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Codes from adjacency matrices of uniform subset graphs

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Abstract Studies of the p-ary codes from the adjacency matrices of uniform subset graphs $\Gamma(n,k,r)$ and their reflexive associates have shown that a particular family of codes defined on the subsets are intimately related to the codes from these graphs. We describe these codes here and examine their relation to some particular classes of uniform subset graphs. In particular we include a complete analysis of the p-ary codes from $\Gamma(n,3,r)$ for $p \geq 5$, thus extending earlier results for $p = 2, 3$.

Keywords Uniform subset graphs · Codes from graphs

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1 Introduction

Studies of codes obtained from the row span over finite fields of adjacency matrices of uniform subset graphs $\Gamma(n,k,r)$ and their reflexive counterparts $R\Gamma(n,k,r)$, for example in [9,14,15,12,11], have shown that a class of codes, the members of which are denoted by W_i, for $0 \leq i \leq k$, and another, W_Π, are present in the ambient space $\mathbb{F}_p^V$ of these codes (where $V = \Omega^{\{k\}}$, the set of subsets of size k of a set Ω of size n, and p is a prime), and closely related to the specific codes from the graphs. Since codes from adjacency matrices of graphs have not been easy to classify, a uniform approach for at least one class of graphs would seem to be desirable.

In this paper we define these W codes for general uniform subset graphs, and obtain some general properties, before we illustrate their use in the p-ary

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codes of the specific case of $\Gamma(n,3,r)$ and $R\Gamma(n,3,r)$, for $p \geq 5$, since the cases $p=2,3$ have already been established in [9,14,15,12,11].

After a general background section, we define the W codes in Section 3, Equations (1), (2), (3), and obtain some further general results about these codes in Sections 4, 5. We then consider their relationship to the codes from the graphs $\Gamma(n,k,r)$ in Section 6, and in particular for $k=3$ in Section 7. Our conclusions for $k=3$, for all primes p, are shown in tables in Sections 8.1 to 8.7. Any computations were done with Magma [3,5].

2 Terminology and background

The notation for designs and codes is as in [2]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{J})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{J}$ is a t-(v,k,λ) design if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The **code $\boldsymbol{C_F(\mathcal{D})}$ of the design** $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the **incidence vector** of $\mathcal{Q}$ by $\boldsymbol{v^{\mathcal{Q}}}$, and if $\mathcal{Q} = \{P\}$ where $P \in \mathcal{P}$, then we will write v^P instead of $v^{\{P\}}$. Thus $C_F(\mathcal{D}) = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. For any $w \in F^{\mathcal{P}}$ and $P \in \mathcal{P}$, $\boldsymbol{w(P)}$ denotes the value of w at P. If $F = \mathbb{F}_p$ then the **$\boldsymbol{p}$-rank** of the design, written **rank$_{\boldsymbol{p}}(\boldsymbol{\mathcal{D}})$**, is the dimension of its code $C_F(\mathcal{D})$; for $F = \mathbb{F}_p$ we usually write $C_p(\mathcal{D})$ for $C_F(\mathcal{D})$.

All the codes here are **linear codes**, and the notation $[n,k,d]_q$ will be used for a q-ary code C of length n, dimension k, and minimum weight d, where the **weight wt($\boldsymbol{v}$)** of a vector v is the number of non-zero coordinate entries. Vectors in a code are also called **words**. The **support**, Supp(v), of a vector v is the set of coordinate positions where the entry in v is non-zero. So $|\mathrm{Supp}(v)| = \mathrm{wt}(v)$. The **dual** code $C^\perp$ is the orthogonal under the standard inner product $(,)$, i.e. $C^\perp = \{v \in F^n \mid (v,c) = 0 \text{ for all } c \in C\}$. The **hull** of a code C is the self-orthogonal code $C \cap C^\perp$. Following [19], a linear code C is LCD (linear with complementary dual) if $\mathrm{Hull}(C) = \{0\}$. The **all-one vector** will be denoted by $\boldsymbol{\jmath}$, and is the vector with all entries equal to 1. If we need to specify the length **m** of the all-one vector, we write $\boldsymbol{\jmath_{\mathbf{m}}}$. A **constant vector** is a non-zero vector in which all the non-zero entries are the same. We call two linear codes **isomorphic** (or permutation isomorphic) if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code C is an isomorphism from C to C. The automorphism group will be denoted by Aut(C), also called the permutation group of C, and denoted by PAut(C) in [13].

