},$$

where B_t is a standard Brownian motion (run at twice the usual speed). By conditioning on T_t we see that the density of X_t is given by

$$\psi_t(x) = \int_0^\infty \frac{1}{(4\pi s)^{n/2}} e^{-|x|^2/4s} \eta^{\alpha/2}(t,s)ds,$$

where $\eta^{\alpha/2}(t,\cdot)$ is the density of T_t. Since $\varphi = \psi_1$, we see that

$$\begin{aligned}\partial_{x_i}\varphi(x) &= \int_0^\infty \frac{1}{(4\pi s)^{n/2}} \frac{x_i}{s} e^{-|x|^2/4s} \eta^{\alpha/2}(1,s)\, ds \\ &= C_n \frac{x_i}{|x|^n} \int_0^\infty u^{\frac{n}{2}-1} e^{-u} \eta^{\alpha/2}\left(1, \frac{|x|^2}{4u}\right) du.\end{aligned}$$

It is known (see e.g. [15]) that there exists a constant C_α, depending only on α, such that

$$\eta^{\alpha/2}(t,s) \le C_\alpha t s^{-1-\alpha/2}. \tag{2.9}$$

Therefore we have

$$|\partial_{x_i}\varphi(x)| \leq \frac{C_\alpha}{|x|^{n+1+\alpha}} \int_0^\infty u^{(n+\alpha)/2} e^{-u} du$$

so (2.8) holds. Similar computations show

$$|\varphi(x)| \leq \frac{C_\alpha}{|x|^{n+\alpha}} \int_0^\infty u^{(n+\alpha-2)/2} e^{-u} du$$

and

$$|\partial_{x_i}\partial_{x_j}\varphi(x)| \leq \frac{C_\alpha}{|x|^{n+\alpha+2}} \int_0^\infty u^{(n+\alpha+2)/2}(u+1) e^{-u} du.$$

Moreover, since φ is smooth, it and all of its all of its partial derivatives are bounded near the origin. Therefore φ satisfies (2.3) and (2.5). ■

We are now poised to prove the theorem.

Case 1. Either i or $j = n+1$:

The fact that $a^{(i,j)}(x,y) = a^{(i,j)}(y)$ depends only on y allows us to use the semigroup property of ψ_y. Note that

$$\varphi_y * \varphi_y = \psi_{y^\alpha} * \psi_{y^\alpha} = \psi_{2y^\alpha} = \varphi_{2^{1/\alpha}y}.$$

Therefore, substituting $w = \bar{x} - \tilde{x}$ we see that

$$\begin{aligned}
|K^{(i,j)}(x,\tilde{x})| &= \left| \int_0^\infty \int_{\mathbb{R}^n} 2ya^{(i,j)}(y)\partial_{x_i}\varphi_y(w)\partial_{x_j}\varphi_y(w-(x-\tilde{x}))dwdy \right| \\
&= \left| \int_0^\infty 2ya^{(i,j)}(y) \int_{\mathbb{R}^n} \partial_{x_i}\varphi_y(w)\partial_{x_j}\varphi_y(w-(x-\tilde{x}))dwdy \right| \\
&= \left| \int_0^\infty 2ya^{(i,j)}(y)\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}y}(x-\tilde{x})dy \right| \\
&\leq \|a^{(i,j)}\|_\infty \int_0^\infty 2y|\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}y}(x-\tilde{x})|dy.
\end{aligned}$$

Likewise,

$$\begin{aligned}
|\partial_{x_k}K^{(i,j)}(x,\tilde{x})| &= \left| \int_0^\infty \int_{\mathbb{R}^n} 2ya^{(i,j)}(y)\partial_{x_i}\varphi_y(w)\partial_{x_k}\partial_{x_j}\varphi_y(w-(x-\tilde{x}))dwdy \right| \\
&\leq \|a^{(i,j)}\|_\infty \int_0^\infty 2y|\partial_{x_i}\partial_{x_j}\partial_{x_k}\varphi_{2^{1/\alpha}y}(x-\tilde{x})|dy.
\end{aligned}$$

Therefore, it suffices to show that there exists a constant $C_{n,\alpha}$ so that $|K(x)| \leq C_{n,\alpha}\frac{1}{|x|^n}$ and $|K'(x)| \leq C_{n,\alpha}\frac{1}{|x|^{n+1}}$ for all $x \neq 0$, where

$$K(x) = \int_0^\infty 2y|\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}y}(x)|dy$$

and

$$K'(x) = \int_0^\infty 2y|\partial_{x_i}\partial_{x_j}\partial_{x_k}\varphi_{2^{1/\alpha}y}(x)|dy.$$

φ_y is homogeneous of order $-n$, so its k-th order partial derivatives are homogeneous of order $-n-k$. Therefore, if we make the substitution $y = |x|t$ we have

$$\begin{aligned} K(x) &= \int_0^\infty 2y|\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}y}(x)|dy \\ &= \int_0^\infty 2|x|t|\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}|x|t}(|x|x')||x|dt \\ &= \int_0^\infty 2|x|t\frac{1}{|x|^{n+2}}|\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}t}(x')||x|dt \\ &= \frac{1}{|x|^n}\int_0^\infty 2t|\partial_{x_i}\partial_{x_j}\varphi_{2^{1/\alpha}t}(x')|dt \end{aligned}$$

where $x' = \frac{x}{|x|}$. Similarly,

$$K'(x) = \frac{1}{|x|^{n+1}}\int_0^\infty 2t|\partial_{x_i}\partial_{x_j}\partial_{x_k}\varphi_{2^{1/\alpha}t}(x')|dt.$$

The lemma will be proved as soon as we bound the above integrals. We have assumed that either i or $j = n+1$, so we need to bound the following four integrals for any $1 \leq k, l \leq n$.

$$\int_0^\infty 2t|\partial_t\partial_t\varphi_{2^{1/\alpha}t}(x')|dt, \qquad \int_0^\infty 2t|\partial_t\partial_{x_k}\varphi_{2^{1/\alpha}t}(x')|dt,$$
$$\int_0^\infty 2t|\partial_t\partial_{x_k}\partial_{x_l}\varphi_{2^{1/\alpha}t}(x')|dt, \qquad \int_0^\infty 2t|\partial_t\partial_t\partial_{x_k}\varphi_{2^{1/\alpha}t}(x')|dt.$$

We will show how to bound the first integral. The other three may be bounded by the exact same method. Recalling that $\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right)$, we see that

$$\partial_t\partial_t\varphi_{2^{1/\alpha}t}(x) = \frac{C_n^{(1)}}{t^{n+2}}\varphi\left(\frac{x}{t}\right) + \frac{C_n^{(2)}}{t^{n+3}}\sum_{i=1}^n x_i\partial_{x_i}\varphi\left(\frac{x}{t}\right) + \frac{C_n^{(3)}}{t^{n+4}}\sum_{i=1}^n\sum_{j=1}^n x_ix_j\partial_{x_i}\partial_{x_j}\varphi\left(\frac{x}{t}\right),$$

where $C_n^{(1)}, C_n^{(2)}$ and $C_n^{(3)}$ are constants depending on n.

Therefore, it suffices to bound

$$\int_0^\infty \frac{t}{t^{n+a}} \partial^\beta \varphi\left(\frac{x'}{t}\right) dt$$

when $a = 2, 3$, or 4 and β is a multi-index with $|\beta| = a - 2$. By (2.3), (2.4), and (2.5), we have

$$\partial^\beta \varphi(x) \le \frac{C_{n,\alpha}}{(1+|x|^2)^{(n+\alpha+|\beta|)/2}},$$

which implies

$$\partial^\beta \varphi\left(\frac{x}{t}\right) \le \frac{C_{n,\alpha} t^{n+\alpha+|\beta|}}{(t^2+|x|^2)^{(n+\alpha+|\beta|)/2}}.$$

Therefore,

$$\begin{aligned} \int_0^\infty \frac{t}{t^{n+a}} \partial^\beta \varphi\left(\frac{x'}{t}\right) dt &\le \int_0^\infty \frac{t}{t^{n+a}} \frac{t^{n+\alpha+|\beta|}}{(t^2+1)^{(n+\alpha+|\beta|)/2}} dt \\ &= \int_0^\infty \frac{t^{\alpha-1}}{(1+t^2)^{(n+\alpha+|\beta|)/2}} dt < \infty. \end{aligned}$$

Case 2. $1 \le i, j \le n$:

