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Rees algebras and iterated Jacobian duals

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REES ALGEBRAS AND ITERATED JACOBIAN DUALS

A Dissertation

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of

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by

Vivek Mukundan

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This thesis is dedicated to my parents.

“ *It is an old maxim of mine that when you have excluded the impossible, whatever remains, however improbable, must be the truth.*”

- Sherlock Holmes

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ABSTRACT

Mukundan, Vivek PhD, Purdue University, August 2016. Rees algebras and Iterated Jacobian duals. Major Professor: Bernd Ulrich.

Consider the rational map $\Psi : \mathbb{P}^{d-1} \xrightarrow{[f_1:\cdots:f_m]} \mathbb{P}^{m-1}$ where the f_i's are homogeneous forms of the same degree in the homogeneous coordinate ring $R = k[x_1, \ldots, x_d]$ of $\mathbb{P}^{d-1}$. Assume that $I = (f_1, \ldots, f_m)$ is a height 2 perfect ideal in the polynomial ring R. In this context, the coordinate ring of the graph of Ψ is the Rees algebra of I and the co-ordinate ring of the image of Ψ is the special fiber ring. We study two settings. The first setting is when I is almost linearly presented. Here we study the ideal defining the graph and the image of Ψ. Whenever possible, we also study invariants such as the Castelnuovo-Mumford regularity and the relation type of the graph of Ψ. In the second setting we impose no constraints on the column degrees of the presentation matrix of I, but the number of generators of I is restricted to $d+1$ (two more than the dimension of the source of Ψ). For this configuration, we study the image of Ψ.

We also introduce a new method, namely "iterated" Jacobian duals, to study the graph of Ψ. This is a generalization of the usual Jacobian duals which are often used to describe the graph of Ψ.

1. INTRODUCTION

The primary focus of this thesis is to find the defining ideal of Rees algebras of ideals in a polynomial ring. As a consequence we also solve the implicitization problem under some conditions.

The Rees algebra is one of the most often studied blow-up algebras. Rees algebras provide an algebraic realization for the concept of blowing up a variety along a subvariety. The search for the implicit equations defining the Rees algebra is a classical and fundamental problem which has been studied for many decades. In low dimensions, the implicitization problem has often been referred to as the moving-curve and moving-surface ideal problem and has significant applications in the area of computer aided geometric design. For example, it can be used to draw a curve/surface near a singularity, compute intersections with parametrized curves/surfaces etc.

Let $I = (f_1, \ldots, f_m)$ be an ideal of R. Then the Rees algebra of I is $\mathcal{R}(I) = R[It] \subseteq R[t]$. Since the Rees algebra is an R-algebra we define an epimorphism $\Phi : R[T_1, \ldots, T_m] \to \mathcal{R}(I)$ given by $\Phi(T_i) = f_i t$. We are interested in finding a complete generating set for the ideal, $\ker \Phi$, "defining" $\mathcal{R}(I)$. The kernel $\ker \Phi$ is called the defining ideal (or defining equations) of $\mathcal{R}(I)$.

To describe the implicitization problem, consider the rational map

$$\Psi : \mathbb{P}_k^{d-1} \xrightarrow{[f_1:\cdots:f_m]} \mathbb{P}_k^{m-1}$$

where the f_i's are homogeneous forms of the same degree in the homogeneous co-ordinate ring $R = k[x_1, \ldots, x_d]$ of $\mathbb{P}_k^{d-1}$. The implicitization problem involves finding the defining ideal of the image of Ψ. A way to solve this problem is to study the Rees algebra of the ideal $I = (f_1, \ldots, f_m)$. This is because the Rees algebra gives the bi-homogeneous co-ordinate ring of the graph of Ψ and the special fiber ring gives the homogeneous co-ordinate ring of the image of Ψ.

The primal nature of $\ker \Phi$ makes it harder to decipher. If one were to compute the kernel directly, devoid of sophistry, the resulting equations may be quite hard to read and the algebraic properties such as Cohen-Macaulayness, relation type etc. of $\mathcal{R}(I)$ are harder to unravel. So the emphasis must be laid on finding *forms* of the defining equations which help in the twin tasks of computing the defining equations and also of studying various algebraic properties of $\mathcal{R}(I)$, without much effort.

The symmetric algebra $\mathrm{Sym}(I)$ and the associated graded ring $\mathrm{gr}_I(R)$ are other blow-up algebras often studied in conjunction with the Rees algebra. An easy observation shows that the map Φ factors through the symmetric algebra and hence to study $\ker \Phi$, we often study $\mathcal{A} = \ker(\mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}(I))$. Let $\mathcal{L} \subseteq R[T_1, \ldots, T_m]$ denote the defining ideal of $\mathrm{Sym}(I)$. When the symmetric algebra and the Rees algebra of an ideal I are isomorphic, the ideal I is said to be of *linear type*. The study of ideals of linear type has been very extensive ([1–3]). The most general theorem characterizing ideals of linear type is by using the theory of approximation complexes developed by Herzog, Simis, Vasconcelos [4].

The starting point of our investigation is the following result of Herzog, Simis and Vasconcelos. These authors show that when the ideal I is strongly Cohen-Macaulay and satisfies

$$\mu(I_P) \leq \mathrm{ht}\ P \text{ for all } P \in V(I) \tag{1.1}$$

then the ideal I is of linear type. The class of strongly Cohen-Macaulay ideals is reasonably large and includes licci ideals such as height two perfect and height three Gorenstein ideals. In the course of proving these results the authors introduced the notion of approximation complexes in [4]. These complexes turned out to be a powerful tool in the study of ideals of linear type and also of Rees algebras in general. The approximation complexes, $\mathcal{M}_\bullet$ and $\mathcal{Z}_\bullet$, “approximate” the resolutions of $\mathrm{Sym}(I)$, $\mathrm{Sym}(I/I^2)$. The acyclicity of these complexes is equivalent, under some conditions, to the ideal I being generated by a d-sequence or a proper sequence, respectively. These complexes also provide information on the Cohen Macaulayness and Gorensteiness of the Rees algebra $\mathcal{R}(I)$ and the associated graded ring $\mathrm{gr}_I(R)$.

Ideals generated by more than $\dim R$ elements cannot be of linear type. To study the Rees algebra of I in this case, Huckaba and Huneke in [5] used a reduction of I that can be generated by $\dim R$ elements. A reduction J of I is closely related to I, in that, $\mathcal{R}(I)$ is module finite over $\mathcal{R}(J)$. The reduction number $r_J(I)$ measures how "closely" the two ideals are related. Even though I is not of linear type, one can hope to salvage algebraic properties of the Rees algebra $\mathcal{R}(I)$ by studying the respective properties of $\mathcal{R}(J)$. So it is also very useful to study the properties of the Rees algebra of the reduction J and then study the transfer of properties between $\mathcal{R}(I)$ and $\mathcal{R}(J)$. When the reduction number is very small, Huneke and Huckaba show that $\mathcal{R}(J)$ is Cohen Macaulay and the property does transfer to $\mathcal{R}(I)$.

Under suitable assumptions, the reduction number of strongly Cohen Macaulay ideals is bounded by $\ell(I) - g + 1$, where $\ell(I)$ and g are the analytic spread and the height of the ideal. In fact $\ell(I) - g + 1$ is the smallest positive number the reduction number $r(I)$ can possibly attain [6]. Such a reduction number is called the *expected reduction number*. Ulrich in [7] shows that the expected reduction number characterizes the Cohen Macaulayness of $\mathcal{R}(I)$ in the case of grade 2 perfect ideals. Morey and Ulrich in [8] extend the characterization, under suitable conditions, by showing that the Rees algebra $\mathcal{R}(I)$ is Cohen Macaulay if and only if the defining ideal of the Rees algebra is of the expected form. This description, as most traditional descriptions of the defining ideal of $\mathcal{R}(I)$, revolves around the notion of *Jacobian Duals*. The Jacobian dual is a matrix, $B(\varphi)$, which dualizes the presentation matrix φ with respect to $I_1(\varphi) = (a_1, ..., a_r)$. In general, it can be shown that the expected form is the smallest possible ideal the defining ideal of $\mathcal{R}(I)$ can possibly be equal to, when I is not of linear type.

The expected form is one such form that supports the earlier emphasis on the shape of the defining ideals and its twin uses. Morey and Ulrich show that when R is a polynomial ring and I is grade 2 perfect ideal with a linear presentation matrix, then the defining ideal of $\mathcal{R}(I)$ is of the expected form. One important assumption which appears in this result is a weakening of condition (1.1), namely,

$$\mu(I_P) \leq \text{ht } P \text{ for all } P \in V(I) \text{ with ht } P \leq d-1 \tag{1.2}$$

In the literature, the above condition (1.2) is called the G_d condition. As a direct consequence of [4, 2.6], this condition means that strongly Cohen Macaulay ideals which satisfy the G_d condition are of linear type locally on the "punctured" spectrum. The presentation matrix of I being linear is a noticeable constraint and hence natural questions on the nature of the defining ideal of $\mathcal{R}(I)$ when the presentation matrix is non-linear can be asked. Such questions bring the focus to the problems being discussed in this thesis. Not all the grade 2 perfect ideals have the property that the defining ideal of the Rees algebra is of the expected form. Thus we focus our attention to the case of grade 2 perfect ideals generated by homogeneous elements of the same degree.

Using the Hilbert-Burch theorem, such an ideal can be realized as the ideal generated by the maximal minors of a $m \times m-1$ matrix with homogeneous entries of constant degree along each column. We first restrict the presentation matrix φ of I to be "almost linearly presented", that is, all but the last column of φ are linear and the last column consists of homogeneous entries of arbitrary degree $n \geq 1$.

When $d = 2$, Kustin, Polini, Ulrich in [9, 2.4] gave a description of the defining ideals of the Rees algebra of grade 2 perfect almost linearly presented ideals. Their work involves a construction of a "well-behaved" ring A mapping onto the Rees algebra, so that the kernel of the map $A \twoheadrightarrow \mathcal{R}(I)$ is a height one prime ideal. The ring A, is the homogeneous coordinate ring of a rational normal scroll built with the presentation matrix φ of I. Describing explicit representatives of divisor classes on rational normal scrolls, they constructed a new height one prime ideal K and prove that $\overline{\ker \Phi} \cong_A K^{(n)}$. Recall that n is the degree of the entries in the last column of the presentation matrix φ. This type of construction is very productive as they go on to give a complete description of the generators of $K^{(n)}$. The latter form of the defining equation was also useful in determining various properties of the Rees algebra like the depth, analytic spread of I, the reduction number of I etc.

One of the problems this thesis discusses is the efforts taken extending the above result to $d > 2$. We were able to construct a close enough ring A and also an appropriate height one prime K to show that $\overline{\ker \Phi} \cong K^{(n)}$. Using this result we also show that $\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d)^n$. Notice that when the presentation matrix φ is linear ($n = 1$), the theory of residual intersections will immediately gives

$$\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d) = \mathcal{L} + I_d(B(\varphi))$$

recovering the result of Morey and Ulrich. A complete generating set of $K^{(n)}$, similar to the one presented in [9, 3.6] resisted generalization to the case of $d > 2$,' mainly due to the uncharacterizable nature of the presentation matrix φ.

The thesis introduces the notion of *iterated Jacobian duals*. It is an attempt to study the generators of $\mathcal{L} : (x_1, \ldots, x_d)^n$. This method extends the notion of Jacobian duals, and helps in constructing generators for $\ker \Phi$. For the presentation matrix φ with $I_1(\varphi) \subseteq (a_1, \ldots, a_r)$, we set $B_1(\varphi) = B(\varphi)$ and we iteratively construct $B_i(\varphi)$ from $B_{i-1}(\varphi)$ (we refer to Chapter 3 for details on the construction). By construction, $\mathcal{L} + I_r(B_i(\varphi)) \subseteq \mathcal{L} + I_r(B_{i+1}(\varphi))$. Though $B_i(\varphi)$ may not be uniquely determined, we prove that $\mathcal{L} + I_r(B_i(\varphi))$ is uniquely determined when $a_1, \ldots, a_r$ is an R-regular sequence.

Theorem 1.0.1 *Let R be a Noetherian ring and φ be a presentation matrix of the ideal I with entries in R. Suppose $I_1(\phi) \subseteq (a_1, \ldots, a_r)$ and $a_1, \ldots, a_r$ is a regular sequence. Then the ideal $\mathcal{L} + I_r(B_i(\phi))$ of $R[T_1, \ldots, T_m]$ is uniquely determined by the matrix ϕ and the regular sequence $a_1, \ldots, a_r$.*

One can also show that $(\mathcal{L}, I_d(B_i(\varphi))) \subseteq \mathcal{L} : (x_1, \ldots, x_d)^i$. Thus it is interesting to study when these two ideals are equal. We present a condition, namely the equality $K^n = K^{(n)}$ in the ring A, for when $\ker \Phi$ coincides with the ideal of the iterated Jacobian dual $(\mathcal{L} + I_d(B_n(\varphi)))$.

Theorem 1.0.2 *Let $R = k[x_1, \ldots, x_d]$ and I be a grade 2 perfect ideal. Suppose that the presentation matrix φ of I is almost linear, that is, all but the last column of φ is linear and the last column consists of homogeneous entries of arbitrary degree $n \geq 1$. Further assume that the ideal I satisfies G_d. If $K^n = K^{(n)}$ then* $\ker \Phi = \mathcal{L} + I_d(B_n(\varphi))$.

Next, under the above hypotheses, we study ideals which satisfy $\mu(I) = d + 1$. Such ideals are also known as ideals of second analytic deviation one. For such ideals one has $K^n = K^{(n)}$ in the ring A and hence $\ker \Phi = (\mathcal{L} + I_d(B_n(\varphi)))$. Furthermore, such an explicit form of the defining ideal also helps to determine the relation type of I, the Castelvnuovo-Mumford regularity and the Cohen-Macaulayness of $\mathcal{R}(I)$.

Theorem 1.0.3 *Let $R = k[x_1, \ldots, x_d]$ and I be a grade 2 perfect ideal. Suppose that the presentation matrix of I is almost linear, that is, all but the last column of φ is linear and the last column consists of homogeneous entries of arbitrary degree $n \geq 1$.Further assume that the ideal I satisfies G_d and $\mu(I) = d + 1$. Then*

(*a*) *the defining ideal of $\mathcal{R}(I)$ satisfies* $\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d)^n = \mathcal{L} + I_d(B_n(\varphi))$.

(*b*) *$\mathcal{R}(I)$ is almost Cohen-Macaulay (i.e.,* $\operatorname{depth} \mathcal{R}(I) = \dim \mathcal{R}(I) - 1$) *and the special fiber ring $\mathcal{F}(I)$ is Cohen-Macaulay.*

(*c*) *the relation type satisfies* $\operatorname{rt}(I) = n(d-1) + 1$ *where* $\operatorname{rt}(I)$ *is defined to be the maximum $\underline{T}$-degree appearing in a homogeneous minimal generating set of* $\ker \Phi$.

(*d*) *the Castelnuovo-Mumford regularity satisfies* $\operatorname{reg} \mathcal{R}(I) = n(d-1)$.

When restated, the above theorem gives explicit generators defining the graph of Ψ (and hence the image of Ψ) when the presentation matrix is almost linear. Under the hypotheses of the previous theorem we attempt to find the equations defining the image of Ψ with no constraints on the presentation matrix. This is the next question we study in this thesis. As a consequence, a method to check when the map Ψ is birational onto its image was found.

The map Ψ is said to be birational onto its image when there exists a rational map $\Upsilon : \operatorname{Im} \Psi \to \mathbb{P}^{d-1}$ such that $\Upsilon \circ \Psi = \mathrm{id}$. The criteria for the rational map Ψ to be birational on to its image is of interest to geometers. Much work has been done on the problem in the case $m = d$ in [10]. The technique of using the Rees algebras as a means to check birationality was emphasized in [11, 12].

Henceforth, we assume that the $d+1 \times d$ presentation matrix φ has homogeneous columns of the same degree e_i, where $1 \leq e_1 \leq \cdots \leq e_{d+1}$. The starting point of our investigation for computing the image of Ψ was the theorem of Jouanolou [13]. A non-constructive proof of the same has been given by Kustin, Polini, Ulrich in [14]. This theorem provides a method to study $\mathcal{A}$ by considering the dual of the symmetric algebra. The ring $B = R[T_1, \cdots, T_{d+1}] = k[x_1, \ldots, x_d][T_1, ..., T_{d+1}]$ is a bigraded algebra as it is naturally endowed with the bigrading $\deg x_i = (1,0), \deg T_j = (0,1)$. Thus both $\mathcal{A}$ and $\mathrm{Sym}(I)$ become bigraded B-modules. This paves the way to define modules $\mathcal{A}_i = \oplus_j \mathcal{A}_{(i,j)}$ and $\mathrm{Sym}(I)_i = \oplus_j \mathrm{Sym}(I)_{(i,j)}$ over the ring $S = k[T_1, \ldots, T_{d+1}]$. The theorem of J.-P. Jouanaolou states that

$$\mathcal{A}_i \cong \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I)_{\delta-i}, S(-d))$$

where $\delta = \sum_j e_j - d$ and $\underline{\mathrm{Hom}}$ denotes the graded dual. With the hypotheses of Theorem 1.0.3, it is well known that the image of Ψ is a hypersurface. Since the defining ideal of the image of Ψ is $\mathcal{A}_0$, we conclude that $\mathcal{A}_0$ is principally generated. By Jouanolou's theorem, notice that $\mathcal{A}_0 \cong \mathrm{Hom}_S(\mathrm{Sym}(I)_\delta, S(-d))$. We first begin by finding a generator of $\mathrm{Hom}_S(\mathrm{Sym}(I)_\delta, S(-d))$. By computing the dimension of $\mathrm{Sym}(I)$, we first notice that $\mathrm{Sym}(I)$ is a complete intersection ring. Thus the Koszul complex $\mathcal{K}_\bullet$ gives a natural bi-homogeneous B-resolution for $\mathrm{Sym}(I)$. From this we extract an S-resolution $(\mathbb{F}_i, \phi_i)_{0 \leq i \leq n-1}$ for $\mathrm{Sym}(I)_\delta$. We use a theorem of Buchsbaum-Eisenbud, to obtain an element in $\mathrm{Hom}_S(\mathrm{Sym}(I)_\delta, S)$. The element in $\mathrm{Hom}_S(\mathrm{Sym}(I)_\delta, S(-d))$ can then be easily recovered by shifting the T-degrees. We first fix a basis for $\wedge^{r_k}\mathbb{F}_k$ and an orientation $\eta \in \wedge^{t_i}\mathbb{F}_i$ where $t_k = \mathrm{rank}\, \mathbb{F}_k$. The orientation η allows us to define an isomorphism $\wedge^s \mathbb{F}_k \cong \wedge^{t_k - s}\mathbb{F}_k^*$. The result in [15, 3.1] now states that for each $1 \le k \le n-2$, there exists unique homomorphism $a_k : R \to \wedge^{r_k}\mathbb{F}_{k-1}$ such that

$$a_{n-1} = \wedge^{r_{n-1}}\phi_{n-1} \text{ and}$$

$$\begin{tikzcd} \wedge^{r_k}F_k \arrow[rr, "\wedge^{r_k}\phi_k"] \arrow[dr, "a_{k+1}^*"'] & & \wedge^{r_k}F_{k-1} \\ & R \arrow[ur, "a_k"'] & \end{tikzcd}$$

where $r_k = \text{rank } \phi_k$. Since $\mathcal{A}_0$ is principally generated, $\text{Sym}(I)_\delta$ is an S-module of rank one and hence $\wedge^{r_1}\mathbb{F}_0 \cong \mathbb{F}_0^*$. For $a_1 \in \wedge^{r_1}\mathbb{F}_0$, we denote $a_1^* \in \mathbb{F}_0^*$ for the image of a_1 under the isomorphism $\wedge^{r_1}\mathbb{F}_0 \cong \mathbb{F}_0^*$. Our candidate for an element in $\text{Hom}_S(\text{Sym}(I)_\delta, S)$ arises from a_1^*. To recover the element $\tilde{a}_1 \in \mathcal{A}_0$ from $a_1^* \in \text{Hom}_S(\text{Sym}(I)_\delta, S)$, we use the method of *Morley Forms* developed by J.-P. Jouanolou. The degree of the element $\tilde{a}_1$ can be compared to a known multiplicity.

Theorem 1.0.4 $\deg \tilde{a}_1 = e(R/(g_1, \ldots, g_{d-1}) : I^\infty)$ *where g_i's are general k-linear combination of the generators of I, namely the f_i's.*

By a theorem of Kustin, Polini and Ulrich in [16, 3.7], one has

$$e(R/(g_1, \ldots, g_{d-1}) : I^\infty) = e(\mathcal{F}(I)) \cdot \alpha$$

where α is the degree of the map Ψ. Also, since the image of Ψ is a hypersurface, notice that $e(\mathcal{F}(I))$ gives the degree of the generator of $\mathcal{A}_0$. From this we conclude that $\tilde{a}_1$ generates $\mathcal{A}_0$ if and only if the map Ψ is birational onto its image. As a consequence, the constructive proof gives us a method to check whether the rational map Ψ is birational onto its image.

Theorem 1.0.5 *Let $\Psi : \mathbb{P}^{d-1} \overset{[f_1:\cdots:f_{d+1}]}{\dashrightarrow} \mathbb{P}^d$ such that $I = (f_1, \ldots, f_{d+1})$ is a grade 2 perfect ideal generated by forms of the same degree. Also assume that I satisfies the G_d condition. Then the following are equivalent:*

1. *Ψ is birational onto its image.*

2. grade $I_1(a_1) > 1$.

3. $\tilde{a}_1$ *generates* $\mathcal{A}_0$.

At the time of writing this thesis, a generalized method to compute $\mathcal{A}_i, i > 0$ has been found extending by the above methods.

This thesis is organized as follows. Basic facts on Commutative Algebra is covered in Chapter 2. We study the defining ideal of Rees algebra of height two perfect ideals which are almost linearly presented in chapter 3. We introduce the notion of iterated Jacobian duals in Chapter 4. In Chapter 5, applying the technique of iterated Jacobian duals, we present a complete generating set for the defining ideal of Rees algebras whose second analytic deviation is one. Finally in Chapter 6, we solve the implicitization problem for a certain class of ideals.

2. PRELIMINARIES

In this chapter we give definitions, examples and results required to make the presentation self contained. Throughout, R will be a Noetherian ring and $I = (f_1, \ldots, f_m)$ an R-ideal.

2.1 Height two perfect ideals and Hilbert-Burch theorem

One of the important conditions we assume as a hypothesis in our results is that the ideals are grade 2 perfect.

Definition 2.1.1 *The grade of a proper ideal is the maximal length of an R-regular sequence in I.*

In a Cohen-Macaulay ring, it can be shown that the notion of height and grade of an ideal coincide. An equivalent definition for the grade of the ring R/I is grade $R/I =$ min $\{i \mid \operatorname{Ext}^i_R(R/I, R) \neq 0\}$. Thus grade $R/I \leq \operatorname{projdim}_R R/I$ for any proper ideal I.

Definition 2.1.2 *I is said to be a perfect ideal when* grade $R/I = \operatorname{projdim}_R R/I$.

Perfect ideals of height 2 are characterized by the Hilbert-Burch theorem.

Theorem 2.1.3 [17, 20.15], (**Hilbert-Burch Theorem**)

(*a*) *If a complex*

$$\mathbb{F} : 0 \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} R \to R/I \to 0$$

is exact where F_2 is free and $F_1 \cong R^n$, then $F_2 \cong R^{n-1}$ and there exists a non zero divisor a such that $I = aI_{n-1}(\phi_2)$. In fact, the ith entry of the matrix ϕ_1 is $(-1)^i a$ times the minor obtained from ϕ_2 by leaving out the ith row. The ideal $I_{n-1}(\phi_2)$ has grade at least two.