The **graphs**, $\Gamma = (V,E)$ with vertex set V and edge set E, discussed here are undirected with no loops, apart from the case where **all** loops are included, in which case the graph is called **reflexive**. The reflexive graph from Γ is denoted by $R\Gamma$. If $x,y \in V$ and x and y are adjacent, we write $x \sim y$, and xy or $[x,y]$ for the **edge** in E that they define. The **set of neighbours** of $x \in V$ is denoted by $N(x)$, and the **valency of** x is $|N(x)|$. Γ is **regular** if all the vertices have the same valency.

An **adjacency matrix** A is a $|V| \times |V|$ symmetric matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices x_i and x_j are adjacent, and $a_{ij} = 0$ otherwise. If $\Gamma = (V, E)$ is a graph with adjacency matrix A then $A + I_{|V|}$ is an adjacency matrix for the reflexive graph $R\Gamma$ from Γ.

The **code** of $\Gamma = (V, E)$ over a finite field F is the row span of an adjacency matrix A over the field F, denoted by $C_F(\Gamma)$ or $C_F(A)$. Similarly for $R\Gamma$. The dimension of the code is the rank of the matrix over F, also written $\mathrm{rank}_p(A)$ if $F = \mathbb{F}_p$, in which case we will speak of the p**-rank** of A or Γ (respectively $R\Gamma$), and write $C_p(\Gamma)$ or $C_p(A)$ for the code (respectively $C_p(R\Gamma)$ or $C_p(A+I_{|V|})$). The ambient space of these codes is $\mathbb{F}_p^V$.

The **uniform subset graph** $\Gamma(n, k, r)$ has for vertices the set of all subsets of size k of a set of size n with two k-subsets x and y defined to be adjacent if $|x \cap y| = r$. Then $\Gamma(n, k, r)$ is regular of valency $\binom{k}{r}\binom{n-k}{k-r}$. The symmetric group S_n always acts on $\Gamma(n, k, r)$, transitively on vertices and edges. Similarly for $R\Gamma(n, k, r)$. We take $n \geq 2k$, and generally $n \geq 2k + 1$.

3 The W codes

We define here the W codes as mentioned, and develop a few of their general properties, but we will restrict mainly to $k = 3$ and mostly leave the further more general study for $k \geq 4$ to a future paper.

Let $\Omega = \{1, \ldots, n\}$, $k \leq \frac{n}{2}$, $V = \Omega^{\{k\}}$. For $0 \leq s \leq n$ let $\Omega^{\{s\}}$ denote the set of s-subsets of Ω. All spans are over $\mathbb{F}_p$ for a given prime p, and all the codes are subcodes of $\mathbb{F}_p^V$, where we fix n, k.

For $\Lambda \in \Omega^{\{s\}}$ and $0 \leq s \leq k$, define in $\mathbb{F}_p^V$,

$$w_\Lambda = \sum_{\substack{\Lambda_1 \in \Omega^{\{k-s\}} \\ \Lambda \cap \Lambda_1 = \emptyset}} v^{\Lambda \cup \Lambda_1}, W_s = \langle w_\Lambda \mid \Lambda \in \Omega^{\{s\}} \rangle, E(W_s) = \langle w_{\Lambda_1} - w_{\Lambda_2} \mid \Lambda_i \in \Omega^{\{s\}} \rangle. \tag{1}$$

Identities 1 *For* $\Lambda \in \Omega^{\{s\}}$ *and* $0 \leq s \leq k$,
(1) $w_\emptyset = \boldsymbol{\jmath}$; *(2)* $\mathrm{wt}(w_\Lambda) = \binom{n-s}{k-s}$; *(3) if* $s = k$, *then* $w_\Lambda = v^\Lambda$, $W_k = \mathbb{F}_p^V$ *and* $E(W_k) = \langle \boldsymbol{\jmath} \rangle^\perp$;
(4) $(k - s)w_\Lambda = \sum_{a \in \Omega \setminus \Lambda} w_{\Lambda \cup \{a\}}$ *for* $0 \leq s \leq k - 1$; *(5)* $\sum_{\Lambda \in \Omega^{\{s\}}} w_\Lambda = \binom{k}{s}\boldsymbol{\jmath}$ *for* $1 \leq s \leq k$;
(6) if $u \in W_s$, $u = \sum_{\Lambda \in \Omega^{\{s\}}} \alpha_\Lambda w_\Lambda$, *then* $u(x) = \sum_{\Lambda \subset x} \alpha_\Lambda$, *a sum of* $\binom{k}{s}$ *terms* α_Λ;
(7) for all $1 \leq s \leq k - 1$, $E(W_s) \subseteq \langle \boldsymbol{\jmath} \rangle^\perp$, *and if* $W_s = E(W_s)$ *then* $p | \binom{n-s}{k-s}$.