Since $a^{i,j}(x,y)$ depends on both x and y, we are unable to use the semi-group property of ψ_y. We are, however, still able to pull out $\|A\|$ and use homogeneity. This again allows us to bound our kernel by the product of $\frac{1}{|x-\tilde{x}|^n}$ and an integral. As in case 1, we start out by substituting $w = x - \tilde{x}$ to see

$$|K^{(i,j)}(x,\tilde{x})| \le \|A\| \int_0^\infty \int_{\mathbb{R}^n} 2y |\partial_{x_i}\varphi_y(w)| |\partial_{x_j}\varphi_y(w-(\tilde{x}-x))| dw dy \text{ and}$$
$$|\partial_{x_k} K^{(i,j)}(x,\tilde{x})| \le \|A\| \int_0^\infty \int_{\mathbb{R}^n} 2y |\partial_{x_i}\varphi_y(w)| |\partial_{x_k}\partial_{x_j}\varphi_y(w-(\tilde{x}-x))| dw dy.$$

Therefore, we need to show that there exists a constant $C_{n,\alpha}$ so that $|K(x)| \le C_{n,\alpha}\frac{1}{|x|^n}$ and $|K'(x)| \le C_{n,\alpha}\frac{1}{|x|^{n+1}}$ for all $x \ne 0$ where now

$$K(x) = \int_0^\infty \int_{\mathbb{R}^n} 2y |\partial_{x_i}\varphi_y(w)| |\partial_{x_j}\varphi_y(w-x)| dw dy$$

and

$$K'(x) = \int_0^\infty \int_{\mathbb{R}^n} 2y |\partial_{x_i}\varphi_y(w)| |\partial_{x_k}\partial_{x_j}\varphi_y(w-x)| dw dy.$$

Using homogeneity and substituting $y = t|x|$ and $w = |x|z$ we see that

$$\begin{aligned}
|K(x)| &= \int_0^\infty \int_{\mathbb{R}^n} 2y|\partial_{x_i}\varphi_y(w)||\partial_{x_j}\varphi_y(w-x)|dwdy \\
&= \int_0^\infty \int_{\mathbb{R}^n} 2t|x||\partial_{x_i}\varphi_{|x|t}(|x|z)||\partial_{x_j}\varphi_{|x|t}(|x|(z-x'))||x|^{n+1}dzdt \\
&= \frac{1}{|x|^n}\int_0^\infty \int_{\mathbb{R}^n} 2t|\partial_{x_i}\varphi_t(z)||\partial_{x_j}\varphi_t(z-x')|dzdt. \qquad (2.10)
\end{aligned}$$

Similarly, we have

$$|K'(x)| = \frac{1}{|x|^{n+1}}\int_0^\infty \int_{\mathbb{R}^n} 2t|\partial_{x_i}\varphi_t(z)||\partial_{x_k}\partial_{x_j}\varphi_t(z-x')|dzdt. \qquad (2.11)$$

Therefore, to complete the proof, we need to show that the integrals in (2.10) and (2.11) are convergent. (Note that a simple rotation of coordinates shows they do not depend on x'.) We will show that the integral in (2.11) converges. The proof that the integral in (2.10) converges is similar.

By (2.4) and (2.5) we know that there exists a constant $C_{n,\alpha}$ so

$$|\partial_{x_i}\varphi(x)| \leq \frac{C_{n,\alpha}|x|}{(1+|x|^2)^{(n+2+\beta)/2}} \qquad (2.12)$$

and

$$|\partial_{x_i}\partial_{x_j}\varphi(x)| \leq \frac{C_{n,\alpha}}{(1+|x|^2)^{(n+2+\beta)/2}}, \qquad (2.13)$$

where $\beta = \min\{\alpha, \frac{1}{2}\}$. (The fact that $\beta \leq \frac{1}{2}$ will be used to see that a certain integral is convergent.) Therefore,

$$|\partial_{x_i}\varphi_t(x)| \leq \frac{C_n t^\beta |x|}{(t^2+|x|^2)^{(n+2+\beta)/2}} \qquad (2.14)$$

and

$$|\partial_{x_i}\partial_{x_j}\varphi_t(x)| \leq \frac{C_n t^\beta}{(t^2+|x|^2)^{(n+2+\beta)/2}}. \qquad (2.15)$$

This allows us to see that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} 2t|\partial_{x_j}\partial_{x_i}\varphi_t(z)||\partial_{x_j}\varphi_t(z-x')|dzdt \\
&\leq \int_{\mathbb{R}^n}\int_0^\infty t\frac{t^\beta}{(|z|^2+t^2)^{(n+2+\beta)/2}}\frac{|x'-z|t^\beta}{(|x'-z|^2+t^2)^{(n+2+\beta)/2}}dtdz,
\end{aligned}$$

so it suffices to show that $g_1(z)$ and $g_2(z)$ are integrable over $\mathbb{R}^n$ for

$$g_1(z) = \int_0^1 t^{1+2\beta} \frac{1}{(|z|^2+t^2)^{(n+2+\beta)/2}} \frac{|x'-z|}{(|x'-z|^2+t^2)^{(n+2+\beta)/2}} dt \quad \text{and}$$
$$g_2(z) = \int_1^\infty t^{1+2\beta} \frac{1}{(|z|^2+t^2)^{(n+2+\beta)/2}} \frac{|x'-z|}{(|x'-z|^2+t^2)^{(n+2+\beta)/2}} dt.$$

If z_n is a sequence converging to z, then

$$\begin{aligned}\lim_{n\to\infty} g(z_n) &= \lim_{n\to\infty} \int_1^\infty t^{1+2\beta} \frac{1}{(|z_n|^2+t^2)^{(n+\beta+2)/2}} \frac{|x'-z_n|}{(|x'-z_n|^2+t^2)^{(n+\beta+2)/2}} dt \\ &= \int_1^\infty t^{1+2\beta} \frac{1}{(|z|^2+t^2)^{(n+2+\beta)/2}} \frac{|x'-z|}{(|x'-z|^2+t^2)^{(n+2+\beta)/2}} dt = g_2(z),\end{aligned}$$

with the middle inequality justified by the dominating convergence theorem applied to $|x'-z|\frac{1}{t^{2n+3}}$. Thus $g_2(z)$ is continuous on $\mathbb{R}^n$. Furthermore, for large z, substituting $t = |z|\tan(\theta)$ allows us to see that

$$\begin{aligned}g_2(z) &\le C_{n,\beta} \int_1^\infty t^{1+2\beta} \frac{|z|}{(|z|^2+t^2)^{n+\beta+2}} dt \\ &\le C_{n,\beta} \int_0^{\pi/2} \frac{|z|^{1+2\beta}\tan^{1+2\beta}(\theta)|z||z|\sec^2(\theta)}{|z|^{2n+4+2\beta}\sec^{2n+4+2\beta}(\theta)} d\theta \\ &= C_{n,\beta} \frac{1}{|z|^{2n+1}} \int_0^{\pi/2} \sin^{1+2\beta}(\theta)\cos^{2n+1}(\theta) d\theta,\end{aligned}$$

so $g_2(z)$ is integrable.

Likewise, we can see that $g_1(z)$ is continuous on $\mathbb{R}^n \setminus \{0, x'\}$ using by applying the dominating convergence theorem with $t^{2\beta+1}|z|^{-n-\beta-2}|x'-z|^{-n-\beta-1}$, and for large z we have

$$g_1(z) \le C_{n,\beta} \frac{1}{|z|^{2n+1}} \int_0^{\pi/2} \sin^{1+2\beta}(\theta)\cos^{2n+1}(\theta) d\theta.$$

Therefore, it remains to show that $g_1(z)$ is integrable near 0 and x'.

If $|z| < 1/2$ and $0 < t < 1$, it is easy to see

$$\frac{|x'-z|}{(|x'-z|^2+t^2)^{(n+2+\beta)/2}} \le C_{n,\beta},$$

so again substituting $t = |z|\tan(\theta)$ we see

$$\begin{aligned}g_1(z) &\le C_{n,\beta} \int_0^1 \frac{t^{1+2\beta}}{(|z|^2+t^2)^{(n+2+\beta)/2}} dt \\ &\le C_{n,\beta} \frac{1}{|z|^{n-\beta}} \int_0^{\pi/2} \sin^{2\beta}(\theta)\cos^{n-\beta-1}(\theta) d\theta.\end{aligned}$$

Since $\beta \leq \frac{1}{2}$, the last integral is finite, so $g_1(z)$ is integrable near 0. A simple change of variables and a nearly identical computation shows that $g_1(z)$ is integrable near x', so therefore $g_1(z)$ is integrable on all of $\mathbb{R}^n$ which completes the proof.