(b) *Conversely, given any* $(n-1) \times n$ *matrix* ϕ_2 *such that* grade $I_{n-1}(\phi_2) \geq 2$ *and a non zero divisor* a*, the map* ϕ_1 *obtained as in part (a) makes* $\mathbb{F}$ *a free resolution of* R/I *with* $I = aI_{n-1}(\phi_2)$*.*

2.2 Strongly Cohen-Macaulay Ideals

Definition 2.2.1 [18, 5.42] *An ideal* I *in a Cohen-Macaulay ring* R *is said to be strongly Cohen-Macaulay if the Koszul homology modules of* I *with respect to one* (*and then to any*) *generating set are Cohen-Macaulay.*

Consider a complete intersection ideal I in a Cohen-Macaulay ring R. Notice that in this case, the Koszul homology modules (except the zeroth) of a regular sequence generating I are all zero and hence I is strongly Cohen-Macaulay. The case of generic complete intersections is discussed below

Recall that I is said to be generically a complete intersection if I_P is generated by a regular sequence for every prime ideal P which is minimal over I.

Proposition 2.2.2 [19, 2.2] *Let* R *be a Cohen-Macaulay local ring, and* I *an ideal of* R *such that*

(a) R/I *is Cohen-Macaulay*

(b) I *is generically a complete intersection*

(c) $\mu(I) = \text{ht } I + 1$.

Then I *is strongly Cohen-Macaulay.*

Other standard examples of strongly Cohen-Macaulay ideals are grade two perfect and grade three Gorenstein ideals. In fact an ideal in the linkage class of a complete intersection (licci) is always strongly Cohen-Macaulay [20].

One of the important questions, commutative algebraist are interested in is the equality of the ordinary and symbolic powers of ideals. A criterion we use to check this equality for strongly Cohen-Macaulay ideals is the following theorem of Simis and Vasconcelos. Recall that for a prime ideal P, the symbolic power $P^{(k)}$ is defined to be $P^k R_P \cap R$. Similarly, for any ideal I, we define the k-th symbolic power of I to be $I^{(k)} = \cap_{P \in \mathrm{Ass}_R(R/I)}(I^k R_P \cap R)$.

Theorem 2.2.3 [21, 3.4] *Let I be an ideal of height g of a Cohen-Macaulay ring R. Assume that I is generically a complete intersection and is strongly Cohen-Macaulay. If*

$$\mu(I_P) \leq \mathrm{ht}\ P - 1 \text{ for all } P \in V(I) \text{ with } g+1 \leq \mathrm{ht}\ P,$$

then $I^i = I^{(i)}$ for all i.

2.3 Fitting Ideals and G-conditions

Fitting ideals of an ideal I are important invariants connected to the ideal I. Let φ be a presentation matrix of I i.e.,

$$R^s \xrightarrow{\varphi} R^m \to I \to 0.$$

For the $m \times s$ matrix φ, let $I_t(\varphi)$ represent the R-ideal generated by all the t by t minors of φ. We set $I_t(\varphi) = R$ for $t \leq 0$ and $I_t(\varphi) = 0$ for $t > \min\{m, s\}$.

Definition 2.3.1 *Define* $\mathrm{Fitt}_i(I) = I_{m-i}(\varphi)$. *This ideal is called the i-th Fitting ideal of I.*

It is well known that $\mathrm{Fitt}_i(I)$ depends only on i and I and not on m, s, φ. Some of the properties of the Fitting ideals are :

Observation 2.3.2 (a) $(\mathrm{ann}(I))^m \subset \mathrm{Fitt}_0(I) \subset \mathrm{ann}(I)$.

(b) *In case R is local,* $\mathrm{Fitt}_i(I) = R$ *if and only if* $\mu(I) \leq i$.

(c) $V(\mathrm{Fitt}_i(I)) = \{P \in \mathrm{Spec}(R) \mid \mu_{R_P}(I_P) > i\}$.

One of the important conditions we use in this thesis concerns the G-conditions

Definition 2.3.3 *I satisfies G_i if $\mu(I_P) \leq \text{ht } P$ for all $P \in V(I)$ with $\text{ht } P < i$.*

We say I satisfies G_∞ if I satisfies G_i for all i. Using (b),(c) in Observation 2.3.2, we obtain a practical method to verify the G_i condition by rewriting it in terms of the Fitting invariants

$$I \text{ satisfies } G_i \text{ if ht } \operatorname{Fitt}_k(I) > k \text{ for all } k < i.$$

We will be using the condition G_d throughout this thesis, where $\dim R = d$.

2.4 Rees Algebra of Ideals

Definition 2.4.1 *The Rees algebra $\mathcal{R}(I)$ of an ideal I is defined to be the subring $R[It] = \bigoplus_{i=0}^{\infty} I^i t^i \subset R[t]$.*

It can be shown that $\mathcal{R}(I) \cong \bigoplus_{i=0}^{\infty} I^i$. To study Rees algebras one often considers their *defining ideal*. Let $I = (f_1, \ldots, f_m)$. There exists an R-epimorphism

$$\Phi : R[T_1, \ldots, T_m] \twoheadrightarrow \mathcal{R}(I)$$
$$T_i \to f_i t$$

Definition 2.4.2 *The kernel,* $\ker \Psi$ *of the epimorphism* Ψ *is called the defining ideal of the Rees algebra $\mathcal{R}(I)$.*

Let φ be a presentation matrix of I, i.e.,

$$R^s \xrightarrow{\varphi} R^m \to I \to 0.$$

We can generate some obvious relations in $\ker \Phi$ using the presentation matrix φ. Let $\mathcal{J}_\varphi = ([T_1 \cdots T_m] \cdot \varphi)$ be the $R[T_1, \ldots, T_m]$-ideal generated by entries of the row vector $[T_1 \cdots T_m] \cdot \varphi$. The generators of $\mathcal{J}_\varphi$ are linear forms in $\ker \Phi$. Such relations are important in the study of the symmetric algebra $\operatorname{Sym}(I)$. The map Φ factors through $\operatorname{Sym}(I)$, and hence to study $\ker \Phi$, it is enough to study $\mathcal{A} = \ker(\operatorname{Sym}(I) \twoheadrightarrow \mathcal{R}(I)) = \overline{\ker \Phi}$.

Remark 2.4.3 *The symmetric algebra* $\mathrm{Sym}(I) \cong R[T_1, \ldots, T_m]/\mathcal{L}$ *where* $\mathcal{L} = \mathcal{J}_\varphi$ *for a presentation matrix* φ *of* I. *The ideal* $\mathcal{L} \subseteq R[T_1, \ldots, T_m]$ *is called the defining ideal of* $\mathrm{Sym}(I)$.

The dimensions of the algebras $\mathcal{R}(I), \mathrm{Sym}(I)$ can be easily computed.

Theorem 2.4.4 *(a) If* $\mathrm{ht}\ I > 0$, $\dim \mathcal{R}(I) = \dim R + 1$.

(b) [22, 2.6] $\dim \mathrm{Sym}(I) = \sup\{\mu(I_P) + \dim R/P \mid P \in \mathrm{Spec}(R)\}$

The relationship between $\mathcal{R}(I)$ and $\mathrm{Sym}(I)$ has been studied extensively. An interesting class forms the ideals satisfying $\mathcal{R}(I) \cong \mathrm{Sym}(I)$.

Definition 2.4.5 *When* $\mathcal{R}(I) \cong \mathrm{Sym}(I)$ *via the map* Φ, *then the ideal* I *is said to be of linear type.*

A classical theorem classifying ideals of linear type is the following theorem of Herzog, Simis and Vasconcelos.

Theorem 2.4.6 [4, 2.6] *Let* R *be a Cohen-Macaulay ring and let* I *be an ideal of positive grade. Assume*

(a) I *satisfies the condition* G_∞.

(b) I *is a strongly Cohen-Macaulay ideal.*

Then ideal I *is of linear type. Further,* $\mathcal{R}(I)$ *is Cohen-Macaulay.*

Example 2.4.7 Let $R = k[x, y]$ and $I = (x^2, y^2)$. Then the ideal I satisfies both the conditions of the above theorem. Thus I is of linear type.

Often, to study $\ker \Phi$, we study the minors of the *Jacobian dual* matrix.

Definition 2.4.8 *Let* φ *be a* $m \times s$ *presentation matrix of* I *and* $I_1(\varphi) = (a_1, \ldots, a_t)$ *be the ideal of entries of* φ. *Write*

$$[T_1 \cdots T_m] \cdot \varphi = [a_1 \cdots a_t] \cdot B(\varphi)$$

where $B(\varphi)$ *is a* $t \times s$ *matrix with linear entries in* $R[T_1, \ldots, T_m]$. *The matrix* $B(\varphi)$ *is called a Jacobian dual of* φ.

Let $R = k[x_1, \ldots, x_d]$ and I be an R-ideal. If the presentation matrix φ of I consists of linear entries, $[a_1, \ldots, a_t] = [x_1, \ldots, x_d]$ then $B(\varphi)$ is a unique matrix. But the matrix $B(\varphi)$ is not always uniquely determined as the following example show.

Example 2.4.9 Let $\varphi = \begin{bmatrix} x & x^2 \\ y & xy + y^2 \\ 0 & xy \end{bmatrix}$ be a presentation matrix of an ideal I. Then

$$\begin{aligned} [T_1\ T_2\ T_3] \cdot \varphi &= [x\ y] \cdot \begin{bmatrix} T_1 & xT_1 + yT_2 + yT_3 \\ T_2 & yT_2 \end{bmatrix} \\ &= [x\ y] \cdot \begin{bmatrix} T_1 & xT_1 \\ T_2 & yT_2 + xT_2 + xT_3 \end{bmatrix} \end{aligned}$$

Both the matrices $\begin{bmatrix} T_1 & xT_1 + yT_2 + yT_3 \\ T_2 & yT_2 \end{bmatrix}$ and $\begin{bmatrix} T_1 & xT_1 \\ T_2 & yT_2 + xT_2 + xT_3 \end{bmatrix}$ are candidates for $B(\varphi)$.

Recall that $\mathcal{L} = ([T_1 \cdots T_m] \cdot \varphi) = ([a_1 \cdots a_t] \cdot B(\varphi))$ is the defining ideal of $\mathrm{Sym}(I)$. Even though there are two candidates B_1, B_2 for $B(\varphi)$, we show, in Lemma 2.4.11, that $(\mathcal{L}, I_t(B_1)) = (\mathcal{L}, I_t(B_2))$ when $a_1, \ldots, a_t$ is an R-regular sequence. Thus in $\mathrm{Sym}(I)$, we have $\overline{I_t(B_1)} = \overline{I_t(B_2)}$.

We first prove a general lemma which make use of *Cramer's Rule*.

Lemma 2.4.10 *Let $[a_1 \cdots a_t]$ be a $1 \times t$ matrix and N be a $t \times t-1$ matrix with entries in R. Now let N_r, $1 \leq r \leq t$, be the $t-1 \times t-1$ submatrix of N obtained by removing the r-th row of N. Set $m_r = \det N_r$. Then, in the ring $R/([a_1 \cdots a_t] \cdot N)$*

$$\overline{a_r} \cdot \overline{m_k} = (-1)^{r-k} \overline{a_k} \cdot \overline{m_r},\ 1 \leq r \leq t,\ 1 \leq k \leq t \tag{2.1}$$

Proof Let $N = (b_{ij})$ and

$$[a_1 \cdots a_{k-1}\ a_{k+1} \cdots a_t] \cdot N_k = [g_1 \cdots g_{t-1}]$$

We make use of Cramer's Rule to see that $a_r \cdot m_k = \det N_{k_r}$, $r \in \{1, \cdots, k-1, k, \cdots, t\}$ where N_{k_r} is a matrix obtained from N_k, by replacing the r-th row by $[g_1 \cdots g_{t-1}]$. But in the ring $R/([a_1 \cdots a_t] \cdot N)$,

$$\overline{g_i} = -\overline{a_k}\overline{b_{ki}}$$

Thus, in this ring, we have $\overline{a_r} \cdot \overline{m_k} = \overline{\det N_{k_r}} = -\overline{a_k} \cdot \overline{m''}$, where m'' is the determinant of the matrix N'' whose rows are equal to that of N_k, except for the r-th row which is replaced by $[b_{k1} \cdots b_{kt}]$. Also, after $r-k-1$ row transposition of the r-th row of N'', we get $m'' = (-1)^{r-k-1} m_r$, where m_r is as described in the statement of the lemma. Putting all these observations together, we get $\overline{a_r} \cdot \overline{m_k} = -\overline{a_k} \cdot \overline{m''} = -\overline{a_k}(-1)^{t-k-1}\overline{m_r} = (-1)^{r-k}\overline{a_k} \cdot \overline{m_r}$. ∎

Proposition 2.4.11 *Let φ be an $m \times s$ presentation matrix of I and $I_1(\varphi) = (a_1, \ldots, a_t)$ such that $a_1, \ldots, a_t$ is an R-regular sequence. Suppose B_1 and B_2 are two matrices with t rows satisfying*

$$([a_1 \cdots a_t] \cdot B_1) = ([a_1 \cdots a_t] \cdot B_2), \tag{2.2}$$

then $(\mathcal{L}, I_t(B_1)) = (\mathcal{L}, I_t(B_2))$ where $\mathcal{L} = ([a_1 \cdots a_t] \cdot B_1) = ([a_1 \cdots a_t] \cdot B_2)$.

Proof Let $E = (a_1, \ldots, a_t)/([a_1 \cdots a_t] \cdot B_1)$ and consider the free presentation

$$F_1 \xrightarrow{[\delta \mid B_1]} F_0 \to E \to 0$$

where δ represents the first differential of the Koszul complex $\mathcal{K}$ on the R-regular sequence $a_1, \ldots, a_t$. Notice that

$$I_t([\delta \mid B_1]) = \mathrm{Fitt}_0(E) = I_t([\delta \mid B_2]) \tag{2.3}$$

as $([a_1 \cdots a_t] \cdot B_1) = ([a_1 \cdots a_t] \cdot B_2)$ and the Fitting ideals do not depend on the presentation matrix.

Thus it suffices to show that

$$\mathcal{L} + I_t([\delta \mid B_1]) \subseteq \mathcal{L} + I_t(B_1) \tag{2.4}$$

as this would imply, using (2.3), that

$$\mathcal{L} + I_t(B_1) = \mathcal{L} + I_t([\delta \mid B_1]) = \mathcal{L} + I_t([\delta \mid B_2])$$

Using similar arguments we can also show $\mathcal{L} + I_t([\delta \mid B_2]) = \mathcal{L} + I_t(B_2)$ proving that $\mathcal{L} + I_t(B_1) = \mathcal{L} + I_t(B_2)$.

Notice that $([a_1 \cdots a_t]\cdot[\delta \mid B_1]) = ([a_1 \cdots a_t]\cdot B_2) = \mathcal{L}$ because $[a_1 \cdots a_t]\cdot\delta = 0$. Now to prove (2.4), it is enough show to that $\overline{I_t([\delta \mid B_1])} \subseteq \overline{I_t(B_1)}$ in the ring $\overline{R} = R/\mathcal{L}$. Since δ is the first Koszul differential, we may assume the columns of δ are of the form $a_j e_k - a_k e_j,\ 1 \leq j, k \leq t$, where $\{e_j\}$ form a basis of R^t.

Now any element of $I_t([\delta \mid B_1])$ involving a column of δ is of the form $\det[\delta' \mid M]$ where M is a $t \times t-1$ submatrix of $[\delta \mid B_1]$ and δ' is a column of δ. Then $\det[\delta' \mid M]$ is of the form

$$(-1)^{k+1}(a_j m_k - (-1)^{j-k} a_k m_j) \tag{2.5}$$

where m_r is the determinant of the submatrix of M obtained by removing the r-th row of M. Now in the ring $R/\mathcal{L}$, using Lemma 2.4.10, we see that elements of the form (2.5) are zero. Thus $\overline{I_t([\delta \mid B_1])} \subseteq \overline{I_t(B_1)}$ in the ring $\overline{R}$ and hence $\mathcal{L} + I_t([\delta \mid B_1]) = \mathcal{L} + I_t(B_2)$. ■

Thus, irrespective of the candidate B_1, B_2 for $B(\varphi)$, the ideal $(\mathcal{L}, I_t(B(\varphi)))$ is uniquely determined when $I_1(\varphi)$ is generated by an R regular sequence. We show in Lemma 2.4.13 that $(\mathcal{L}, I_t(B(\varphi))) \subset \mathcal{L} : (a_1, \ldots, a_t)$.

Another useful lemma which we use is the following

Lemma 2.4.12 *Let $a_1, \ldots, a_t$ be elements in R. Let B be an $t \times t$ matrix with entries in R and $[L_1 \cdots L_t] = [a_1 \cdots a_t] \cdot B$. Let m_{ij} be the minor of B obtained by deleting the i-th row and j-th column. Then in the ring $A/(L_1, \ldots, L_{t-1})$*

$$a_i \det B = (-1)^{i+t} m_{it} L_t$$

Proof Using Cramer's rule, $a_i \det B$ is the determinant of a matrix C, obtained by replacing the i-th row by $[L_1 \cdots L_t]$. To compute $\det C$, we expand along the

i-th row to see that $\det C = \sum_{j=1}^{t} (-1)^{i+k} m_{ik} L_j$. But in the ring $A/(L_1, \ldots, L_{t-1})$, $a_i \det B = (-1)^{i+t} m_{it} L_t$. ■

Lemma 2.4.13 *Let B be a $t \times s$ matrix with entries in R and $[L_1 \cdots L_s] = [a_1 \cdots a_t] \cdot B$. Then $a_i \cdot I_d(B) \subseteq (L_1, \ldots, L_s)$ for all i.*

Proof The proof is immediate from the Cramers rule or Lemma 2.4.12. ■

2.5 Residual Intersection

Some of the proofs presented in this thesis use the theory of residual intersections.

Definition 2.5.1 [23, 1.1] *Let R be a Cohen-Macaulay local ring and let $\mathfrak{a} = (a_1, \ldots, a_t)$ be an R-ideal and $\mathfrak{b} = (b_1, \ldots, b_s) \subseteq \mathfrak{a}$ with $\mathfrak{b} \neq \mathfrak{a}$. Set $J = \mathfrak{b} : \mathfrak{a}$.*

(a) If $\operatorname{ht} J \geq s \geq \operatorname{ht} \mathfrak{a}$, then J is said to be an s-residual intersection of $\mathfrak{a}$ (with respect to $\mathfrak{a}$).

(b) If furthermore, $\mathfrak{a}_P = \mathfrak{b}_P$ for all $P \in V(\mathfrak{a})$ with $\operatorname{ht} P \leq s$, then we say J is a geometric s-residual intersection of I.

The Cohen-Macaulayness of residual intersections are well documented. The following is a result of Huneke, Ulrich .

Theorem 2.5.2 [23, 5.3] *Let R be a local Gorenstein ring, I a strongly Cohen-Macaulay ideal satisfying G_∞. Let $J = (L_1, \ldots, L_s) : I$ be any s-residual intersection of I. Then R/J is Cohen-Macaulay.*

When I is a complete intersection, a complete generating set for $J = (L_1, \ldots, L_s) : I$ can be found. Such a generating set can be found using the techniques of generic residual intersection. In fact complete resolutions of J have been worked out by Bruns, Kustin and Miller.

Theorem 2.5.3 [24, 4.8] *Let R be a Cohen-Macaulay local ring and $I = (a_1, \ldots, a_t)$, with $a_1, \ldots, a_t$ a regular sequence. Let $(L_1, \ldots, L_s) \subset I$ and ψ be a $t \times s$ matrix with entries in R so that $[L_1 \cdots L_s] = [a_1 \cdots a_t] \cdot \psi$. If $(L_1, \ldots, L_s) : I$ is an s-geometric residual intersection of I, then $(L_1, \ldots, L_s) : I = (L_1, \ldots, L_s) + I_t(\psi)$.*

2.6 Relation type and Regularity of $\mathcal{R}(I)$

We now define two important invariants namely relation type and regularity of the Rees algebra.

Definition 2.6.1 *The relation type* $\operatorname{rt}(I)$ *is defined to be the maximum $\underline{T}$-degree appearing in a homogeneous minimal generating set of the defining ideal of the Rees algebra.*

For the regularity, we use the definition as in [25]. Let $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}_n$ be a finitely generated standard graded ring over a Noetherian commutative ring $\mathcal{S}_0$. For any graded $\mathcal{S}$-module M we denote by M_n, the homogeneous component of degree n of M, and define

$$a(M) := \begin{cases} \max\{n \ : \ M_n \neq 0\} & \text{if } M \neq 0 \\ -\infty & \text{if } M = 0 \end{cases}$$

Let $\mathcal{S}_+$ be the ideal generated by the homogeneous elements of positive degree of $\mathcal{S}$.

Definition 2.6.2 *For $i \geq 0$, set $a_i(\mathcal{S}) := a(H^i_{\mathcal{S}_+}(\mathcal{S}))$, where $H^i_{\mathcal{S}_+}(.)$ denotes the i-th local cohomology functor with respect to the ideal $\mathcal{S}_+$. The Castelnuovo-Mumford regularity of S is defined as the number* $\operatorname{reg} \mathcal{S} = \max\{a_i(\mathcal{S}) + i \ : \ i \geq 0\}$

We also make use of Castelnuovo-Mumford regularity on short exact sequences.

Theorem 2.6.3 [17, 20.19] *If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely graded $\mathcal{S}$-modules, then*

(a) $\operatorname{reg} A \leq \max\{\operatorname{reg} B, \operatorname{reg} C + 1\}$.

(b) $\operatorname{reg} B \leq \max\{\operatorname{reg} A, \operatorname{reg} C\}$.

(c) reg $C \leq \max\{\text{reg } A - 1, \text{reg } B\}$.

(d) *If* A *has finite length, then* reg $B = \max\{\text{reg } A, \text{reg } C\}$.

2.7 Buchsbaum-Eisenbud Multipliers

The theorem we discuss in this section concerns the relationship between the ideals of minors of the matrices appearing in a free resolution. We will be using this theorem to a great effect in Chapter 6.

An oriented free module is by definition a free module F with a fixed generator $\eta \in \wedge^r F$ where $r = \text{rank } F$. η is called the orientation of F. The orientation of F determines an isomorphism $\wedge^k F \cong \wedge^{r-k} F$ for $0 \leq k \leq r$.

Theorem 2.7.1 [15, 3.1] *Let* $(\mathbb{F}_\bullet, \phi_\bullet)$ *be the free resolution*

$$0 \to F_n \xrightarrow{\phi_n} F_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

where F_i *are oriented free modules of finite rank. Let* $r_i = \text{rank } \phi_i$. *Then*

(a) *for each* k, $1 \leq k < n$, *there exists a unique homomorphism* $a_k : R \to \wedge^{r_k} F_{k-1} = \wedge^{r_{k-1}} F^*_{k-1}$ *such that*

(i) $a_n = \wedge^{r_n} \phi_n : R = \wedge^{r_n} F_n \to \wedge^{r_n} F_{n-1}$.

(ii) *for each* $k < n$, *the diagram*

$$\begin{tikzcd} \wedge^{r_k} F_k \arrow[rr, "\wedge^{r_k} f_k"] \arrow[dr, "a^*_{k+1}"'] & & \wedge_{r_k} F_{k-1} \\ & R \arrow[ur, "a_k"'] & \end{tikzcd}$$

commutes.

(b) *For all* $k > 1$, $\sqrt{I_1(a_k)} = \sqrt{I_{r_k}(\phi_k)}$.

Example 2.7.2 Let $R = k[x, y]$ and $I = (x^2, xy, y^2, z^2)$. Consider the resolution of I

$$0 \to R^2 \xrightarrow{\phi_2} R^5 \xrightarrow{\phi_1} R^4 \xrightarrow{[x^2\ \ xy\ \ y^2\ \ z^2]} I$$

where

$$\phi_1 = \begin{bmatrix} -y & 0 & -z^2 & 0 & 0 \\ x & -y & 0 & -z^2 & 0 \\ 0 & x & 0 & 0 & -z^2 \\ 0 & 0 & x^2 & xy & y^2 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \\ -y & 0 \\ x & -y \\ 0 & x \end{bmatrix}$$

Let $r_i = \text{rank } \phi_1$. Thus $r_1 = 3, r_2 = 2$. We use the following notation for an ordered basis of $\wedge^r F$. For any R-free module F and an ordered basis $\{e_1, \ldots, e_n\}$ of F, let $\mathbb{B}$ denote the basis $\{(-1)^{i_1+\cdots+i_r-1} e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq n\}$ of $\wedge^r F$. Further we arrange the set $\mathbb{B}$ by decreasing lexicographic order on the index set $\{(i_1, \ldots, i_r) \mid 1 \leq i_1 < \cdots < i_r \leq n\} \subseteq \mathbb{N}^r$.