From Identities 1 (4), we see that if $p \nmid (k - s)$ then $W_s \subset W_{s+1}$, and thus if $p \nmid 2, 3, \ldots k - 1$ then

$$\langle \boldsymbol{\jmath} \rangle = W_0 \subset W_1 \subset W_2 \subset \ldots W_{k-1} \subset W_k = \mathbb{F}_p^V,$$

which is true, for example, if $p > k$.

For the code W_{Π} we make use of partitions of subsets of size $2k$ of Ω. For the $2k$-set $\Lambda = \{a_{1,1}a_{1,2}, a_{2,1}, a_{2,2} \ldots, a_{k,1}, a_{k,2}\}$ let

$$[[a_{1,1}a_{1,2}], [a_{2,1}, a_{2,2}], \ldots, [a_{k,1}, a_{k,2}]]$$

be the partition π, then the word w_π, of weight 2^k is given by

$$w_\pi = \sum \pm v^{\{a_{1,i_1}, a_{2,i_2}, \ldots, a_{k,i_k}\}} \tag{2}$$

where all the subsets of Λ of k-sets of the form $\{a_{1,i_1}, a_{2,i_2}, \ldots, a_{k,i_k}\}$ where $i_j \in \{1,2\}$, are present, with the sign being determined by giving $\{a_{1,1}, a_{2,1}, \ldots, a_{k,1}\}$ the sign "+", and then demanding that any other k-set in the support with intersection of size $k-1$ with this set will have sign "−", and then applying this in general to get the signs on all the 2^k vertices. Then

$$W_{\Pi} = \langle w_\pi \mid \pi \text{ a partition of } \Lambda \subset \Omega,\ \ |\Lambda| = 2k\rangle. \tag{3}$$

For example, if $k = 3$, $\Lambda = \{a_1, a_2, b_1, b_2, c_1, c_2\}$, $\pi = [[a_1, a_2], [b_1, b_2], [c_1, c_2]]$, then with

$$X = \{\{a_1, b_1, c_1\}, \{a_1, b_2, c_2\}, \{a_2, b_1, c_2\}, \{a_2, b_2, c_1\}\}$$
$$X^c = \{\{a_2, b_2, c_2\}, \{a_2, b_1, c_1\}, \{a_1, b_2, c_1\}, \{a_1, b_1, c_2\}\},$$

$$w_\pi = \sum_{x \in X} v^x - \sum_{x \in X^c} v^x.$$

Alternatively the words can be defined inductively. For example, from $k = 3$ to $k = 4$, with the extra partition set $[d_1, d_2]$ by adjoining d_1 to all the elements of the sets X and X^c, keeping the same signs, and then do the same with d_2, but switching the signs. Another interpretation takes the 2^k vertices in the support of w_π as the vertices of the k-cube, $\mathcal{Q}_k$, i.e. the Hamming graph $H(k, 2)$, with alternate signs on the vertices.

Lemma 1 *For all n, k, p, for all $1 \le s \le k-1$, $W_{\Pi} \subseteq W_s^{\perp}$.*

Proof: Clear. ■

3.1 W_1

In W_1 we write, for $a \in \Omega$, w_a instead of $w_{\{a\}}$. We have $\mathrm{wt}(w_a) = \binom{n-1}{k-1}$.