■

We end this section by remarking that if i or $j = n+1$, then the integral in (2.11) is divergent. This is why we need the assumption that $a^{i,j}(x,y) = a^{i,j}(y)$ in that case.

2.2 Remarks

Examining the proof of theorem 1.7.1, we see that the only facts we used were the homogeneity of $\varphi_y(x)$, the fact that $\widehat{\varphi}$ is "small enough" to cause T_A to be bounded on $L^2(\mathbb{R}^n)$, and the bounds (2.3), (2.4), and (2.5). This immediately gives us the following corollary.

Corollary 1 *Let $\phi \in C^2(\mathbb{R}^n)$ satisfy (2.3), (2.4), and (2.5), and for $y > 0$, let $\phi_y = \frac{1}{y^n}\phi\left(\frac{x}{y}\right)$. Assume that there exists a constant C such that for all $\xi' \in S^{n-1}$*

$$\int_0^\infty t\widehat{\phi}(t\xi')^2 dt < C. \tag{2.16}$$

Let $A(x,y)$ be as in theorem 1.7.1. Consider the kernel

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} 2yA(\bar{x},y)\nabla\phi_y(\bar{x}-\tilde{x})\nabla\phi_y(\bar{x}-x)d\bar{x}dy,$$

where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_y)$. Then the operator

$$T_A f(x) = \int_{\mathbb{R}^n} K(x,\tilde{x})f(\tilde{x})d\tilde{x}$$

is a CZ operator.

The key to proving lemma (2) was the fact that we could write the α-stable process as B_{T_t} where T_t is the $\alpha/2$ stable subordinator and B_t is an independent Brownian motion (run at twice the usual speed). This motivates the following question. Let T_t be a subordinator, let B_t be an independent Brownian motion, and let $X_t = B_{T_t}$.

Under what conditions on T_t does the density of X_1 satisfy the conditions of corollary 1?

If $X_t = B_{T_t}$ is any such process, called subordinate Brownian motion in the literature, it is well known (see for example [32]) that there exists a function $\Phi : [0,\infty) \to [0,\infty)$, called the Laplace exponent of T_t, such that

$$e^{-\lambda T_t} = e^{-t\Phi(\lambda)} \tag{2.17}$$

and that the Lévy symbol of X_t is given by

$$\rho(\xi) = -\Phi\left(|\xi|^2\right).$$

Inspecting the proofs of lemma 1 and lemma 2, we see that in order for the density of X_1 to satisfy the conditions of corollary 1, it suffices to have a bound similar to (2.9) on the density of T_1, and for $\Phi(\lambda)$ to increase fast enough as $\lambda \to \infty$ for the integrals in the proofs to converge. We summarize this in the following corollary.

Corollary 2 *Let $X_t = B_{T_t}$ where T_t is a subordinator and B_t is an independent Brownian motion run at twice the usual speed. Let ϕ denote the density of X_1, and for $y > 0$, let $\phi_y(x) = \frac{1}{y^n}\phi\left(\frac{x}{y}\right)$. Let Φ be the Laplace exponent of T_t and assume that there exists some $\delta > 0$ so that*

$$\Phi(\lambda) \geq O(\lambda^\delta), \quad \text{as } \lambda \to \infty. \tag{2.18}$$

Further assume that there exist a constants C and $\gamma > 0$ such that the density of T_1, $\eta(1,\cdot)$, satisfies

$$\eta(1,s) \leq Cs^{-1-\gamma/2} \tag{2.19}$$

for all $s > 0$. Let $A(x,y)$ be as in theorem 1.7.1 and consider the kernel

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} 2yA(\bar{x},y)\nabla\phi_y(\bar{x}-\tilde{x})\nabla\phi_y(\bar{x}-x)d\bar{x}dy,$$

where $\nabla = (\partial_{x_1},\ldots,\partial_{x_n},\partial_y)$. Then the operator

$$T_Af(x) = \int_{\mathbb{R}^n} K(x,\tilde{x})f(\tilde{x})d\tilde{x}$$

is a CZ operator.

An interesting example of subordinate Brownian motion is provided by the so called relativistic α-stable processes. For $0 < \alpha < 2$, $M > 0$, there exists a Lévy process, $(X_t)_{t\geq 0}$ with symbol $\rho(\xi) = (|\xi|^2 + M^{2/\alpha})^{\alpha/2} - M$ and infinitesimal generator

$$M - (-\Delta + M^{2/\alpha})^{\alpha/2}.$$

When $\alpha = 1$, this operator reduces to free-relativistic Hamiltonian which has been intensely studied because of its applications to relativistic quantum mechanics. For further background information on this process, we refer the reader to [20], [12], and the references provided in therein.

In [37] it is shown that T_t, the subordinator for X_t^m, has density

$$\eta^{m,\alpha/2}(t,s) = e^{mt} e^{-m^{2/\alpha}s} \eta^{\alpha/2}(t,s), \tag{2.20}$$

and Laplace exponent

$$\Phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m. \tag{2.21}$$

Therefore, we readily see that the conditions of corollary 2 are satisfied.

The motivation of this chapter was to answer questions left open in [11] and [8]. Are the operators considered in those papers weak-type $(1,1)$ in addition to being strong-type (p,p) for $1 < p < \infty$? Proving that these operators are CZ shows that the answer to this question is, in fact, yes. However, CZ operators are also known to satisfy a number of other desirable properties. For example, they boundedly map the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ to the space of functions with bounded mean oscillation (BMO). More precisely, if T is any CZ operator, then there exist universal constants C_n and C_n', which depend only on n, so that

$$\|T\|_{H^1\to L^1} \leq C_n(\kappa + \|T\|_{L^2\to L^2})$$

and

$$\|T\|_{L^\infty\to BMO} \leq C_n'(\kappa + \|T\|_{L^2\to L^2})$$

where κ is as in (1.2), (1.3) and (1.4). For details on this topic, see [26, ch. 8].

Another interesting property of CZ operators is that they are bounded on certain weighted L^p spaces. A weight is a function $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ which is positive almost everywhere. The associated space $L^p(w)$, $1 \le p < \infty$, is the collection of functions f on $\mathbb{R}^n$ such that

$$\|f\|^p_{L^p(w)} = \int_{\mathbb{R}^m} |f(x)|^p w(x) dx < \infty.$$

The Muckenhoupt characteristic of w is defined as

$$\|w\|_{A_p} = \sup_Q \frac{1}{|Q|} \int_Q w dx \cdot \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} dx \right)^{p-1}, \tag{2.22}$$

with the supremum taken over all cubes, Q. Note that when $p = 2$ this becomes

$$\|w\|_{A_2} = \sup_Q \frac{1}{|Q|} \int_Q w dx \cdot \frac{1}{|Q|} \int_Q w^{-1} dx.$$

w is said to be an A_p weight if $\|w\|_{A_p}$ is finite. In this case, it is well known (see for example [26, ch. 9]) that if T is a CZ operator, then there exists a constant $C_{n,p,T,w}$, depending on the n, p, T, and w, such that

$$\|Tf\|_{L^p(w)} \le C_{n,p,T,w} \|f\|_{L^p(w)},$$

for all $f \in L^p(w)$ when $1 < p < \infty$. (A corresponding weak-type result holds when $p = 1$.)

Recently, in [28], Hytönen proved the so called "A_2 conjecture," that $C_{n,2,T,w}$ depends linearly on $\|w\|_2$, i.e., there exists a constant $C_{n,2,T}$ such that

$$\|Tf\|_{L^2(w)} \le C_{n,2,T} \|w\|_{A_2} \|f\|_{L^2(w)},$$

for all $f \in L^2(w)$. Combining this with a result of Dragičević, Grafakos, Pereyra, and Petermichl [21] shows that

$$\|Tf\|_{L^p(w)} \le C_{n,p,T} \|w\|_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(w)}$$

for all $f \in L^p(w)$. For more information weighted L^p spaces and the A_2 conjecture, see [26, ch. 9] and [28].

The operators considered in [11] are generalized in [1] by taking the projections of martingales transforms involving more general Lévy processes in place of Brownian motion. These more general operators are shown in these papers to obey the same "$p^* - 1$" strong-type bound for $1 < p < \infty$ as the operators from [11]. In the current paper we have shown that the operators considered in [11] are CZ operators, and therefore are also weak-type (1,1). It would be interesting to know if the same is true of the operators studied in [1].

