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Homological properties of determinantal arrangements

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ABSTRACT

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We study a certain family of hypersurface arrangements known as determinantal arrangements. Determinantal arrangements are a union of varieties defined by minors of a matrix of indeterminates. In particular, we investigate determinantal arrangements using the 2-minors of a $2 \times n$ generic matrix (which can be thought of as natural extensions of braid arrangements), and prove certain statements about their freeness.

We also study the topology of these objects. We construct a fibration for the complement of free determinantal arrangements, and use this fibration to prove statements about their homotopy groups. Furthermore, we show that the Poincaré polynomial of the complement factors nicely.

1. Introduction

An important aspect of a divisor D in a complex manifold X is its *singular locus*. While near a typical smooth point of a divisor, an appropriate coordinate system makes the pair (X, D) look like the pair $(\mathbb{C}^n, \mathrm{Var}(x_1 = 0))$; the singular locus of D is the set of points where this is not so. This thesis explores a family of divisors known as *determinantal arrangements* (which are unions of determinantal varieties) and furthers our understanding of their singular locus.

The singularities of D are the points where the tangent space to D is the same as the tangent space to X, in which case there is no well-defined normal direction to D in X. Concretely, if a di+ 36visor D is defined by the equation $f = 0$ for a reduced holomorphic f, then the singular locus is exactly the set of points where the gradient of f is zero. This allows one to study the singular locus of D using algebraic methods by looking at the behavior of the gradient of f.

Our goal is to determine whether or not the singular locus of a divisor is "well-behaved," and in particular, whether or not the singular loci of determinantal arrangements are "well-behaved." While there are different ways to characterize the behavior of the singular locus, we take an algebraic approach. Using this approach we say that a divisor is "well-behaved" or *free* if its singular locus has the shortest possible free resolution. That is, the linear dependencies of the entries of the gradient have no relations between themselves and hence form a free module. Normal crossing divisors are examples of free divisors.

Example 1.0.1 Consider a normal crossing divisor D in $\mathbb{C}^3$. Locally, D behaves like $\mathrm{Var}(xyz) = \{(x, y, z) \in \mathbb{C}^3 | xyz = 0\}$. The gradient of xyz is (yz, xz, xy), which has linear dependencies generated by $v_1 = (x, 0, -z)$, $v_2 = (0, y, -z)$. Note also that $(x, -y, 0)$ is also a relation between the entries of the gradient, but this relation is can be realized as $v_1 - v_2$, so we will not include it in the generating set. Now, v_1 and v_2 have no relations between themselves, hence D is a free divisor.

One can study divisors using flows (or vector fields) on X. A *logarithmic* flow along D on X is a vector field that is tangent to D. Near a smooth point of D, any point of D can flow to any other point of D in a logarithmic flow, but this may not be the case near singularities. Algebraically, a logarithmic flow is a vector field that is perpendicular at each point of D to the gradient, and thus gives us a formula in terms of derivations. The *module of logarithmic derivations* $\mathrm{Der}_X(-\log D) := \{\theta \in \mathrm{Der}_X | \theta(\mathcal{O}_X(-D)) \subseteq \mathcal{O}_X(-D)\}$. If $\mathrm{Der}_X(-\log D)$ is locally free, then D is called a free divisor. Note that this notion of free divisors coincide with the description given earlier.

Free divisors were first introduced by Saito [1], and were motivated by his study of the discriminants of versal deformations of isolated hypersurface singularities. The study of free divisors arising from discriminants of versal deformations has since been the source of many advances in the theory of singularities (see [2–6]).

Aside from versal deformations, free divisors show up naturally in many different settings. The theory of free divisors has been looked at extensively in the setting of hyperplane arrangements. In fact, many of the well-known hyperplane arrangements (such as braid arrangements and all Coxeter arrangements) are free (see [7]).

Interestingly, freeness can also give us topological information. In particular, Terao proves in [8] that for a free hyperplane arrangement, the Poincaré polynomial of its complement is determined by the degrees of the vector fields in the basis of the module of logarithmic derivations.

It seemed natural to wonder how freeness is connected to the topology of more complicated divisors. This motivated the study of free divisors in arrangements of more general hypersurfaces. For example, Schenck and Tohăneanu [9] give conditions for when an arrangement of lines and conics on $\mathbb{P}^2$ is free.

The focus of this thesis is on determinantal arrangements, which are unions of hypersurfaces defined by the minors of generic matrices. These hypersurfaces are particularly interesting to us because they have nice combinatorial structures. Buchweitz and Mond [10] showed that the arrangement defined by the product of the maximal minors of a $n \times (n+1)$ matrix of indeterminates is free. More recently, Damon and Pike [11] showed that certain determinantal arrangements coming from symmetric, skew-symmetric and square generic matrices are free and have complements that are $K(\pi, 1)$. In both of these cases, the arrangements turn out to be linear free divisors (i.e. the basis for $\mathrm{Der}_X(-\log D)$ is generated by linear vector fields). The vector fields arising in these situations correspond to matrix group actions on the generic matrix which stabilize the divisor D. Many interesting determinantal arrangements, however, are not linear free divisors as our next example shows.

Example 1.0.2 Let M be the 2×4 matrix of indeterminates

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix},$$

and for $i < j$, let Δ_{ij} be the 2-minor of M using the i-th and j-th columns, $\Delta_{ij} = x_i y_j - x_j y_i$. Let f be the product $f = \prod_{i<j} \Delta_{ij}$. Then $\mathrm{Der}_X(-\log f)$ is free with basis consisting of 7 linear derivations (coming from $\mathrm{SL}(2, \mathbb{C})$-action, column and row scaling actions on M), and one derivation of degree 5: $\theta = \Delta_{24}\Delta_{34}\left(x_1 \frac{\partial}{\partial x_4} + y_1 \frac{\partial}{\partial y_4}\right)$.

In this thesis, we study the determinantal arrangement analog of the braid arrangement. These arrangements are defined by the maximal minors of $2 \times n$ generic matrices. In Chapter 3, we examine the freeness of certain families of determinantal arrangements. In particular, we prove in Theorem 3.1.4, that our analog of the braid arrangement is indeed free. In Chapter 4, study the topology of free determinantal arrangements, and show that the Poincaré polynomials of their complements factors over $\mathbb{Q}$.

2. Background

In this chapter, we introduce some basic notations and definitions for the objects we will be studying, and for the tools we will be using. We also look at some basic results and examples which motivated our study of determinantal arrangements.

2.1 Logarithmic Derivations

Let D be a divisor in a complex manifold X. Our goal is to understand the behavior of the singular locus of D, so we look locally at the singularities. Therefore, we will assume that $X = \mathbb{C}^n$ and $D = \mathrm{Var}(f)$ for some $f \in S := \mathbb{C}[x_1, \ldots, x_n]$. In most cases that we are interested in, f will be a homogeneous polynomial.

Recall that a derivation θ on S over $\mathbb{C}$ is a $\mathbb{C}$-linear map $\theta : S \to S$ satisfying Leibniz's rule: $\theta(fg) = \theta(f)g + f\theta(g)$. Let $\mathrm{Der}_{\mathbb{C}}(S)$ denote the collection of all derivations on S over $\mathbb{C}$. Note that $\mathrm{Der}_{\mathbb{C}}(S)$ is a free S-module with a basis $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$, and can be thought of as the collection of holomorphic vector fields on X.

We are interested in vector fields on X tangent along the divisor D.

Definition 2.1.1 The *module of logarithmic derivations along* $D = \mathrm{Var}(f)$ is the S-module

$$\mathrm{Der}_X(-\log f) := \{\theta \in \mathrm{Der}_{\mathbb{C}}(S) | \theta(f) \subseteq (f)\}.$$

We may also write this module as $\mathrm{Der}_X(-\log D)$.

Note that $\mathrm{Der}_X(-\log f)$ is a submodule of $\mathrm{Der}_{\mathbb{C}}(S)$.

Since D is often a union of hypersurfaces, the following proposition will allows us to consider logarithmic derivations on the different components.

Proposition 2.1.1 *(i) If D is a union of two hypersurfaces* $\mathrm{Var}(f)$ *and* $\mathrm{Var}(g)$ *where $f, g \in S$ are relatively prime, then*

$$\theta \in \mathrm{Der}_X(-\log D) \Leftrightarrow \theta \in \mathrm{Der}_X(-\log f) \cap \mathrm{Der}_X(-\log g).$$

(ii) Suppose $D = \mathrm{Var}(f^2)$, then $\theta \in \mathrm{Der}_X(-\log D) \Leftrightarrow \theta \in \mathrm{Der}_X(-\log f)$.