Lemma 2 *For all $n \ge 2k$, $\dim(W_1) = n$ if $p \nmid k$ and $\dim(W_1) = n - 1$ if $p | k$.*

Proof: Let $u = \sum_a \alpha_a w_a = 0$. Then for $x = \{a_1, \ldots, a_{k-1}, a_k\}$, $u(x) = \sum_{i=1}^k \alpha_{a_i} = 0$. For $y = \{a_1, \ldots, a_{k-1}, b_k\}$ we have $\sum_{i=1}^{k-1} \alpha_{a_i} + \alpha_{b_k} = 0$, so subtracting gives $\alpha_{a_k} = \alpha_{b_k}$ and hence all the coefficients are the same, $\alpha_a = \alpha$ for all a. So $u = \alpha \sum_a w_a = \alpha k \boldsymbol{\jmath} = 0$, and hence $\alpha k = 0$. If $p \not| k$ then $\alpha = 0$ and the w_a are linearly independent and $\dim(W_1) = n$; if $p|k$ then $\sum_a w_a = 0$, and $\dim(W_1) \leq n-1$ and spanned by the w_i for $i < n$, say. Suppose $u = \sum_{i=1}^n \alpha_i w_i = 0$, where $\alpha_n = 0$. If $x = \{a_1, \ldots, a_{k-1}, a_k\}$ then $u(x) = \sum_{i=1}^k \alpha_{a_i} = 0$. If $a_i \neq n$ for $1 \leq i \leq k$, and $y = \{a_1, \ldots, a_{k-1}, n\}$ then we have $\alpha_{a_k} = 0$ for all a_k and so the w_i for $i < n$ are linearly independent. ■

Lemma 3 *For $p = 2$, $k \geq 2$, $n \geq 2k$, the non-zero words of W_1 have weight n_r for $1 \leq r \leq \lfloor n/2 \rfloor$ where*

$$n_r = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \binom{r}{2i+1} \binom{n-r}{k-(2i+1)} \tag{4}$$

and also of weight $\binom{n}{k} - n_r$ if k is odd. There are $\binom{n}{r}$ of weight n_r.

Proof: Since $\sum_{a \in \Omega} w_a = k\boldsymbol{\jmath}$, we have $\boldsymbol{\jmath} \in W_1$ for k odd. Otherwise $\sum_{a \in \Omega} w_a = 0$, and in either case we need only take up to $\lfloor n/2 \rfloor$ sums of the w_a, as long as we take the differences with $\boldsymbol{\jmath}$ as well, in the odd case.

Arguing exactly as in [14, Lemma 6], taking, for $\Delta = \{a_1, \ldots, a_r\}$ and adding the corresponding vectors w_{a_i}, we have, for $w = \sum_{i=1}^r w_{a_i} \in W_1$,

$$w = \sum_{i=1}^r \Big(\sum_{x_1,\ldots,x_{k-1} \notin \Delta} v^{\{a_i, x_1, \ldots, x_{k-1}\}} + \sum_{i_1 \neq i;} \sum_{x_1,\ldots,x_{k-2} \notin \Delta} v^{\{a_i, a_{i_1}, x_1, \ldots, x_{k-2}\}} +$$

$$\sum_{i_1, i_2 \neq i;} \sum_{x_1,\ldots,x_{k-3} \notin \Delta} v^{\{a_i, a_{i_1}, a_{i_2}, x_1, \ldots, x_{k-3}\}} + \ldots + \sum_{i_1,\ldots,i_{k-1} \neq i} v^{\{a_i, a_{i_1}, \ldots, a_{i_{k-1}}\}} \Big),$$

and counting gives the stated formula. If k is odd then the vectors $\boldsymbol{\jmath} - w$ must also be included. ■

Lemma 4 *For $n \geq 2k$, $k \geq 3$, p prime, the minimum weight of $W_1^\perp$ is 4. For $k = 2$, the minimum weight of $W_1^\perp$ is 4 for $p > 3$, and 3 for $p = 2$.*

Proof: For $k \geq 3$, the word

$$v^{\{1,..,k-2,k-1,k\}} + v^{\{1,..,k-2,k+1,k+2\}} - v^{\{1,..,k-2,k-1,k+1\}} - v^{\{1,..,k-2,k,k+2\}}$$

is clearly in $W_1^\perp$. Obviously there can be no words of weight 1 or 2 in $W_1^\perp$ and that there can be no words of weight 3 follows from a simple argument.

For $k = 2$, the word of weight 4 is $v^{\{k-1,k\}} + v^{\{k+1,k+2\}} - v^{\{k-1,k+1\}} - v^{\{k,k+2\}}$, but for $p = 2$ the word $v^{\{1,2\}} + v^{\{1,3\}} + v^{\{2,3\}}$ is in $W_1^\perp$. ■

Lemma 5 *For $n \geq 5$, $k \geq 2$, $n \geq 2k$, any p, $W_1 = E(W_1)$ if and only if $p|k$ and $p \nmid n$.*

If $k = 3$ and $p \geq 5$, then $E(W_1) \subset W_1$.