3. A Method of Rotations for Lévy Multipliers

3.1 The proof of theorem 1.8.1

The main idea of the proof is to use a method of rotations to write T_{m_r} as the weighted average of multipliers which can be studied using the Marcinkiewicz multiplier theorem.

Proof We first observe (see [26] Appendix D, p. 443) that

$$\int_{\mathbb{S}^{n-1}} |\xi \cdot \theta|^r d\sigma(\theta) = A_{n,r}|\xi|^r, \tag{3.1}$$

where $A_{n,r} = \frac{1}{2\pi^{(n-1)/2}} \frac{\Gamma(\frac{1+r}{2})}{\Gamma(\frac{n+r}{2})}$. Therefore,

$$m_r(\xi) = A_{n,r}^{-1} \int_{\mathbb{S}^{n-1}} \frac{|\xi \cdot \theta|^r}{|\xi|^r} \varphi(\theta) d\sigma(\theta). \tag{3.2}$$

Now for $\theta \in \mathbb{S}^{n-1}$, we let $m_\theta(\xi) = \frac{|\xi \cdot \theta|^r}{|\xi|^r}$. Using (3.2), we may write T_{m_r} as a weighted average of the T_{m_θ}'s. More precisely, we shall prove the following lemma.

Lemma 3 *For all $f \in C_0^\infty(\mathbb{R}^n)$,*

$$T_{m_r} f(x) = A_{n,r}^{-1} \int_{\mathbb{S}^{n-1}} T_{m_\theta} f(x) \varphi(\theta) d\sigma(\theta),$$

for almost every x.

Proof Let f and $g \in C_0^\infty(\mathbb{R}^n)$. Then by Plancherel's theorem, Fubini's theorem, and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&A_{n,r}^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} T_{m_\theta}f(x)\varphi(\theta)d\sigma(\theta)g(x)dx\\
=&A_{n,r}^{-1}\int_{\mathbb{S}^{n-1}}\varphi(\theta)\int_{\mathbb{R}^n} T_{m_\theta}f(x)g(x)dxd\sigma(\theta)\\
=&A_{n,r}^{-1}\int_{\mathbb{S}^{n-1}}\varphi(\theta)\int_{\mathbb{R}^n} m_\theta(\xi)\widehat{f}(\xi)\bar{\widehat{g}}(\xi)d\xi d\sigma(\theta)\\
=&A_{n,r}^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} m_\theta(\xi)\varphi(\theta)d\sigma(\theta)\widehat{f}(\xi)\bar{\widehat{g}}(\xi)d\xi\\
=&\int_{\mathbb{R}^n}\widehat{T_{m_r}f}(\xi)\bar{\widehat{g}}(\xi)d\xi\\
=&\int_{\mathbb{R}^n} T_{m_r}f(x)g(x)dx.
\end{aligned}$$

■

We will also need to estimate the L^p boundedness of the operators T_{m_θ}. This is accomplished by the following lemma.

Lemma 4 *There exists* $0 < C_n < \infty$ *such that*

$$\|T_{m_\theta}f\|_p \le C_n(p^*-1)^{6n}\|f\|_p,$$

for all $f \in L^p(\mathbb{R}^n)$*.* C_n *depends only on* n *and, in particular, does not depend on* r *or* θ*.*

Before proving lemma 4, we will first show how it is used to give a simple proof of Theorem 1.8.1. By Minkowski's integral inequality,

$$\begin{aligned}
\|T_{m_r}f\|_p &= A_{n,r}^{-1}\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{S}^{n-1}}\varphi(\theta)T_{m_\theta}f(x)d\sigma(\theta)\right|^p dx\right)^{1/p}\\
&\le A_{n,r}^{-1}\int_{\mathbb{S}^{n-1}}\left(\int_{\mathbb{R}^n}|\varphi(\theta)|^p|T_{m_\theta}f(x)|^p dx\right)^{1/p} d\sigma(\theta)\\
&= A_{n,r}^{-1}\int_{\mathbb{S}^{n-1}}|\varphi(\theta)|\left(\int_{\mathbb{R}^n}|T_{m_\theta}f(x)|^p dx\right)^{1/p} d\sigma(\theta)\\
&\le A_{n,r}^{-1}\int_{\mathbb{S}^{n-1}}\|T_{m_\theta}f\|_p d\sigma(\theta)\\
&\le A_{n,r}^{-1}C_n(p^*-1)^{6n}\omega_{n-1}\|f\|_p,
\end{aligned}$$

where ω_{n-1} is the surface area of $\mathbb{S}^{n-1}$. Therefore, theorem 1.8.1 is proved. ■

We shall now prove lemma 4

Proof For θ in $\mathbb{S}^{n-1}$, let R be a rotation such that $R\theta = e_1$ and for $f \in L^p$ let $g(x) = f(R^{-1}x)$. Then a simple change for variables shows that $T_{m_\theta} f(x) = T_{m_{e_1}} g(Rx)$. Therefore, it suffices to show that

$$\|T_{m_{e_1}} f\|_p \leq C_n (p^* - 1)^{6n} \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n).$$

To prove this, we will show that m_{e_1} satisfies the assumptions of theorem 1.3.1 and that we can take K to be independent of r in (1.8). Note that it follows from [39, p. 110] that for each fixed r, $T_{m_{e_1}}$ is a Marcinkiewicz multiplier, but it takes considerably more work to show that K can be taken to be independent of r in (1.8). $m_{e_1}(\xi)$ is even in each ξ_i so it suffices to restrict attention to the region where all ξ_i are positive. Noting that for all $A_1, \ldots, A_k > 0$

$$\int_{A_1}^{2A_1} \cdots \int_{A_k}^{2A_k} \frac{1}{\xi_{i_1} \xi_{i_2} \ldots \xi_{i_k}} d\xi_{i_k} \ldots d\xi_{i_1} = \log(2)^k,$$

we see that it suffices to prove there exists C independent of r such that

$$|\partial_{i_1} \ldots \partial_{i_k} m_{e_1}(\xi)| \leq \frac{C}{\xi_{i_1} \xi_{i_2} \ldots \xi_{i_k}}$$

or equivalently that

$$\xi_{i_1} \xi_{i_2} \ldots \xi_{i_k} |\partial_{i_1} \ldots \partial_{i_k} m_{e_1}(\xi)| \leq C. \tag{3.3}$$

The left hand side of (3.3) is homogeneous of order zero, so it suffices to bound this quantity on the portion of the unit sphere where all $\xi_i \geq 0$. To do this, we will make use of two elementary lemma's which involve the use of Lagrange multipliers to bound polynomials on ellipses.

Lemma 5 *Let $a, b, c, d > 0$. The maximum value of*

$$f(x, y) = x^a y^b$$

subject to the constraints $cx^2 + dy^2 = 1$, $x, y \geq 0$, *is given by*

$$\frac{\left(\frac{a}{c}\right)^{a/2}\left(\frac{b}{d}\right)^{b/2}}{(a+b)^{(a+b)/2}}.$$

Proof It is easy to check using the method of Lagrange multipliers to show that f is maximized when

$$x^2 = \frac{a}{c(a+b)} \quad \text{and} \quad y^2 = \frac{b}{d(a+b)}.$$

The result follows immediately. ∎

Lemma 6 *Let* $1 < k \leq n$, *then the maximum value of* $f(x,y,z) = (k-1)x^{2k}y^r + (n-k)x^{2k-2}y^r z^2$ *subject to the constraint that* $g(x,y,z) = (k-1)x^2 + y^2 + (n-k)z^2 = 1, x, y, z \geq 0$ *is*

$$\frac{(2k)^k}{(k-1)^{k-1}}\left(\frac{r}{2k+r}\right)^{r/2}\frac{1}{(2k+r)^k}.$$

Proof If $k = n$ then,

$$f(x,y,z) = f(x,y) = (n-1)x^{2n}y^r \quad \text{and} \quad g(x,y,z) = g(x,y) = (n-1)x^2 + y^2,$$

so the result follows from lemma 5. If $1 < k < n$, the method of Lagrange multipliers can be used to show that at any point at which f achieves a local maximum, $z = 0$. Therefore, the result again follows from lemma 5. ∎

Now, in order to verify that m_r satisfies (3.3), we consider three cases.