(iii) Suppose $D = \mathrm{Var}(f_1^{i_1} \cdots f_k^{i_k})$, where $f_j \in S$ are pairwise relatively prime and $i_j \in \mathbb{N}_{>0}$ for $j = 1, \dots k$, then $\theta \in \mathrm{Der}_X(-\log D) \Leftrightarrow \theta \in \bigcap_{j=1}^{k} \mathrm{Der}_X(-\log f_j)$.

Proof (i) If $\theta \in \mathrm{Der}_X(-\log f) \cap \mathrm{Der}_X(-\log g)$, then by Leibniz's rule, $\theta(fg) = \theta(f)g + f\theta(g)$. Since both summands on the right are divisible by fg, $\theta \in \mathrm{Der}_X(-\log fg)$.

On the other hand, if $\theta \in \mathrm{Der}_X(-\log fg)$, then

$$\begin{aligned} \theta(fg) &= \alpha fg \qquad \text{for some } \alpha \in S \\ \theta(f)g + f\theta(g) &= \alpha fg \end{aligned}$$

If we divide both sides by f, we can write the above as $\frac{\theta(f)}{f} g = \alpha g - \theta(g)$. Since f and g are relatively prime, f must divide $\theta(f)$ so $\theta(f) \in \mathrm{Der}_X(-\log f)$. Similarly, $\theta(g) \in \mathrm{Der}_X(-\log g)$.

(ii) For any $\theta \in \mathrm{Der}_{\mathbb{C}}(S)$, we have $\theta(f^2) = 2f\theta(f)$. Thus f divides $\theta(f)$ if and only if f^2 divides $\theta(f^2)$.

(iii) This is a consequence of (i) and (ii).

■

2.2 Freeness

We say that the divisor $D = \mathrm{Var}(f)$ in X is *free* if its singular locus is well-behaved in an algebraic sense. Let $J = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ be the Jacobian ideal of f. We say that the singular locus of D is well-behaved if its coordinate ring $S/(J+(f))$ has the shortest possible free resolution:

$$0 \leftarrow S/(J+(f)) \leftarrow S \leftarrow S^{n+1} \leftarrow \mathrm{Syz}(J+(f)) \leftarrow 0.$$

Note that for any $\theta = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i} \in \mathrm{Der}_X(-\log f)$ where $\alpha_i \in S$, we have $\theta(f) = \beta f$ for some $\beta \in S$. Therefore $\sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i} = \beta f$, which gives us a syzygy on $J+(f)$. Similarly, each syzygy on $J+(f)$ gives us an element in $\mathrm{Der}_X(-\log f)$ so these modules are in 1-to-1 correspondence. Thus we use the following definition for free divisor:

Definition 2.2.1 A divisor $D = \mathrm{Var}(f)$ in X is *free* if $\mathrm{Der}_X(-\log f)$ is a free S-module.

To determine whether or not a divisor is free, one can try to find a basis for $\mathrm{Der}_X(-\log f)$. Given elements in $\mathrm{Der}_X(-\log f)$, one can check whether or not they form a basis using *Saito's criterion* [1]:

Theorem 2.2.1 (Saito) *A divisor $D = \mathrm{Var}(f)$ is free if and only if there exists n elements*

$$\theta_j = \sum_{i=1}^{n} \alpha_{ij} \frac{\partial}{\partial x_i} \in \mathrm{Der}_X(-\log f)$$

such that $\det((\alpha_{ij})) = c \cdot f$ for some non-zero $c \in \mathbb{C}$.

2.3 Examples from Hyperplane Arrangements

The theory of logarithmic derivations and free divisors has been studied extensively for hyperplane arrangements (see [7]). While not all hyperplane arrangements are free divisors, many of the classically arising arrangements are indeed free. Since the determinantal arrangements we will be studying are natural extensions of the braid arrangements, we use braid arrangements as a motivating example.

Example 2.3.1 The braid arrangement in $\mathbb{C}^n$, defined by $f = \prod_{1\le a<b\le n}(x_a - x_b)$, is free. To show this, we simply find n elements in $\mathrm{Der}_X(-\log f)$ and use Saito's criterion (Theorem 2.2.1) to show that they form a basis.

Consider the derivations

$$\theta_j = \sum_{i=1}^{n} x_i^j \frac{\partial}{\partial x_i} \text{ for } j = 0, \dots, n-1.$$

For $1 \le a < b \le n$, we have that $\theta_j(x_a - x_b) = x_a^j - x_b^j$ is divisible by $(x_a - x_b)$, so by Proposition 2.1.1, we know that $\theta_j \in \mathrm{Der}_X(-\log f)$.

To check that these logarithmic derivations form a basis, we calculate the determinant of

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}$$

which is precisely f. By Theorem 2.2.1, the braid arrangement is a free divisor in $\mathbb{C}^n$.

Freeness is often tied to the topology and combinatorics of the object. A landmark result of Terao relates the freeness of a central hyperplane arrangement (i.e. an arrangement such that each hyperplane passes through the origin) to the Poincaré polynomial of the complement of that arrangement [8]:

Theorem 2.3.2 (Terao) *Let $\mathcal{A} \subset \mathbb{C}^n$ be a free central hyperplane arrangement and suppose that* $\mathrm{Der}_{\mathbb{C}^n}(-\log \mathcal{A}) \cong \bigoplus_{i=1}^{n} S(-b_i)$, *then*

$$\mathrm{Poin}(\mathbb{C}^n \setminus \mathcal{A}, t) = \prod_{i=1}^{n}(1 + b_i t).$$

Example 2.3.3 Using Theorem 2.3.2, we can compute the Poincaré polynomial for the complement of the braid arrangement that was described in Example 2.3.1. Since $\mathrm{Der}_X(-\log f)$ is generated in degrees $0, \dots, n-1$,

$$\mathrm{Poin}(\mathbb{C}^n \setminus \mathrm{Var}(f)) = (1+t)(1+2t)\cdots(1+(n-1)t).$$

A famous open problem in the study of hyperplane arrangements is Terao's conjecture relating the freeness of a hyperplane arrangement to its combinatorics:

Conjecture 2.3.4 (Terao) *The freeness of a hyperplane arrangement $\mathcal{A}$ depends only on its lattice of interesection $L_{\mathcal{A}}$.*

Graphic arrangements are examples where Terao's conjecture is true. These arrangements are a union of a subcollection of the hyperplanes in the braid arrangement. As their name suggests, one can associate a graph to each graphic arrangement.

Let $\mathcal{A} \subset \mathbb{C}^n$ be a graphic arrangement, then the graph associated to $\mathcal{A}$ has n vertices labeled $x_1, \dots, x_n$. For each hyperplane $\mathrm{Var}(x_a - x_b) \subseteq \mathcal{A}$, we have an edge between vertices x_a and x_b.

Definition 2.3.1 Let G be a graph with vertex set V and edge set E, and let W be a subset of V. The *induced subgraph on* W is the subgraph of G consisting of every edge in E whose endpoints lie in W.

We say that a graph is *chordal* if every cycle of length 4 or greater has chord (i.e. an edge between two nonconsecutive vertices). While this description is easy to visualize, it will be more helpful to use the following characterization of chordal due to Fulkerson and Gross [12]:

Definition 2.3.2 A graph G is *chordal* if and only if there exists an ordering of vertices, such that for each vertex v, the induced subgraph on v and its neighbors that occur before it in the sequence is a complete graph. This ordering of vertices is called the *reverse perfect elimination ordering.*

For each graphic arrangement $\mathcal{A}$, its lattice of intersections $L_{\mathcal{A}}$ is exactly the lattice of contractions of the graph associated to $\mathcal{A}$. Since chordal graphs have a supersolvable lattice of contractions [13] and since hyperplane arrangements with supersolvable lattice of intersections is free [7], we know that a graphic arrangements associated to chordal graphs are free. In fact, one can prove the following [14]:

Theorem 2.3.5 (Kung-Schenck) *Let $\mathcal{A} \subset \mathbb{C}^n$ be a graphic arrangement, then $\mathcal{A}$ is free if and only if its associated graph is chordal. Moreover, if k is the length of the longest chord-free induced cycle, then the projective dimension of* $\mathrm{Der}_{\mathbb{C}^n}(-\log \mathcal{A})$ *is bounded below:*

$$\mathrm{pdim}_S(\mathrm{Der}_{\mathbb{C}^n}(-\log \mathcal{A})) \geq k - 3.$$

Using Theorem 2.3.5 we can quickly determine whether or not a graphic arrangement is free.

Example 2.3.6 Consider the graphic arrangement in $\mathbb{C}^4$ defined by $f = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)$. Since the graph associated to this arrangement is the cyclic graph on four vertices, it is not free.