Now by Theorem 2.7, .

$$a_2 = \wedge^2 \phi_2 = \begin{bmatrix} x^2 \\ xy \\ y^2 \\ 0 \\ xz^2 \\ yz^2 \\ -xz^2 \\ -yz^2 \\ 0 \\ z^4 \end{bmatrix}$$

Thus using the identification $\wedge^3 R^5 \cong \wedge^2 R^5$, we have

$$a_2^* = \begin{bmatrix} -z^4 & 0 & yz^2 & -xz^2 & -yz^2 & xz^2 & 0 & -y^2 & xy & -x^2 \end{bmatrix}$$

Also

$$\wedge^3\phi_1 = \begin{bmatrix} -x^2z^4 & 0 & x^2yz^2 & -x^3z^2 & -x^2yz^2 & x^3z^2 & 0 & -x^2y^2 & x^3y & -x^4 \\ -xyz^4 & 0 & xy^2z^2 & -x^2yz^2 & -xy^2z^2 & x^2yz^2 & 0 & -xy^3 & x^2y^2 & -x^3y \\ -y^2z^4 & 0 & y^3z^2 & -xy^2z^2 & -y^3z^2 & xy^2z^2 & 0 & -y^4 & xy^3 & -x^2y^2 \\ -z^6 & 0 & yz^4 & -xz^4 & -yz^4 & xz^4 & 0 & -y^2z^2 & xyz^2 & -x^2z^2 \end{bmatrix}$$

By Theorem 2.7, we have the following commutative diagram

$$\begin{tikzcd} \wedge^3 R^5 \arrow[rr, "\wedge^3\phi_1"] \arrow[dr, "a_2^*"'] & & \wedge^3 R^4 \\ & R \arrow[ur, "a_1"'] & \end{tikzcd} .$$

Thus a_1 can be computed considering a column (first column), and dividing its entries by the corresponding column (first entry) of a_2^*. Thus

$$a_1 = \begin{bmatrix} x^2 \\ xy \\ y^2 \\ z^2 \end{bmatrix} .$$

3. ALMOST LINEARLY PRESENTED IDEALS

In this chapter we study the defining ideal of the Rees algebra of height two perfect ideals. Further we require the presentation matrix of the ideal to be almost linear. The methods we discuss in this chapter are a generalization of the work presented in [9]. This is joint work with Jacob A. Boswell.

We follow the following setting in this chapter

Setting 3.0.1 *Let*

(a) $R = k[x_1, \ldots, x_d]$

(b) $I = (f_1, \ldots, f_m)$ *be a height two perfect ideal where* $f_1, \ldots, f_m$ *are homogeneous polynomials of the same degree.*

(c) I *satisfies the* G_d *condition, that is,*

$$\mu(I_P) \leq \operatorname{ht} P \text{ for all } P \in V(I) \text{ with } \operatorname{ht} P < d \tag{3.1}$$

(d) Since I *is height two perfect ideal, it can be generated by the maximal minors of an* $m \times m-1$ *matrix* φ *with homogeneous entries of constant degree along each column (Theorem 2.1.3). Further, assume that the presentation matrix* φ *is almost linear, that is, all but the last column of* φ *is linear and the last column consists of homogeneous entries of arbitrary degree* $n \geq 1$*.*

Remark 3.0.2 Since I satisfies the G_d condition, we assume $\mu(I) = m > d$. For, if $\mu(I) \leq d$, then I satisfies the G_∞ condition and hence I will be of linear type by Theorem 2.4.6.

Let

$$\Phi : B = R[T_1, \ldots, T_m] \twoheadrightarrow \mathcal{R}(I)$$
$$T_i \to f_i t$$

We attempt to compute a generating set for $\ker \Phi$ (called the defining ideal of $\mathcal{R}(I)$).

Let $\mathcal{L} = (L_1, \ldots, L_{m-1})$ where $[T_1, \ldots, T_m] \cdot \varphi = [L_1, \ldots, L_{m-1}]$. This is the defining ideal of the symmetric algebra $\mathrm{Sym}(I)$.

Remark 3.0.3 Setting $\deg x_i = (1, 0)$ and $\deg T_j = (0, 1)$, we see that B is a bigraded ring. With respect to this grading, both $\ker \Phi$ and $\mathcal{L}$ are bi-homogeneous.

Theorem 3.0.4 *One has* $\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d)^\infty$.

Proof Let $0 \neq y \in (x_1, \ldots, x_d)$. Since I satisfies the G_d condition, we first show that I_y satisfies G_∞ in R_y. Since I satisfies G_d, ht $\mathrm{Fitt}_i(I) > i$ for $i < d$. Now ht $\mathrm{Fitt}_{d-1}(I) > d-1$ and $\mathrm{Fitt}_{d-1}(I)$ is homogeneous. Hence $\mathrm{Fitt}_{d-1}(I)$ is an $(x_1, \ldots, x_d)$-primary ideal. Thus $\mathrm{Fitt}_{d-1}(I_y) = R_y$. Also ht $\mathrm{Fitt}_i(I_y) > i$ for $i < d-1$. Thus I_y satisfies G_∞ in R_y. Using Theorem 2.4.6 we have I_y is of linear type in R_y ($\mathcal{L}_y = \ker \Phi_y$). Thus $y^s \mathcal{L} \subseteq \ker \Phi$ for some s. Since this is true for all $y \in (x_1, \ldots, x_d)$ and B is Noetherian, $\ker \Phi \subseteq \mathcal{L} : (x_1, \ldots, x_d)^t$ for some $t >> 0$. Also, $\mathcal{L} : (x_1, \ldots, x_d)^i \subseteq \ker \Phi$ for all i as $(x_1, \ldots, x_d)^i \not\subset \ker \Phi$ and $\ker \Phi$ is a prime ideal. Thus $\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d)^\infty$. ■

At the end of this chapter we present the index of saturation of $\mathcal{L} : (x_1, \ldots, x_d)^\infty$ in the setting of 3.0.1. We now follow the construction as in [9] to find a form of $\ker \Phi$. We first construct a Cohen-Macaulay ring A close to $\mathcal{R}(I)$ such that $\mathcal{R}(I)$ is A modulo a prime ideal of height one.

Notation 3.0.5 *(a) Let φ' denote the matrix obtained by deleting the last column of φ. Since φ is an almost linear matrix, φ' is a linear matrix.*

(b) $J = (L_1, \ldots, L_{m-2}) + I_d(B(\varphi')) \subset B$ where $[L_1 \cdots L_{m-2}] = [x_1 \cdots x_d] \cdot B(\varphi')$ and $B(\varphi')$ is a matrix with linear entries in $k[T_1, \ldots, T_m]$.

(c) Define $A = B/J$

(d) Let N be the matrix obtained by deleting the last row of $B(\varphi')$

(e) *Define* $K = J + I_{d-1}(N) + (x_d)$

(f) *Let* $\mathfrak{m}$ *denote the ideal* $\overline{(x_1, \ldots, x_d)}$ *in* A.

Observation 3.0.6 If φ is almost linear and $m > d$, $I_1(\varphi') \supset I_2(\varphi) \supset I_{m-d+1}(\varphi) = \text{Fitt}_{d-1}(I)$. Recall that φ' is obtained from φ by removing the last column. Since ht $\text{Fitt}_{d-1}(I) \geq d$ by the G_d condition, $I_1(\varphi) = (x_1, \ldots, x_d)$.

Notice that $B(\varphi)$ is a $d \times m-1$ matrix and $B(\varphi')$ is a $d \times m-2$ matrix.

We use the following two theorems to prove that the ring A is a Cohen-Macaulay ring of dimension $d+2$.

Theorem 3.0.7 [26, 2.2] *The ideal* $(L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d)^\infty = ([x_1, \ldots, x_d] \cdot B(\varphi')) : (x_1, \ldots, x_d)^\infty$ *is a prime ideal in* B*, of height* $m-2$.

Theorem 3.0.8 [26, 2.4] *The ideal* $I_d(B(\varphi'))$*is a prime ideal in* $k[T_1, \ldots, T_m]$*, of height* $m-d-1$ *and* $(L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d)$ *is a geometric residual intersection of the ideal* $(x_1, \ldots, x_d)$ *in* B. *Furthermore,*

$$\begin{aligned}(L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d)^\infty &= (L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d) \\ &= (L_1, \ldots, L_{m-2}) + I_d(B(\varphi')).\end{aligned}$$

Lemma 3.0.9 *The following statements are true.*

(a) *The ring* A *is a Cohen-Macaulay domain of dimension* $d+2$.

(b) *The ideals* $\overline{K}$ *and* $\mathfrak{m}$ *are Cohen-Macaulay* A*-ideals of height one.*

Proof (a) By Theorem 3.0.4, we have $\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d)^\infty = (L_1, \ldots, L_{m-1}) : (x_1, \ldots, x_d)^\infty$ and $\ker \Phi$ is a prime ideal of height $m-1$. By Theorem 3.0.7, $(L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d)^\infty$ is a prime ideal of height $m-2$ and by Theorem 3.0.8

$$\begin{aligned}(L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d)^\infty &= (L_1, \ldots, L_{m-2}) : (x_1, \ldots, x_d) \\ &= (L_1, \ldots, L_{m-2}) + I_d(B(\varphi')) \\ &= J\end{aligned}$$

Further, since J is a geometric residual intersection of $(x_1, \ldots, x_d)$ in B (Theorem 3.0.8), A is a Cohen-Macaulay ring (Theorem 2.5.2). Also, since J is a prime ideal of height $m-2$, we have A is a Cohen-Macaulay domain of dimension $d+2$.
(b) Notice that $J + (x_1, \ldots, x_d) = I_d(B(\varphi')) + (x_1, \ldots, x_d)$. Again using Theorem 3.0.8, $I_d(B(\varphi'))$ is a prime ideal of height $m-d-1$. As $m-d-1$ is of maximal possible height, $I_d(B(\varphi'))$ is Cohen-Macaulay. Notice that $x_1, \ldots, x_d$ is a regular sequence on $I_d(B(\varphi'))$ and hence $I_d(B(\varphi'))+(x_1, \ldots, x_d)$ is Cohen-Macaulay of height $m-1$. Now as J is a prime ideal that is homogeneous with respect to $(x_1, \ldots, x_d)$ and $(T_1, \ldots, T_m)$, it follows that $J + (x_1, \ldots, x_d)$ is a prime ideal. But notice that $J + (x_1, \ldots, x_d) = I_d(B(\varphi')) + (x_1, \ldots, x_d)$. Thus $\mathfrak{m}$ is a height 1 prime ideal in A.

Recall that $K = J + I_{d-1}(N) + (x_d)$. But notice that K can also be generated by $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_{d-1}(N) + (x_d)$ where $[\tilde{L}_1, \ldots \tilde{L}_{m-2}] = [x_1 \cdots x_{d-1}] \cdot N$. Since J is a prime ideal of height $m-2$ and $x_d \in K \backslash J$, K has height at least $m-1$. By Krull's Altitude Theorem $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_{d-1}(N)$ is of height at least $m-2$. Notice that

$$(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_{d-1}(N) \subseteq (\tilde{L}_1, \ldots, \tilde{L}_{m-2}) : (x_1, \ldots, x_{d-1})$$

Thus $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) : (x_1, \ldots, x_{d-1})$ is a residual intersection and hence $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) : (x_1, \ldots, x_{d-1})$ is Cohen-Macaulay (Theorem 2.5.2). Using [8, 1.5,1.8] we get

$$(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) : (x_1, \ldots, x_{d-1}) = (\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_{d-1}(N)$$

Also, the generators $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_{d-1}(N)$ do not involve the variable x_d. Thus x_d is regular on $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_{d-1}(N)$ and hence K is Cohen-Macaulay of height $m-1$. This shows that $\overline{K}$ is a height 1 prime ideal in A. ■

Lemma 3.0.10 *The following statements are true*

(a) $\mathfrak{m}^i = \mathfrak{m}^{(i)}$.

(b) $\overline{K}^{(i)} = (x_d)^i :_A \mathfrak{m}^{(i)}$.

(c) $\mathfrak{m}^{(i)} = (x_d)^i :_A \overline{K}^{(i)}$.

Proof (a) Temporarily setting $\deg x_i = 1$ and $\deg T_j = 0$, we get $\mathrm{gr}_{\mathfrak{m}}(A) \cong A$, a domain. Thus $\mathfrak{m}^i = \mathfrak{m}^{(i)}$.

(b) Using Lemma 2.4.13, we see that $\mathfrak{m}\overline{K} \subseteq (\overline{x_d})$. Thus $\mathfrak{m}^i\overline{K}^i \subseteq (\overline{x_d})^i$. After localizing at any prime P with ht $P = 1$, we see $\mathfrak{m}^{(i)}\overline{K}^{(i)} \subseteq (\overline{x_d})^i$. Thus

$$\mathfrak{m}^{(i)} \subseteq (\overline{x_d})^i : \overline{K}^{(i)}. \tag{3.2}$$

We now show that $\mathfrak{m}^i$ and $\overline{K}^{(i)}$ do not share an associated prime. Recall that $K = (\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_d(B) + (x_d)$ and $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) : (x_1, \ldots, x_{d-1}) = (\tilde{L}_1, \ldots, \tilde{L}_{m-2}) + I_d(B)$ is of height $m-2$. Now suppose $I_d(B) = 0$, then $(\tilde{L}_1, \ldots, \tilde{L}_{m-2}) : (x_1, \ldots, x_{d-1}) = (\tilde{L}_1, \ldots, \tilde{L}_{m-2})$ is of height $m-2$. Thus $(x_1, \ldots, x_{d-1})$ contains a regular element on $B/(\tilde{L}_1, \ldots, \tilde{L}_{m-2})$. This would imply $m-2 < d-1$ showing $m \leq d$, a contradiction. Thus $I_{d-1}(N) \neq 0$. By degree considerations, $I_{d-1}(N) \not\subset J$ as the generators of $I_{d-1}(N)$ have T-degree $d-1$ in $k[T_1, \ldots, T_m]$. Thus $K \not\subset (x_1, \ldots, x_d) + J$. Since $\mathfrak{m}$ is the unique associated prime of $\mathfrak{m}^i$ and $\overline{K} \not\subset \mathfrak{m}$, $\mathfrak{m}^i$ and $\overline{K}^{(i)}$ do not share an associated prime. Thus

$$(\overline{x_d})^i : \overline{K}^{(i)} \subseteq \mathfrak{m}^{(i)}.$$

Hence $\overline{K}^{(i)} = (x_d)^i :_A \mathfrak{m}^{(i)}$.

(c) The proof is analogous to that of (b). ∎

Since $L_{m-1} \in (x_1, \ldots, x_d)^n$ (recall L_{m-1} is the form coming from the non-linear column of φ), $\overline{L_{m-1}} \in \mathfrak{m}^n = (\overline{x_d})^n : K^{(n)}$. Thus $\overline{L_{m-1}K}^{(n)} \subseteq (\overline{x_d})^n$. We are ready define the A-ideal $\mathcal{D} = \frac{\overline{L_{m-1}K}^{(n)}}{\overline{x_d}^n}$. Also, $\mathcal{D} \subseteq \overline{\ker \Phi}$ as $L_{m-1} \in \ker \Phi, x_d \notin \ker \Phi$ and $\ker \Phi$ is a prime ideal.

Theorem 3.0.11 *In the setting of 3.0.1, $\mathcal{D} = \overline{\ker \Phi}$ in the ring A, where $\overline{\ker \Phi}$ denotes the image of $\ker \Phi$ in the ring A.*

Proof A module M satisfies the *Serre's Condition* S_2 if

$$\text{depth } M_P \geq \min\{2, \dim M_P\}$$

for every $P \in \mathrm{Supp}(M)$.

It is well known, that in a Cohen-Macaulay ring, proper ideal of positive height is unmixed of height one if and only if it satisfies the S_2 condition. Since A is Cohen-Macaulay, $\overline{K}^{(n)}$ is unmixed of height one and hence satisfies S_2. Since $\mathcal{D} \cong K^{(n)}$, $\mathcal{D}$ is also a height one unmixed ideal.

Since $\mathcal{D} \subseteq \overline{\ker\Phi}$ it suffices to show that these ideals are equal locally at the associated primes of $\mathcal{D}$. Recall that the associated primes of $\mathcal{D}$ are of height one.

As $\overline{K} \not\subset \mathfrak{m}$ we have $\overline{K}^{(i)}_{\mathfrak{m}} = A_{\mathfrak{m}}$. Also, $\overline{\ker\Phi} \not\subset \mathfrak{m}$. Thus

$$\begin{aligned} \mathfrak{m}^n_{\mathfrak{m}} &= (\overline{x_d})^n_{\mathfrak{m}} : \overline{K}^{(n)}_{\mathfrak{m}} \\ &= (\overline{x_d})^n_{\mathfrak{m}} : A_{\mathfrak{m}} \\ &= (\overline{x_d})^n_{\mathfrak{m}}. \end{aligned}$$

As $\mathfrak{m}_{\mathfrak{m}} = (x_d)_{\mathfrak{m}}$, the only $\mathfrak{m}$-primary ideals of A are $\mathfrak{m}^{(i)}$ (and hence $\mathfrak{m}^i$). Thus $(\overline{L_{m-1}})_{\mathfrak{m}} = \mathfrak{m}^i_{\mathfrak{m}} = (\overline{x_d})^i_{\mathfrak{m}}$ for some i. As L_{m-1} has T-degree n, $i \leq n$. Thus

$$\mathcal{D}_{\mathfrak{m}} = \frac{(\overline{L_{m-1}})_{\mathfrak{m}} \overline{K}^{(n)}_{\mathfrak{m}}}{\overline{x_d}^n_{\mathfrak{m}}} = \frac{(\overline{x_d}^i)_{\mathfrak{m}}}{(\overline{x_d}^n)_{\mathfrak{m}}} \supset A_{\mathfrak{m}} = \overline{\ker\Phi}_{\mathfrak{m}}$$

For a height one prime $P \neq \mathfrak{m}$,

$$\begin{aligned} \overline{K}_P &= (\overline{x_d}))_P : \mathfrak{m}_P \\ &= (\overline{x_d}))_P : A_P \\ &= (\overline{x_d}))_P. \end{aligned}$$

Therefore

$$\mathcal{D}_P = \frac{(\overline{L_{m-1}})_P \overline{K}^{(n)}_P}{\overline{x_d}^n_P} = (\overline{L_{m-1}})_P = \overline{\ker\Phi}_P$$

The last equality is due to the G_d condition (linear type in the punctured spectrum). Thus $\mathcal{D}_P = A_P$ for all height one prime $P \in \operatorname{Spec}(A)$. ∎

Corollary 3.0.12 *In the setting of 3.0.1,* $\ker\Phi = \mathcal{L} : (x_1, \ldots, x_d)^n$. *Further, n is the smallest possible integer for which this holds.*

Proof Notice that $\mathcal{D}\mathfrak{m}^n \subseteq (\overline{L_{m-1}})$ in A. Thus $\mathcal{D} \subseteq \overline{\mathcal{L}} : \mathfrak{m}^n$. Thus $\overline{\ker\Phi} \subseteq \overline{\mathcal{L}} : \mathfrak{m}^n$. This shows $\ker\Phi \subseteq (\mathcal{L} + I_d(B(\varphi'))) : (x_1, \ldots, x_d)^n$. Since $(\mathcal{L} + I_d(B(\varphi')))$ is bi-homogeneous and $(x_1, \ldots, x_d) \cdot I_d(B(\varphi')) \subseteq \mathcal{L}$ it follows that

$$\ker\Phi \subseteq \mathcal{L} : (x_1, \ldots, x_d)^n.$$

As $\mathcal{L} : (x_1, \ldots, x_d)^\infty = \ker\Phi \subseteq \mathcal{L} : (x_1, \ldots, x_d)^n$, we have $\ker\Phi = \mathcal{L} : (x_1, \ldots, x_d)^n$.

Assume there exists an $i \in \mathbb{N}$ with $i < n$ so that $\ker\Phi = \mathcal{L} : (x_1, \ldots, x_d)^i$. Then in the ring A, $\mathfrak{m}^i\overline{\ker\Phi} \subseteq (\overline{L_{m-1}})$. Localizing at the prime $\mathfrak{m}$, we obtain $\mathfrak{m}^i_{\mathfrak{m}} \subseteq (\overline{L_{m-1}})_{\mathfrak{m}}$. As $L_{m-1} \in (x_1, \ldots, x_d)^i$ $(i < n)$, we have $\mathfrak{m}^i_{\mathfrak{m}} = (\overline{L_{m-1}})_{\mathfrak{m}}$. Similarly we can show that $(\overline{L_{m-1}})_{\mathfrak{m}} \subseteq \mathfrak{m}^n_{\mathfrak{m}}$. Thus $\mathfrak{m}^i = \mathfrak{m}^{(i)} = (\overline{L_{m-1}})_{\mathfrak{m}} \subseteq \mathfrak{m}^n \subseteq \mathfrak{m}^i$, which is a contradiction. ∎

Notice that a generating set for $\ker\Phi$ can be completely determined, if a generating set for $K^{(n)}$ can be found. If $d = 2$, a generating set for $K^{(n)}$ is given in [9]. In general, it is hard to compute a generating set for $K^{(n)}$, but for computational purposes, one can use Lemma 3.0.10. We will present attempts to construct a generating set for $\mathcal{L} : (x_1, \ldots, x_d)^n$ in Chapter 4 and for the special case of $K^{(n)} = K^n$ in Chapter 5.

4. ITERATED JACOBIAN DUALS

In the previous section we proved that $\ker \Phi = \mathcal{L} : (x_1, \ldots, x_d)^n$ in the setting of 3.0.1. An attempt to construct a generating set for $\mathcal{L} : (x_1, \ldots, x_d)^n$ led to the conception of *iterated Jacobian duals*. This notion generalizes Jacobian dual matrices. The minors of these matrices help us construct more generators for $\ker \Phi$, especially those which are not of the expected form. This is joint work with Jacob Boswell.

First we define the iterated Jacobian dual of an arbitrary matrix ϕ, in a Noetherian ring R. We then apply the setting of 3.0.1 and present a condition for the equality of the ideal arising from iterated Jacobian duals and the defining ideal of $\mathcal{R}(I)$.

4.1 Constructing the Iterated Jacobian dual

Let R be a Noetherian ring. Consider a presentation

$$R^s \xrightarrow{\phi} R^m \to \operatorname{coker} \phi \to 0.$$

Assume $I_1(\phi) \subseteq (a_1, \ldots, a_t)$. Then there exists a $t \times s$ matrix $B(\phi)$, called a *Jacobian dual* of ϕ, with linear entries in $R[T_1, \ldots, T_m]$ such that the following condition is satisfied

$$[T_1 \cdots T_m] \cdot \phi = [a_1 \cdots a_t] \cdot B(\phi). \tag{4.1}$$

The existence of $B(\phi)$ is clear, but it may not be uniquely determined. In a polynomial ring, the uniqueness of $B(\phi)$ depends on the linearity of the matrix ϕ.

Let $\mathcal{L}$ denote the ideal defining the symmetric algebra $\operatorname{Sym}(\operatorname{coker} \phi)$.

Definition 4.1.1 *Set $B_1(\phi) = B(\phi)$ and $\mathcal{L}_1 = \mathcal{L}$. Suppose $(B_1(\phi), \mathcal{L}_1), \ldots,$ $(B_{i-1}(\phi), \mathcal{L}_{i-1})$ have been constructed inductively such that, for $1 \leq j \leq i-1$,*

$B_j(\phi)$ are matrices with t rows having homogeneous entries of constant degree in $R[T_1,\dots,T_m]$ along each column and $\mathcal{L}_j = ([a_1 \cdots a_t] \cdot B_j(\phi)),\ 1 \le j \le i-1$.

We now construct $(B_i(\phi), \mathcal{L}_i)$. Let

$$\mathcal{L}_{i-1} + (I_t(B_{i-1}(\phi)) \cap (a_1,\dots,a_t)) = \mathcal{L}_{i-1} + (u_1,\dots,u_l)$$

where $u_1,\dots,u_l$ are homogeneous in $R[T_1,\dots,T_m]$. Then there exists a matrix C having homogeneous entries of constant degree in $R[T_1,\dots,T_m]$ along each column such that

$$[u_1 \cdots u_l] = [a_1 \cdots a_t] \cdot C. \tag{4.2}$$

Define $B_i(\phi)$, an i-th iterated Jacobian dual of ϕ, to be

$$B_i(\phi) = [B_{i-1}(\phi) \mid C] \tag{4.3}$$

where $\mid$ represents matrix concatenation. Now set $\mathcal{L}_i = ([a_1 \cdots a_t] \cdot B_i(\phi))$.