Case 1 $1 \notin \{i_1, \ldots, i_k\}$:

By direct computation,

$$|\partial_{i_1} \ldots \partial_{i_k} m_{e_1}(\xi)| = r(r+2)\ldots(r+2k-2)\frac{\xi_1^r \xi_{i_1} \ldots \xi_{i_k}}{|\xi|^{r+2k}}.$$

Therefore, we need to bound

$$r(r+2)\ldots(r+2k-2)\xi_1^r \xi_{i_1}^2 \ldots \xi_{i_k}^2$$

on the portion of the unit sphere where all coordinates are non-negative. By symmetry, it is clear that this last term is maximized when $\xi_{i_1} = \xi_{i_2} = \ldots = \xi_{i_k}$ and $\xi_i = 0$, whenever $i \notin \{i_1, \ldots, i_k, 1\}$. Therefore, we are lead to the two-dimensional optimization problem of maximizing

$$f(x, y) = x^{2k} y^r,$$

subject to the constraint that $g(x, y) = kx^2 + y^2 = 1$. By lemma 5, the maximal value of f subject to this constraint is less than

$$C_k \left(\frac{1}{2k + r} \right)^k.$$

Therefore, on the unit sphere

$$r(r + 2) \ldots (r + 2k - 2) \xi_j^r \xi_{i_1}^2 \ldots \xi_{i_k}^2 \leq C_k \frac{r(r + 2) \ldots (r + 2k - 2)}{(2k + r)^k} \leq C_k.$$

Case 2 $k = 1, i_1 = 1$:

Differentiating, we see

$$|\xi_1 \partial_1 m_{e_1}(\xi)| = r \frac{\xi_1^r}{|\xi|^{r+2}} (\xi_2^2 + \ldots + \xi_n^2),$$

and (3.3) can be verified by repeating the arguments of case 1.

Case 3 $k > 1$ *and* $1 \in \{i_1, \ldots, i_k\}$:

Without loss of generality, we may assume $i_k = 1$. Carrying out the computations, we see

$$\begin{aligned} & |\partial_{i_1} \ldots \partial_{i_{k-1}} \partial_1 m(\xi)| \\ = & \left| \frac{r(r + 2) \ldots (r + 2k - 4) r \xi_{i_1} \ldots \xi_{i_{k-1}} \xi_1^{r-1}}{|\xi|^{r+2k-2}} - \frac{r(r + 2) \ldots (r + 2k - 2) \xi_{i_1} \ldots \xi_{i_{k-1}} \xi_1^{r+1}}{|\xi|^{r+2k}} \right| \\ = & \frac{r(r + 2) \ldots (r + 2k - 4) \xi_{i_1} \ldots \xi_{i_{k-1}} \xi_1^{r-1}}{|\xi|^{r+2k}} \left| r(\xi_2^2 + \xi_3^2 + \ldots + \xi_n^2) - (2k - 2) \xi_1^2 \right|. \end{aligned}$$

Therefore, it suffices to show that there exists C_k such that

$$r(r + 2) \ldots (r + 2k - 4) \xi_{i_1}^2 \ldots \xi_{i_{k-1}}^2 \xi_1^{r+2} < C_k$$

and

$$r(r+2)\dots(r+2k-4)r\xi_{i_1}^2\dots\xi_{i_{k-1}}^2\xi_1^r(\xi_2^2+\dots+\xi_n^2) < C_k,$$

whenever $|\xi| = 1$ and all $\xi_i \geq 0$. This can be done by using lemmas 5 and 6 in a manner similar to cases 1 and 2. ■

Remark 4 *In the case that $r = 2k$ is an even integer, we have that $T_{e_1} = R_1^{2k}$, the $2k$-th order Riesz transform in direction 1. Dimension free estimates for this operator were obtained by Iwaniec and Martin in [30] using a method that compared polynomials of the Riesz transforms to polynomials of the complex Riesz transforms and then in turn estimated the complex Riesz transforms by comparing them to the iterated Beurling-Ahlfors transform.*

Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ the complex Riesz transforms are defined by

$$C_j = R_j + iR_{n+j}$$

for $1 \leq j \leq n$. For a polynomial $p(x) = \sum_{|\beta|\leq m} c_\beta x^\beta$, $p(\mathbf{R})$ and $p(\mathbf{C})$ are defined by

$$p(\mathbf{R}) = \sum_{|\beta|\leq m} c_\beta \mathbf{R}^\beta \quad \text{and} \quad p(\mathbf{C}) = \sum_{|\beta|\leq m} c_\beta \mathbf{C}^\beta,$$

where $\mathbf{R}^\beta = R_1^{\beta_1} \circ \dots \circ R_n^{\beta_n}$ and $\mathbf{C}^\beta = C_1^{\beta_1} \circ \dots \circ C_n^{\beta_n}$. Iwaniec and Martin show that if p_{2k} is a homogeneous polynomial of degree $2k$ we have that

$$\|p_{2k}(\mathbf{R})\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq \|p_{2k}(\mathbf{C})\|_{L^p(\mathbb{C}^n)\to L^p(\mathbb{C}^n)} \leq \frac{2\Gamma(n+k)\|B^k\|_p}{k\pi^n\Gamma(k)} \int_{\mathbb{S}^{2n-1}} |p_{2k}(z)|d\sigma(z),$$

where $\|B^k\|_p$ is the norm of the k-th iterated Beurling-Ahlfors transform on $L^p(\mathbb{C})$. Picking $p(x) = x_1^{2k}$ and computing the integral on the right-hand side using the formulas in Appendix D of [26], we see

$$\|R_1^{2k}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq \|B^k\|_p.$$

The L^p boundedness of B^k was studied by Dragicevic, Petermichl, and Volberg in [22] where they showed that

$$C_1 k^{1-2/p^*} p^* \leq \|B^k\|_p \leq C_2 k^{1-2/p^*} p^*.$$

Combining this with (4) gives

$$\|R_1^{2k}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leq C_2 k^{1-2/p^*} p^*.$$

Therefore,

$$\|T_{m_r} f\|_p \leq C_n \frac{\Gamma(\frac{n+r}{2})}{\Gamma(\frac{n+1}{2})} \left(\frac{r}{2}\right)^{1-2/p^*} p^* \|f\|_p.$$

Like the bound obtained in theorem 1.8.2, this bound is linear in p*. Futhermore, with* p *fixed it has order* $r^{(n+1)/2-2/p^*}$ *as* $r \to \infty$*, which is slightly better than the bound obtained in theorem 1.8.2. However, this bound has the disadvantage of only being valid when* r *is an even integer whereas the bound obtained in theorem 1.8.2 is valid for all sufficiently large* r*.*

3.2 The proof of theorem 1.8.2

Proof It is clear that $\|m_r\|_\infty \leq 1$, so by (1.9) it suffices to show that

$$\left(\sup_{R>0} R^{-n+2|\beta|} \int_{R<|\xi|<2R} |\partial^\beta m_r(\xi)|^2 d\xi\right)^{1/2} \leq C_n r^{|\beta|}$$

for all multi-indexes with $|\beta| \leq n_0$. But since m_r is homogeneous of order zero, we can make a change of variables and then use polar coordinates to see that

$$\begin{aligned}
\sup_{R>0} R^{-n+2|\beta|} \int_{R<|\xi|<2R} |\partial^\beta m_r(\xi)|^2 d\xi &= \int_{1<|\xi|<2} |\partial^\beta m_r(\xi)|^2 d\xi \\
&= \int_1^2 t^{n-1} \int_{\mathbb{S}^{n-1}} |\partial^\beta m_r(t\xi')|^2 d\sigma(\xi) dt \\
&= \int_1^2 t^{n-1-2|\beta|} dt \int_{\mathbb{S}^{n-1}} |\partial^\beta m_r(\xi')|^2 d\sigma(\xi) \\
&\leq C_n \int_{\mathbb{S}^{n-1}} |\partial^\beta m_r(\xi)|^2 d\sigma(\xi),
\end{aligned}$$

where $\xi' = \frac{\xi}{|\xi|}$. Therefore, it suffices to show that for all multi-indexes β with $|\beta| \leq n_0$,

$$\left(\int_{\mathbb{S}^{n-1}} |\partial^\beta m_r(\xi)|^2 d\sigma(\xi)\right)^{1/2} \leq C_n r^{|\beta|}. \tag{3.4}$$

As in (3.1), we see that

$$m_r(\xi) = C_n \frac{\Gamma(\frac{r+n}{2})}{\Gamma(\frac{r+1}{2})} n_r(\xi),$$

where

$$n_r(\xi) = \int_{\mathbb{S}^{n-1}} \frac{|\xi \cdot \theta|^r}{|\xi|^r} \varphi(\theta) d\sigma(\theta).$$