On the other hand, the graphic arrangement in $\mathbb{C}^4$ defined by $g = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)(x_1 - x_3)$ is free, because the associated graph is chordal.

2.4 Determinantal Arrangements

While hyperplane arrangements have been studied extensively, not much is known for arrangements of more general hypersurfaces. The focus of this thesis is on determinantal arrangements.

Definition 2.4.1 Let M be an $m \times n$ matrix of indeterminates. A *determinantal arrangement on* M is a union of hypersurfaces defined by the minors of M.

This thesis examines determinantal arrangements on a $2 \times n$ generic matrix:

$$M = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

These arrangements are natural extensions of braid arrangements and graphic arrangements. When looking at the arrangement using every minor of M, i.e. the arrangement defined by

$$f = \prod_{1 \leq i < j \leq n} (x_i y_j - x_j y_i),$$

one can consider points on this arrangement as a selection of n vectors in $\mathbb{C}^2$ such that two of these vectors are linearly dependent. Whereas the braid arrangement can be thought of as a selection of n points in $\mathbb{C}$ such that two of these points are the same.

As with graphic arrangements, we can associate these determinantal arrangements to graphs:

Definition 2.4.2 Let G be a graph on n vertices $\{v_1, \ldots, v_n\}$. The determinantal arrangement $\mathcal{A}_G$ associated to G is a union of hypersurfaces $\mathrm{Var}(x_i y_j - x_j y_i)$ for each edge in G between vertices v_i and v_j.

3. Freeness of Determinantal Arrangements

In this chapter, we prove statements about the freeness of determinantal arrangements on a $2 \times n$ generic matrix. In Theorem 3.1.4, we show that our analog of the braid arrangement is a free divisor in $\mathbb{C}^{2n}$ for $n \geq 3$. We provide elements of the module of logarithmic derivations and show that these elements form a basis using Saito's criterion. In Theorem 3.2.1, we prove that free determinantal arrangements must come from chordal graphs and we give bounds on the projective dimension of the module of logarithmic derivations for non-chordal arrangements.

Note that when $n = 2$, the determinantal variety $\mathrm{Var}(x_1y_2 - x_2y_1)$ is not a free divisor in $\mathbb{C}^4$. The singular locus consists of the origin which is codimension 4, so S/J cannot possibly have projective dimension 2. For $n \geq 3$, we are able to take union of hypersurfaces which gives us singular loci of codimension 2, in which case we might have a free divisor.

3.1 Determinantal Braid Arrangement

In this section we prove that the determinantal arrangement on

$$M = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

using every 2-minor is free. Let Δ_{ij} denote the minor of M using the i-th and j-th column with $i < j$. With this notation, we can write our determinantal arrangement as the vanishing of

$$f = \prod_{1 \leq i < j \leq n} (x_iy_j - x_jy_i) = \prod_{1 \leq i < j \leq n} \Delta_{ij}.$$

Since this is a natural extension of the braid arrangement, we refer to this divisor in $\mathbb{C}^{2n}$ as the *determinantal braid arrangement*.

To prove that the determinantal braid arrangement is free, we will use the following two lemmas.

Lemma 3.1.1 *Let A be a block matrix $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ with blocks of size $n \times n$ with entries in $\mathbb{C}(z_1, \ldots, z_n)$. If A_1 and A_3 are diagonal matrices with nonzero entries, then* $\det(A) = \det(A_1A_4 - A_3A_2)$.

Proof Consider the block matrix $B = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_3^{-1} \end{pmatrix}$, then

$$BA = \begin{pmatrix} I_n & A_1^{-1}A_2 \\ I_n & A_3^{-1}A_4 \end{pmatrix}.$$

Using elementary row operations, we find

$$\det(BA) = \det \begin{pmatrix} I_n & A_1^{-1}A_2 \\ 0 & A_3^{-1}A_4 - A_1^{-1}A_2 \end{pmatrix}.$$

Now, let C be the block matrix $C = \begin{pmatrix} I_n & 0 \\ 0 & A_1A_3 \end{pmatrix}$, then

$$\det(CBA) = \det \begin{pmatrix} I_n & A_1^{-1}A_2 \\ 0 & A_1A_4 - A_3A_2 \end{pmatrix} = \det(A_1A_4 - A_3A_2).$$

Since $\det(CBA) = \det(A)$, we have $\det(A) = \det(A_1A_4 - A_3A_2)$. ■

Lemma 3.1.2 *For each $n \in \mathbb{Z}_{>0}$, let $s_{i,j,k} \in \mathbb{C}[z_1, \ldots, z_n]$ be a degree k symmetric polynomial on the variables $z_i, \ldots, z_n$ that is degree one in each variable omitting the variable z_j, given by*

$$s_{i,j,k} = \sum_{\substack{\alpha_m \neq j \\ i \leq \alpha_1 < \cdots < \alpha_k \leq n}} z_{\alpha_1} z_{\alpha_2} \cdots z_{\alpha_k},$$

and let $s_{i,j,0} = 1$.

Let A_i denote the $(n+1-i)\times(n+1-i)$ matrix $(s_{i,j,k})$, where the row index j ranges from i to n, and the column index k ranges from 0 to $n-i$. Then

$$\det(A_i) = \left(\prod_{i<s\leq n}(z_i - z_s)\right)\det(A_{i+1}).$$

Proof We start by explicitly writing A_i:

$$A_i = \begin{pmatrix} 1 & (z_{i+1}+z_{i+2}+\cdots+z_n) & \cdots & (z_{i+1}z_{i+2}\cdots z_n) \\ 1 & (z_i+z_{i+2}+\cdots+z_n) & \cdots & (z_i z_{i+2}\cdots z_n) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (z_i+z_{i+1}+\cdots+z_{n-1}) & \cdots & (z_i z_{i+1}\cdots z_{n-1}) \end{pmatrix}.$$

Subtracting the first row from all other rows gives us the following matrix:

$$\begin{pmatrix} 1 & (z_{i+1}+z_{i+2}+\ldots+z_n) & (z_{i+1}z_{i+2}+z_{i+1}z_{i+3}+\cdots+z_{n-1}z_n) & \cdots & (z_{i+1}z_{i+2}\cdots z_n) \\ 0 & (z_i - z_{i+1}) & (z_i - z_{i+1})(z_{i+2}+z_{i+3}+\cdots+z_n) & \cdots & (z_i - z_{i+1})(z_{i+2}z_{i+3}\cdots z_n) \\ 0 & (z_i - z_{i+2}) & (z_i - z_{i+2})(z_{i+1}+z_{i+3}+\cdots+z_n) & \cdots & (z_i - z_{i+2})(z_{i+1}z_{i+3}\cdots z_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (z_i - z_n) & (z_i - z_n)(z_{i+1}+z_{i+2}+\cdots+z_{n-1}) & \cdots & (z_i - z_n)(z_{i+1}z_{i+2}\cdots z_{n-1}) \end{pmatrix}.$$

We factor the lower right $(n-i)\times(n-i)$ submatrix as

$$\begin{pmatrix} (z_i - z_{i+1}) & & & \\ & (z_i - z_{i+2}) & & \\ & & \ddots & \\ & & & (z_i - z_n) \end{pmatrix}\begin{pmatrix} 1 & (z_{i+2}+z_{i+3}+\cdots+z_n) & \cdots & (z_{i+2}z_{i+3}\cdots z_n) \\ 1 & (z_{i+1}+z_{i+3}+\cdots+z_n) & \cdots & (z_{i+1}z_{i+3}\cdots z_n) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (z_{i+1}+z_{i+2}+\cdots+z_{n-1}) & \cdots & (z_{i+1}z_{i+2}\cdots z_{n-1}) \end{pmatrix}$$

$$= \begin{pmatrix} (z_i - z_{i+1}) & & & \\ & (z_i - z_{i+2}) & & \\ & & \ddots & \\ & & & (z_i - z_n) \end{pmatrix} A_{i+1}.$$

Since our elementary row operations do not change the determinant, we have that $\det(A_i) = \left(\prod_{i<s\leq n}(z_i - z_s)\right)\det(A_{i+1})$. ■

Remark 3.1.3 We can compute the determinant of A_1 in Lemma 3.1.2 inductively to find

$$\det(A_1) = \prod_{1\leq i<j\leq n}(z_i - z_j).$$

We now prove the main result of this section:

Theorem 3.1.4 *Let G be the complete graph on n vertices for $n \geq 3$. The determinantal arrangement $\mathcal{A}_G$ is a free divisor in $X = \mathbb{C}^{2n}$.*

Proof $\mathcal{A}_G$ is the determinantal braid arrangement defined by the vanishing of

$$f = \prod_{1 \leq i < j \leq n} \Delta_{ij}.$$

We explicitly list elements of $\mathrm{Der}_X(-\log f)$, and use Saito's criterion to show that this list forms a basis for $\mathrm{Der}_X(-\log f)$.