Proof: Recall that $k\jmath = \sum_{i=1}^n w_i$.

Suppose $p|k$. Then $w_1 = -\sum_{i=2}^n w_i = \sum_{i=2}^n (w_1 - w_i) - (n-1)w_1$. Thus $nw_1 = \sum_{i=2}^n (w_1 - w_i)$, so if $p \nmid n$ then $w_1 \in E(W_1)$, so $W_1 = E(W_1)$.

Suppose $w_1 \in E(W_1)$. Then $w_1 = \sum_{i=2}^n \alpha_i(w_1 - w_i)$, for some $\alpha_i \in \mathbb{F}_p$, and thus $u = (\sum_{i=2}^n \alpha_i - 1)w_1 - \sum_{i=2}^n \alpha_i w_i = 0$. Thus $u(x) = 0$ for every $x \in V$. For $x = \{1, a_2, \ldots, a_{k-1}, a_k\}$ and $y = \{1, a_2, \ldots, a_{k-1}, b_k\}$ this gives

$$(\sum_{i=2}^n \alpha_i - 1) - \sum_{i \in \{a_2, \ldots, a_k\}} \alpha_i = 0 \text{ and } (\sum_{i=2}^n \alpha_i - 1) - \sum_{i \in \{a_2, \ldots, b_k\}} \alpha_i = 0.$$

Thus $\alpha_i = \alpha$, a constant for all $i \geq 2$, and $w_1 = \alpha \sum_{i=2}^n (w_1 - w_i)$ where we assume $\alpha \neq 0$.

Thus $u = ((n-1)\alpha - 1)w_1 - \alpha \sum_{i=2}^n w_i = ((n-1)\alpha - 1)w_1 - \alpha(\sum_{i=1}^n w_i - w_1) = ((n-1)\alpha - 1)w_1 - \alpha k\jmath + \alpha w_1 = 0$, and so $(n\alpha - 1)w_1 = \alpha k\jmath$. Both sides of this equation must be zero, so $p|k$ and $n = \alpha^{-1}$, so that $p \nmid n$.

The last statement follows. $\square$

3.2 W_2

In W_2 we write, for $a, b \in \Omega$, $w_{a,b}$ for $w_{\{a,b\}}$. Recall that $\mathrm{wt}(w_{a,b}) = \binom{n-2}{k-2}$.

Lemma 6 *For $k = 3$, all $n \geq 6$, if $p > 3$ then W_2 is a $[\binom{n}{3}, \binom{n}{2}, n-2]_p$ code and for $n \geq 8$ any word of weight $n-2$ is a scalar multiple of $w_{a,b}$ for some $a, b \in \Omega$. For $n = 7$ and $p \geq 3$, the word with support*

$$\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{5,6,7\}\}$$

is in W_2.

Proof: For $k = 3$, if $u = \sum_{a,b} \alpha_{a,b} w_{a,b}$, then $u(\{a,b,c\}) = \alpha_{a,b} + \alpha_{a,c} + \alpha_{b,c}$.

If $u = 0$ then $\alpha_{a,b} = -(\alpha_{a,c} + \alpha_{b,c}) = -(\alpha_{a,d} + \alpha_{b,d})$, and thus $2\alpha_{a,b} = -(\alpha_{a,c} + \alpha_{b,c} + \alpha_{a,d} + \alpha_{b,d}) = -(-\alpha_{c,d} - \alpha_{c,d}) = 2\alpha_{c,d}$. So if $p \neq 2$, $\alpha_{a,b} = \alpha_{c,d} = \alpha_{a,e}$ for all a, b, c, d, e, and $\alpha_{a,b} = \alpha$ a constant.

Thus $u = \sum_{a,b} \alpha w_{a,b} = \binom{k}{2}\alpha\jmath = 3\alpha\jmath$ since $k = 3$, and thus if $p \neq 3$, $u = 0$ only if $\alpha = 0$. So $\dim(W_2) = \binom{n}{2}$ as asserted.