We will show that

$$\left(\int_{\mathbb{S}^{n-1}} |\partial^\beta n_r(\xi)|^2 d\sigma(\xi) \right)^{1/2} \leq C_n r^{|\beta|} \frac{\Gamma(\frac{r-n_0+1}{2})}{\Gamma(\frac{r-n_0+n}{2})},$$

and so (3.4) will follow by observing that Sterling's formula implies that there exists C_n such that for all $r \geq n_0$

$$\frac{\Gamma(\frac{r+n}{2})}{\Gamma(\frac{r+1}{2})} \frac{\Gamma(\frac{r-n_0+1}{2})}{\Gamma(\frac{r-n_0+n}{2})} \leq C_n.$$

For all $\theta \in \mathbb{S}^{n-1}$, let $m_\theta(\xi) = \frac{|\xi \cdot \theta|^r}{|\xi|^r}$ so that

$$\partial^\beta n_r(\xi) = \int_{\mathbb{S}^{n-1}} \partial^\beta m_\theta(\xi) \varphi(\theta) d\sigma(\theta).$$

We note that it suffices to show that for all $|\beta| \leq n_0$,

$$|\partial^\beta m_\theta(\xi)| \leq C_n r^{|\beta|} |\xi \cdot \theta|^{r-n_0}. \tag{3.5}$$

For then we see that

$$\begin{aligned}
\left(\int_{\mathbb{S}^{n-1}} |\partial^\beta n_r(\xi)|^2 d\sigma(\xi) \right)^{1/2} &\leq C_n r^{|\beta|} \left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\partial^\beta m_\theta(\xi)| d\sigma(\theta) \right)^2 d\sigma(\xi) \right)^{1/2} \\
&\leq C_n r^{|\beta|} \left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\xi \cdot \theta|^{r-n_0} d\sigma(\theta) \right)^2 d\sigma(\xi) \right)^{1/2} \\
&= C_n r^{|\beta|} \frac{\Gamma(\frac{r-n_0+1}{2})}{\Gamma(\frac{r-n_0+r}{2})}.
\end{aligned}$$

Let $g_\theta(\xi) = |\xi \cdot \theta|^r$ and $h(\xi) = |\xi|^{-r}$ so that $m_\theta(\xi) = g_\theta(\xi) h(\xi)$. By Leibniz's rule

$$\begin{aligned}
|\partial^\beta m_\theta(\xi)| &= \left| \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma g_\theta(\xi) \partial^\delta h(\xi) \right| \\
&\leq C_n \sum_{\gamma \leq \beta} |\partial^\gamma g_\theta(\xi)| |\partial^\delta h(\xi)|,
\end{aligned}$$

where $\delta = \beta - \gamma$.

Letting $\gamma = (\gamma_1, \ldots, \gamma_i)$ and $\delta = (\delta_1, \ldots, \delta_j)$, we see that when $|\theta| = |\xi| = 1$

$$\begin{aligned} |\partial^\gamma g_\theta(\xi)| &= r(r-1)\ldots(r-i+1)|\xi \cdot \theta|^{(r-i)} \, |\theta_{\gamma_1} \ldots \theta_{\gamma_i}| \\ &\leq r^i |\xi \cdot \theta|^{r-n_0} \end{aligned} \tag{3.6}$$

and

$$|\partial_\delta h(\xi)| = r(r+1)\ldots(r+j-1)|\xi|^{-r-2j} \left|\xi_{\delta_1} \ldots \xi_{\delta_j}\right| \leq C_n r^j. \tag{3.7}$$

(3.5) follows immediately which completes the proof. ■

Remark 5 *If we inspect the proof of theorem 1.8.2, we will see that if $r > n+1$, it follows from (3.6) and (3.7), that m_r is multiplier which satisfies the estimate*

$$|\xi|^{|\beta|} |\partial^\beta m_r(\xi)| \leq C_r \tag{3.8}$$

for all multi-indexes with $|\beta| \leq n+1$. Therefore, by a result of McConnell [34], m_r may be obtained using martingale transforms with respect to a Cauchy process.

3.3 The Method of Rotation for other Lévy Multipliers

We have seen that the Lévy multipliers which arise from martingale transforms with respect to α-stable processes can be studied analytically using the method of rotations. This approach has the disadvantage that it does not allow us to obtain as good of constants as those that are obtained through probabilistic methods. However, it has the advantage of allowing us to remove the restriction that $\alpha < 2$ and thereby obtain a larger class of operators which are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. It is natural to wonder if this method can be applied to study the multipliers which arise from other Lévy processes and if so will it again let us remove restrictions on any relevant parameters.

Let $(X_t)_{t\geq 0}$ be a Lévy process whose Lévy measure ν is rotationally-symmetric and absolutely continuous with respect to the Lebesgue measure. Write ν in polar coordinates as $d\nu = v(r)drd\sigma(\theta)$ for some function $v(r)$. Let φ be a bounded function on $\mathbb{R}^n$ that is homogeneous of order zero, and consider the multiplier given by

$$m_\nu(\xi) = \frac{\int_{\mathbb{R}^n}(\cos(\xi\cdot z)-1)\varphi(z)d\nu(z)}{\int_{\mathbb{R}^n}(\cos(\xi\cdot z)-1)d\nu(z)}.$$

Let $\rho(\xi)$ be the Lévy exponent corresponding to the Lévy triple $(0,0,\nu)$. Since the ν is symmetric, $\rho(\xi)$ is real, and therefore

$$\int_{\mathbb{R}^n}(\cos(\xi\cdot z)-1)d\nu(z) = \rho(\xi). \tag{3.9}$$

To examine the numerator define $L:\mathbb{R}\to\mathbb{R}$ by

$$L(x) = \int_0^\infty (\cos(rx)-1)v(r)dr. \tag{3.10}$$

Then, we have that

$$\int_{\mathbb{R}^n}(\cos(\xi\cdot z)-1)\varphi(z)d\nu(z) = \int_{\mathbb{S}^{n-1}} L(\xi\cdot\theta)\varphi(\theta)d\sigma(\theta). \tag{3.11}$$

Therefore, combining (3.9) and (3.11) we see that the multiplier which arises as the projection of martingale transforms with respect to X_t is given by

$$m_\nu(\xi) = \int_{\mathbb{S}^{n-1}} \frac{L(\xi\cdot\theta)}{\rho(\xi)}\varphi(\theta)d\sigma(\theta).$$

Similarly to section 3.1, we set $m_\theta(\xi) = \frac{L(\xi\cdot\theta)}{\rho(\xi)}$ so that

$$m_\nu(\xi) = \int_{\mathbb{S}^{n-1}} m_\theta(\xi)\varphi(\theta)d\sigma(\theta).$$

Then repeating the arguments of section 3.1, we see that if $T_{m_{e_1}}$ is bounded on $L^p(\mathbb{R}^n)$, then T_{m_r} is bound on $L^p(\mathbb{R}^n)$.

More generally, we have the following corollary.

Corollary 3 *For any function $L:\mathbb{R}\to\mathbb{R}$, let $AL(\xi) = \int_{\mathbb{S}^{n-1}} L(\xi\cdot\theta)d\sigma(\theta)$. If $m_{e_1}(\xi) = \frac{L(\xi_1)}{AL(\xi)}$ is an L^p multiplier for some $1<p<\infty$, then for all $\varphi\in L^\infty(\mathbb{S}^{n-1})$.*

$$m_L(\xi) = \frac{\int_{\mathbb{S}^{n-1}} L(\xi\cdot\theta)\varphi(\theta)d\sigma(\theta)}{\int_{\mathbb{S}^{n-1}} L(\xi\cdot\theta)d\sigma(\theta)}.$$

is also an L^p multiplier. In particular, if for some $C_{n,p}>0$,

$$\|T_{m_{e_1}}f\|_p \le C_{n,p}\|f\|_p \quad \textit{for all } f\in L^p,$$

then

$$\|T_{m_L}f\|_p \le \omega_{n-1}C_{n,p}\|f\|_p \quad \textit{for all } f\in L^p.$$

Consider now, for $0 < \beta < \alpha < 2$, the so-called "mixed-stable" process defined by, $Z_t = X_t + aY_t$ where X_t is a rotationally-symmetric α-stable process, Y_t is an independent rotationally symmetric β-stable process, and $a > 0$. Z_t is a Lévy process with exponent $\rho(\xi) = -(|\xi|^\alpha + a^\beta|\xi|^\beta)$ and Lévy measure

$$d\nu(z) = (C_{n,\alpha} r^{-1-\alpha} + C_{n,\beta} a^\beta r^{-1-\beta}) dr d\sigma(\theta).$$

In this case, by an argument similar to the α-stable case, the corresponding multiplier is given by

$$m_{\alpha,\beta}(\xi) = \frac{\int_{\mathbb{S}^{n-1}} (C_{n,\alpha}|\xi\cdot\theta|^\alpha + C_{n,\beta,a}|\xi\cdot\theta|^\beta)\varphi(\theta)d\sigma(\theta)}{\int_{\mathbb{S}^{n-1}} (C_{n,\alpha}|\xi\cdot\theta|^\alpha + C_{n,\beta,a}|\xi\cdot\theta|^\beta)d\sigma(\theta)}.$$

It is already known that $m_{\alpha,\beta}$ is an L^p multiplier for $1 < p < \infty$ by the results of [9] and [5]. However, the method of rotations allows us to to remove the restriction that $0 < \beta < \alpha < 2$. More precisely, we can prove the following.