We first consider the following linear derivations:

$$\alpha = \sum_{k=1}^{n} x_k \frac{\partial}{\partial y_k}$$

$$\beta = \sum_{k=1}^{n} y_k \frac{\partial}{\partial x_k}$$

$$\gamma = \sum_{k=1}^{n} y_k \frac{\partial}{\partial y_k}.$$

To show that these derivations are actually elements of $\mathrm{Der}_X(-\log f)$, by Proposition 2.1.1, it is enough to show that these derivation sends each Δ_{ij} to the ideal (Δ_{ij}):

$$\begin{aligned}
\alpha(\Delta_{ij}) &= \left(\sum_{k=1}^{n} x_k \frac{\partial}{\partial y_k}\right)(x_i y_j - x_j y_i) \\
&= \left(x_i \tfrac{\partial}{\partial y_i} + x_j \tfrac{\partial}{\partial y_j}\right)(x_i y_j - x_j y_i) \\
&= -x_i x_j + x_j x_i \\
&= 0.
\end{aligned}$$

Since α stabilizes each (Δ_{ij}), $\alpha \in \mathrm{Der}_X(-\log f)$.

Similarly,

$$\begin{aligned}
\beta(\Delta_{ij}) &= \left(\sum_{k=1}^{n} y_k \frac{\partial}{\partial x_k}\right)(x_i y_j - x_j y_i) \\
&= \left(y_i \tfrac{\partial}{\partial x_i} + y_j \tfrac{\partial}{\partial x_j}\right)(x_i y_j - x_j y_i) \\
&= y_i y_j - y_j y_i \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\gamma(\Delta_{ij}) &= \left(\sum_{k=1}^{n} y_k \frac{\partial}{\partial y_k}\right)(x_i y_j - x_j y_i) \\
&= \left(y_i \tfrac{\partial}{\partial y_i} + y_j \tfrac{\partial}{\partial y_j}\right)(x_i y_j - x_j y_i) \\
&= -y_i x_j + y_j x_i \\
&= \Delta_{ij},
\end{aligned}$$

thus $\beta, \gamma \in \mathrm{Der}_X(-\log f)$.

We also have n linear derivations of the form

$$\theta_k = x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k}$$

for $k = 1, 2 \ldots, n$. When we apply θ_k to Δ_{kj}, we get

$$\begin{aligned}
\theta_k(\Delta_{kj}) &= \left(x_k \tfrac{\partial}{\partial x_k} + y_k \tfrac{\partial}{\partial y_k}\right)(x_k y_j - x_j y_k) \\
&= x_k y_j - y_k x_j \\
&= \Delta_{kj},
\end{aligned} .$$

Similarly, $\theta_k(\Delta_{ik}) = \Delta_{ik}$. When $i, j \neq k$, $\theta_k(\Delta_{ij}) = 0$, thus θ_k stabilizes each (Δ_{ij}). This shows that $\theta_k \in \mathrm{Der}_X(-\log f)$.

Finally, we have $n-3$ elements of degree $n+1$. For $k = 4, 5, .., n$, let τ_k be a bijection of sets from $\{1, \ldots, n-4\}$ to $\{4, \ldots, k-1, k+1, \ldots n\}$, and let S_{n-4} be the symmetric group acting on the numbers $\{1, \ldots, n-4\}$. For $m = 0, 1, \ldots, n-4$, define

$$a_{m,k} = \frac{1}{m!(n-4-m)!} \sum_{\sigma \in S_{n-4}} x_{(\tau_k \circ \sigma)(1)} \cdots x_{(\tau_k \circ \sigma)(m)} y_{(\tau_k \circ \sigma)(m+1)} \cdots y_{(\tau_k \circ \sigma)(n-4)}.$$

We now define the degree $n+1$ derivations:

$$\varphi_m = \sum_{k=4}^{n} a_{m,k} \Delta_{2k} \Delta_{3k} \left(x_1 \frac{\partial}{\partial x_k} + y_1 \frac{\partial}{\partial y_k}\right).$$

To check that $\varphi_m \in \mathrm{Der}_X(-\log f)$, we must consider several cases of $\varphi_m(\Delta_{ij})$. If $i, j < 4$, then $\varphi_m(\Delta_{ij}) = 0$. Now, suppose that $i < 4$ and $j \geq 4$, then

$$\begin{aligned}
\varphi_m(\Delta_{ij}) &= \left(\sum_{k=4}^{n} a_{m,k} \Delta_{2k} \Delta_{3k} \left(x_1 \frac{\partial}{\partial x_k} + y_1 \frac{\partial}{\partial y_k}\right)\right)(x_i y_j - x_j y_i) \\
&= a_{m,j} \Delta_{2j} \Delta_{3j} \left(x_1 \tfrac{\partial}{\partial x_j} + y_1 \tfrac{\partial}{\partial y_j}\right)(x_i y_j - x_j y_i) \\
&= a_{m,j} \Delta_{2j} \Delta_{3j} \left(-x_1 y_i + y_1 x_i\right).
\end{aligned}$$

When $i = 2, 3$, $\varphi_m(\Delta_{ij}) \in (\Delta_{ij})$, and when $i = 1$, $\varphi_m(\Delta_{ij}) = 0$.

Finally, if $i, j \geq 4$,

$$\begin{aligned}
\varphi_m(\Delta_{ij}) &= \left(\sum_{k=4}^{n} a_{m,k}\Delta_{2k}\Delta_{3k}\left(x_1\frac{\partial}{\partial x_k} + y_1\frac{\partial}{\partial y_k}\right)\right)(\Delta_{ij}) \\
&= \left(a_{m,i}\Delta_{2i}\Delta_{3i}\left(x_1\tfrac{\partial}{\partial x_i} + y_1\tfrac{\partial}{\partial y_i}\right) + a_{m,j}\Delta_{2j}\Delta_{3j}\left(x_1\tfrac{\partial}{\partial x_j} + y_1\tfrac{\partial}{\partial y_j}\right)\right)(\Delta_{ij}) \\
&= a_{m,i}\Delta_{2i}\Delta_{3i}\left(x_1y_j - y_1x_j\right) + a_{m,j}\Delta_{2j}\Delta_{3j}\left(-x_1y_i + y_1x_i\right) \\
&= a_{m,i}\Delta_{2i}\Delta_{3i}\Delta_{1j} - a_{m,j}\Delta_{2j}\Delta_{3j}\Delta_{1i}.
\end{aligned}$$

Note that the terms of $a_{m,i}$ and $a_{m,j}$ are nearly identical except the factors of x_j are replaced with factors of x_i, and the factors of y_j are replaced with factors of y_i. Thus if we match up the terms in $a_{m,i}$ and $a_{m,j}$ and remove the common factors, we only need to show that $x_j\Delta_{2i}\Delta_{3i}\Delta_{1j} - x_i\Delta_{2j}\Delta_{3j}\Delta_{1i}$ and $y_j\Delta_{2i}\Delta_{3i}\Delta_{1j} - y_i\Delta_{2j}\Delta_{3j}\Delta_{1i}$ are divisible by Δ_{ij}. Using Plücker relations, we can write:

$$\begin{aligned}
x_j\Delta_{2i}\Delta_{3i}\Delta_{1j} - x_i\Delta_{2j}\Delta_{3j}\Delta_{1i} &= x_j\Delta_{3i}(\Delta_{1j}\Delta_{2i}) - x_i\Delta_{2j}\Delta_{3j}\Delta_{1i} \\
&= x_j\Delta_{3i}(\Delta_{1i}\Delta_{2j} - \Delta_{12}\Delta_{ij}) - x_i\Delta_{2j}\Delta_{3j}\Delta_{1i} \\
&= \Delta_{1i}\Delta_{2j}(x_j\Delta_{3i} - x_i\Delta_{3j}) - x_j\Delta_{3i}\Delta_{12}\Delta_{ij} \\
&= \Delta_{1i}\Delta_{2j}(x_jx_3y_i - x_jx_iy_3 - x_ix_3y_j + x_ix_jy_3) \\
&\qquad - x_j\Delta_{3i}\Delta_{12}\Delta_{ij} \\
&= \Delta_{1i}\Delta_{2j}(x_jx_3y_i - x_ix_3y_j) - x_j\Delta_{3i}\Delta_{12}\Delta_{ij} \\
&= \Delta_{1i}\Delta_{2j}(-x_3\Delta_{ij}) - x_j\Delta_{3i}\Delta_{12}\Delta_{ij} \in (\Delta_{ij}),
\end{aligned}$$