By construction, $B_{i-1}(\phi)$ is a submatrix of $B_i(\phi)$ and $\mathcal{L}_{i-1} \subseteq \mathcal{L}_i$. As with the Jacobian dual matrix $B(\phi)$, the iterated Jacobian dual matrices $B_i(\phi)$ are also not uniquely determined. Further, notice that the generating set $(u_1,\dots,u_l)$ need not be unique. Thus we can construct different candidates for $B_i(\phi)$ of different sizes. Suppose

$$\mathcal{L}_{i-1} + (I_r(B_{i-1}(\phi)) \cap (a_1,\dots,a_t)) = \mathcal{L}_{i-1} + (u_1,\dots,u_l) = \mathcal{L}_{i-1} + (v_1,\cdots,v_p) \tag{4.4}$$

and suppose B and B' satisfy

$$\begin{aligned} [u_1 \cdots u_l] &= [a_1 \cdots a_t] \cdot C \quad \text{and } B = [B_{i-1}(\phi) \mid C] \\ [v_1 \cdots v_p] &= [a_1 \cdots a_t] \cdot C' \quad \text{and } B' = [B_{i-1}(\phi) \mid C'] \end{aligned} \tag{4.5}$$

For our purposes, we show in Theorem 4.1.2 that $\mathcal{L}+I_t(B) = \mathcal{L}+I_t(B')$ when $a_1,\dots,a_t$ is a R-regular sequence. Thus the ideal of an iterated Jacobian dual, $\mathcal{L}+I_t(B_{i+1}(\phi))$, depends only on the matrix ϕ and the regular sequence $a_1,\dots,a_t$. Also, in the construction, we assume that t should not be "too big", otherwise the matrix $B(\phi)$ (and hence $B_i(\phi)$) may have rows of zeros, which would trivialize the construction.

Remark 4.1.1 $\mathcal{L}_1 = \mathcal{L}$ is a well defined $R[T_1, \ldots, T_m]$-ideal because it is the ideal defining the symmetric algebra $\mathrm{Sym}(\operatorname{coker} \phi)$. Assume that $\mathcal{L}_j,\ 1 \leq j \leq i-1$ are well defined ideals. The candidates for $B_i(\phi)$, namely B and B', are constructed with the generators, $(u_1, \ldots, u_l)$ and $(v_1, \ldots, v_p)$ respectively. Now (4.4) guarantees that

$$([a_1 \cdots a_t] \cdot B) = ([a_1 \cdots a_t] \cdot B')$$

showing that $\mathcal{L}_i$ is a well-defined $R[T_1, \cdots, T_m]$-ideal.

Now using Lemma 2.4.11, proved in the preliminaries section, we show the uniqueness of the ideal of an iterated Jacobian dual $\mathcal{L} + I_r(B_i(\phi))$.

Theorem 4.1.2 *Let R be a Noetherian ring and ϕ be a $m \times s$ matrix with entries in R. Suppose $I_1(\phi) \subseteq (a_1, \ldots, a_r)$ and $a_1, \ldots, a_r$ is a regular sequence. Then the ideal $\mathcal{L} + I_r(B_i(\phi))$ of $R[T_1, \ldots, T_m]$ is uniquely determined by the matrix ϕ and the regular sequence $a_1, \ldots, a_r$.*

Proof Since the construction of the iterated Jacobian dual is inductive, we prove this result using induction. Using Lemma 2.4.11, we see that $\mathcal{L} + I_r(B_1(\phi))$ is a well defined ideal, proving the initial step of the induction hypothesis. Now suppose that $\mathcal{L} + I_r(B_j(\phi)),\ 1 \leq j \leq i-1$ are well defined ideals. Now, if B, B' are two matrices which satisfies (4.5), then we show that

$$\mathcal{L} + I_r(B) = \mathcal{L} + I_r(B')$$

Since $\mathcal{L} \subseteq \mathcal{L}_i$, clearly $\mathcal{L} + I_t(B) \subseteq \mathcal{L}_i + I_t(B)$. Also $I_t(B) \supseteq I_t(B_{i-1}(\phi))$. Now notice that

$$\begin{aligned}
\mathcal{L}_i &\subseteq ([a_1 \cdots a_t] \cdot B_{i-1}(\phi)) + ([a_1 \cdots a_t] \cdot C) \\
&= \mathcal{L}_{i-1} + (u_1, \ldots, u_l) \\
&\subseteq \mathcal{L}_{i-1} + I_t(B_{i-1}(\phi)) \\
&\subseteq \mathcal{L}_{i-1} + I_t(B)
\end{aligned}$$

Since $I_t(B) \supseteq I_t(B_{i-1}(\phi)) \supseteq I_t(B_{i-2}(\phi)) \supseteq \cdots \supseteq I_t(B_1(\phi))$, we show, successively, that

$$\mathcal{L}_i \subseteq \mathcal{L}_{i-1} + I_t(B) \subseteq \mathcal{L}_{i-2} + I_t(B) \subseteq \cdots \subseteq \mathcal{L} + I_t(B)$$

Thus $\mathcal{L}_i + I_t(B) \subseteq \mathcal{L} + I_t(B)$ and hence $\mathcal{L}_i + I_t(B) = \mathcal{L} + I_t(B)$. Similarly we can show that $\mathcal{L}_i + I_t(B) = \mathcal{L} + I_t(B')$

Now it is enough to show $\mathcal{L}_i + I_t(B) = \mathcal{L}_i + I_t(B')$. Since $\mathcal{L}_i = ([a_1 \cdots a_t] \cdot B) = ([a_1 \cdots a_t] \cdot B')$, we now use Lemma 2.4.11 to show the result.

■

Now, since $\mathcal{L} + I_r(B_i(\phi)) \subseteq \mathcal{L} + I_t(B_{i+1}(\phi))$ and $R[T_1, \cdots, T_m]$ is Noetherian, the procedure stops after a certain number of iterations. In fact, when R is a polynomial ring and ϕ is linear, the procedure stops after the first iteration.

Remark 4.1.3 Using Lemma 2.4.13, we can see that $\mathcal{L}+I_t(B_1(\phi)) \subseteq \mathcal{L} : (a_1, \ldots, a_t)$. Notice that $([a_1, \cdots, a_t] \cdot B_2(\phi)) \subseteq \mathcal{L} + I_t(B_1(\phi))$. Using Lemma 2.4.13 again,

$$(a_1, \ldots, a_t) \cdot I_t(B_2(\phi)) \subseteq \mathcal{L} + I_t(B(\phi)) \subseteq \mathcal{L} : (a_1, \ldots, a_t).$$

Thus $\mathcal{L} + I_t(B_2(\phi)) \subseteq \mathcal{L} : (x_1, \ldots, x_d)^2$. Iteratively, we can show that

$$\mathcal{L} + I_r(B_i(\phi)) \subseteq \mathcal{L} : (a_1, \ldots, a_t)^i.$$

It is still unclear when the two ideals are equal or if their respective index of stabilizations are related.

4.2 Ideals of Codimension two

In the previous section we showed that the ideal of an iterated Jacobian dual $\mathcal{L}+I_t(B_i(\phi))$ are uniquely determined. We now apply the notion of iterated Jacobian duals to the Setting of 3.0.1.

Let $R = k[x_1, \ldots, x_d]$ and I, a grade 2 perfect ideal generated by forms of the same degree. Further, I satisfies the G_d condition. Let φ be the presentation matrix of I. Assume $\mu(I) = m > d$. If φ is linear, then the defining ideal of the Rees algebra $\mathcal{R}(I)$ has the *expected form* $\mathcal{L}+I_d(B(\varphi))$ (see [8]). If φ is not linear, it is interesting to study when the defining ideal of $\mathcal{R}(I)$ and $\mathcal{L}+I_d(B_i(\varphi))$ coincide. Such a form of the defining ideals is easier to compute and has advantages when computing invariants such as relation type, regularity etc.

In [27], the author presents a condition as to when $\ker\Phi$ equals the expected form. An analogous condition is presented below for the ideal of an iterated Jacobian dual.

Remark 4.2.1 *Let $R = k[x_1, \ldots x_d]$ be a polynomial ring with the homogeneous maximal ideal $\mathfrak{M}$ and I be a grade 2 perfect ideal with presentation matrix φ. Assume I satisfies the G_d condition and let $I_1(\varphi) \subseteq (a_1, \ldots, a_d)$ where $a_1, \ldots, a_d$ form an R-regular sequence. If*

(a) $\operatorname{ht}\,(I_d(B_n(\varphi)) + \mathfrak{M})/\mathfrak{M}R[T_1, \ldots, T_m] \geq m - d$ *and*

(b) $\mathcal{L} + I_d(B_n(\varphi))$ *is unmixed,*

then $\ker\Phi = \mathcal{L} + I_d(B_n(\varphi))$.

Proof The proof of the remark is identical to the one presented in [27, 3.1], but for ease of reference we present the proof.

We have to show that $\mathfrak{L} = \mathcal{L} + I_d(B_n(\varphi))$ is a prime ideal of height $m-1$. Let P be an associated prime of $\mathfrak{L}$ and $\mathfrak{p} = P \cap R$.

If $\mathfrak{M} \neq \mathfrak{p}$, then $I_\mathfrak{p}$ is of linear type and hence $\mathcal{L}_\mathfrak{p} = m-1$. Since $\mathcal{L}_\mathfrak{p} \subseteq \mathfrak{L}_\mathfrak{p} \subseteq P_\mathfrak{p}$, we have $\operatorname{ht}\,P \geq m-1$.

Now suppose $\mathfrak{M} = \mathfrak{p}$, then $I_d(B_n(\varphi)) + \mathfrak{M} \subseteq \mathfrak{L} + \mathfrak{M} \subseteq P$. Further, using (a) in the hypotheses and the fact that $a_1, \ldots, a_d$ is an R-regular sequence, $\operatorname{ht}\,(I_d(B_n(\varphi)) + \mathfrak{M}) \geq m$. Thus $\operatorname{ht}\,P \geq m$.

Therefore if P is a minimal prime of $\mathfrak{L}$, then $\operatorname{ht}\,P \geq m-1$ and hence $\operatorname{ht}\,\mathfrak{L} = m-1$. Further, if $\mathfrak{L} \subseteq P$ and $\mathfrak{M} = P \cap R$, then $\operatorname{ht}\,\mathfrak{L} \neq \operatorname{ht}\,P$.

Using (b) in the hypothesis, all the minimal primes P of $\mathfrak{L}$ are of height $m-1$. In particular, $\mathfrak{M} \not\subseteq P$. Thus there exists an $a \in \mathfrak{M}$ which is regular modulo $\mathfrak{L}$. To show $\mathfrak{L}$ is a prime ideal we show $\mathfrak{L}_a$ is a prime ideal. But by the G_d condition, I_a is of linear type and hence $\mathcal{L}_a = \ker \Phi_a$, a prime ideal. Thus $\mathfrak{L}_a = \mathcal{L}_a$ is a prime ideal. ■

Not much information is available on the unmixed nature of $\mathcal{L} + I_d(B_n(\varphi))$, but its believed to be strong enough for the above remark to be of practical use.

Now in the setting of 3.0.1, we put our efforts to search for a condition for the equality of $\ker \Phi$ and $\mathcal{L} + I_t(B_n(\varphi))$. When $d = 2$, Kustin, Polini and Ulrich present a complete generating set for $\ker \Phi$ in [9]. From this generating set we see that $\ker \Phi$ and $\mathcal{L} + I_t(B_n(\varphi))$ are not always equal. A search for a condition led us to Corollary 4.2.3.

Henceforth, we assume that $\overline{(\ )}$ denote the image in the ring A.

Theorem 4.2.2 *Let A, K be as defined in Notation 3.0.5. Then in the setting of 3.0.1, one has $\frac{\overline{L_{m-1}K^n}}{\overline{x_d}^n} \subseteq \overline{\mathcal{L} + I_d(B_n(\varphi))}$ in the ring A.*

Proof Write $D_i = \frac{\overline{L_{m-1}K^i}}{\overline{x_d}^i}$ and $D'_i = \overline{\mathcal{L} + I_d(B_i(\varphi))}$. Clearly $D_i \subseteq D_{i+1}$ and $D'_i \subseteq D'_{i+1}$.

As in Notation 3.0.5, $K = (\tilde{L}_1, \cdots, \tilde{L}_{m-2}, I_{d-1}(B), x_d)$ where B is a submatrix of $B(\varphi')$. Now let $B(\varphi) = (b_{ij}),\ 1 \leq i \leq d,\ 1 \leq j \leq m-1$.

We prove the containment $D_i \subseteq D'_i,\ 1 \leq i \leq n$, by induction. Let $i = 1$. As $\overline{\tilde{L}_i} \in (\overline{x_d})$ in the ring A, it is clear that $\frac{\overline{L_{m-1}}\overline{\tilde{L}_i}}{\overline{x_d}} \in (\overline{L_{m-1}}) = \overline{\mathcal{L}}$. Now let w be a $d-1 \times d-1$ minor of B. For ease of notation, assume that w is the determinant of the submatrix of B obtained from the first $d-1$ rows and the first $d-1$ columns of B. Consider M, a submatrix of $B(\varphi)$ obtained from the first d rows and column indices belonging to the set $\{1, \ldots, d-1, m-1\}$. Using Lemma 2.4.12, we have $(-1)^{d+m-1}\overline{L_{m-1}} \cdot \overline{w} = \overline{\det(M)} \cdot \overline{x_d}$ in the ring A. Thus we have $\frac{\overline{L_{m-1}}\overline{w}}{\overline{x_d}} = \overline{\det(M)} \in \overline{I_d(B_1(\varphi))}$ proving the initial step of induction $D_1 \subseteq D'_1$.

Now assume that the result is true for $1 \leq i < n$. Consider $\frac{\overline{L_{m-1}}\overline{w_1}\cdots\overline{w_n}}{\overline{x_d}^n} \in D_n$ with $w_1, \ldots, w_n \in K$. It is enough to show that $\frac{\overline{L_{m-1}}\overline{w_1}\cdots\overline{w_n}}{\overline{x_d}^n} \in D'_n$.

By induction hypothesis, we have $\frac{\overline{L_{m-1}}\overline{w_1}\cdots\overline{w_{n-1}}}{\overline{x_d}^{n-1}} = \overline{w'} \in D'_{n-1}$. Thus $\frac{\overline{L_{m-1}}\overline{w_1}\cdots\overline{w_n}}{\overline{x_d}^{n}} = \frac{\overline{w'}\overline{w_n}}{\overline{x_d}}$. If $w' \in \mathcal{L}$, then $\overline{w'} \in (\overline{L_{m-1}})$ in the ring A. Thus by induction hypothesis, $\frac{\overline{w'}\overline{w_n}}{\overline{x_d}} \in D_1 \subseteq D'_1 \subseteq D'_n$.

If $w' \in I_d(B_{n-1}(\varphi))$ and $I_d(B_{n-1}(\varphi)) \cap (x_1, \ldots, x_d) = (0)$, then $w' \in I_d(B(\varphi')) \subseteq J$. Thus $\overline{w'} = 0$ and in this case, $\frac{\overline{w'}\overline{w_n}}{\overline{x_d}} = 0$ (recall that A is a domain and $n \geq 2$).

Now, assume $w' \in I_d(B_{n-1}(\varphi)) \cap (x_1, \ldots, x_d) = (u_1, \ldots, u_l)$. It is enough to show that $\frac{\overline{u_p}\overline{w_n}}{\overline{x_d}} \in D'_n$, $1 \leq p \leq l$. So, let $w' = u_p$ for some $p \in \{1, \cdots, l\}$.

Recall that $w_n \in K$. Now, if $\overline{w_n} \in (\overline{x_d}) \subseteq K$, then $\frac{\overline{w'}\overline{w_n}}{\overline{x_d}} \in (\overline{w'}) \subseteq D'_{n-1} \subseteq D'_n$. Thus assume that $w_n \in I_{d-1}(B)$. Rewrite

$$w' = \sum_{k=1}^{d} x_k w'_k \text{ for some } w'_k \in B. \tag{4.6}$$

Now

$$\frac{\overline{w'}\overline{w_n}}{\overline{x_d}} = \frac{\sum_{k=1}^{d} \overline{x_k}\overline{w'_k}\overline{w_n}}{\overline{x_d}}.$$

For ease of notation assume that $w_n = \det M$ where M is a $d \times d-1$ submatrix consisting of the first d rows and the first $d-1$ columns of $B(\phi)$. Hence in the ring A, using Lemma 2.4.10, we have $\overline{x_k}\overline{w_n} = (-1)^{k-d}\overline{x_d}\overline{\det M_k}$ where M_k is the submatrix of M obtained by removing the k-th row. Thus,

$$\begin{aligned}\frac{\overline{w'}\overline{w_n}}{\overline{x_d}} &= \frac{\sum_{k=1}^{d} \overline{x_k}\overline{w'_k}\overline{w_n}}{\overline{x_d}} \\ &= \frac{\sum_{k=1}^{d} (-1)^{k-d}\overline{x_d}\overline{\det M_k w'_k}}{\overline{x_d}} \\ &= \sum_{k=1}^{d} (-1)^{k-d}\overline{\det M_k w'_k}\end{aligned}$$

which is the determinant of the $d \times d$ matrix

$$H = \left[\begin{array}{c|c} & w'_1 \\ M & \vdots \\ & w'_d \end{array}\right].$$

Notice that $\overline{\det H} \in \overline{\mathcal{L} + I_d(B_n(\varphi))} = D'_n$. The decomposition in (4.6) is not unique. Thus a different decomposition in (4.6) leads to a different choice of H. But Theorem 4.1.2 shows that irrespective of the decomposition in (4.6), $\overline{\det H} \in D'_n$. ■

Corollary 4.2.3 *In the setting of 3.0.1, if* $\overline{K}^{(n)} = \overline{K}^n$*, then the defining ideal of the Rees algebra satisfies* $\ker \Phi = \mathcal{L} + I_d(B_n(\varphi))$.

Proof If $\overline{K}^{(n)} = \overline{K}^n$, then $\overline{\ker \Phi} = \frac{\overline{L_{m-1}K}^{(n)}}{\overline{x_d}^n} = \frac{\overline{L_{m-1}K}^n}{\overline{x_d}^n} \subseteq \overline{\mathcal{L} + I_d(B_n(\varphi))} \subseteq \overline{\ker \Phi}$. ■

Interestingly, the above corollary states that, $\mathcal{L} : (x_1, \ldots, x_d)^n = \mathcal{L} + I_d(B_n(\varphi))$ and the index of stabilization of the ideal of an iterated Jacobian dual is n.

Remark 4.2.4 In Theorem 4.2.2, the ideals D_1 and D'_1 are actually equal. We already showed the inclusion $D_1 \subset D'_1$. To show the reverse inequality $D'_1 \subseteq D_1$, notice that $x_d \in K$ and hence $\overline{L_{m-1}} = \frac{\overline{L_{m-1}x_d}}{\overline{x_d}} \in D_1$ showing that $\overline{\mathcal{L}} \subseteq D_1$. Now let $w \in I_d(B_1(\varphi))$. Since $I_d(B(\varphi')) \subseteq J$, we can assume that $w \notin I_d(B(\varphi'))$. Now in the ring A, $\overline{wx_d} = \overline{L_{m-1}w'}$. Thus $\overline{w} = \frac{\overline{L_{m-1}w'}}{\overline{x_d}} \in D_1$.

A natural question is whether $D_i = D'_i$ for $1 \leq i \leq n$?. The answer is affirmative, if a slight change is made in the construction of the iterated Jacobian duals. The change is described as follows. Instead of considering all the minors of $I_t(B_i(\varphi)) \cap (x_1, \ldots, x_d)$ to construct C in (4.2), we consider a special subset of minors. These minors are determinants of submatrices all but one of whose columns are columns of $B(\varphi')$ and the last column is that of $B_{i-1}(\varphi)$. This type of construction has been independently studied by Cox,Hoffman and Wang [28] in the case of $d = 2,\ m = 3$.

In the setting of 3.0.1, we showed that $I_1(\varphi) = (x_1, \ldots, x_d)$ (Observation 3.0.6). But in general, the iterated Jacobian dual is defined to be constructed with any generating set containing $I_1(\varphi)$. The generating set need not even be homogeneous and this feature was explored in the case of $d = 2$ by Hong, Simis and Vasconcelos in [29].

We now present some examples on how to construct the iterated Jacobian duals.

Example 4.2.5 Consider a matrix

$$\varphi = \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & x_2 & x_1^2 \\ 0 & x_3 & x_3^2 \end{bmatrix}$$

in a polynomial ring $R = k[x_1, x_2, x_3]$. Since grade $I_3(\varphi) \geq 2$, the converse of the Hilbert-Burch Theorem, guarantees the existence of a grade 2 perfect ideal I whose presentation matrix is φ. Also, the G_3 condition is satisfied as ht $\mathrm{Fitt}_3(I) =$ ht $I_1(\varphi) = 3$, $\mathrm{Fitt}_2(I) =$ ht $I_2(\varphi) \geq 3$. We now construct candidates for iterated Jacobian duals.

$$B_1(\varphi) = B(\varphi) = \begin{bmatrix} T_1 & T_2 & xT_3 \\ T_2 & T_3 & 0 \\ T_3 & T_4 & zT_4 \end{bmatrix}.$$

To construct $B_2(\varphi)$, we have to construct $(\det B_1(\varphi)) = I_3(B(\varphi))$.

$$\begin{aligned} \det B_1(\varphi) &= -xT_3^3 - zT_2^2T_4 + zT_1T_3T_4 + xT_2T_3T_4 \\ &= x(-T_3^3 + T_2T_3T_4) + z(-T_2^2T_4 + T_1T_3T_4) \\ &= x(-T_3(T_3^2 - T_2T_4)) + y(0) + z(T_4(-T_2^2 + T_1T_3)) \end{aligned}$$

We can construct $B_2(\varphi)$ using $\det B_1(\varphi)$.

$$B_2(\varphi) = \begin{bmatrix} T_1 & T_2 & xT_3 & -T_3(T_3^2 - T_2T_4) \\ T_2 & T_3 & 0 & 0 \\ T_3 & T_4 & zT_4 & T_4(-T_2^2 + T_1T_3) \end{bmatrix}.$$

We already know that $\mathcal{L} + I_3(B_2(\varphi)) \subseteq \mathcal{L} : (x_1, x_2, x_3)^2 = \ker \Phi$ (Remark 4.1.3). In the next section we will show that the defining ideal of the Rees algebra $\mathcal{R}(I)$ satisfies $\ker \Phi = \mathcal{L} + I_3(B_2(\varphi))$.

Example 4.2.6 Let $R = k[x_1, x_2]$. Let I be a grade 2 perfect ideal whose presentation matrix

$$\varphi = \begin{bmatrix} x_1 & 0 & x_1^2 \\ x_2 & x_1 & x_2^2 \\ 0 & x_2 & x_1^2 + x_2^2 \\ 0 & 0 & x_1^2 + x_2^2 + x_1 x_2 \end{bmatrix}.$$

One candidate for the first iterated Jacobian dual is

$$B_1(\varphi) = \begin{bmatrix} T_1 & T_2 & x_1T_1 + x_1T_3 + x_1T_4 + x_2T_4 \\ T_2 & T_3 & x_2T_2 + x_2T_3 + x_2T_4 \end{bmatrix}.$$

Notice that $I_2(B_1(\varphi)) \cap (x_1, x_2) = (d_1, d_2)$ where

$$d_1 = x_1(-T_1T_2 - T_2T_3 - T_2T_4) + x_2(T_1T_2 + T_1T_3 + T_1T_4 - T_2T_4)$$
$$d_2 = x_1(-T_1T_3 - T_3^2 - T_3T_4) + x_2(T_2^2 + T_2T_3 + T_2T_4 - T_3T_4)$$

We construct $B_2(\varphi)$ using d_1, d_2 to get

$$B_2(\varphi) = \begin{bmatrix} T_1 & T_2 & x_1T_1 + x_1T_3 + x_1T_4 + x_2T_4 & -T_1T_2 - T_2T_3 - T_2T_4 & -T_1T_3 - T_3^2 - T_3T_4 \\ T_2 & T_3 & x_2T_2 + x_2T_3 + x_2T_4 & T_1T_2 + T_1T_3 + T_1T_4 - T_2T_4 & T_2^2 + T_2T_3 + T_2T_4 - T_3T_4 \end{bmatrix}.$$

Using [9, 3.6], one can show that $f = T_2^2 + T_1T_2 + T_3^2 + T_1T_3 + T_3T_4 + T_1T_4 - T_2T_4 \in \ker \Phi$, but it is clear that $f \notin \mathcal{L} + I_2(B_2(\varphi))$. Subsequent iterations of the Jacobian dual do not produce an element of bi-degree $(0, 2)$ (we refer to Remark 3.0.3 for the grading scheme on B). Thus $\mathcal{L} + I_2(B_2(\varphi)) \neq \mathcal{A}$.