Corollary 4 *Let $0 < r < s < \infty$, let $C_r, C_s > 0$, and let $\varphi \in L^\infty(\mathbb{R}^n)$. Then $m_{r,s}$ defined by*

$$m_{r,s}(\xi) = \frac{\int_{\mathbb{S}^{n-1}} (C_r|\xi\cdot\theta|^r + C_s|\xi\cdot\theta|^s)\varphi(\theta)d\sigma(\theta)}{\int_{\mathbb{S}^{n-1}} (C_r|\xi\cdot\theta|^r + C_s|\xi\cdot\theta|^s)d\sigma(\theta)}.$$

is an L^p multiplier, for all $1 < p < \infty$ and

$$\|T_{m_{r,s}} f\|_p \leq C_{n,r,s}(p^* - 1)^{6n}\|f\|_p \quad \textit{for all } f \in L^p(\mathbb{R}^n).$$

Proof As in the proof of theorem 1.8.1, the integral in the denominator can be computed directly and

$$\int_{\mathbb{S}^{n-1}} (C_r|\xi\cdot\theta|^r + C_s|\xi\cdot\theta|^s)d\sigma(\theta) = C_r'|\xi|^r + C_s'|\xi|^s.$$

Therefore, in light of corollary 3 it suffices to show that

$$m_{e_1}(\xi) = \frac{C_r|\xi_1|^r + C_s|\xi_1|^s}{C_r'|\xi|^r + C_s'|\xi|^s}$$

is an Marcinkiewicz multiplier. As in the proof of lemma 4, we restrict attention to the region where all ξ_i are non-negative, and check that m_{e_1} satisfies (3.3). We already know that $\frac{|\xi_1|^r}{|\xi|^r}$ satisfies (3.3) so it suffices to show that

$$n(\xi) = \frac{1 + a|\xi_1|^t}{b + c|\xi|^t}$$

satisfies (3.3) for all $a, b, c, t > 0$ since it is easy to check using Leibniz's rule that the product of two multipliers which satisfy (3.3) is again a multiplier satisfying (3.3).

Applying Faá di Bruno's formula to the function $g(h(\xi))$, where $h(\xi) = |\xi|^2$ and $g(x) = \frac{1}{b+cx^{t/2}}$, we see that $\partial_{i_1} \ldots \partial_{i_k} \frac{1}{b+c|\xi|^t}$ is a finite linear combination of terms of the form

$$\left(\frac{|\xi|^t}{b+c|\xi|^t}\right)^i \frac{\xi_{i_1} \ldots \xi_{i_k}}{|\xi|^{2k}} \frac{1}{b+c|\xi|^t}, \quad 0 \leq i \leq k.$$

(3.3) then follows easily which completes the proof. ∎

Consider again the relativistic α-stable process which was introduced in section 2.2. Here we will show that the multipliers which arise from taking the projections of martingale transforms with respect to this process can be studied using the method of rotations. Unfortunately, unlike in the case of the mixed stable processes, the fact that $0 < \alpha < 2$ will play a crucial role in the proof. Therefore, we will not be able to remove that restriction and obtain a larger class of operators.

Corollary 5 *Let $0 < \alpha < 2$, $M > 0$, and $\varphi \in L^\infty(\mathbb{R}^n)$ homogeneous of order zero. Let $d\nu(z) = r^{-1-\alpha}\phi(r)drd\theta$ be the Lévy measure corresponding to the relativistic α-stable process with mass M and let L be defined as in (3.10). Then $\frac{L(\xi_1)}{\rho(\xi)}$ is a Marcinkiewicz multiplier and therefore, by corollary 3*

$$m_\nu = \frac{\int_{\mathbb{R}^n}(1-\cos(\xi\cdot\theta))\varphi(\theta)d\nu(z)}{\int_{\mathbb{R}^n}(1-\cos(\xi\cdot\theta))d\nu(z)}$$

is an L^p multiplier and

$$\|T_{m_\nu} f\|_p \leq C_{n,\alpha}(p^*-1)^{6n}\|f\|_p.$$

This is of course a weaker version of results already proven in [9] and [5], but nevertheless, it is interesting to observe that this result can also be obtained analytically.

Proof In [20], it is shown that the Lévy measure corresponding to X_t can be written in polar coordinates by

$$d\nu(z) = r^{-1-\alpha}\phi(r)drd\sigma(\theta)$$

where $\phi(r)$ is a bounded positive function that that satisfies

$$\phi(r) \leq Ce^{-r}r^{(n+\alpha-1)/2} \tag{3.12}$$

when $r \geq 1$.

Now, by Faá di Bruno's formula, $\partial_{i_1} \dots \partial_{i_k} \frac{1}{\rho(\xi)}$ is a finite linear combination of terms with the form

$$\frac{\xi_{i_1} \dots \xi_{i_k} (|\xi|^2 + M^{2/\alpha})^{\frac{\alpha}{2}j-k}}{((|\xi|^2 + M^{2/\alpha})^{\frac{\alpha}{2}} - M)^{j+1}}, \quad 0 \leq j \leq k. \tag{3.13}$$

Therefore, we see that $\frac{1}{\rho(\xi)}$ is infinitely differentiable on $\mathbb{R}^n \setminus \{0\}$ and

$$\left| \partial_{i_1} \dots \partial_{i_k} \frac{1}{\rho(\xi)} \right| \leq O\left(\frac{1}{|\xi|^{\alpha+k}} \right) \quad \text{as } |\xi| \to \infty.$$

Near 0, each term in (3.13) is bounded above by

$$\begin{aligned} &C_{M,n,\alpha} \frac{1}{(|\xi|^2 + M^{2/\alpha})^{\frac{\alpha}{2}} - M)^{j+1}} \\ &\leq C_{M,n,\alpha} \frac{1}{(|\xi|^2 + M^{2/\alpha})^{\frac{\alpha}{2}} - M)} \leq O\left(\frac{1}{|\xi|^2} \right) \quad \text{as } |\xi| \to 0. \end{aligned}$$

It is easy to check using the dominated convergence theorem, the mean value theorem and the fact that $r^{k-\alpha}\phi(r)$ is integrable on $(0, \infty)$ for all $k \geq 1$, that L is infinitely differentiable on $(0, \infty)$. Therefore, in order to show that $\frac{L(\xi_1)}{\rho(\xi)}$ is a Marcinkiewicz multiplier it suffices to show that

$$|L(\xi)| \leq C_\alpha \min\{|\xi|^\alpha, |\xi|^2\} \tag{3.14}$$

and

$$|L'(\xi)| \leq C_\alpha \min\{|\xi|^{\alpha-1}, |\xi|\}. \tag{3.15}$$

For then it will follow that $\frac{L(\xi_1)}{\rho(\xi)}$ satisfies (3.3) since

$$\left| \xi_{i_1} \dots \xi_{i_k} \partial_{i_1} \dots \partial_{i_k} \frac{L(\xi_1)}{\rho(\xi)} \right|$$

is a continuous function on $\mathbb{R}^n \setminus \{0\}$ which is bounded near the origin and as $|\xi| \to \infty$.