and similarly,

$$\begin{aligned}
y_j\Delta_{2i}\Delta_{3i}\Delta_{1j} - y_i\Delta_{2j}\Delta_{3j}\Delta_{1i} &= y_j\Delta_{3i}(\Delta_{1j}\Delta_{2i}) - y_i\Delta_{2j}\Delta_{3j}\Delta_{1i} \\
&= y_j\Delta_{3i}(\Delta_{1i}\Delta_{2j} - \Delta_{12}\Delta_{ij}) - y_i\Delta_{2j}\Delta_{3j}\Delta_{1i} \\
&= \Delta_{1i}\Delta_{2j}(y_j\Delta_{3i} - y_i\Delta_{3j}) - y_j\Delta_{3i}\Delta_{12}\Delta_{ij} \\
&= \Delta_{1i}\Delta_{2j}(y_jx_3y_i - y_jx_iy_3 - y_ix_3y_j + y_ix_jy_3) \\
&\qquad - y_j\Delta_{3i}\Delta_{12}\Delta_{ij} \\
&= \Delta_{1i}\Delta_{2j}(-y_jx_iy_3 + y_ix_jy_3) - y_j\Delta_{3i}\Delta_{12}\Delta_{ij} \\
&= \Delta_{1i}\Delta_{2j}(-y_3\Delta_{ij}) - y_j\Delta_{3i}\Delta_{12}\Delta_{ij} \in (\Delta_{ij}).
\end{aligned}$$

Since φ_m stabilizes each (Δ_{ij}), $\varphi_m \in \mathrm{Der}_X(-\log f)$.

It remains to show that the collection $\{\alpha, \beta, \gamma, \theta_1, \ldots, \theta_n, \varphi_0, \ldots, \varphi_{n-4}\}$ form a basis. By Theorem 2.2.1, these derivations form a basis if and only if the determinant of the coefficient matrix is a nonzero constant multiple of f. With our elements, we have the coefficient matrix:

$$\begin{pmatrix}
y_1 & & & x_1 & & & & & & & & \\
y_2 & & & & x_2 & & & & & & & \\
y_3 & & & & & x_3 & & & & & & \\
y_4 & & & & & & x_4 & & & a_{0,4}\Delta_{24}\Delta_{34}x_1 & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}x_1 \\
\vdots & & & & & & & \ddots & & \vdots & \ddots & \vdots \\
y_n & & & & & & & & x_n & a_{0,n}\Delta_{2n}\Delta_{3n}x_1 & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}x_1 \\
& x_1 & y_1 & y_1 & & & & & & & & \\
& x_2 & y_2 & & y_2 & & & & & & & \\
& x_3 & y_3 & & & y_3 & & & & & & \\
& x_4 & y_4 & & & & y_4 & & & a_{0,4}\Delta_{24}\Delta_{34}y_1 & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}y_1 \\
& \vdots & \vdots & & & & & \ddots & & \vdots & \ddots & \vdots \\
& x_n & y_n & & & & & & y_n & a_{0,n}\Delta_{2n}\Delta_{3n}y_1 & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}y_1
\end{pmatrix}.$$

By swapping rows of this matrix, we can write our matrix as a triangular block matrix (note that while swapping rows might change the sign of the determinant, that will not matter in checking Saito's criterion):

$$
\left(\begin{array}{cccccc|cccccc}
y_1 & & & x_1 & & & & & & & & \\
y_2 & & & & x_2 & & & & & & & \\
y_3 & & & & & x_3 & & & & & & \\
 & x_1 & y_1 & y_1 & & & & & & & & \\
 & x_2 & y_2 & & y_2 & & & & & & & \\
 & x_3 & y_3 & & & y_3 & & & & & & \\
\hline
y_4 & & & & & & x_4 & & & a_{0,4}\Delta_{24}\Delta_{34}x_1 & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}x_1 \\
\vdots & & & & & & & \ddots & & \vdots & \ddots & \vdots \\
y_n & & & & & & & & x_n & a_{0,n}\Delta_{2n}\Delta_{3n}x_1 & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}x_1 \\
 & x_4 & y_4 & & & & y_4 & & & a_{0,4}\Delta_{24}\Delta_{34}y_1 & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}y_1 \\
 & \vdots & \vdots & & & & & \ddots & & \vdots & \ddots & \vdots \\
 & x_n & y_n & & & & & & y_n & a_{0,n}\Delta_{2n}\Delta_{3n}y_1 & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}y_1
\end{array}\right).
$$

Denote the previous matrix by N, with blocks $N = \left(\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right)$. Since N is a triangular block matrix, $\det(N) = \det(A)\det(D)$. One can compute

$$\det(A) = \Delta_{12}\Delta_{13}\Delta_{23}. \tag{3.1}$$

To find the determinant of D, we organize the matrix into more blocks:

$$
D = \left(\begin{array}{c|c} D_1 & D_2 \\ \hline D_3 & D_4 \end{array}\right) = \left(\begin{array}{ccc|ccc}
x_4 & & & a_{0,4}\Delta_{24}\Delta_{34}x_1 & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}x_1 \\
 & \ddots & & \vdots & \ddots & \vdots \\
 & & x_n & a_{0,n}\Delta_{2n}\Delta_{3n}x_1 & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}x_1 \\
\hline
y_4 & & & a_{0,4}\Delta_{24}\Delta_{34}y_1 & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}y_1 \\
 & \ddots & & \vdots & \ddots & \vdots \\
 & & y_n & a_{0,n}\Delta_{2n}\Delta_{3n}y_1 & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}y_1
\end{array}\right).
$$

Using Lemma 3.1.1, we know that $\det(D)$ and $\det(D_1D_4 - D_3D_2)$ agree over the field of fractions $\mathbb{C}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ but since these determinants are polynomials we must have $\det(D) = \det(D_1D_4 - D_3D_2)$. Now,

$$
\begin{aligned}
D_1D_4 - D_3D_2 &= \begin{pmatrix} a_{0,4}\Delta_{24}\Delta_{34}(y_1x_4 - x_1y_4) & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}(y_1x_4 - x_1y_4) \\ \vdots & \ddots & \vdots \\ a_{0,n}\Delta_{2n}\Delta_{3n}(y_1x_n - x_1y_n) & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}(y_1x_n - x_1y_n) \end{pmatrix} \\
&= -\begin{pmatrix} a_{0,4}\Delta_{24}\Delta_{34}\Delta_{14} & \cdots & a_{n-4,4}\Delta_{24}\Delta_{34}\Delta_{14} \\ \vdots & \ddots & \vdots \\ a_{0,n}\Delta_{2n}\Delta_{3n}\Delta_{1n} & \cdots & a_{n-4,n}\Delta_{2n}\Delta_{3n}\Delta_{1n} \end{pmatrix} \\
&= -\begin{pmatrix} \Delta_{24}\Delta_{34}\Delta_{14} & & \\ & \ddots & \\ & & \Delta_{2n}\Delta_{3n}\Delta_{1n} \end{pmatrix} \begin{pmatrix} a_{0,4} & \cdots & a_{n-4,4} \\ \vdots & \ddots & \vdots \\ a_{0,n} & \cdots & a_{n-4,n} \end{pmatrix} \\
&=: -D_5D_6.
\end{aligned}
$$

Observe that

$$\det(D_5) = \prod_{i=1}^{3}\prod_{j=4}^{n} \Delta_{ij}, \tag{3.2}$$

therefore it remains to show that $\det(D_6)$ is a nonzero constant multiple of the product of all minors using the last $n-3$ columns of M.

We show that each Δ_{ij} for $i, j \geq 4$ divides $\det(D_6)$ by showing that $\det(D_6)$ vanishes on $\mathrm{Var}(\Delta_{ij})$. Now, Δ_{ij} vanishes when columns i and j of M are scalar multiples of each other. Write $x_j = cx_i$ and $y_j = cy_i$. In rows i and j of D_6, we have $a_{m,j} = ca_{m,i}$. Since these rows are scalar multiples of each other, $\det(D_6)$ vanishes here which implies that each Δ_{ij} divides $\det(D_6)$. The degree of the product of the minors, $2\begin{pmatrix} n-3 \\ 2 \end{pmatrix} = (n-3)(n-4)$, is the same as the degree of $\det(D_6)$, hence $\det(D_6)$ is a constant multiple of the product of the minors. To check that $\det(D_6)$ is not identically zero, we substitute $y_k = 1$ for $k = 4, \ldots, n$ into D_6 to get the matrix in Lemma 3.1.2 on the variables $x_4, \ldots, x_n$. Using Remark 3.1.3, we see that if $x_4 \neq x_5 \neq \cdots \neq x_n$, then $\det(D_6) \neq 0$.