5. SECOND ANALYTIC DEVIATION ONE IDEALS

In this chapter we present a generating set for $\ker \Phi$ of ideals whose second analytic deviation is one, in terms of iterated Jacobian duals. Further, properties like depth, Cohen-Macaulayness and Castelnuovo-Mumford regularity of the Rees algebra are also studied. This is joint work with Jacob Boswell.

5.1 The defining ideal of the Rees algebra $\mathcal{R}(I)$ if $\mu(I) = d + 1$

The rest of this chapter assumes the setting of 3.0.1. The *special fiber ring* $\mathcal{F}(I)$ is defined as

$$\mathcal{F}(I) \cong \mathcal{R}(I)/(x_1, \ldots, x_d)\mathcal{R}(I).$$

The dimension of $\mathcal{F}(I)$ is called *analytic spread* and is denoted by $\ell(I)$. It is known that $\operatorname{ht} I \leq \ell(I) \leq \dim R = d$.

In this chapter we further assume that $\mu(I) = d+1$. Since I is of maximal analytic spread ($\ell(I) = d$, see for example [30]), the second analytic deviation $\mu(I) - \ell(I)$ is 1.

Remark 5.1.1 Since $\dim \operatorname{Sym}(I) = d + 1$ [22], $\operatorname{ht} \mathcal{L} = d$. Now the presentation matrix φ of I is an $d + 1 \times d$ matrix and hence $\mathcal{L}$ is d-generated. Thus $\operatorname{Sym}(I)$ is a complete intersection ring.

Observation 5.1.2 *Let A, K be as defined in Notation 3.0.5. Then in the setting of 3.0.1, $\overline{K}$ is generically a complete intersection and strongly Cohen-Macaulay in the ring A.*

Proof Recall that the A-ideals $\overline{K}$ and $\mathfrak{m}$ are Cohen-Macaulay of height one (Lemma 3.0.9). To prove $\overline{K}$ is generically a complete intersection, we localize $\overline{K}$ at height one primes $P \in V(\overline{K})$ of A and then show $\overline{K}_P$ is a complete intersection in A_P. Now let $P \in V(\overline{K})$. Since $\mathfrak{m}$ is not an associated prime of $\overline{K}$, we have $\mathfrak{m}_P = A_P$. Since $(\overline{x_d})_P :_{A_P} \overline{K}_P = \mathfrak{m}_P = A_P$ (Lemma 3.0.10) showing that $\overline{K}_P = (\overline{x_d})_P$. Thus $\overline{K}$ is generically a complete intersection.

Following Notation 6.0.2, notice that $\overline{K} = (\overline{w}, \overline{x_d})$, where $I_{d-1}(N) = (w)$, is an almost complete intersection ideal of height one in the Cohen-Macaulay ring A. Also, $A/\overline{K}$ is Cohen-Macaulay (Lemma 3.0.9). Thus $\overline{K}$ is strongly Cohen-Macaulay (Theorem 2.2.2). ■

Lemma 5.1.3 *Let $B(\varphi')$ be as defined in Setting 3.0.5 and $\mu(I) = d+1$, then* ht $I_{d-1}B(\varphi') = 2$.

Proof We know that A is a complete intersection domain (Remark 5.1.1) of dimension $d+2$. Since $\mu(I) = d+1$, we see that $I_d(B(\varphi')) = 0$. Thus $A = S/(L_1, \ldots, L_{m-2})$. Also, $[L_1 \cdots L_{m-2}] = [x_1 \cdots x_d] \cdot B(\varphi')$ and hence A can be viewed as a symmetric algebra $A \cong \mathrm{Sym}_{k[\underline{T}]}(\mathrm{coker}\, B(\varphi'))$ over the ring $k[T_1, \ldots, T_{d+1}]$. Since A is a domain we see that

$$\begin{aligned} d+2 = \dim A &= \dim \mathrm{Sym}_{k[\underline{T}]}(\mathrm{coker}\, B(\varphi')) \\ &= \mathrm{rank}\ \mathrm{coker}\, B(\varphi') + \dim k[\underline{T}]. \end{aligned}$$

Thus rank coker $B(\varphi') = 1$. Using [31, 6.8,6.6] we see that grade $I_{d-1}(B(\varphi')) \geq 2$. Since $B(\varphi')$ is a $d \times d-1$ matrix, ht $I_{d-1}(B(\varphi')) \leq 2$. Thus ht $I_{d-1}(B(\varphi')) = 2$. ■

Theorem 5.1.4 *Let $R = k[x_1, \cdots, x_d]$ be a polynomial ring and let I be a grade 2 perfect R-ideal whose presentation matrix φ is almost linear. If I satisfies G_d and $\mu(I) = d+1$, then the defining ideal of $\mathcal{R}(I)$ satisfies*

$$\mathcal{A} = \mathcal{L} + I_d(B_n(\varphi)) = \mathcal{L} : (x_1, \cdots, x_d)^n$$

where n is the degree of the entries of the last column of φ. Furthermore, the special fiber ring $\mathcal{F}(I) \cong k[T_1, \ldots, T_{d+1}]/(\mathfrak{f})$ where $\deg \mathfrak{f} = n(d-1)+1$.

Proof It suffices to show that $\overline{K}^n = \overline{K}^{(n)}$ (Corollary 4.2.3). To prove this equality, we use Theorem 2.2.3. Using Observation 5.1.2, K is strongly Cohen-Macaulay. So we have to show

$$\mu(\overline{K}_P) \leq \text{ht } P - 1 \text{ for all } P \in V(\overline{K}), \text{ with ht } P = 2. \tag{5.1}$$

Let $\overline{K} = (\overline{w}, \overline{x_d})$ and

$$(w) = I_{d-1}(B) \subset I_{d-1}(B(\varphi')) = (w, w'_1, \cdots w'_{d-1}).$$

Now let $P \in V(\overline{K})$ such that ht $P = 2$. If $P \notin V(\overline{(x_1, \ldots, x_d)})$, then $\overline{K}_P = (\overline{x_d})_P$ and hence (5.1) is trivially satisfied.

Now suppose $P \in V(\overline{(x_1, \ldots, x_d)})$. Observe that, since ht $I_{d-1}(B(\varphi')) = 2$ (Observation 5.1.3), we have ht $(x_1, \ldots, x_d, I_{d-1}(B(\varphi')) = d + 2$. Thus ht $\overline{(x_1, \ldots, x_d, I_{d-1}(B(\varphi'))} = 3$ in A and hence $P \not\supset \overline{I_{d-1}(B(\varphi'))}$. Using Lemma 2.4.10, we have $\overline{x_i} \cdot \overline{w} = (-1)^{i-d}\overline{x_d} \cdot \overline{w'_i}$. Since $\overline{w} \in \overline{K} \subseteq P$, we have $\overline{w'_i} \notin P$ for some $1 \leq i \leq d-1$. Thus $\overline{x_d} \in (\overline{w})_P$ in A_P and hence $\overline{K}_P = (\overline{w})_P$. This proves (5.1) in this case too.

We now prove the statement on the special fiber ring. Notice that

$$\begin{aligned}(x_1, \ldots, x_d) + \ker \Phi &= (x_1, \ldots, x_d) + \mathcal{L} + I_d(B_n(\varphi)) \\ &= (x_1, \ldots, x_d) + (f')\end{aligned}$$

where $f' \in I_d(B_n(\varphi))$. Notice that f' is an element which has x-degree equals zero. Any element in $I_d(B_1(\varphi))$ has bidegree $(n-1, d)$. Subsequently, any element in $I_d(B_2(\varphi))$ having the least x-degree has bidegree $(n-2, 2d-1)$. Continuing like this, the element in $I_d(B_n(\varphi))$ having the least x-degree (equal zero) has bi-degree $(0, n(d-1)+1)$. Thus the degree of f' is also $(0, n(d-1)+1)$. Now let $\mathfrak{f} = \overline{f'}$ where $^{-}$ represents the image in the ring $k[T_1, \ldots, T_{d+1}]$.

■

Corollary 5.1.5 *Let I be the ideal defining a set of 11 points in $\mathbb{P}^2$. Then for a general choice of points, the defining equations of the Rees algebra satisfy* $\ker \Phi = \mathcal{L} + I_3(B_2(\varphi))$, *where φ is a presentation matrix of I.*

Proof From the discussion in [32, 1.2], we see that for a general choice of $11 = \binom{4+1}{2} + 1$ points, the ideal I is a grade two perfect ideal satisfying the G_3 condition. Further, the 4×3 presentation matrix φ of I is almost linear with the last column consisting of quadratic entries. Thus the defining ideal of the Rees algebra satisfies $\ker \Phi = \mathcal{L} + I_3(B_2(\varphi))$. ■

Example 5.1.6 *In Example 4.2.5, $\overline{K} = (\overline{T_1T_3} - \overline{T_2^2}, \overline{x_2})$, an almost complete intersection in the domain A. By the above theorem, $\mathcal{A} = \mathcal{L} + I_3(B_2(\varphi)) = \mathcal{L} : (x_1, x_2, x_3)^2$.*

5.2 Depth, Relation type and Regularity

We begin by constructing a series of short exact sequences which play an important role in computing the invariants such as depth and regularity of the Rees algebra.

Recall that $\mathfrak{m}$ denotes the ideal $(x_1, \ldots, x_d)$ and $\mathfrak{n}$, the homogeneous maximal ideal of A. As in the above theorem, $\overline{K} = (\overline{w}, \overline{x_d})$, where $(w) = I_{d-1}(B)$. Also $\overline{K}$ is a Cohen-Macaulay ideal and $\mathfrak{m}A = (\overline{x_d}) : \overline{K}$, which gives the exact sequence of bi-graded A-modules

$$0 \to \mathfrak{m}A(0, -(d-1)) \to A(-1, 0) \oplus A(0, -(d-1)) \to \overline{K} \to 0. \tag{5.2}$$

Apply Sym( ) to the above short exact sequence and consider the n-th degree component to obtain

$$\mathfrak{m}A(0, -(d-1)) \otimes \mathrm{Sym}_{n-1}(A(-1, 0) \oplus A(0, -(d-1))) \xrightarrow{\sigma}$$
$$\mathrm{Sym}_n(A(-1, 0) \oplus A(0, -(d-1))) \to \mathrm{Sym}_n(\overline{K}) \to 0.$$

Due to rank reasons $\ker \sigma$ is torsion. But the source of σ is a torsion-free module and hence σ is injective. Thus we have an exact sequence

$$0 \to \mathfrak{m}A(0, -(d-1)) \otimes \mathrm{Sym}_{n-1}(A(-1, 0) \oplus A(0, -(d-1))) \to$$
$$\mathrm{Sym}_n(A(-1, 0) \oplus A(0, -(d-1))) \to \mathrm{Sym}_n(\overline{K}) \to 0. \tag{5.3}$$

Notice that $\overline{K}$ is strongly Cohen-Macaulay (Observation 5.1.2). Also $\overline{K}$ satisfies the G_∞ condition. Thus $\overline{K}$ is an A-ideal of linear type (Theorem 2.4.6). Therefore $\mathrm{Sym}_n(\overline{K}) \cong \overline{K}^n$.

Thus sequence (5.3) now reads

$$0 \to \bigoplus_{i=0}^{n-1} \mathfrak{m}A(-i, -(n-i)(d-1)) \to \bigoplus_{i=0}^{n} A(-i, -(n-i)(d-1)) \to \overline{K}^n \to 0. \quad (5.4)$$

Recall that a Noetherian local ring $\mathcal{S}$ is said to be *almost* Cohen-Macaulay when depth $\mathcal{S} = \dim \mathcal{S} - 1$.

Theorem 5.2.1 *Assume the setting of Theorem 5.1.4. Further, let $n > 1$. Then* depth $\mathcal{F}(I) =$ depth $\mathcal{R}(I) = d$, *i.e the Rees algebra $\mathcal{R}(I)$ is almost Cohen-Macaulay and the special fiber ring $\mathcal{F}(I)$ is Cohen-Macaulay.*

Proof From the short exact sequence,

$$0 \to \mathfrak{m}A \to A \to A/\mathfrak{m}A \to 0 \quad (5.5)$$

we have depth $\mathfrak{m}A = d+2$. Now from (5.4) we have depth $\overline{K}^n \geq d+1$. The sequence

$$0 \to \overline{\ker \Phi} \to A \to \mathcal{R}(I) \to 0$$

and the isomorphism $\overline{\ker \Phi} = \frac{\overline{L_{m-1}K^{(n)}}}{\overline{x_d}^n} \cong K^{(n)}$ now implies that depth $\mathcal{R}(I) \geq d$. The Rees algebra $\mathcal{R}(I)$ is not a Cohen-Macaulay ring unless $\mathcal{A} = \mathcal{L} : (x_1, \ldots, x_d)$ [33, 4.5]. Recall that the defining ideal of the Rees algebra is also of the form $\mathcal{L} : (x_1, \ldots, x_d)^n$. Also, n is minimal by Corollary 3.0.12. Thus when $n > 1$, the Rees algebra is not a Cohen-Macaulay ring and hence we conclude depth $\mathcal{R}(I) = d$.

Since $\mathcal{F}(I) \cong k[T_1, \ldots, T_{d+1}]/(f)$ (Theorem 5.1.4), we have depth $\mathcal{F}(I) = d$. ∎

We compute the regularity of $\mathcal{R}(I)$ with respect to $\mathfrak{M} = (x_1, \ldots, x_d)$, $\mathfrak{N} = (x_1, \ldots, x_d, T_1, \ldots, T_{d+1})$ and $(T_1, \ldots, T_{d+1})$. For convenience, we let $(\underline{T}) = (T_1, \ldots, T_{d+1})$. When computing reg $_{\mathfrak{M}}\mathcal{R}(I)$ we set $\deg x_i = 1$, $\deg T_i = 0$. Analogously the grading scheme is set for reg $_{\mathfrak{N}}\mathcal{R}(I)$ and reg $_{(\underline{T})}\mathcal{R}(I)$.

Theorem 5.2.2 *In the setting of Theorem 5.1.4,*

$$\mathrm{rt}(I) = \mathrm{reg}\ \ \mathcal{F}(I) + 1 = \mathrm{reg}\ _{(\underline{T})}\mathcal{R}(I) + 1 = n(d-1) + 1$$

Furthermore, $\mathrm{reg}\ _{\mathfrak{M}}\mathcal{R}(I) \leq n-1$ *and* $\mathrm{reg}\ _{\mathfrak{N}}\mathcal{R}(I) \leq (n+1)(d-1)$

Proof Since $\overline{\ker\Phi} = \frac{\overline{g}\overline{K}^{(n)}}{\overline{x_d}^n} = \frac{\overline{g}\overline{K}^n}{\overline{x_d}^n}$, the relation type, $\mathrm{rt}(I)$, is easily computed by considering the $(\underline{T})$-degrees of the generating set of $\overline{K}^n$. Thus $\mathrm{rt}(I) = n(d-1)+1$.

The statement $\mathrm{reg}\ \mathcal{F}(I) = n(d-1)$ is clear as $\mathcal{F}(I) \cong k[T_1, \ldots, T_{d+1}]/(f)$ where $\deg f = n(d-1)+1$ (Theorem 5.1.4).

It is well known that $\mathrm{rt}(I) - 1 \leq \mathrm{reg}_{(\underline{T})}\mathcal{R}(I)$. Therefore, in order to show the equality $\mathrm{reg}_{(\underline{T})}\mathcal{R}(I)+1 = n(d-1)+1$, it is enough to show that $\mathrm{reg}_{(\underline{T})}\mathcal{R}(I) \leq n(d-1)$.

To compute the regularities we make use of exact sequences (5.4) and (5.5). Notice that $A/\mathfrak{M}A \cong k[T_1, \ldots, T_{d+1}]$. Since A is a complete intersection domain defined by forms which are linear in both the $x_1, \ldots, x_d$ and $T_1, \ldots, T_{d+1}$ variables, we have

$$\mathrm{reg}_{(\underline{T})}A = \mathrm{reg}_{(\underline{T})}A/\mathfrak{M}A = 0$$

$$\mathrm{reg}_{\mathfrak{M}}A = \mathrm{reg}_{\mathfrak{M}}A/\mathfrak{M}A = 0$$

$$\mathrm{reg}_{\mathfrak{N}}A = d-1,\ \mathrm{reg}_{\mathfrak{N}}A/\mathfrak{M}A = 0$$

Thus from (5.5) we have,

$$\mathrm{reg}_{(\underline{T})}\mathfrak{M}A \leq 1,\ \mathrm{reg}_{\mathfrak{M}}\mathfrak{M}A \leq 1,\ \mathrm{reg}_{\mathfrak{N}}\mathfrak{M}A = d-1.$$

Let $M = \bigoplus_{i=0}^{n-1} \mathfrak{M}A(-i, -(n-i)(d-1))$ and $N = \bigoplus_{i=0}^{n} A(-i, -(n-i)(d-1))$. Thus

$$\begin{aligned} &\mathrm{reg}_{(\underline{T})}M \leq n(d-1)+1 && \mathrm{reg}_{(\underline{T})}N = n(d-1) \\ &\mathrm{reg}_{\mathfrak{M}}M \leq n && \mathrm{reg}_{\mathfrak{M}}N = n \\ &\mathrm{reg}_{\mathfrak{N}}M = (n+1)(d-1) && \mathrm{reg}_{\mathfrak{N}}N = (n+1)(d-1). \end{aligned}$$

Now using (5.4) we have

$$\begin{aligned} \mathrm{reg}_{(\underline{T})}\overline{K}^n &\leq n(d-1) \\ \mathrm{reg}_{\mathfrak{M}}\overline{K}^n &\leq n \\ \mathrm{reg}_{\mathfrak{N}}\overline{K}^n &\leq (n+1)(d-1). \end{aligned} \tag{5.6}$$

Next, consider the short exact sequence

$$0 \to \overline{\ker \Phi} \to A \to \mathcal{R}(I) \to 0.$$

We now use the bigraded isomorphism $\overline{\ker \Phi} \cong \overline{K}^n(0, -1)$ and the inequalities in (5.6) to show

$$\begin{aligned} \operatorname{reg}_{(\underline{T})}\mathcal{R}(I) &\leq n(d-1) \\ \operatorname{reg}_{\mathfrak{M}}\mathcal{R}(I) &\leq n-1 \\ \operatorname{reg}_{\mathfrak{N}}\mathcal{R}(I) &\leq (n+1)(d-1). \end{aligned}$$

■

6. IMAGES OF CERTAIN RATIONAL MAPS

In this chapter we study the blow-up algebras associated to the rational map

$$\Psi : \mathbb{P}^{d-1} \xrightarrow{[f_1:\cdots:f_m]} \mathbb{P}^{m-1}$$

where the f_i's are homogeneous forms of the same degree in the homogeneous coordinate ring $R = k[x_1, \ldots, x_d]$ of $\mathbb{P}^{d-1}$. The implicitization problem involves finding the implicit equations defining the image of the rational map Ψ. Recall that the coordinate ring of the image of Ψ is the special fiber ring $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ where $\mathfrak{m} = (x_1, \ldots, x_d)$. We concern ourselves with the case when $m = d+1$ and $I = (f_1, \ldots, f_{d+1})$ is a grade 2 perfect ideal satisfying the G_d condition in $R = k[x_1, \ldots, x_d]$. In low dimensions, the implicitization problem has been referred to as the moving curve and moving surface ideal problem [34]. This is joint work with Youngsu Kim.

We first present a condition when the map Ψ is birational onto its image. It is a constructive method that uses the Buchsbaum-Eisenbud Multipliers (we refer to Section 2.7). We also find the defining ideal of the image of Ψ when the map Ψ is birational onto its image. These defining ideals have been computed before by J.-P. Jouanolou [13], but the methods presented here are different from his. Jouanolou's method involves finding the MacRae invariant of a graded components of $\mathrm{Sym}(I)$, whereas we find the Buchsbaum-Eisenbud multiplier of a different component of $\mathrm{Sym}(I)$. Both methods uses the minors of the matrices in the respective resolutions. But as the resolutions we consider are smaller, the ideal of minors we need are smaller and hence, computationally simpler.

We now define the setting of the problem

Setting 6.0.1 *Let*

(a) $\Psi : \mathbb{P}^{d-1} \overset{[f_1:\cdots:f_{d+1}]}{\dashrightarrow} \mathbb{P}^d$ *where f_i's are homogeneous forms of the same degree $\mathfrak{d}$ in the homogeneous coordinate ring $R = k[x_1, \ldots, x_d]$ of $\mathbb{P}^{d-1}$*

(b) $I = (f_1, \ldots, f_{d+1})$ *is a grade two perfect ideal satisfying the G_d condition in R.*

(c) *the homogeneous coordinate ring of $\mathbb{P}^d$ is $S = k[T_1, \ldots, T_{d+1}]$.*

(d) *the homogeneous coordinate ring of $\mathbb{P}^{d-1} \times \mathbb{P}^d$ is $B = R[T_1, \ldots, T_{d+1}]$.*

As in the previous sections, the homogeneous $d+1 \times d$ presentation matrix φ of I consists of homogeneous entries of constant degree along each column. In this chapter we do not impose any constraints on the degrees of the columns of φ. Recall that the defining ideal of the Rees algebra $\mathcal{R}(I)$ is the kernel $\ker \Phi$ of the epimorphism $\Phi : B \twoheadrightarrow \mathcal{R}(I)$ where $\Phi(T_i) = f_i t$. Since the map Φ factors thorough the symmetric algebra and the defining ideal of the symmetric algebra is well understood, it suffices to find the kernel $\mathcal{A} = \ker(\mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}(I))$.

We use the following notation in this chapter

Notation 6.0.2 (a) *Let the degrees of the entries of the i-th column of φ be $\mathfrak{d}_i$ and $1 \leq \mathfrak{d}_1 \leq \cdots \leq \mathfrak{d}_d$.*

(b) *In B, set $\deg x_i = (1, 0)$ and $\deg T_j = (0, 1)$ making B a bi-graded algebra.*

(c) *The B-modules $\mathcal{A}$ and $\mathrm{Sym}(I)$ are also bi-graded. Let $\mathcal{A}_{(i,j)}$ and $\mathrm{Sym}(I)_{(i,j)}$ represent the (i, j)-th bi-homogeneous component of $\mathcal{A}$ and $\mathrm{Sym}(I)$ respectively.*

(d) *Let $\mathrm{Sym}(I)_i = \bigoplus_j \mathrm{Sym}(I)_{(i,j)}$ and $\mathcal{A}_i = \bigoplus_j \mathcal{A}_{(i,j)}$ which are also S-modules.*

Recall that the defining ideal of $\mathrm{Sym}(I)$ is $\mathcal{L} = (L_1, \ldots, L_d)$ where $[T_1 \cdots T_{d+1}] \cdot \varphi = [L_1, \ldots, L_d]$. In the setting of 6.0.1 we have $\dim \mathrm{Sym}(I) = d+1$, and hence $\mathrm{ht}\, \mathcal{L} = d$. Thus $\mathrm{Sym}(I)$ is a complete intersection ring. A natural choice of a B-free resolution of $\mathrm{Sym}(I)$ is the Koszul complex $\mathcal{K}_\bullet(L_1, \ldots, L_d; B)$ on the generating set $L_1, \ldots, L_d$.