For the statement about the minimum weight, the proofs of the minimum weight and support of Propositions 4,5 of [15] hold for all odd p, as can be checked. For $n = 7$ and $p \geq 3$, it can be verified that

$$w_{1,2} + w_{1,3} + w_{1,4} + w_{2,3} + w_{2,4} + w_{3,4} + w_{5,6} + w_{5,7} + w_{6,7} - \jmath$$
$$= 2(v^{\{1,2,3\}} + v^{\{1,2,4\}} + v^{\{1,3,4\}} + v^{\{2,3,4\}} + v^{\{5,6,7\}}),$$

of weight $5 = n - 2$. $\blacksquare$

Note 1 For $k = 3$, $n \geq 7$, $p = 2, 3$ are covered in [12,11,14,15]. For $n = 6$, for $p = 2$, [12, Lemma 1] holds also for $n = 6$, giving the dimension as $\binom{n-1}{2} = 10$. For $p = 3$, the proof of Lemma 6 above shows that $\dim(W_2) = \binom{n}{2} - 1$.

Lemma 7 *For $n \geq 2k$, $k \geq 4$, and any prime p:*

$$dim(W_2) = \begin{cases} \binom{n}{2} \ \textit{if}\ p \nmid (k-1),\ p \nmid \binom{k}{2}; \\ \binom{n}{2} - 1 \ \textit{if}\ p \nmid (k-1),\ p | \binom{k}{2}; \\ \binom{n}{2} - n \ \textit{if}\ p | (k-1),\ p | \binom{k}{2}; \\ \binom{n}{2} - n + 1 \ \textit{if}\ p | (k-1),\ p \nmid \binom{k}{2},\ \textit{i.e.}\ p = 2, k \equiv 3 \pmod 4. \end{cases}$$

Proof: (1): Let $p \nmid (k-1)$.
Suppose $u = \sum_{a,b} \alpha_{a,b} w_{a,b} = 0$. Then for $x \in V$, $x = \{a_1, \ldots, a_k\}$, $u(x) = \sum_{a,b \in x} \alpha_{a,b} = 0$. For $y = \{a_1, \ldots, a_s\} \in \Omega^{\{s\}}$, any $s \leq k$, $a \notin y$, let $f_y(a) = \sum_{i=1}^{s} \alpha_{a,a_i}$.

Now take $s = k - 1$, $x = y \cup \{a\}$. Then

$$u(x) = 0 = \sum_{b,c \in y} \alpha_{b,c} + f_y(a),$$

so $f_y(a)$, depends only on y and not on a, so we write it now as f_y. Let $z = \{b_1, \ldots, b_{k-1}\}$ be such that $y \cap z = \emptyset$. Then $f_z = \sum_{i=1}^{k-1} \alpha_{a_j,b_i}$ for $j = 1, \ldots, k-1$. Summing these $k-1$ equations gives $(k-1)f_z = (k-1)f_y$ and hence $f_z = f_y$ if $p \nmid (k-1)$. Thus in this case $f_y = \beta$, a constant for all $(k-1)$-subsets y. Taking now $z = \{a_1, \ldots, a_{k-2}, b\}$, for $b \neq a_{k-1}$, and $a \notin y, z$, we have

$$\beta = \sum_{i=1}^{k-2} \alpha_{a,a_i} + \alpha_{a,a_{k-1}} = \sum_{i=1}^{k-2} \alpha_{a,a_i} + \alpha_{a,b},$$

so that $\alpha_{a,a_{k-1}} = \alpha_{a,b}$, and hence $\alpha_{i,j} = \alpha$, a constant for all i, j.

Thus $u = \alpha \sum_{a,b} w_{a,b} = \alpha \binom{k}{2} \boldsymbol{\jmath} = 0$, so if $p \nmid \binom{k}{2}$ then $\alpha = 0$ and $\dim(W_2) = \binom{n}{2}$. If $p | \binom{k}{2}$ but $p \nmid (k-1)$ then $\dim(W_2) = \binom{n}{2} - 1$.
(2): Let $p | (k-1)$.
Then $\sum_{i=2}^{n} w_{1,i} = 0$, so $W_2 = \langle w_{a,b} \mid a, b \in \Omega \setminus \{n\} \rangle$, and $\dim(W_2) \leq \binom{n-1}{2}$. Since $\sum_{a,b} w_{a,b} = \binom{k}{2} \boldsymbol{\jmath}$, if $p | \binom{k}{2}$ then $\sum_{a,b} w_{a,b} = \sum_{a,b \neq n} w_{a,b} + \sum_{a,n} w_{a,n} = 0$, so since $\sum_{a,n} w_{a,n} = 0$, the $\{w_{a,b} \mid a, b \neq n\}$ are not linearly independent. We show that removing another arbitrary $w_{a,b}$, say $w_{n-2,n-1}$ would produce a linearly independent set.