With equations (3.1) and (3.2), we find $\det(N) = (-1)^{n-3}\det(A)\det(D_5)\det(D_6)$ is a constant multiple of the product of all of the minors of M. By Saito's criterion,

$\{\alpha, \beta, \gamma, \theta_1, \ldots, \theta_n, \varphi_0, \ldots, \varphi_{n-4}\}$ form a basis for $\mathrm{Der}_X(-\log f)$, hence our determinantal arrangement is free. ■

We believe that our work with determinantal arrangements on $2 \times n$ generic matrices only scratches the surface of a broader class of free divisors. For example, suppose that M is an $m \times n$ matrix of indeterminates. In the case where $m = 3$ and $n = 4$, one knows that the arrangement is a linear free divisor (see [10], [15]). However, in the next case, $m = 3$ and $n = 5$, we already do not know whether or not the arrangement is free. More generally, one can ask:

Question 3.1.5 Let M be the $m \times n$ matrix of indeterminates with $n > m > 2$, and let f be the product of all maximal minors of M. Is the arrangement defined by f free?

3.2 Free Determinantal Arrangements and Chordal Graphs

One can also consider determinantal arrangements defined by subgraphs of the complete graph. Much like hyperplane arrangements, we find that the freeness of the determinantal arrangement is related to whether or not the graph is chordal.

Theorem 3.2.1 *If a determinantal arrangement $\mathcal{A}_G$ is free, then G is chordal. Moreover, if G has a chord-free induced cycle of length k, then*

$$\mathrm{pdim}(\mathrm{Der}_X(-\log \mathcal{A}_G)) \geq k - 3.$$

Proof Suppose that G is not chordal, then G has an chord-free induced cycle of length k where $4 \leq k \leq n$. We can reorganize the columns of M so that this chord-free induced cycle occurs on the first k vertices of $\mathcal{A}_G$. To show that $\mathcal{A}$ is not free, we localize to a neighborhood of the point $p = \begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & 2 & \cdots & n-k \end{pmatrix}$. We will consider our divisor in the local ring $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]_{\mathfrak{m}_p}$ where $\mathfrak{m}_p$ is the maximal ideal associated to the point p. In this local ring, Δ_{ij} is a unit if i or j

is greater than k. Thus, around p, $\mathcal{A}_G$ looks like $\mathrm{Var}(\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k})$ whose associated graph is the cyclic graph on k vertices.

We show that p is in the non-free locus of $\mathrm{Var}(\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k})$. In our local ring, x_i is a unit for all i, thus

$$\mathrm{Var}(\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k}) = \mathrm{Var}\left(\frac{x_1^{k-2}x_k^{k-2}}{x_2^2x_3^2\cdots x_{k-1}^2}\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k}\right).$$

But,

$$\begin{aligned}
&\tfrac{x_1^{k-2}x_k^{k-2}}{x_2^2x_3^2\cdots x_{k-1}^2}\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k}\\
&\quad= \tfrac{x_1^{k-2}x_k^{k-2}}{x_2^2x_3^2\cdots x_{k-1}^2}(x_1y_2-x_2y_1)(x_2y_3-x_3y_2)\cdots(x_{k-1}y_k-x_ky_{k-1})(x_1y_k-x_ky_1)\\
&\quad= \left(\tfrac{x_1x_k}{x_2}y_2-x_ky_1\right)\left(\tfrac{x_1x_k}{x_3}y_3-\tfrac{x_1x_k}{x_2}y_2\right)\cdots\left(x_1y_k-\tfrac{x_1x_k}{x_{k-1}}y_{k-1}\right)(x_1y_k-x_ky_1).
\end{aligned}$$

Now, making a change of coordinates

$$\begin{array}{ccc}
z_1 & \leftrightarrow & x_ky_1\\
z_2 & \leftrightarrow & \frac{x_1x_k}{x_2}y_2\\
\vdots & \vdots & \vdots\\
z_{k-1} & \leftrightarrow & \frac{x_1x_k}{x_{k-1}}y_{k-1}\\
z_k & \leftrightarrow & x_1y_k
\end{array},$$

we have that $\mathrm{Var}(\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k}) = \mathrm{Var}((z_2-z_1)(z_3-z_2)\cdots(z_k-z_{k-1})(z_k-z_1))$. Since our point p corresponds to $z_i = 0$ for the cyclic graphic arrangement $\mathrm{Var}((z_2-z_1)(z_3-z_2)\cdots(z_k-z_{k-1})(z_k-z_1))$, we know that p is in the non-free locus of $\mathrm{Var}(\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k})$, and thus $\mathcal{A}_G$ is not free. Moreover, this is a generic hyperplane arrangement so by Rose and Terao [16],

$$\mathrm{pdim}(\mathrm{Der}_X(-\log(\Delta_{12}\Delta_{23}\cdots\Delta_{(k-1)k}\Delta_{1k}))) = k-3.$$

Since localization is an exact functor, $\mathrm{pdim}(\mathrm{Der}_X(-\log{(A)_G})) \geq k-3$. ■

Remark 3.2.2 The converse of Theorem 3.2.1 is not true. For example, for any chordal graph with a vertex v of degree 2, if the induced subgraph v with its neighbors is not a cycle then the corresponding determinantal arrangement is not free. In this case, the arrangement locally behaves like $f = \Delta_{12}\Delta_{13}$, and one can check that this arrangement is not free. However, evidence suggests that many of the arrangements with chordal graphs are indeed free. For example, direct computations of small cases suggest that arrangements corresponding to doubly-connected (graphs that remain connected after removing any single vertex) chordal graphs are free.

4. Topology of Determinantal Arrangements

In this chapter, we investigate the topology of the complements of free determinantal arrangements. We exploit the combinatorial structure of free determinantal arrangements to construct a fibration for the complement. In Theorem 4.1.2, we use this fibration to show that the higher homotopy groups of the complement behave like the homotopy groups of S^3. In Theorems 4.2.2 and 4.2.3, we show that the Poincaré polynomial factors over $\mathbb{Q}$ and give the explicit Poincaré polynomial for the complement of the determinantal braid arrangement.

4.1 Fibration of the Complement

Let $\mathcal{A}_n = \operatorname{Var}\left(\prod_{1\leq i<j\leq n} \Delta_{ij}\right)$ denote the determinantal braid arrangement on a $2 \times n$ generic matrix. Now, consider the arrangement in the ambient space of $2 \times n$ matrices with coefficients in $\mathbb{C}$. Let $U_n = \mathbb{C}^{2n} \setminus \mathcal{A}$ be the complement of the arrangement. To study the topology of U_n, consider the fibration $p : U_n \to U_{n-1}$, where p is the projection onto the first $(n-1)$ columns. This map is well defined because the columns of U_n are pairwise linearly independent, and so the first $(n-1)$ columns is also pairwise linearly independent. The fiber of this map is a selection of a last column that is linearly independent from the first $(n-1)$ columns. Thus the fibers are homotopy equivalent to $\mathbb{C}^2$ minus $(n-1)$ lines through the origin.

This fibration can also be generalized to any free determinantal arrangement on a $2 \times n$ generic matrix M. From Theorem 3.2.1, we know that the graph associated to the arrangement is chordal. From Definition 2.3.2, we can order the vertices such that for each vertex v, the induced subgraph on v and its neighbors that occur before it in the sequence is a complete graph. Without loss of generality, assume that our free determinantal arrangement is associated to a graph G with vertices $\{v_1, \ldots, v_n\}$

labeled according to the reverse perfect elimination ordering. Let G_k denote the induced subgraph on $\{v_1, \ldots, v_k\}$, then for $k = 2, \ldots, n$, let $U_k = \mathbb{C}^{2k} \setminus \mathcal{A}_{G_k}$. Now, for each $k = 3, \ldots, n$, we have a fibration $p_k : U_k \to U_{k-1}$ where p_k is the projection onto the first $(k-1)$ columns. This map is well defined because for each $\mathrm{Var}(\Delta_{ij}) \subset \mathcal{A}_{G_{k-1}}$, we must have $\mathrm{Var}(\Delta_{ij}) \subset \mathcal{A}_{G_k}$ from the way the arrangements are defined. Suppose that induced subgraph on v_k and its neighbors in G_k is the complete graph on m vertices, then the fibers, F_k, of p_k are homotopy equivalent to $\mathbb{C}^2$ minus $(m-1)$ lines through the origin. We also have a fibration $p_2 : U_2 \to \mathbb{C}^2 \setminus \{0\}$ with fibers homotopy equivalent to $\mathbb{C}^2 \setminus \mathbb{C}$.