Since the co-ordinate ring of the image of Ψ is $\mathcal{F}(I) \cong \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ we attempt to find the defining ideal of $\mathcal{F}(I)$. Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r] & B = R[T_1, \ldots, T_{d+1}] \arrow[r, two heads] & \mathcal{R}(I) \\ 0 \arrow[r] & \mathfrak{K} \arrow[r] \arrow[u, hook] & S = k[T_1, \ldots, T_{d+1}] \arrow[r, two heads] \arrow[u, hook] & \mathcal{F}(I) \arrow[u, hook] \end{tikzcd} \tag{6.1}$$

Thus the defining ideal of $\mathcal{F}(I)$ is a subset of the defining ideal of $\mathcal{R}(I)$. In fact, the defining ideal of $\mathcal{F}(I)$ consists of homogeneous elements of the defining ideal of $\mathcal{R}(I)$ that have x-degree zero. In the above diagram $\mathcal{J}$ is the defining ideal of $\mathcal{R}(I)$ and $\mathfrak{K}$ is the defining ideal of $\mathcal{F}(I)$. Thus $\mathcal{J}/\mathcal{L} = \mathcal{A}$ and $\mathcal{A}_0 = \mathcal{J}_0 = \mathfrak{K}$. Thus we infer that to study the defining ideal of $\mathcal{F}(I)$ it is enough to study $\mathcal{A}_0$.

The starting point of our investigation is the following result of J.-P.Jouanolou. He used Morley forms to prove the theorem that will be presented in the next section. A non-constructive proof of the same was given by Kustin, Polini and Ulrich in [14, 2.4], whose generalization is what we discuss below.

Theorem 6.0.3 *There is an isomorphism of bi-graded B modules*

$$\mathcal{A} \cong \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I), S)(-\delta, -d) \tag{6.2}$$

where $\delta = \sum_{i=1}^{d} \mathfrak{d}_i - d$. Here $\underline{\mathrm{Hom}}$ represents graded Hom.

Proof Since I is a grade 2 perfect ideal satisfying G_d, [4, 3.6] shows that

$$\mathcal{A} = 0 :_{\mathrm{Sym}(I)} \mathfrak{m}^{\infty} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I)).$$

We now consider the Koszul complex $\mathcal{K}_{\bullet}(L_1, \ldots, L_d; B)$. Notice that the L_i's are bi-homogeneous elements of bi-degree $(\mathfrak{d}_i, 1)$. Now

$$\mathcal{K}_{\bullet}(L_1, \ldots, L_d; B) : 0 \to K_d \xrightarrow{\partial_d} K_{d-1} \to \cdots \to K_1 \to K_0$$

is a bi-graded B-free resolution of $\mathrm{Sym}(I)$. For the module of the right hand side of (6.2) one has

$$\begin{aligned} \underline{\mathrm{Hom}}_S(\mathrm{Sym}(I), S) &\cong \underline{\mathrm{Hom}}_S(\mathcal{K}_{\bullet}(L_1, \ldots, L_d; B), S) \\ &\cong \mathcal{K}_{\bullet}(L_1, \ldots, L_d; \underline{\mathrm{Hom}}(B, S))(\sum_{i=1}^{d} \mathfrak{d}_i, d). \end{aligned} \tag{6.3}$$

The second isomorphism is due to the self duality of the Koszul complex. We now realize $\underline{\mathrm{Hom}}(B, S)$ as the highest local cohomology of B using the following series of isomorphisms.

$$\begin{aligned}
\mathrm{H}^d_{\mathfrak{m}}(B) &\cong \mathrm{H}^d_{\mathfrak{m}}(R \otimes_R B) \\
&\cong \mathrm{H}^d_{\mathfrak{m}}(R) \otimes_R B && (B \text{ is } R \text{ flat}) \\
&\cong \mathrm{H}^d_{\mathfrak{m}}(R) \otimes_k S && (B \cong R \otimes_k S) \\
&\cong \underline{\mathrm{Hom}}_k(R, k)[d] \otimes_k S && (\text{Serre Duality}) \\
&\cong \underline{\mathrm{Hom}}_S(B, S)(d, 0). && (6.4)
\end{aligned}$$

Next we decompose the Koszul complex $\mathcal{K}_\bullet(L_1, \ldots, L_d; B)$ into short exact sequences

$$\begin{aligned}
&0 \to J_0 \to K_0 \to \mathrm{Sym}(I) \to 0 \\
&0 \to J_1 \to K_1 \to J_0 \to 0 \\
&\vdots \\
&0 \to J_{d-2} \to K_{d-2} \to J_{d-1} \to 0 \\
&0 \to K_d \to K_{d-1} \to J_{d-3} \to 0.
\end{aligned}$$

Applying the local cohomology functor to the above short exact sequences, we see

$$\begin{aligned}
&\mathrm{H}^0_{\mathfrak{m}}(K_0) \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I)) \to \mathrm{H}^1_{\mathfrak{m}}(J_0) \to \mathrm{H}^1_{\mathfrak{m}}(K_0) = 0, \\
&\mathrm{H}^1_{\mathfrak{m}}(K_1) = 0 \to \mathrm{H}^1_{\mathfrak{m}}(J_0) \to \mathrm{H}^2_{\mathfrak{m}}(J_1) \to \mathrm{H}^2_{\mathfrak{m}}(K_1) = 0, \\
&\vdots \\
&\mathrm{H}^{d-2}_{\mathfrak{m}}(K_{d-2}) = 0 \to \mathrm{H}^{d-3}_{\mathfrak{m}}(J_{d-3}) \to \mathrm{H}^{d-1}_{\mathfrak{m}}(J_{d-2}) \to \mathrm{H}^{d-1}_{\mathfrak{m}}(K_{d-2}) = 0, \text{ and} \\
&\mathrm{H}^{d-1}_{\mathfrak{m}}(K_{d-1}) = 0 \to \mathrm{H}^{d-1}_{\mathfrak{m}}(J_{d-2}) \to \mathrm{H}^d_{\mathfrak{m}}(K_d) \xrightarrow{\rho} \mathrm{H}^d_{\mathfrak{m}}(K_{d-1}).
\end{aligned}$$

The map ρ is the differential of the Koszul complex $\mathcal{K}_\bullet(L_1, \ldots, L_d, \mathrm{H}^d_{\mathfrak{m}}(B))$. Notice that $\mathrm{H}^0_{\mathfrak{m}}(K_0) = \mathrm{H}^1_{\mathfrak{m}}(K_1) = \cdots = \mathrm{H}^{d-1}_{\mathfrak{m}}(K_{d-2}) = 0$ because the modules K_i are free R modules and grade $\mathfrak{m} = d$. Thus

$$\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I)) \cong \mathrm{H}^1_{\mathfrak{m}}(J_0) \cong \cdots \cong \mathrm{H}^{d-1}_{\mathfrak{m}}(J_{d-2}) \cong \mathrm{H}_d(\mathcal{K}_\bullet(L_1, \ldots, L_d; \mathrm{H}^d_{\mathfrak{m}}(B)))$$

But

$$\begin{aligned}\mathrm{H}_d(\mathcal{K}_\bullet(L_1,\ldots,L_d;\mathrm{H}^d_{\mathfrak{m}}(B))) &\cong \mathrm{H}_d(\mathcal{K}_\bullet(L_1,\ldots,L_d;\underline{\mathrm{Hom}}(B,S)(d,0))) \text{ by (6.4)}\\ &\cong \underline{\mathrm{Hom}}(\mathrm{Sym}(I),S)(-\delta,-d).\end{aligned}$$

■

Remark 6.0.4 From the above theorem, we conclude that $\mathcal{A}_0 \cong \mathrm{Hom}(\mathrm{Sym}(I)_\delta, S(-d))$.

We compute $\mathrm{Hom}(\mathrm{Sym}(I)_\delta, S)$, but we can easily recover $\mathrm{Hom}(\mathrm{Sym}(I)_\delta, S(-d))$ by shifting the $\underline{T}$-degree.

Now notice the Koszul complex $\mathcal{K}_\bullet(L_1,\ldots,L_d;B)$ is

$$0 \to B\left(-\sum_{i=1}^{d}\mathfrak{d}_i, -d\right) \to \bigoplus_{1\le i_1<\cdots<i_{d-1}\le d} B\left(\sum_{j=1}^{d-1}\mathfrak{d}_{i_j}, -(d-1)\right) \to \cdots$$
$$\cdots \to \bigoplus_{1\le i\le d} B(-\mathfrak{d}_i,-1) \to B \to \mathrm{Sym}(I) \to 0$$

From this B-resolution $\mathcal{K}_\bullet(L_1,\ldots,L_d;B)$ of $\mathrm{Sym}(I)$ we extract an S-resolution for $\mathrm{Sym}(I)_\delta$

$$\mathbb{F}: 0 \to F_{n-1} \xrightarrow{\phi_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} \mathrm{Sym}(I)_\delta \to 0 \tag{6.5}$$

where

$$F_i = \bigoplus_{1\le j_1<j_2<\cdots<j_i\le d} S^{\binom{\delta-(\mathfrak{d}_{j_1}+\cdots+\mathfrak{d}_{j_i})+d-1}{d-1}}(-i) = \bigoplus_{1\le k_1<k_2<\cdots<k_{d-i}\le d} S^{\binom{\mathfrak{d}_{k_1}+\cdots+\mathfrak{d}_{k_{d-i}}-1}{d-1}}(-i)$$

and $n \le d$.

Let

$$\begin{aligned} r_i &= \mathrm{rank}\ \phi_i\\ t_i &= \mathrm{rank}\ F_i = \sum_{1\le k_1<k_2<\cdots<k_{d-i}\le d}\binom{\mathfrak{d}_{k_1}+\cdots+\mathfrak{d}_{k_{d-i}}-1}{d-1}\end{aligned} \tag{6.6}$$

Remark 6.0.5 Notice that from (6.5), if $n \le i \le d-1$, then

$$\sum_{1\le k_1<k_2<\cdots<k_{d-i}\le d}\binom{\mathfrak{d}_{k_1}+\cdots+\mathfrak{d}_{k_{d-i}}-1}{d-1} = 0$$

Remark 6.0.6 Since rank $\mathrm{Sym}(I)_\delta = 1$, notice that $r_1 = \text{rank } \phi_1 = \text{rank } F_0 - 1 = t_0 - 1$. Also, $\sum_{j=0}^{n-1}(-1)^j t_j = 1$ and $r_i + r_{i+1} = \text{rank } F_i$ [35].

Applying the functor $\mathrm{Hom}(\_, S)$ to (6.5) we get

$$\mathrm{Hom}(\mathrm{Sym}(I)_\delta, S) = \ker \phi_1^* \tag{6.7}$$

where $\phi_1^* : F_0^* \to F_1^*$. When $\mu(I) = d+1$, it is well know that the image of Ψ is a hypersurface (see for example [30, 2.4]). Thus $\mathcal{A}_0 = (\mathfrak{a})$ is principally generated. Using Remark 6.0.4, this in turn shows that rank $\mathrm{Sym}(I)_\delta = 1$ and that $\ker \phi_1^*$ is generated by one element.

Before we explain the process of constructing an element in $\ker \phi_1^*$, we define a crucial isomorphism $\wedge^k F_i \cong \wedge^{t_i - k} F_i^*$ for $k \le t_i$ which is used throughout the chapter. And this isomorphism is explicitly defined by fixing an "*orientation*". We briefly mention the method below.

Basis for F_i and orientation of $\wedge^{t_i} F_i$: Let $\{e_1^i, \ldots, e_{t_i}^i\}$ denote the ordered basis for F_i. For an ordered set $\nu = \{j_1, \ldots, j_t\} \subseteq \{1, \ldots, t_i\}$, let e_ν^i denote the element $e_{j_1}^i \wedge \cdots \wedge e_{j_t}^i$. Fix the *orientation* $e_1^i \wedge \cdots \wedge e_{t_i}^i \in \wedge^{t_i} F_i$ for each F_i, which defines an isomorphism $\wedge^{t_i} F_i \xrightarrow{\cong} R$. Using this orientation we define the isomorphism $\wedge^k F_i \cong \wedge^{t_i - k} F_i^*$ for each $k \le t_i$. Consider an ordered subset $\nu \subseteq \{1, \ldots, t_i\}$ of cardinality k and let ν^c denote its complement in $\{1, \ldots, t_i\}$. Since $\wedge^k F_i \otimes \wedge^{t_i - k} F_i \cong \wedge^{t_i} F_i$, every element $e_\nu^i \in \wedge^k F_i$ defines a map

$$\begin{aligned} \wedge^{t_i - k} F_i \to \wedge^{t_i} F_i &\xrightarrow{\cong} R \\ e_{\nu^c}^i \to e_{\nu^c}^i \wedge e_\nu^i &= (-1)^\chi \cdot e_1^i \wedge \cdots \wedge e_{t_i}^i \\ e_\mu^i &\to 0 \end{aligned} \tag{6.8}$$

where χ is the number of permutations required to convert $e_{\nu^c}^i \wedge e_\nu^i$ into $e_1^i \wedge \cdots \wedge e_{t_i}^i$ and $\mu \neq \nu^c$ is a subset of $\{1, \ldots, t_i\}$ of cardinality $t_i - k$. This map is nothing but the map $((-1)^\chi \cdot e_{\nu^c}^i)^* \in \wedge^{t_i - k} F_i^*$. Thus for every element $e_\nu^i \in \wedge^k F_i$, we have a unique element $((-1)^\chi \cdot e_{\nu^c}^i)^* \in \wedge^{t_i - k} F_i^*$and hence we have the isomorphism $\wedge^k F_i \cong \wedge^{t_i - k} F_i$.

We now explain the process of constructing an element in $\mathrm{Hom}_S(\mathrm{Sym}(I)_\delta, S) \cong \ker \phi_1^*$. First notice that $\phi_1 : F_1 \to F_0$ induces a map $F_0^* \otimes F_1 \to S$, which by dualizing, gives

$$S \xrightarrow{\widetilde{\phi_1}} F_0 \otimes F_1^*.$$

Let

$$d_i^{\phi_1} : \wedge^{r_1+i} F_0 \to \wedge^{r_1+i+1} F_0 \otimes F_1^*$$

be the composite map

$$\wedge^{r_1+i} F_0 = \wedge^{r_1+i} F_0 \otimes S \xrightarrow{\mathrm{id} \otimes \widetilde{\phi_1}} \wedge^{r_1+i} F_0 \otimes F_0 \otimes F_1^* \xrightarrow{m \otimes \mathrm{id}} \wedge^{r_1+i+1} F_0 \otimes F_1^*$$

where $m : \wedge^{r_1+i} F_0 \otimes F_0 \to \wedge^{r_1+i+1} F_0$ is the usual multiplication in the exterior algebra $\bigwedge F_0$.

Lemma 6.0.7 [15, 3.2] *The following statements are true.*

(*a*) *The composition*

$$\wedge^i F_0 \otimes \wedge^{r_1} F_1 \xrightarrow{m(\mathrm{id} \otimes \wedge^{r_1} \phi_1)} \wedge^{r_1+i} F_0 \xrightarrow{d_i^{\phi_1}} \wedge^{r_1+i+1} F_0 \otimes F_1^*$$

is zero.

(*b*) *The following diagram commutes*

$$\begin{tikzcd}
\wedge^{r_1+i} F_0 \arrow[r, "d_i^{\phi_1}"] \arrow[d, "\cong"] & \wedge^{r_1+i+1} F_0 \otimes F_1^* \arrow[d, "\cong"] \\
\wedge^{t_0-r_1-i} F_0^* \arrow[r, "(\mathrm{id} \otimes \phi_1)^* m^*"] & \wedge^{t_0-r_1-i-1} F_0^* \otimes F_1^*
\end{tikzcd}$$

Proposition 6.0.8 *The image of each column of* $\wedge^{r_1} \phi_1$ *under the isomorphism* $\wedge^{r_1} F_0 \cong F_0^*$ *is in* $\ker \phi_1^*$.

Proof In statement (b) of the Lemma 6.0.7, substituting $i = 0$, we get

$$(\mathrm{id} \otimes \phi_1)^* m^* : \wedge^{t_0-r_1} F_0^* \to \wedge^{t_0-r_1-1} F_0^* \otimes F_1^*.$$

Since by Remark 6.0.6, we have $r_1 = t_0 - 1$, we get

$$(\mathrm{id} \otimes \phi_1)^* m^* : \wedge^1 F_0^* \cong F_0^* \to \wedge^0 F_0^* \otimes F_1^* = S \otimes F_1^* \cong F_1^*,$$

which is nothing but the map ϕ_1^*. Similarly, we can show that

$$m(\mathrm{id} \otimes \wedge^{r_1} \phi_1) : \wedge^i F_0 \otimes \wedge^{r_1} F_1 = \wedge^0 F_0 \otimes \wedge^{r_1} F_1 \cong \wedge^{r_1} F_1 \to \wedge^{r_1+i} F_0 = \wedge^{r_1} F_0$$

is the map $\wedge^{r_1}\phi_1$. Thus statement (a) of Lemma 6.0.7 now says that when $i = 0$, the image of each column of $\wedge^{r_1}\phi_1$ under the isomorphism $\wedge^{r_1} F_0 = \wedge^{t_0 - r_1} F_0^* = \wedge^1 F_0^* \cong F_0^*$ is in $\ker \phi_1^*$. ■

Notice that this shows that each column of $\wedge^{r_1}\phi_1$ is a candidate for the element of $\ker \phi_1^*$. But for each fixed column, the entries of the column may have a common factor. One of the ways to wean out the common factors is to use the method of Buchsbaum-Eisenbud multipliers. The theorem of Buchsbaum-Eisenbud (Theorem 2.7.1) guarantees the existence of unique homomorphisms $a_k : R \to \wedge^{r_k} F_{k-1}$ for $1 \leq k \leq n-1$ such that

(a) $a_{n-1} = \wedge^{r_{n-1}} \phi_{n-1}$

(b) for each $k < n-1$, the diagram

$$\begin{tikzcd} \wedge^{r_k} F_k \arrow[rr, "\wedge^{r_k}\phi_k"] \arrow[dr, "a_{k+1}^*"'] & & \wedge^{r_k} F_{k-1} \\ & S \arrow[ur, "a_k"'] & \end{tikzcd} \tag{6.9}$$

commutes.

(c) Further, $\sqrt{I_1(a_i)} = \sqrt{I_{r_i}(\phi_i)}$ for $1 \leq i \leq n-1$.

How to implement Buchsbaum-Eisenbud multipliers: Notice that $a_{n-1} = \wedge^{r_{n-1}}\phi_{n-1}$. Thus a_{n-1} is a column matrix consisting of signed maximal minors of ϕ_{n-1}. Next consider the diagram

$$\begin{tikzcd} \wedge^{r_{n-2}}F_{n-2} \arrow[rr, "\wedge^{r_{n-2}}\phi_{n-2}"] \arrow[dr, "a^*_{n-1}"'] & & \wedge^{r_{n-2}}F_{n-3} \\ & S \arrow[ur, "a_{n-2}"'] & \end{tikzcd}$$

Using a_{n-1} we can construct the dual map

$$a^*_{n-1} : \wedge^{r_{n-2}}F_{n-2} \cong \wedge^{t_{n-2}-r_{n-2}}F^*_{n-2} = \wedge^{r_{n-1}}F^*_{n-2} \to S^*$$

Thus the entries of a^*_{n-1} are still maximal minors of ϕ_{n-1} but in a different order. Now let

$$a^*_{n-1} = [a^*_{n-1,1} \cdots a^*_{n-1,l_{n-1}}] \qquad \text{and} \qquad a_{n-2} = \begin{bmatrix} a_{n-2,1} \\ \vdots \\ a_{n-2,l_{n-2}} \end{bmatrix}$$

Then the above commutative diagram says that

$$\wedge^{r_{n-2}}\phi_{n-2} = a_{n-2} \cdot a^*_{n-1} = \begin{bmatrix} a_{n-2,1}a^*_{n-1,1} & a_{n-2,1}a^*_{n-1,2} & \cdots & a_{n-2,1}a^*_{n-1,l_{n-1}} \\ a_{n-2,2}a^*_{n-1,1} & a_{n-2,2}a^*_{n-1,2} & \cdots & a_{n-2,2}a^*_{n-1,l_{n-1}} \\ \vdots & & \vdots & \vdots \\ a_{n-2,l_{n-2}}a^*_{n-1,1} & a_{n-2,l_{n-2}}a^*_{n-1,2} & \cdots & a_{n-2,l_{n-2}}a^*_{n-1,l_{n-1}} \end{bmatrix} \tag{6.10}$$

Not all the entries of a^*_{n-1} can be zero, as this would imply $\wedge^{r_{n-2}}\phi_{n-2}$ is zero, a contradiction to the fact that rank $\phi_{n-2} = r_{n-2}$. Assume that the p-th entry of a^*_{n-1} is non zero. Now to recover a_{n-2} we consider the a nonzero column of $\wedge^{r_{n-2}}\phi_{n-2}$, say the pth column, and divide it by $a^*_{n-1,p}$.

We iteratively keep using the commutative diagram in (6.9) to get a_1. By abuse of notation, we identify the map a_1 with the element $a_1(1) \in \wedge^{r_1}F_0$.

Proposition 6.0.9 *In the setting of 6.0.1, let $a_1^* \in F_0^*$ denote the element under the isomorphism $\wedge^{r_1} F_0 \cong F_0^*$. Then $a_1^* \in \ker \phi_1^*$*

Proof Notice that a_1 is a column of $\wedge^{r_1}\phi_1$ with common factors (coming from a_2^*) removed. And the image of each column of $\wedge^{r_1}\phi_1$ under the isomorphism $\wedge^{r_1} F_0 \cong F_0^*$ is in $\ker \phi_1^*$ (by Proposition 6.0.8). Thus $a_1^* \in \ker \phi_1^*$ because $a_2^* \neq 0$.

Alternate Proof: We showed in Proposition 6.0.8 that the composition of the maps

$$\wedge^{r_1} F_1 \xrightarrow{\wedge^{r_1}\phi_1} \wedge^{r_1} F_0 \cong F_0^* \xrightarrow{\phi_1^*} F_1^*$$

is zero. If $\eta : \wedge^{r_1} F_0 \cong F_0^*$, then $\phi_1^* \circ \eta(\wedge^{r_1}\phi_1) = 0$. Combining this with Theorem 2.7.1 we get

$$\begin{tikzcd}
\wedge^{r_1} F_1 \arrow[rr, "\wedge^{r_1}\phi_1"] \arrow[dr, "a_2^*"'] & & \wedge^{r_1} F_0 \cong F_0^* \arrow[r, "\phi_1^*"] & F_1^* \\
& S \arrow[ur, "a_1"'] & &
\end{tikzcd}$$

Let J denote the image of a_2^* in S. Also, $J\,\mathrm{Im}(\phi_1^* \circ \eta(a_1)) = 0$. Since $\sqrt{J} = \sqrt{I_{r_2}(\phi_2)}$ and grade $I_{r_1}(\phi_1) \geq 2$, J contains a nonzero divisor in S. Now since $J\,\mathrm{Im}(\phi_1^* \circ \eta(a_1)) = 0$ in a free module F_1^*, we have $\phi_1^* \circ \eta(a_1) = 0$. By definition, $a_1^* = \eta(a_1)$ and hence $a_1^* \in \ker \phi_1^*$. ■

6.1 Birationality and Defining ideal of the image

Since $\ker \phi_1^* \cong \mathcal{A}_0$, and $a_1^* \in \ker \phi_1^*$ (Proposition 6.0.9), we can recover the corresponding element $\mathfrak{b} \in \mathcal{A}_0 = (\mathfrak{a})$ using the method of *Morley forms* developed by J.-P. Jouanolou. An explicit description of $\mathfrak{b}$ is presented in Theorem 6.2.6 and the method of Morley forms is explained in the next Section 6.2.

Recall the grading of B and $\mathrm{Sym}(I)$ in Notation 6.0.2. Since $\mathrm{Sym}(I)_0 = S$, we have $\mathcal{F}(I) \cong S/\mathcal{A}_0$. Clearly, $\deg \mathfrak{a} = e(\mathcal{F}(I))$ where $e(\_)$ represents the Hilbert-Samuel multiplicity.