Thus assuming $p | (k-1)$ and $p | \binom{k}{2}$, let $u = \sum_{a,b} \alpha_{a,b} w_{a,b} = 0$, where $\alpha_{i,n} = \alpha_{n-1,n-2} = 0$ for all i. Then if $x = \{1, \ldots, k\}$, $u(x) = \sum_{1 \leq i,j \leq k} \alpha_{i,j} = 0$. Similarly, for $y = \{1, \ldots, k-1, n\}$ we have $\sum_{1 \leq i,j \leq k-1} \alpha_{i,j} = 0$, so for any $(k-1)$-set z not containing n, we have $\sum_{a,b \in z} \alpha_{a,b} = 0$. This also implies that $\sum_{i=1}^{k-1} \alpha_{i,k} = 0$, i.e. in the notation above, for any $(k-1)$-set z not containing n, $f_z(a) = 0$ for all $a \notin z$. Thus if $z_1 = \{a_1, \ldots a_{k-2}, a\}$ and $z_2 = \{a_1, \ldots a_{k-2}, b\}$, none of the a_i, a, b equal to n, then $\sum_{i=1}^{k-2} \alpha_{a_i,b} + \alpha_{a,b} =$

$0 = \sum_{i=1}^{k-2} \alpha_{a_i,b} + \sum_{i,j} \alpha_{a_i,a_j}$, so that $\alpha_{a,b} = \sum_{i,j} \alpha_{a_i,a_j}$, and is thus a constant over $\Omega \setminus \{n\}$. Since $\alpha_{n-2,n-1} = 0$ this shows that $\alpha_{a,b} = 0$ for all $a, b \in \Omega \setminus \{n\}$, and thus $\dim(W_2) = \binom{n}{2} - n$.

If $p \nmid \binom{k}{2}$ then clearly $p = 2$ and $k \equiv 3 \pmod 4$. Let $u = \sum_{a,b} \alpha_{a,b} w_{a,b} = 0$, where $\alpha_{i,n} = 0$ for all i. The identical argument to the above yields that $\alpha_{a,b} = \alpha$ is a constant. Thus $u = \alpha \sum_{a,b \neq n} w_{a,b} = 0$, and since $w = \sum_{a,b \neq n} w_{a,b}$ has $w(x) = 1$ if $n \notin x$, $w \neq 0$, so $\alpha = 0$ and we have the stated result. ■

Lemma 8 *For $n \geq 7$, $k \geq 3$, $n \geq 2k$, any p, $W_2 = E(W_2)$ if and only if*

$$p \Big| \binom{k}{2} \text{ and } p \nmid \binom{n}{2}, \quad \text{or,} \quad p|(k-1) \text{ and } p \nmid (n-1).$$

For $k = 3$, $p \geq 5$, $E(W_2) \neq W_2$, and if $n \geq 9$ the minimum weight of $E(W_2)$ is $2(n-3)$.

Proof: Suppose $W_2 = E(W_2)$. Then $w_{1,2} = \sum_{\{c,d\} \neq \{1,2\}} \alpha_{c,d}(w_{1,2} - w_{c,d})$, for some $\alpha_{c,d} \in \mathbb{F}_p$. Thus

$$u = w_{1,2}(1 - \sum_{c,d} \alpha_{c,d}) + \sum_{c,d} \alpha_{c,d} w_{c,d} = \sum_{a,b} \beta_{a,b} w_{a,b} = 0$$

where $\beta_{1,2} = 1 - \sum_{c,d} \alpha_{c,d}$, and $\beta_{c,d} = \alpha_{c,d}$ otherwise.

From the proof of Lemma 7, if $p \nmid (k-1)$ then $\beta_{c,d} = \beta$, a constant for all $\{c, d\}$. Thus $(1 - (\binom{n}{2} - 1)\beta) = \beta$, so that $\binom{n}{2}\beta = 1$. Thus $\binom{n}{2} = \beta^{-1} \neq 0$. We have $u = \beta \sum_{a,b} w_{a,b} = \beta\binom{k}{2}\boldsymbol{\jmath} = 0$, so we must have $p|\binom{k}{2}$. Here we have, conversely, that if $p|\binom{k}{2}$ then $\sum_{a,b} w_{a,b} = \binom{k}{2}\boldsymbol{\jmath} = 0$, so $\sum_{a,b} w_{a,b} = 0$ and $w_{1,2} = -\sum_{\{a,b\} \neq \{1,2\}} w_{a,b} = \sum_{\{a,b\} \neq \{1,2\}}(w_{1,2} - w_{a,b}) - (\binom{n}{2} - 1)w_{1,2}$, so $\binom{n}{2}w_{1,2} = \sum_{\{a,b\} \neq \{1,2\}}(w_{1,2} - w_{a,b})$ and thus if $p \nmid \binom{n}{2}$, and $p|\binom{k}{2}$, then $W_2 = E(W_2)$.