Making a change of variables, we see that

$$\begin{aligned} |L(x)| &= \left| \int_0^\infty (\cos(rx) - 1) r^{-1-\alpha} \phi(r) dr \right| \\ &= |x|^\alpha \left| \int_0^\infty (\cos(s) - 1) s^{-1-\alpha} \phi\left(\frac{s}{|x|}\right) ds \right| \leq C_\alpha |x|^\alpha, \end{aligned}$$

where the last inequality uses the boundedness of ϕ. On the other hand we can use the inequality $|\cos(x) - 1| \leq x^2$, along with (3.12) and the boundedness of ϕ to see that

$$\begin{aligned} |L(x)| &= \left| \int_0^\infty (\cos(rx) - 1) r^{-1-\alpha}) \phi(r) dr \right| \\ &\leq |x|^2 \left| \int_0^\infty r^{1-\alpha} \phi(r) dr \right| \leq C_\alpha |x|^2. \end{aligned}$$

This proves (3.14). Note that the fact that $0 < \alpha < 2$ is needed in order for this integral to converge.

To prove (3.15) observe that

$$L'(x) = \int_0^\infty \sin(rx) r^{-\alpha} \phi(r) dr.$$

Using the fact that $|\sin(x)| \leq |x|$, it follows that $|L'(x)| \leq C_\alpha |x|$ by mimicing the above arguments. To obtain the other part of (3.15) we make a change of variables, and use the fact that φ is decreasing to see

$$\begin{aligned} |L'(x)| &= |x|^{\alpha-1} \left| \int_0^\infty \sin(t) t^{-\alpha} \varphi\left(\frac{t}{x}\right) dt \right| \\ &= |x|^{\alpha-1} \sum_{n=0}^\infty (-1)^n \int_{n\pi}^{(n+1)\pi} \left| \sin(t) t^{-\alpha} \varphi\left(\frac{t}{x}\right) \right| dt \\ &\leq |x|^{\alpha-1} \left| \int_0^\pi \sin(t) t^{-\alpha} \varphi\left(\frac{t}{x}\right) dt \right| \\ &\leq C_\alpha |x|^{\alpha-1}. \end{aligned}$$

This completes the proof of corollary (5). ∎

REFERENCES

REFERENCES

[1] D. Applebaum and R. Bañuleos. Martingale transforms and Lévy processes on Lie groups. *Indiana University Mathematics Journal*, 63:1109–1138, 2014.

[2] K. Astala, T. Iwaniec, and G. Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*. Princeton University Press, 2009.

[3] R. Bañuelos. Martingale transforms and related singular integrals. *Transactions of the American Math. Soc.*, 293(2):547–563, 1986.

[4] R. Bañuelos. The foundational inequalities of D.L. Burkholder and some of their ramifications. *Illinois J. Math.*, 54:789–868, 2010.

[5] R. Bañuelos, A. Bielaszewski, and K. Bogdan. Fourier multipliers for non-symmetric Lévy processes. *Marcinkiewicz Centenary Volume*, 95:9–25, 2011.

[6] R. Bañuelos and A. Osękowski. Sharp martingale inequalities and applications to Riesz transforms on manifolds, Lie groups and Gauss space. *J. Functional Analysis*, 269(6):1652–1713.

[7] R. Bañuelos and F. Baudoin. Martingale transforms and their projection operators on manifolds. *Potential Analysis*, 38:1071–1089, 2013.

[8] R. Bañuelos and G. Wang. Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms. *Duke Math. J.*, 80(3):575–600, 1995.

[9] R. Bañuelos and K. Bogdan. Lévy processes and Fourier multipliers. *J. Funct. Anal.*, 250:197–213, 2007.

[10] R. Bañuelos and P. Janakiraman. L^p-bounds for the Beurling-Ahlfors transform. *Trans. Amer. Math. Soc.*, 360:3604–3612, 2008.

[11] R. Bañuelos and P. Méndez-Hernández. Space-time Brownian motion and the Beurling-Ahlfors transform. *Indiana University Math J.*, 52:981–990, 2003.

[12] R. Bañuelos and S. Yolcu. Heat trace of non-local operators. *J. Long. Math. Soc. (2)*, 87(3):304–318, 2014.

[13] J. Bertoin. *Lévy Processes.* Cambridge University Press, Cambridge, 1996.

[14] R. M. Blumenthal and R. K. Getoor. *Markov Processes and Potential Theory.* Pure Appl. Math, Academic Press, New York, 1968.

[15] K. Bogdan, A. Stós, and P. Sztonyk. Harnack inequality for symmetric stable processes on d-sets. *Studia Math.*, 158:163–198, 2003.

[16] A. Borichev, P. Janakiraman, and A. Volberg. Subordination by orthogonal martingales in L^p and zeros of Laguerre polynomials. *Duke Math. J.*, 162(5):889–924, 2013.

[17] D. L. Burkholder. Martingale transforms. *Ann. Math. Statist.*, 37:1494–1504, 1966.

[18] D. L. Burkholder. Boundary value problems and sharp inequalities for martingale transforms. *Ann. Probab.*, 12:647–702, 1984.

[19] D. L. Burkholder. Sharp inequalities for martingales and stochastic integrals. *Asterique*, 157-158:75–94, 1988.

[20] Z.-Q. Chen, P. Kim, and R. Song. Sharp heat kernel estimates for relativistic stable processes on open sets. *Annals of Probability*, 40(1):213–244, 2012.

[21] O. Dragičević, L. Grafakos, M. Pereya, and S. Petermichl. Extrapolation and sharp norm estimates for the classical operators on weighted Lebesgue spaces. *Publ. Mat.*, 49(1):73–91, 2005.

[22] O. Dragičević, S. Petermichl, and A. Volberg. A rotation method which gives linear L^p estimates for powers of the Ahlfors-Beurling operator. *J. Math, Pures Appl.*, 86 no.6:494–509, 2006.

[23] R. Durrett. *Brownian Motion and Martingales in Analysis.* Wadsworth, Belmont, CA, 1984.

[24] A. Osękowski. Sharp logarithmic inequalities for Riesz transforms. *J. Funct. Anal.*, 263:89–108, 2012.

[25] S. Geiss, S. Mongomery-Smith, and E. Saksman. On singular integral and martingale transforms. *Trans. Amer. Math. Soc.*, 362:555–575, 2010.

[26] L. Grafakos. *Classical and Modern Fourier Analysis.* Pearson Education, Inc. New Jersey, 2004.

[27] R. Gundy and N. Varopoulos. Les transformations de Riesz et les integrales stochastiques. *C. R. Acad. Sci. Paris Ser. A*, 289:13–16, 1979.

[28] T. P. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Annals of Math.*, 175 Issue 3:1473–1506, 2012.

[29] T. Iwaniec. Extremal inequalities in Sobolev spaces and quasiconfromal mappings. *Z. Anal, Anwendungen*, 6:1–16, 1982.

[30] T. Iwaniec and G. Martin. The Riesz transforms and related singular integrals. *J. Reine Agnew. Math.*, 473:25–58, 1996.

[31] P. Janakiraman. Weak-type estimates for singular integrals and the Riesz transforms. *Indiana University Math. J.*, 53(2):533–555, 2004.

[32] P. Kim, R. Song, and Z. Vondracek. Potential theory of subordinate Brownian motion revisited. *Stochastic Analysis and Applications to Finance–Essays in Honour of Jia-an Yan, World Scientific*, pages 243–290, 2012.

[33] O. Lehto. Remarks on the integrability of the derivative of quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A. Mat.*, 371:3–8, 1965.

[34] T. McConnell. On Fourier multiplier transformations of Banach-valued functions. *Trans. Amer. Math. Soc.*, 285:739–757, 1984.

[35] F. Nazarov and A. Volberg. Heat extension of the Beurling operator and estimates for its norm. *St. Petersburg Math. J.*, 15:563–573, 2004.

[36] P. E. Protter. *Stochastic Integration and Differential Equations*, volume **21**. Stoch. Model. Appl. Probab, Springer-Verlag, Berlin, second edition, 2004.

[37] M. Ryznar. Estimates of Green function for relativistic α-stable processes. *Potential Analysis*, 17:1–23, 2002.

[38] K. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1968.

[39] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, 1979.

[40] E. M. Stein. Some results in harmonic analysis in R^n for $n \to \infty$. *Bull. Amer. Math. Soc. (N.S.)*, 9:71–73, 1983.

[41] G. Wang. Differential subordination and strong differential subordination for continuous-time martingales and related sharp inequalities. *Ann. Probab.*, 23(2):552–551, 1995.

VITA

VITA

Michael Perlmutter was born in Boston, Massachusetts on March 5, 1988. He grew up in nearby Sherborn, Massachusetts. He attended Tufts University in Medford, Massachusetts from 2006 to 2010, where he earned a bachelor's degree in mathematics. In the fall of 2010, he enrolled in graduate school at Purdue University.