Note that p_k is only a fibration when the graph associated to the arrangement is chordal. If the graph is not chordal, the fibers are not homotopy equivalent.

Example 4.1.1 Consider the cyclic arrangement on 4 vertices: $f = \Delta_{12}\Delta_{23}\Delta_{34}\Delta_{14}$, we can follow our procedure of projecting the complement onto the first three columns, however, some fibers look like $\mathbb{C}^2$ minus 2 lines (when the first and third column are linearly independent) and other fibers look like $\mathbb{C}^2$ minus 1 line (when the first and third column are linearly dependent).

When the graph is a chordal, this is no longer an issue since all of the relevant columns are guaranteed to be linearly independent. We now use this fibration to prove statements about the topology of U_n.

Theorem 4.1.2 *Let G be a chordal graph on n vertices labeled according to the reverse perfect elimination ordering. Let G_k denote the induced subgraph of G on the first k vertices, let $U_k = \mathbb{C}^{2k} \setminus \mathcal{A}_{G_k}$, and let p_k be the fibration described above with fibers F_k. Then for $k = 2, \ldots, n$ the following sequence is exact:*

$$0 \to \pi_1(F_k) \to \pi_1(U_k) \to \pi_1(U_{k-1}) \to 0.$$

Furthermore, $\pi_i(U_k) \cong \pi_i(S^3)$ for $i \geq 2$.

Proof Denote $U_1 = \mathbb{C}^2 \setminus 0$, then for each fibration $p_k : U_k \to U_{k-1}$ for $k = 2, \ldots n$, consider the homotopy long exact sequence (note that our spaces are path-connected, so the reduced homotopy π_0 is zero):

$$\cdots \to \pi_2(F_k) \to \pi_2(U_k) \to \pi_2(U_{k-1}) \to \pi_1(F_k) \to \pi_1(U_k) \to \pi_1(U_{k-1}) \to 0. \qquad (4.1)$$

By Proposition 5.6 in [7], every central line arrangement is $K(\pi, 1)$, so since each F_k is a central line arrangement $\pi_i(F_k) = 0$ for $i \geq 2$ for each k. From (4.1), $\pi_i(U_k) \cong \pi_i(U_{k-1})$ for $i \geq 3$. Since $U_1 = \mathbb{C}^2 \setminus \{0\} \cong S^3$, for each k, $\pi_i(U_k) \cong \pi_i(S^3)$ for $i \geq 3$.

Furthermore, consider the segment

$$0 \cong \pi_2(F_k) \to \pi_2(U_k) \to \pi_2(U_{k-1}). \qquad (4.2)$$

When $k = 2$, the group on the right in (4.2) is $\pi_2(S^3) = 0$, which implies that $\pi_2(U_2) = 0$. By induction on k, $\pi_2(U_k) = 0$ for all k.

Plugging in $\pi_2(U_{k-1}) = 0$ into (4.1) we get the short exact sequence

$$0 \to \pi_1(F_k) \to \pi_1(U_k) \to \pi_1(U_{k-1}) \to 0.$$

■

4.2 Poincaré Polynomial of the Complement

Inspired by Theorem 2.3.2 we attempted to find a connection between the generators of the module of logarithmic derivations and the Poincaré polynomial of the complement of a free determinantal arrangement. Unfortunately, there is not a nice relation like the one given by Terao, but we are still able to calculate the Poincaré polynomial nonetheless.

To calculate the Poincaré polynomial, we will be using the cohomology Serre spectral sequence for the fibration described in this chapter. Since the terms on the E_2 page are calculated using with local coefficients, we show that the fundamental group on the base space induces the trivial monodromy action on the cohomology of the fiber. This allows us to use constant coefficients to describe the terms on the E_2 page of the spectral sequence.

As a first step, we will try to understand the fundamental group of our determinantal arrangement complements.

Lemma 4.2.1 *Let G be the complete graph on n vertices and let $U_n = \mathbb{C}^{2n} \setminus \mathcal{A}_G$, then $\pi_1(U_n)$ is generated by loops $\gamma : [0, 2\pi] \to U_n$ given by*

$$\gamma(t) = \begin{pmatrix} e^{it} & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \frac{1}{2} + e^{-it} & \frac{1}{3} + e^{-it} & \cdots & \frac{1}{n-1} + e^{-it} \end{pmatrix},$$

$\delta : [0, 2\pi] \to U_n$ given by

$$\delta(t) = \begin{pmatrix} e^{it} & 1 & 1 & \cdots & 1 \\ e^{it} & 2 & 3 & \cdots & n \end{pmatrix},$$

and loops constructed by permuting the columns of γ and δ.

Proof We will proceed by induction on n. For the base case $n = 2$, we know that U_2 is $\mathrm{GL}(2, \mathbb{C})$, and we know that $\pi_1(\mathrm{GL}(2, \mathbb{C})) = \mathbb{Z}$ and is generated by $\gamma(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix}$ (which can also be continuously deformed into $\delta(t)$).

Assume that the lemma is true for $\pi_1(U_n)$. To find the generators for $\pi_1(U_{n+1})$, we use Theorem 4.1.2. The short exact sequence

$$0 \to \pi_1(F_{n+1}) \to \pi_1(U_{n+1}) \to \pi_1(U_n) \to 0$$

tells us that $\pi_1(U_{n+1})$ is generated by lifts of generators in $\pi_1(U_n)$ and the images of generators from $\pi_1(F_{n+1})$.

To lift the generators from $\pi_1(U_n)$, we simply add a last column to γ, δ, and their permutations. For γ and its permutations, we add on the column $\begin{pmatrix} 1 \\ \frac{1}{n} + e^{-it} \end{pmatrix}$. For δ and its permutations, we add on the column $\begin{pmatrix} 1 \\ n+1 \end{pmatrix}$.

It remains to look at the images of generators from $\pi_1(F_{n+1})$. Recall that F_{n+1} is the complement of a central line arrangement. Pick a coordinate system so that one of the lines is $\mathrm{Var}(x_1)$. Consider the Hopf bundle $h : \mathbb{C}^2 \to \mathbb{CP}^1$ with fiber $\mathbb{C}^*$. Note that the h restricted to the $\mathbb{C}^2 \setminus \mathrm{Var}(x_1)$ has image isomorphic to $\mathbb{C}$, therefore $h : \mathbb{C}^2 \setminus \mathrm{Var}(x_1) \to \mathbb{C}$ is a trivial bundle. Now, if we restrict h further to F_{n+1}, we see that its image is isomorphic to $\mathbb{C}$ with $n-1$ points removed. So we have that F_{n+1} is homotopy equivalent to $(\mathbb{C} \setminus (n-1) \text{ points}) \times \mathbb{C}^*$ which has homotopy type $\left(\bigvee_{n-1} S^1\right) \times S^1$. Therefore $\pi_1(F_{n+1})$ is generated by the meridians around $n-1$ lines, and a loop around the origin.

The image of the meridian around the line generated by the first column is homotopic to the loop:

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & e^{it} \\ 1 & \frac{1}{2} + e^{-it} & \frac{1}{3} + e^{-it} & \cdots & \frac{1}{n} + e^{-it} & 0 \end{pmatrix}$$

for $t \in [0, 2\pi]$. The $(1, n+1)$-minor of this loop is e^{it}, thus it is a meridian to the subvariety $x_1 y_{n+1} - x_{n+1} y_1 = 0$. For $2 \le j \le n$, the $(j, n+1)$-minor is $\frac{1}{j}e^{it} - 1$, and all other minors are constant, thus this loop contracts to a point in the complements of the subvarieties $x_j y_k - x_k y_j = 0$ for $j, k \ne 1, (n+1)$.