We can compute the multiplicity $e(\mathcal{F}(I))$ using the following formula

Theorem 6.1.1 [16, 2.4]

$$e(\mathcal{F}(I)) = \frac{1}{[k[R_{\mathfrak{d}}] : k[f_1, \ldots, f_{d+1}]]} \cdot e\left(\frac{R}{(g_1, \ldots, g_{d-1}) : I^\infty}\right) \tag{6.11}$$

where $g_1, \ldots g_{d-1}$ are general k-linear combinations of the generators $f_1, \ldots, f_{d+1}$ of I,

Notice that $k[f_1, \ldots, f_{d+1}] \cong \mathcal{F}(I)$ is the coordinate ring of the image of Ψ. It is well known that the map Ψ is birational onto its image if and only if $[k[R_{\mathfrak{d}}] : k[f_1, \ldots, f_{d+1}]] = 1$.

In an attempt to show that $\mathfrak{b}$ generates $\mathcal{A}_0$, we compare $\deg \mathfrak{b}$ with $e(\mathcal{F}(I)) = \deg \mathfrak{a}$. Once we show that $\deg \mathfrak{b} = \deg \mathfrak{a}$, then $\mathfrak{b}$ generates $\mathcal{A}_0$. The degree of the entries of a_1 and the degree of the element $\mathfrak{b}$ are related by the formula

$$\text{degree of the entries of } a_1 + d = \deg \mathfrak{b} \tag{6.12}$$

where the extra d comes from Morley forms (Observation 6.2.7).

Theorem 6.1.2 *In the setting of 6.0.1,*

$$\deg \mathfrak{b} = e\left(\frac{R}{(g_1, \ldots, g_{d-1}) : I^\infty}\right)$$

where $g_1, \ldots, g_{d-1}$ are general k-linear combinations of $f_1, \ldots, f_{d+1}$.

Proof We first use (6.12) to compute $\deg \mathfrak{b}$. Recall that $\operatorname{rank} \phi_i = r_i, \operatorname{rank} F_i = t_i$ and that the entries of the matrices ϕ_i are linear. Note that the entries of a_{n-1} are $r_{n-1} \times r_{n-1}$ minors of ϕ_{n-1} and hence the degree of the entries of a_{n-1} is r_{n-1}. Now to construct a_{n-2} we considered a column of $\wedge^{r_{n-2}} \phi_{n-2}$ and divided it by a non zero entry of a^*_{n-1}. Thus the degree of the entries of a_{n-2} is the difference between the degree of the entries of $\wedge^{r_{n-2}} \phi_{n-2}$ and the degree of the entries of a^*_{n-1}. The degree of the entries of $\wedge^{r_{n-2}} \phi_{n-2}$ is r_{n-2}. Thus

$$\text{degree of the entries of } a_{n-2} = r_{n-2} - r_{n-1}$$

Iteratively, we can compute

$$\text{degree of the entries of } a_1 = \sum_{i=1}^{n-1}(-1)^{i-1}r_i \tag{6.13}$$

Since $r_i = \sum_{j=i}^{n-1}(-1)^{j-i}t_j$ and $\sum_{j=0}^{n-1}(-1)^j t_j = 1$ (Remark 6.0.6), we have

$$r_i = \sum_{j=0}^{i-1}(-1)^{i+j+1}t_j + (-1)^i.$$

Now using (6.13) we get

$$\begin{aligned}
\deg \mathfrak{b} &= \text{degree of the entries of } a_1 + d\\
&= \sum_{i=1}^{n-1}(-1)^{i-1}r_i + d\\
&= \sum_{i=1}^{n-1}(-1)^{i-1}\left(\sum_{j=0}^{i-1}(-1)^{i+j+1}t_j + (-1)^i\right) + d\\
&= \sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\big((-1)^j t_j - 1\big) + d\\
&= \sum_{k=1}^{n-1}(-1)^{n-1-k}\cdot k\cdot t_{n-1-k} + (n-1)(-1) + d\\
&= \sum_{k=1}^{n-1}(-1)^{n-1-k}\cdot k\cdot t_{n-1-k} + (d-n) + 1
\end{aligned} \tag{6.14}$$

Now we compute the multiplicity on the right hand side of the result. By the general choice of $g_1, \ldots, g_{d-1}$ and since I satisfies the G_d condition, the ideal $(g_1, \ldots, g_{d-1}) : I$ is a residual intersection of height $d-1$ [36]. Further $(g_1, \ldots, g_{d-1}) : I + I$ has height d [36]. It is known that $(g_1, \ldots, g_{d-1}) : I^\infty = (g_1, \ldots, g_{d-1}) : I$ [19, 3.1]. Recall that $\mu(I) = d+1$ and consider the module $M = I/(g_1, \ldots, g_{d-1})$. Let φ'' be the $2 \times d$ homogeneous presentation matrix of M.

Notice that $\sqrt{\mathrm{Fitt}_0(M)} = \sqrt{\operatorname{ann} M} = \sqrt{(g_1, \ldots, g_{d-1}) : I}$ has height $d-1$. Hence $\operatorname{ht} I_2(\varphi'') = d-1$ and therefore $I_2(\varphi'') = \operatorname{ann} M$ by [37]. Thus $(g_1, \ldots, g_{d-1}) : I = I_2(\varphi'')$ (see also [19]).

Now, the Eagon-Northcott complex gives a minimal free resolution for $N = R/((g_1, \dots, g_{d-1}) : I)$. We first construct the Eagon-Northcott complex $\mathrm{EN}(\varphi'')$ where $\varphi'' : R^d \to R^2$. The column degrees of φ and that of φ'' are the same. Then $\mathrm{EN}(\varphi'')$ is

$$0 \to R^{d-1}(-(\mathfrak{d}_1 + \cdots + \mathfrak{d}_d)) \to \bigoplus_{1 \le j_1 < \cdots < j_{d-1} \le d} R^{d-2}(-(\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{d-1}})) \to \cdots$$
$$\cdots \to \bigoplus_{1 \le j_1 < j_2 \le d} R(-(\mathfrak{d}_{j_1} + \mathfrak{d}_{j_2})) \to R \to 0$$

Since the $e(N) = e(N(-1))$, we consider $\mathrm{EN}(\varphi'')(-1)$, which gives the resolution for $N(-1)$. Now the Hilbert series of $N(-1)$ is

$$\frac{\sum_{i=1}^{d-1} (-1)^i \cdot i \cdot \sum_{1 \le j_1 < \cdots < j_{i+1} \le d} t^{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1} + t^{-1}}{(1-t)^d}$$

Let $p(t) = \sum_{i=1}^{d-1} (-1)^i \cdot i \cdot \sum_{1 \le j_1 < \cdots < j_{i+1} \le d} t^{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1} + t^{-1}$. Then

$$\begin{aligned} e(N) &= e(N(-1)) \\ &= (-1)^{d-1} \frac{p^{d-1}(1)}{(d-1)!} \\ &= (-1)^{d-1} \left[\sum_{i=1}^{d-1} (-1)^i \cdot i \cdot \sum_{1 \le j_1 < \cdots < j_{i+1} \le d} \binom{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1}{d-1} + (-1)^{d-1} \right] \\ &= \sum_{i=1}^{d-1} (-1)^{d-1+i} \cdot i \cdot \sum_{1 \le j_1 < \cdots < j_{i+1} \le d} \binom{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1}{d-1} + 1 \\ &= \sum_{i=1}^{d-1} (-1)^{d-1-i} \cdot i \cdot \sum_{1 \le j_1 < \cdots < j_{i+1} \le d} \binom{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1}{d-1} + 1 \qquad (6.15) \end{aligned}$$

Notice that for $i < d - (n-1)$ and $1 \le j_1 < \cdots < j_{i+1} \le d$, $\binom{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1}{d-1} = 0$ (Remark 6.0.5). Thus (6.15) becomes

$$\begin{aligned} e(N) &= \sum_{i=d-(n-1)}^{d-1} (-1)^{d-1-i} \cdot i \cdot \sum_{1 \le j_1 < \cdots < j_{i+1} \le d} \binom{\mathfrak{d}_{j_1} + \cdots + \mathfrak{d}_{j_{i+1}} - 1}{d-1} + 1 \\ &= \sum_{i=d-(n-1)}^{d-1} (-1)^{d-1-i} \cdot i \cdot t_{d-i-1} + 1 \qquad (6.16) \end{aligned}$$

Notice that $\sum\limits_{i=d-(n-1)}^{d-1}(-1)^{d-1-i}t_{d-i-1} = 1$ (Remark 6.0.6). Thus (6.16) becomes

$$e(N) = \sum_{i=d-(n-1)}^{d-1} (-1)^{d-1-i} \cdot (i-(d-n)) \cdot t_{d-i-1} + (d-n) + 1 \tag{6.17}$$

By a change of indices, we get

$$e(N) = \sum_{k=1}^{n-1} (-1)^{n-1-k} \cdot k \cdot t_{n-1-k} + (d-n) + 1 \tag{6.18}$$

Notice that (6.18) is exactly the same as (6.14). ■

Now we present the main theorem of this section.

Theorem 6.1.3 *In the setting of 6.0.1, the following statements are equivalent.*

(a) Ψ *is birational onto its image.*

(b) $\mathfrak{b}$ *is a principal generator of* $\mathcal{A}_0$*.*

(c) grade $I_1(a_1) =$ grade $I_1(a_1^*) > 1$.

Proof $(a) \Leftrightarrow (b)$ The rational map Ψ is birational onto its image, if and only if $[k[R_{\mathfrak{d}}] : k[f_1, \ldots, f_{d+1}]] = 1$. Thus using Theorem 6.1.1 and Theorem 6.1.2, Ψ is birational onto its image if and only if $\deg \mathfrak{b} = e(\mathcal{F}(I))$.
$(b) \Leftrightarrow (c)$ This is an immediate consequence of the fact that $\mathfrak{b}$ generates $\mathcal{A}_0$ if and only if a_1^* generates $\ker \phi_1^*$. Now use the Buchsbaum-Eisenbud criterion [35] for exactness of complexes. ■

Now we present an example which uses Theorem 6.1.3

Example 6.1.4 Let $\Psi' : \mathbb{P}^2 \xrightarrow{[f_0:f_1:f_2:f_3]} \mathbb{P}^3$ such that

$$f_0 = x^2y^4 - x^4yz + y^3z^3 - yz^5 \qquad f_1 = -x^3y^3 - x^3z^3 + xz^5$$
$$f_2 = x^5y + x^3yz^2 - xy^3z^2 \qquad f_3 = -x^3y^3 + xy^5 + x^5z - x^3z^3$$

Let $I = (f_0, f_1, f_2, f_3)$ be an ideal in $R = k[x, y, z]$ (coordinate ring of $\mathbb{P}^2$). Consider a homogeneous presentation matrix φ of I

$$\varphi = \begin{bmatrix} x & 0 & x^3 \\ y & x^2 & y^3 \\ z & y^2 & z^3 \\ 0 & z^2 & x^2y \end{bmatrix}$$

One can easily check that I is a height two perfect ideal satisfying G_3 in R. Let $S = k[T_0, T_1, T_2, T_3]$ (coordinate ring of $\mathbb{P}^2$) and $B = R[T_0, T_1, T_2, T_3]$ (coordinate ring of $\mathbb{P}^2 \times \mathbb{P}^3$). Let $[L_1\ L_2\ L_3] = [T_0\ T_1\ T_2\ T_3] \cdot \varphi$. Notice that $\delta = 1+2+3-3 = 3$. Using the Koszul complex $\mathcal{K}_\bullet(L_1, L_2, L_3; B)$ we can extract an S resolution of $\mathrm{Sym}(I)_\delta = \mathrm{Sym}(I)_3$.

$$0 \to S \xrightarrow{\phi_2} S^{10} \xrightarrow{\phi_1} S^{10} \to \mathrm{Sym}(I)_3 \to 0$$

where

$$\phi_1 = \begin{bmatrix} T_0 & 0 & 0 & 0 & 0 & 0 & T_1 & 0 & 0 & T_0 \\ T_1 & T_0 & 0 & 0 & 0 & 0 & 0 & T_1 & 0 & T_3 \\ T_2 & 0 & T_0 & 0 & 0 & 0 & 0 & 0 & T_1 & 0 \\ 0 & T_1 & 0 & T_0 & 0 & 0 & T_2 & 0 & 0 & 0 \\ 0 & T_2 & T_1 & 0 & T_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_2 & 0 & 0 & T_0 & T_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_1 & 0 & 0 & 0 & T_2 & 0 & T_1 \\ 0 & 0 & 0 & T_2 & T_1 & 0 & 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & T_2 & T_1 & 0 & T_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & T_3 & T_2 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} T_1 \\ 0 \\ 0 \\ T_2 \\ 0 \\ T_3 \\ -T_0 \\ -T_1 \\ -T_2 \\ 0 \end{bmatrix}$$

Notice that, by Theorem 2.7, $a_2 = \wedge^1\phi_2$. By construction, the entries of a_2^* and ϕ_2 are the same. We wish to construct a_1^*. Now, the first column of $\wedge^9\phi_1$ is

$$\begin{bmatrix}
-T_1^5T_2^4 + 3T_0^2T_1^2T_2^5 + T_1^3T_2^6 - T_1^7T_2T_3 - T_0^2T_1^4T_2^2T_3 + T_1^5T_2^3T_3 + T_0^2T_1^2T_2^4T_3 - T_1T_2^7T_3 - 3T_0^2T_1^4T_2T_3^2 - 2T_1^3T_2^4T_3^2 - T_1^5T_2T_3^3 \\
3T_0T_1^4T_2^4 - T_0^3T_1T_2^5 + T_0T_1T_2^7 + T_0T_1^6T_2T_3 + T_0^3T_1^3T_2^2T_3 + 2T_0T_1^4T_2^3T_3 + 2T_0T_1^3T_2^4T_3 + 2T_0^3T_1^3T_2T_3^2 + T_0T_1^5T_2T_3^2 \\
-2T_0T_1^5T_2^3 - 2T_0^3T_1^2T_2^4 - T_0T_1^3T_2^5 - T_0T_1^2T_2^6 - 3T_0T_1^5T_2^2T_3 - T_0^3T_1^2T_2^3T_3 - 2T_0T_1^4T_2^3T_3 + T_0T_1T_2^6T_3 + T_0^3T_1^4T_3^2 - T_0T_1^6T_3^2 + 2T_0T_1^3T_2^3T_3^2 + T_0T_1^5T_3^3 \\
T_1^6T_2^3 - 3T_0^2T_1^3T_2^4 - T_1^4T_2^5 - T_0^2T_1^5T_2T_3 - T_0^4T_1^2T_2^2T_3 - 2T_0^2T_1^3T_2^3T_3 - 2T_0^2T_1^2T_2^4T_3 + T_1^2T_2^6T_3 - T_0^4T_1^2T_2T_3^2 - 2T_0^2T_1^4T_2T_3^2 + T_1^4T_2^3T_3^2 + T_0^2T_1T_2^4T_3^2 + T_0^2T_1^3T_2T_3^3 \\
-T_1^7T_2^2 + T_0^4T_1T_2^4 + T_1^5T_2^4 - T_0^2T_1T_2^6 - T_1^3T_2^5T_3 - T_0^4T_1^3T_3^2 + T_0^2T_1^5T_3^2 - T_1^5T_2^2T_3^2 - T_0^2T_1^2T_2^3T_3^2 - T_0^2T_1^4T_3^3 \\
T_1^8T_2 + 2T_0^2T_1^5T_2^2 + T_0^4T_1^2T_2^3 - T_1^6T_2^3 + T_0^2T_1^3T_2^4 + 2T_0^2T_1^2T_2^5 + 3T_0^2T_1^5T_2T_3 + T_0^4T_1^2T_2^2T_3 + 2T_0^2T_1^4T_2^2T_3 + T_1^4T_2^4T_3 - T_0^2T_1T_2^5T_3 + T_1^6T_2T_3^2 - T_0^2T_1^3T_2^2T_3^2 \\
-3T_0T_1^5T_2^3 + T_0^3T_1^2T_2^4 - T_0T_1^2T_2^6 + T_0^3T_1^4T_2T_3 + T_0^5T_1T_2^2T_3 - T_0T_1^4T_2^3T_3 - T_0^3T_1T_2^4T_3 + 3T_0^3T_1^3T_2T_3^2 - 2T_0T_1^3T_2^3T_3^2 - 2T_0^3T_1^2T_2T_3^3 \\
2T_0T_1^6T_2^2 + 2T_0^3T_1^3T_2^3 + T_0T_1^4T_2^4 + T_0T_1^3T_2^5 + 2T_0^3T_1^3T_2^2T_3 + T_0T_1^5T_2^2T_3 + 3T_0^3T_1^2T_2^3T_3 - T_0T_1^2T_2^5T_3 + T_0^5T_1^2T_3^2 - T_0^3T_1^4T_3^2 + T_0T_1^4T_2^2T_3^2 - T_0^3T_1T_2^3T_3^2 + T_0^3T_1^3T_3^3 \\
-T_0T_1^7T_2 - 2T_0^3T_1^4T_2^2 - T_0^5T_1T_2^3 - 2T_0T_1^5T_2^3 - T_0T_1^4T_2^4 + T_0^3T_1T_2^5 - 2T_0^3T_1^4T_2T_3 - T_0T_1^6T_2T_3 - 3T_0^3T_1^3T_2^2T_3 + 2T_0T_1^3T_2^4T_3 + 2T_0^3T_1^2T_2^2T_3^2 \\
3T_0T_1^6T_2^2 - T_0^3T_1^3T_2^3 + T_0T_1^5T_2^3 - 3T_0^3T_1^2T_2^4 - T_0^3T_1^5T_3 + T_0T_1^7T_3 - T_0^5T_1^2T_2T_3 + T_0^3T_1^4T_2T_3 - 3T_0T_1^4T_2^3T_3 + T_0^3T_1T_2^4T_3 - T_0T_1^6T_3^2 - T_0^3T_1^3T_2T_3^2
\end{bmatrix}$$

The first entry of a_2^* is the first entry of ϕ_2 which equals T_1. Dividing the above column by T_1 we get

$$a_1 = \begin{bmatrix}
-T_1^4T_2^4 + 3T_0^2T_1T_2^5 + T_1^2T_2^6 - T_1^6T_2T_3 - T_0^2T_1^3T_2^2T_3 + T_1^4T_2^3T_3 + T_0^2T_1T_2^4T_3 - T_2^7T_3 - 3T_0^2T_1^3T_2T_3^2 - 2T_1^2T_2^4T_3^2 - T_1^4T_2T_3^3 \\
3T_0T_1^3T_2^4 - T_0^3T_2^5 + T_0T_2^7 + T_0T_1^5T_2T_3 + T_0^3T_1^2T_2^2T_3 + 2T_0T_1^3T_2^3T_3 + 2T_0T_1^2T_2^4T_3 + 2T_0^3T_1^2T_2T_3^2 + T_0T_1^4T_2T_3^2 \\
-2T_0T_1^4T_2^3 - 2T_0^3T_1T_2^4 - T_0T_1^2T_2^5 - T_0T_1T_2^6 - 3T_0T_1^4T_2^2T_3 - T_0^3T_1T_2^3T_3 - 2T_0T_1^3T_2^3T_3 + T_0T_2^6T_3 + T_0^3T_1^3T_3^2 - T_0T_1^5T_3^2 + 2T_0T_1^2T_2^3T_3^2 + T_0T_1^4T_3^3 \\
T_1^5T_2^3 - 3T_0^2T_1^2T_2^4 - T_1^3T_2^5 - T_0^2T_1^4T_2T_3 - T_0^4T_1T_2^2T_3 - 2T_0^2T_1^2T_2^3T_3 - 2T_0^2T_1T_2^4T_3 + T_1T_2^6T_3 - T_0^4T_1T_2T_3^2 - 2T_0^2T_1^3T_2T_3^2 + T_1^3T_2^3T_3^2 + T_0^2T_2^4T_3^2 + T_0^2T_1^2T_2T_3^3 \\
-T_1^6T_2^2 + T_0^4T_2^4 + T_1^4T_2^4 - T_0^2T_2^6 - T_1^2T_2^5T_3 - T_0^4T_1^2T_3^2 + T_0^2T_1^4T_3^2 - T_1^4T_2^2T_3^2 - T_0^2T_1T_2^3T_3^2 - T_0^2T_1^3T_3^3 \\
T_1^7T_2 + 2T_0^2T_1^4T_2^2 + T_0^4T_1T_2^3 - T_1^5T_2^3 + T_0^2T_1^2T_2^4 + 2T_0^2T_1T_2^5 + 3T_0^2T_1^4T_2T_3 + T_0^4T_1T_2^2T_3 + 2T_0^2T_1^3T_2^2T_3 + T_1^3T_2^4T_3 - T_0^2T_2^5T_3 + T_1^5T_2T_3^2 - T_0^2T_1^2T_2^2T_3^2 \\
-3T_0T_1^4T_2^3 + T_0^3T_1T_2^4 - T_0T_1T_2^6 + T_0^3T_1^3T_2T_3 + T_0^5T_2^2T_3 - T_0T_1^3T_2^3T_3 - T_0^3T_2^4T_3 + 3T_0^3T_1^2T_2T_3^2 - 2T_0T_1^2T_2^3T_3^2 - 2T_0^3T_1T_2T_3^3 \\
2T_0T_1^5T_2^2 + 2T_0^3T_1^2T_2^3 + T_0T_1^3T_2^4 + T_0T_1^2T_2^5 + 2T_0^3T_1^2T_2^2T_3 + T_0T_1^4T_2^2T_3 + 3T_0^3T_1T_2^3T_3 - T_0T_1T_2^5T_3 + T_0^5T_1T_3^2 - T_0^3T_1^3T_3^2 + T_0T_1^3T_2^2T_3^2 - T_0^3T_2^3T_3^2 + T_0^3T_1^2T_3^3 \\
-T_0T_1^6T_2 - 2T_0^3T_1^3T_2^2 - T_0^5T_2^3 - 2T_0T_1^4T_2^3 - T_0T_1^3T_2^4 + T_0^3T_2^5 - 2T_0^3T_1^3T_2T_3 - T_0T_1^5T_2T_3 - 3T_0^3T_1^2T_2^2T_3 + 2T_0T_1^2T_2^4T_3 + 2T_0^3T_1T_2^2T_3^2 \\
3T_0T_1^5T_2^2 - T_0^3T_1^2T_2^3 + T_0T_1^4T_2^3 - 3T_0^3T_1T_2^4 - T_0^3T_1^4T_3 + T_0T_1^6T_3 - T_0^5T_1T_2T_3 + T_0^3T_1^3T_2T_3 - 3T_0T_1^3T_2^3T_3 + T_0^3T_2^4T_3 - T_0T_1^5T_3^2 - T_0^3T_1^2T_2T_3^2
\end{bmatrix}$$

Macaulay2 computations show that the grade $I_1(a_1) =$ grade $I_1(a_1^*) = 2$ and hence, by Theorem 6.1.3 Ψ' is birational onto its image.

6.2 Morley Forms

The objective of Morley forms is to define an explicit isomorphism between $\mathcal{A}_i$ and $\underline{\mathrm{Hom}}(\mathrm{Sym}(I)_{\delta-i}, S(-d))$. Thus given an element in $\underline{\mathrm{Hom}}(\mathrm{Sym}(I)_{\delta-i}, S(-d))$, we can recover the element in $\mathcal{A}_i$ through this isomorphism. It was conceptualized by J.-P.Jouanolou in [13, 38].

Morley forms are graded component of the determinant of a matrix in $\mathrm{Sym}(I) \otimes_S \mathrm{Sym}(I)$. We now proceed to define Morley forms. Throughout this section we assume the setting of 6.0.1. Recall that $R = k[x_1, \ldots, x_d], S = k[T_1, \ldots, T_{d+1}], B = R \otimes_k S$. The symmetric algebra has a presentation

$$\begin{aligned} \operatorname{Sym}(I) \cong B/\mathcal{L} \text{ where } \mathcal{L} = (L_1, \ldots, L_d) \\ \text{and } [L_1 \cdots L_d] = [T_1 \cdots T_{d+1}] \cdot \varphi \end{aligned}$$

Consider $B \otimes_S B$ and $L_i \otimes 1 - 1 \otimes L_i \in (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_d \otimes 1 - 1 \otimes x_d)$ for $1 \leq i \leq d$. Let D be a $d \times d$ matrix such that

$$[L_1 \otimes 1 - 1 \otimes L_1 \cdots L_d \otimes 1 - 1 \otimes L_d] = [x_1 \otimes 1 - 1 \otimes x_1 \cdots x_d \otimes 1 - 1 \otimes x_d] \cdot D \tag{6.19}$$

Notice that $\det D \in B \otimes_S B$. Consider the natural epimorphisms

$$\begin{aligned} \Gamma :& B \twoheadrightarrow \operatorname{Sym}(I) \\ \Gamma \otimes \Gamma :& B \otimes_S B \twoheadrightarrow \operatorname{Sym}(I) \otimes_S \operatorname{Sym}(I) \end{aligned}$$

We set $\Delta = (\Gamma \otimes \Gamma)(\det H)$. We now impose the grading scheme on $\operatorname{Sym}(I) \otimes_S \operatorname{Sym}(I)$

$$\begin{aligned} \deg x_i \otimes 1 &= (1, 0, 0) & \deg 1 \otimes x_i &= (0, 1, 0) \\ \deg T_j \otimes 1 &= (0, 0, 1) & \deg 1 \otimes T_j &= (0, 0, 1) \end{aligned}$$

We now rewrite

$$\begin{aligned} \Delta = \sum_{i=0}^{\delta} \operatorname{morl}_{(i,\delta-i,d)} \text{ where} \\ \operatorname{morl}_{(i,\delta-i,d)} \in \operatorname{Sym}(I)_{(i,d)} \otimes \operatorname{Sym}(I)_{(\delta-i,d)} \end{aligned}$$

The tri-homogeneous elements $\{\operatorname{morl}_{(i,\delta-i,d)} \mid 0 \leq i \leq \delta\}$ are the *Morley forms* associated to the regular sequence $L_1, \ldots, L_d$ in B.

The *Sylvester element* syl is defined as $\overline{\det B(\varphi)} \in \operatorname{Sym}(I)$. Since $L_1, \ldots, L_d$ is a regular sequence, $(L_1, \ldots, L_d) : (x_1, \ldots, x_d) = (L_1, \ldots, L_d) + \det B(\varphi)$. Thus syl is a nonzero element of bi-degree (δ, d) in $\operatorname{Sym}(I)$. Since $\mathcal{A}_\delta \cong \operatorname{Hom}(\operatorname{Sym}(I)_0, S(-d)) \cong S(-d)$, $\mathcal{A}_\delta$ is generated by syl. Thus this defines an isomorphism

$$\mu_{\text{syl}} : S \to \mathcal{A}_\delta \text{ where } \mu_{\text{syl}}(a) = a \cdot \text{syl}.$$

If $u \in \mathrm{Hom}(\mathrm{Sym}(I)_{\delta-i}, S)$ then $(1 \otimes u)(\mathrm{morl}_{(i,\delta-i,d)}) \in \mathrm{Sym}(I)$ as the map u induces

$$\mathrm{Sym}(I) \otimes_S \mathrm{Sym}(I)_{\delta-i} \xrightarrow{1\otimes u} \mathrm{Sym}(I) \otimes_S S = \mathrm{Sym}(I).$$

In Theorem 6.2.2, it will be shown that the image $(1 \otimes u)(\mathrm{morl}_{(i,\delta-i,d)}) \in \mathcal{A}_i$. This defines a homomorphism

$$\begin{aligned} \nu_1 : \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i}, S) &\to \mathcal{A}_i \\ \nu_1(u) &= (1 \otimes u)(\mathrm{morl}_{(i,\delta-i,d)}) \end{aligned} \tag{6.20}$$

Due to the multiplication map $\mathcal{A}_i \otimes_S \mathrm{Sym}(I)_{\delta-i} \to \mathcal{A}_\delta$, every element $a \in \mathcal{A}_i$ defines an element $\mu_a \in \mathrm{Hom}(\mathrm{Sym}(I)_{\delta-i}, \mathcal{A}_\delta)$ where $\mu_a(b) = a \cdot b$. Thus we can define the map

$$\begin{aligned} \nu_2 : \mathcal{A}_i &\to \mathrm{Hom}_S(\mathrm{Sym}(I)_{\delta-i}, S) \\ \nu_2(a) &= \mu_{\mathrm{syl}}^{-1} \circ \mu_a \end{aligned} \tag{6.21}$$

First we list a few facts about Morley forms.

Lemma 6.2.1 *The following statements hold*

(a) $\mathrm{morl}_{(\delta,0,d)} = \alpha_1 \cdot \mathrm{syl} \otimes 1 \in \mathrm{Sym}(I)_\delta \otimes_S \mathrm{Sym}(I)_0$, *for some* $\alpha_1 \in k$.

(b) $\mathrm{morl}_{(0,\delta,d)} = 1 \otimes \alpha_2 \cdot \mathrm{syl} \in \mathrm{Sym}(I)_0 \otimes_S \mathrm{Sym}(I)_\delta$ *for some* $\alpha_2 \in k$.

(c) *If* $b \in \mathrm{Sym}(I)_k$, *then* $(b \otimes 1)\mathrm{morl}_{(i,\delta-i,d)} = (1 \otimes b)\mathrm{morl}_{(i+k,\delta-i-k,d)} \in \mathrm{Sym}(I)_{i+k} \otimes_S \mathrm{Sym}(I)_{\delta-i}$

The proof of the above lemma is analogous to the proof presented in [14, 4.1]. The following theorem of Jouanolou shows that ν_1 and ν_2 are inverses of each other.

Theorem 6.2.2 [38, 3.11] *Let* $0 \leq i \leq \delta$. *Then the following statements are true*

(a) *If* $u \in \mathrm{Hom}(\mathrm{Sym}(I)_{\delta-i}, S)$, *then* $(1 \otimes u)(\mathrm{morl}_{(i,\delta-i,d)}) \in \mathcal{A}_i$.

(b) *The homomorphisms* ν_1, ν_2 *defined in* (6.20) *and* (6.21) *are inverses of each other (up to multiplication by a unit in* k*).*

Since $\mathrm{Sym}(I)_{\delta-i}$ is minimally generated by $x_1^{m_1}\cdots x_{d-1}^{m_{d-1}}x_d^{\delta-i-\sum_{j=1}^{d-1}m_j}$ where $m_j\geq 0$ and $\sum_{j=1}^{d-1}m_j\leq \delta-i$, there exist elements $q_{m_1\cdots m_{d-1}}\in \mathrm{Sym}(I)_i$ such that

$$\mathrm{morl}_{(i,\delta-i,d)} = \sum_{m_1+\cdots+m_{d-1}\leq\delta-i} q_{m_1\cdots m_{d-1}}\otimes x_1^{m_1}\cdots x_{d-1}^{m_{d-1}}x_d^{\delta-i-\sum_{j=1}^{d-1}m_j} \tag{6.22}$$

An explicit description of $q_{m_1\cdots m_{d-1}}$ for the ideals we consider is not available in the literature. Once such a description is made available, then for $u\in \mathrm{Hom}(\mathrm{Sym}(I)_{\delta-i},S)$,

$$\nu_1(u) = (1\otimes u)(\mathrm{morl}_{(i,\delta-i,d)}) = \sum_{m_1+\cdots+m_{d-1}\leq\delta-i} q_{m_1\cdots m_{d-1}}\cdot u\left(x_1^{m_1}\cdots x_{d-1}^{m_{d-1}}x_d^{\delta-i-\sum_{j=1}^{d-1}m_j}\right)$$

Recall that we are interested in recovering the element $\mathfrak{b}\in\mathcal{A}_0$ from $a_1^*\in\ker\phi_1^*$. So let $u_a\in\mathrm{Hom}(\mathrm{Sym}(I)_\delta,S)$ represent the element a_1^*. Then

$$\nu_1(u_a) = (1\otimes u_a)(\mathrm{morl}_{(0,\delta,d)}) = \sum_{m_1+\cdots+m_{d-1}\leq\delta} q_{m_1\cdots m_{d-1}}\cdot u_a\left(x_1^{m_1}\cdots x_{d-1}^{m_{d-1}}x_d^{\delta-\sum_{j=1}^{d-1}m_j}\right) \tag{6.23}$$

We are going to describe $q_{m_1\cdots m_{d-1}}$ explicitly in Theorem 6.2.5.

Notation 6.2.3 *For non negative integers* $\underline{t}=t_1,t_2,\ldots,t_{d-1}$*, let* $p(\underline{t},j)=\mathfrak{d}_j-\sum_{k=1}^{d-1}t_k$*. Now write*

$L_j = \sum_{t_1+\cdots+t_{d-1}\leq\mathfrak{d}_j} C^{(j)}_{t_1t_2\cdots t_{d-1}}x_1^{t_1}\cdots x_{d-1}^{t_{d-1}}x_d^{p(\underline{t},j)}$ *where* $C^{(j)}_{t_1t_2\cdots t_{d-1}}\in S$ *and* $1\leq j\leq d$.

Lemma 6.2.4 *Let* L_j *be as defined in Notation 6.2.3. Then*

$$\begin{aligned}
&L_j\otimes 1-1\otimes L_j = \\
&(x_1\otimes 1-1\otimes x_1)\cdot\left(\sum_{t_{11}+\cdots+t_{1d-1}\leq\mathfrak{d}_j} C^{(j)}_{t_{11}t_{12}\cdots t_{1d-1}}\sum_{\alpha_1=0}^{t_{11}-1} x_1^{t_{11}-1-\alpha_1}\otimes x_1^{\alpha_1}\cdots x_{d-1}^{t_{1d-1}}x_d^{p(\underline{t_1},j)}\right)\\
&+(x_2\otimes 1-1\otimes x_2)\cdot\left(\sum_{t_{21}+\cdots+t_{2d-1}\leq\mathfrak{d}_j} C^{(j)}_{t_{21}}t_{\cdots t_{2d-1}}\sum_{\alpha_2=0}^{t_{22}-1} x_1^{t_{21}}x_2^{t_{22}-1-\alpha_2}\otimes x_2^{\alpha_2}\cdots x_{d-1}^{t_{2d-1}}x_d^{p(\underline{t_2},j)}\right)\\
&\qquad\vdots\\
&+(x_d\otimes 1-1\otimes x_d)\cdot\left(\sum_{t_{d1}+\cdots+t_{dd-1}\leq\mathfrak{d}_j} C^{(j)}_{t_{d1}t_{d2}\cdots t_{dd-1}}\sum_{\alpha_d=0}^{p(\underline{t_d},j)-1} x_1^{t_{d1}}x_2^{t_{d2}}\cdots x_{d-1}^{t_{dd-1}}x_d^{p(\underline{t_d},j)-1-\alpha_d}\otimes x_d^{\alpha_d}\right)
\end{aligned} \tag{6.24}$$

Proof Notice that

$$(x_i \otimes 1 - 1 \otimes x_i)\left(\sum_{\alpha_i=0}^{t_{ii}-1} x_1^{t_{i1}} \cdots x_i^{t_{ii}-1-\alpha_i} \otimes x_i^{\alpha_i} \cdots x_{d-1}^{t_{id-1}} x_d^{p(\underline{t_i},j)}\right) =$$

$$\left(\sum_{\alpha_i=0}^{t_{ii}-1} x_1^{t_{i1}} \cdots x_i^{t_{ii}-\alpha_i} \otimes x_i^{\alpha_i} \cdots x_{d-1}^{t_{id-1}} x_d^{p(\underline{t_i},j)}\right) - \left(\sum_{\alpha_i=0}^{t_{ii}-1} x_1^{t_{i1}} \cdots x_i^{t_{ii}-1-\alpha_i} \otimes x_i^{\alpha_i+1} \cdots x_{d-1}^{t_{id-1}} x_d^{p(\underline{t_i},j)}\right)$$

$$= x_1^{t_{i1}} \cdots x_i^{t_{ii}} \otimes x_{i+1}^{t_{ii+1}} \cdots x_d^{p(\underline{t_i},j)} - x_1^{t_1} \cdots x_{i-1}^{t_{ii-1}} \otimes x_i^{t_{ii}} \cdots x_d^{p(\underline{t_i},j)}$$

Now use the right hand side of (6.24) and change of index to get

$$\begin{aligned}
&\sum_{t_1+\cdots+t_{d-1}\le \mathfrak{d}_j} C^{(j)}_{t_1t_2\cdots t_{d-1}}(x_1^{t_1} \otimes x_2^{t_2} \cdots x_d^{p_{\underline{t},j}} - 1 \otimes x_1^{t_1} \cdots x_d^{p_{\underline{t},j}}) \\
+&\sum_{t_1+\cdots+t_{d-1}\le \mathfrak{d}_j} C^{(j)}_{t_1t_2\cdots t_{d-1}}(x_1^{t_1} x_2^{t_2} \otimes x_3^{t_3} \cdots x_d^{p_{\underline{t},j}} - x_1^{t_1} \otimes x_2^{t_2} \cdots x_d^{p_{\underline{t},j}}) \\
&\vdots \\
+&\sum_{t_1+\cdots+t_{d-1}\le \mathfrak{d}_j} C^{(j)}_{t_1t_2\cdots t_{d-1}}(x_1^{t_1} \cdots x_d^{p_{\underline{t},j}} \otimes 1 - x_1^{t_1} \cdots x_{d-1}^{t_{d-1}} \otimes x_d^{p_{\underline{t},j}}) \\
=&L_j \otimes 1 - 1 \otimes L_j
\end{aligned}$$

■

Recall that $\Delta = (\Gamma \otimes \Gamma)(\det D)$ where $D = (d_{rs})$ is the matrix defined in (6.19). By Lemma 6.2.4 the j-th column of the matrix, constructed with L_j, is

$$\begin{bmatrix}
\sum_{t_{11}+\cdots+t_{1d-1}\le \mathfrak{d}_j} C^{(j)}_{t_{11}t_{12}\cdots t_{1d-1}} \sum_{\alpha_1=0}^{t_{11}-1} x_1^{t_{11}-1-\alpha_1} \otimes x_1^{\alpha_1} \cdots x_{d-1}^{t_{1d-1}} x_d^{p(\underline{t_1},j)} \\
\sum_{t_{21}+\cdots+t_{2d-1}\le \mathfrak{d}_j} C^{(j)}_{t_{21}t_{22}\cdots t_{2d-1}} \sum_{\alpha_2=0}^{t_{22}-1} x_1^{t_{21}} x_2^{t_{22}-1-\alpha_2} \otimes x_2^{\alpha_2} \cdots x_{d-1}^{t_{2d-1}} x_d^{p(\underline{t_2},j)} \\
\vdots \\
\sum_{t_{d1}+\cdots+t_{dd-1}\le \mathfrak{d}_j} C^{(j)}_{t_{d1}t_{d2}\cdots t_{dd-1}} \sum_{\alpha_d=0}^{p(\underline{t_d},j)-1} x_1^{t_{d1}} x_2^{t_{d2}} \cdots x_{d-1}^{t_{dd-1}} x_d^{p(\underline{t_d},j)-1-\alpha_d} \otimes x_d^{\alpha_d}
\end{bmatrix}$$

Notice that $\mathrm{morl}_{(0,\delta,d)} \in \mathrm{Sym}(I)_0 \otimes \mathrm{Sym}(I)_\delta$ and hence the entries of D which contribute to $\mathrm{morl}_{(0,\delta,d)}$ are entries which are in $B_0 \otimes B_i$. Thus $\mathrm{morl}_{(0,\delta,d)} = (\Gamma \otimes \Gamma)(\det D')$ where the column j of D' is

$$\begin{bmatrix} \sum_{t_{11}+\cdots+t_{1d-1} \leq \mathfrak{d}_j} C^{(j)}_{t_{11}t_{12}\cdots t_{1d-1}} \cdot 1 \otimes x_1^{t_{11}-1} x_2^{t_{12}} \cdots x_{d-1}^{t_{1d-1}} x_d^{p(\underline{t_1},j)} \\ \sum_{t_{22}+\cdots+t_{2d-1} \leq \mathfrak{d}_j} C^{(j)}_{0t_{22}\cdots t_{2d-1}} \cdot 1 \otimes x_2^{t_{22}-1} x_3^{t_{23}} \cdots x_{d-1}^{t_{2d-1}} x_d^{p(\underline{t_2},j)} \\ \vdots \\ \sum_{t_{d-1d-1} \leq \mathfrak{d}_j} C^{(j)}_{00\cdots 0t_{d-1d-1}} \cdot 1 \otimes x_{d-1}^{t_{d-1d-1}} x_d^{p(\underline{t_{d-1}},j)} \\ C^{(j)}_{00\cdots 0} \cdot 1 \otimes x_d^{t_{dd}-1} \end{bmatrix} \tag{6.25}$$

Theorem 6.2.5 *Let L_j be as defined in Notation 6.2.3. The description of $q_{m_1 m_1 \cdots q_{m_{d-1}}}$ in* (6.23) *is*

$$q_{m_1 m_2 \ldots m_{d-1}} = \sum_{\sigma \in \mathrm{S}_d} \mathrm{sgn}(\sigma) \sum C^{(\sigma(1))}_{s_{11}s_{12}\ldots s_{1d-1}} C^{(\sigma(2))}_{0s_{22}\cdots s_{2d-1}} \cdots C^{(\sigma(d))}_{00\cdots 0}$$

where the second summation satisfies the following conditions

$$\textit{for } 1 \leq l \leq d-1, \ \sum_{r=1}^{l} s_{rl} = m_l + 1$$
$$\textit{for } 1 \leq l \leq d-1, \ \sum_{r=l}^{d-1} s_{lr} \leq \mathfrak{d}_{\sigma(l)}$$

Proof To compute $q_{m_1 m_2 \ldots m_{d-1}}$, we compute the determinant of $D' = (d'_{uv})$. If $\dim R = d = 1$, the the matrix D' is a 1×1 matrix and hence the result is clear. Now assume that the above result is true for $d-1 \times d-1$ matrix of the form D'.

We compute the determinant by expanding along the first row to get

$$\det D' = \sum_{v=1}^{d} (-1)^v d'_{1v} D'_{1v}$$

where D'_{1v} is the determinant of the submatrix of D' obtained by deleting row 1 and column v. In fact D'_{1v} is the determinant of the submatrix of D' involving exactly $d-1$ variables, namely, $x_2, x_3, \ldots, x_d$. Thus by induction hypothesis, we have

$$D'_{1v} = \sum_{m_2+\cdots m_{d-1} \leq \delta - \mathfrak{d}_v} Q_{m_2\cdots m_{d-1}} \cdot 1 \otimes x_2^{m_2} \cdots x_{d-1}^{m_{d-1}} x_d^{\delta-\mathfrak{d}_v-\sum_{k=2}^{d-1} m_k} \text{ and}$$

$$Q_{m_2\cdots m_{d-1}} = \sum_{\tau\in S_X} \operatorname{sgn}(\tau) \sum C^{(\tau(1))}_{0t_{22}\cdots t_{2d-1}} \cdots C^{(\tau(v-1))}_{00t_{v-1v-1}\cdots t_{v-1d-1}} C^{(\tau(v+1))}_{00t_{vv}\cdots t_{vd-1}} \cdots C^{(\tau(d))}_{00\cdots 0}$$

where the second summation satisfies

$$X = \{1, \ldots, d\} \backslash \{v\}$$

$$\text{for } 2 \leq l' \leq d-1, \sum_{r'=2}^{l'} t_{r'l'} = m_l + 1$$

$$\text{for } 2 \leq l' \leq d-1, \sum_{s'=l'}^{d-1} t_{l's'} \leq \mathfrak{d}_{\tau(l')}$$

Using (6.25),

$$d'_{1v} = \sum_{t_{11}+\cdots+t_{1d-1}\leq \mathfrak{d}_v} C^{(v)}_{t_{11}t_{12}\cdots t_{1d-1}} \cdot 1 \otimes x_1^{t_{11}-1} x_2^{t_{12}} \cdots x_{d-1}^{t_{1d-1}} x_d^{p(\underline{t_1}, v)} \tag{6.26}$$

Thus

$$\operatorname{morl}_{(0,\delta,d)} = \det D'$$

$$\sum_{m_1+\cdots+m_{d-1}\leq\delta} q_{m_1\cdots m_{d-1}} \otimes x_1^{m_1} \cdots x_{d-1}^{m_{d-1}} x_d^{\delta-\sum_{j=1}^{d-1} m_j} = \sum_{v=1}^{d} (-1)^v d'_{1v} D'_{1v}$$

Notice that x_1 does not appear in the description of D'_{1v} and hence $t_{11} - 1 = m_1$ in (6.26). Hence

$$q_{m_1\cdots m_{d-1}} = \sum_{v=1}^{d} (-1)^v \sum_{P''} C^{(v)}_{m_1+1t_{12}\cdots t_{1d-1}} \cdot Q_{m_2-t_{12}\cdots m_{d-1}-t_{1d-1}}$$

where the second summation satisfies

$$\text{for } 2 \leq l \leq d-1, 0 \leq t_{1l} \leq m_l + 1$$

$$m_1 + 1 + \sum_{l=2}^{d-1} t_{1l} \leq \mathfrak{d}_v \tag{6.27}$$

Thus

$$q_{m_1\cdots m_{d-1}} = \sum_{v=1}^{d}(-1)^v \sum C^{(v)}_{m_1+1t_{12}\cdots t_{1d-1}} \cdot \sum_{\tau\in S_X} \operatorname{sgn}(\tau)$$
$$\sum C^{(\tau(1))}_{0t_{22}\cdots t_{2d-1}} \cdots C^{(\tau(v-1))}_{00t_{v-1v-1}\cdots t_{v-1d-1}} C^{(\tau(v+1))}_{00t_{vv}\cdots t_{vd-1}} \cdots C^{(\tau(d))}_{00\cdots 0}$$

where the second summation satisfies (6.27) and the fourth summation satisfies

$$\begin{aligned} X &= \{1,\ldots,d\}\backslash\{v\} \\ \text{for } 2 \le l \le d-1, \sum_{r=2}^{l} t_{rl} &= m_l - t_{1l} + 1 \\ \text{for } 2 \le l \le d-1, \sum_{s=l}^{d-1} t_{ls} &\le \mathfrak{d}_{\tau(l)} \end{aligned} \tag{6.28}$$

Combining the conditions in (6.27) and (6.28) we get the result. ■

Theorem 6.2.6 *Consider $a_1^* \in \ker\phi_1^*$ and let $u_a \in \operatorname{Hom}_s(\operatorname{Sym}(I)_\delta, S)$ represent the element a_1^*. Then the description of $\mathfrak{b} \in \mathcal{A}_0$ corresponding to the element a_1^* is*

$$\mathfrak{b} = \sum_{m_1+\cdots+m_{d-1}\le\delta} q_{m_1\cdots m_{d-1}} \cdot u_a\left(x_1^{m_1}\cdots x_{d-1}^{m_{d-1}} x_d^{\delta-\sum_{j=1}^{d-1} m_j}\right)$$

where $q_{m_1\cdots m_{d-1}}$ is as described in Theorem 6.2.5.

Proof Notice that $\operatorname{Sym}(I)_\delta$ is minimally generated by $x_1^{m_1}\cdots x_{d-1}^{m_{d-1}} x_d^{\delta-\sum_{j=1}^{d-1} m_j}$ where m_i's are non negative integers. Thus using Theorem 6.2.2, we have

$$\begin{aligned} \mathfrak{b} &= \nu_1(u_a) = (1\otimes u_a)(\operatorname{morl}_{(0,\delta,d)}) \\ &= (1\otimes u_a)\left(\sum_{m_1+\cdots+m_{d-1}\le\delta} q_{m_1\cdots m_{d-1}} \otimes x_1^{m_1}\cdots x_{d-1}^{m_{d-1}} x_d^{\delta-\sum_{j=1}^{d-1} m_j}\right) \\ &= \sum_{m_1+\cdots+m_{d-1}\le\delta} q_{m_1\cdots m_{d-1}} \cdot u_a\left(x_1^{m_1}\cdots x_{d-1}^{m_{d-1}} x_d^{\delta-\sum_{j=1}^{d-1} m_j}\right) \end{aligned}$$

■

Observation 6.2.7 From Theorem 6.2.5, notice that $q_{m_1 m_2 \cdots m_{d-1}} \in \mathrm{Sym}(I)_0$ is of bi-degree $(0, d)$. Thus using using the above proposition, we see that

$$\begin{aligned} \deg \mathfrak{b} &= \text{degree of the entries of } a_1^* + d \\ &= \text{degree of the entries of } a_1 + d \end{aligned}$$

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VITA

VITA

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