The images of the meridians around other lines are simply the loop above with its columns permuted, so these loops are permutations of γ in $\mathbb{C}^{2(n+1)}$

The image of a loop around the origin is homotopic to the loop:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & e^{it} \\ 2 & 3 & \cdots & n+1 & e^{it} \end{pmatrix}$$

for $t \in [0, 2\pi]$, so this loop is a permutation of δ in $\mathbb{C}^{2(n+1)}$. ■

Theorem 4.2.2 *Let G be the complete graph on n vertices. Let $U_n = \mathbb{C}^{2n} \setminus \mathcal{A}_G$, then*

$$\mathrm{Poin}(U_n, t) = (1+t^3)(1+t)^{n-1} \prod_{k=1}^{n-2} (1+kt).$$

Proof We proceed by induction on n. For the base case $n = 2$, the complement U_2 is $\mathrm{GL}(2,\mathbb{C})$. The fibration $p_2 : U_2 \to \mathbb{C}^2 \setminus \{0\}$, where p_2 is the projection onto the first column of a matrix in $\mathrm{GL}(2,\mathbb{C})$, with fibers homotopy equivalent to $\mathbb{C}^2$ minus a line. The base space $\mathbb{C}^2 \setminus \{0\}$ is homotopy equivalent to S^3, and the fiber is homotopy equivalent to S^1. Considering the cohomology Serre spectral sequence,

$$E_2^{p,q} \cong H^p(S^3, H^q(S^1)),$$

we do not have to worry about local coefficients, because S^3 is simply connected. Since the target for $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is always zero for $r \geq 2$, the spectral sequence collapses at the E_2-page. Thus,

$$\mathrm{Poin}(U_2, t) = \mathrm{Poin}(S^3, t) \cdot \mathrm{Poin}(S^1, t) = (1 + t^3)(1 + t).$$

Similarly, the fibration $p_n : U_{n+1} \to U_n$, where p_n is the projection onto the first n columns, with fiber F_n homotopy equivalent to $\mathbb{C}^2$ minus n lines. The cohomology Serre spectral sequence gives us

$$E_2^{p,q} \cong H^p(U_n, \mathcal{H}^q(F_n)) \Rightarrow H^{p+q}(U_{n+1}). \tag{4.3}$$

To show that we can use constant coefficients again, we show that the action of the fundamental group of the base on the homology of the fiber is the identity. Consider the loops γ and δ as defined in Lemma 4.2.1. We can permute the columns of γ and δ to get all of the generators of $\pi_1(U_n)$, thus it is enough to understand how these two loops act on the cohomology of the fiber. Since our fiber is the complement of a central arrangement of lines, elements of $H^1(F)$ generate $H^2(F)$ via the cup product [17]. Hence it is enough to understand how γ acts on $H^1(F)$.

Now, denote the columns of γ by v_j for $j = 1, \ldots, n$. Our fiber is $\mathbb{C}^2 \setminus \bigcup_{j=1}^{n} \mathrm{span}(v_j)$. We can consider the loops in the fiber given by $\alpha_1 = v_1 + \varepsilon \begin{pmatrix} 0 \\ e^{i\theta} \end{pmatrix}$ and $\alpha_j = v_j + \varepsilon \begin{pmatrix} e^{i\theta} \\ 0 \end{pmatrix}$ for $j \geq 2$ and $0 \leq \theta \leq 2\pi$. For ε sufficiently small, the loops α_j are meridians to the lines $\mathbb{C}v_j$. These meridians can be contracted in the complements $\mathbb{C}^2 \setminus \mathbb{C}v_k$ for $k \neq j$, therefore they generate H^1.

Since γ and δ are globally defined on U_n and since α_j at the start and end points of the loops are the same, the action of γ on $H^1(F)$ is the identity. Thus, in equation (4.3), $E_2^{p,q} \cong H^p(U_n, H^q(F))$.

Since $\mathrm{Var}(f)$ has $\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ components, $\dim(H^1(U_{n+1})) = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \frac{n(n+1)}{2}$. Now,

$$\dim(E_\infty^{1,0}) + \dim(E_\infty^{0,1}) = \dim H^1(U_{n+1}) = \frac{n(n+1)}{2}.$$

Note that, $E_r^{1,0}$ is not the target of d_r for any r, therefore $E_2^{1,0} \cong E_3^{1,0} \cong \cdots \cong E_\infty^{1,0}$. Using the induction hypothesis, we can calculate $\dim(E_\infty^{1,0})$ to be the coefficient of t in $\mathrm{Poin}(U_n, t)$, thus

$$\dim(E_\infty^{1,0}) = (n-1) + \sum_{k=1}^{n-2} k = \frac{(n-1)n}{2}.$$

To compute the Poincaré polynomial for F, we use Theorem 2.3.2. Note that the module of logarithmic derivations for a central line arrangement is free with a basis consisting of the Euler vector field (which has degree 1), and another of vector field of degree $n-1$ (by Saito's criterion). Thus $\mathrm{Poin}(F,t) = (1+t)(1+(n-1)t)$, which implies that $\dim(E_2^{0,1}) = n$.

Now,

$$\frac{n(n+1)}{2} = \dim(E_\infty^{1,0}) + \dim(E_\infty^{0,1}) \leq \dim(E_\infty^{1,0}) + \dim(E_2^{0,1}) = \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2},$$

thus we must have $\dim(E_\infty^{0,1}) = \dim(E_2^{0,1})$, and hence $d_r(E_r^{0,1}) = 0$, for all $r \geq 2$.

Since elements of $H^1(F)$ generate $H^2(F)$ and differentials on cup products are derivations, $d_r(E_r^{0,2}) = 0$ for all $r \geq 2$. Any element of $E_2^{p,q}$ can be written as a linear combination of products of $\alpha \in E_2^{p,0}$ and $\beta \in E_2^{0,q}$, hence $d_2(\alpha\beta) = \beta d_2(\alpha) + \alpha d_2(\beta) = 0$. Inductively, $d_r = 0$ for $r \geq 2$, thus $E_2^{p,q} \cong E_\infty^{p,q}$. Furthermore,

$$\begin{aligned} \mathrm{Poin}(U_{n+1}, t) &= \mathrm{Poin}(U_n, t) \cdot \mathrm{Poin}(F, t) \\ &= \left((1+t^3)(1+t)^{n-1} \prod_{k=1}^{n-2} (1+kt) \right) ((1+t)(1+(n-1)t)) \\ &= (1+t^3)(1+t)^n \prod_{k=1}^{n-1} (1+kt). \end{aligned}$$

■

Following the same proof as in Theorem 4.2.2 and using the fibration described earlier in this chapter we have the following result:

Theorem 4.2.3 *Let G be a chordal graph, then Poincaré polynomial of $U = \mathbb{C}^{2n} \setminus \mathcal{A}_G$ factors over $\mathbb{Q}$ into a product of a cubic with $2|\mathcal{A}_G| - 3$ linear terms.*

REFERENCES

REFERENCES

[1] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[2] E. J. N. Looijenga. *Isolated singular points on complete intersections*, volume 77 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1984.

[3] J. W. Bruce. Functions on discriminants. *J. London Math. Soc. (2)*, 30(3):551–567, 1984.

[4] V. M. Zakalyukin. Reconstructions of fronts and caustics depending on a parameter, and versality of mappings. In *Current problems in mathematics, Vol. 22*, Itogi Nauki i Tekhniki, pages 56–93. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983.

[5] D. van Straten. A note on the discriminant of a space curve. *Manuscripta Math.*, 87(2):167–177, 1995.

[6] Hiroaki Terao. Discriminant of a holomorphic map and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 30(2):379–391, 1983.

[7] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[8] Hiroaki Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Invent. Math.*, 63(1):159–179, 1981.

[9] Hal Schenck and Ştefan O. Tohăneanu. Freeness of conic-line arrangements in $\mathbb{P}^2$. *Comment. Math. Helv.*, 84(2):235–258, 2009.

[10] Ragnar-Olaf Buchweitz and David Mond. Linear free divisors and quiver representations. In *Singularities and computer algebra*, volume 324 of *London Math. Soc. Lecture Note Ser.*, pages 41–77. Cambridge Univ. Press, Cambridge, 2006.

[11] James Damon and Brian Pike. Solvable group representations and free divisors whose complements are $K(\pi,1)$'s. *Topology Appl.*, 159(2):437–449, 2012.

[12] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs. *Pacific J. Math.*, 15:835–855, 1965.

[13] R. P. Stanley. Supersolvable lattices. *Algebra Universalis*, 2:197–217, 1972.

[14] Joseph P. S. Kung and Hal Schenck. Derivation modules of orthogonal duals of hyperplane arrangements. *J. Algebraic Combin.*, 24(3):253–262, 2006.

[15] Michel Granger, David Mond, Alicia Nieto-Reyes, and Mathias Schulze. Linear free divisors and the global logarithmic comparison theorem. *Ann. Inst. Fourier (Grenoble)*, 59(2):811–850, 2009.

[16] Lauren L. Rose and Hiroaki Terao. A free resolution of the module of logarithmic forms of a generic arrangement. *J. Algebra*, 136(2):376–400, 1991.

[17] Egbert Brieskorn. Sur les groupes de tresses [d'après V. I. Arnol′d]. In *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401*, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973.

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