If $p|(k-1)$ then $\sum_{i=2}^{n} w_{1,i} = 0$, so $w_{1,2} = -\sum_{i=3}^{n} w_{1,i} = \sum_{i=3}^{n}(w_{1,2} - w_{1,i}) - (n-2)w_{1,2}$. Thus $(n-1)w_{1,2} = \sum_{i=3}^{n}(w_{1,2} - w_{1,i})$, so $w_{1,2} \in E(W_2)$ if $p \nmid (n-1)$.

Finally we show that if $p|(k-1)$ and $p|(n-1)$ then $W_2 \neq E(W_2)$. We can assume that $W_2 = \langle w_{a,b} \mid a, b < n \rangle$. Thus if $w_{1,2} = \sum_{\{c,d\} \neq \{1,2\}} \alpha_{c,d}(w_{1,2} - w_{c,d})$, for some $\alpha_{c,d} \in \mathbb{F}_p$, we take $\alpha_{c,d} = 0$ if c or d is n. Thus

$$u = w_{1,2}(1 - \sum_{c,d} \alpha_{c,d}) + \sum_{c,d} \alpha_{c,d} w_{c,d} = \sum_{a,b} \beta_{a,b} w_{a,b} = 0$$

where $\beta_{1,2} = 1 - \sum_{c,d} \alpha_{c,d}$, and $\beta_{c,d} = \alpha_{c,d}$ otherwise, and $\beta_{a,n} = 0$ for all $1 \leq a \leq n-1$. As in the proof of Lemma 7, we have $f_y(a) = \sum_{i=1}^{k-1} \beta_{a,a_i}$, where $y = \{a_1, \ldots, a_{k-1}\}$, is independent of a, so taking $a = n$ gives $f_y = 0$ for any y not containing n. So inside $\Omega \setminus \{n\}$, taking $z = \{a_1, \ldots, a_{k-2}, b_{k-1}\}$, and $a \notin y, z$ we have $f_y(a) = f_z(a) = 0$, so $\beta_{a,a_k} = \beta_{a,b_k}$, and it follows that $\beta_{a,b} = \beta$ for all $a, b \in \Omega \setminus \{n\}$. Thus $\beta_{1,2} = 1 - \sum_{c,d} \beta = \beta$ where c, d ranges over

$\Omega \setminus \{n\}$, and $\{c,d\} \neq \{1,2\}$. Thus $(1-\beta(\binom{n}{2}-n)) = \beta$, so $1 = (n-1)\beta(\frac{n}{2}-1)$, which is not possible if $p|(n-1)$. So here $W_2 \neq E(W_2)$.

If $k = 3$ and $p \geq 5$ then clearly we have $W_2 \neq E(W_2)$. The statement about the minimum weight follows from [15, Proposition 6], since that proof is purely combinatorial and does not rely on the value of p. ■

Lemma 9 *For all p, all $k \geq 2$, and $n \geq 2k+1$, $\dim(W_{k-1}) \geq \binom{n-1}{k-1}$. For $p = 2$ we have equality.*

Proof: It is easy to see that $S = \{w_\Lambda \mid |\Lambda| = k-1, n \notin \Lambda\}$ is a linearly independent set for any p, and thus $\dim(W_{k-1}) \geq \binom{n-1}{k-1}$. For $p = 2$, for any $\Lambda \in \Omega^{\{k-2\}}$, by Identities 1(4), $(k-(k-2))w_\Lambda = \sum_{a \notin \Lambda} w_{\Lambda \cup \{a\}} = 0$, so $W_{k-1} = \langle w_\Lambda \mid |\Lambda| = k-1, n \notin \Lambda \rangle$, and thus $\dim(W_{k-1}) \leq \binom{n-1}{k-1}$, so that we have equality. ■

4 W_Π

In [14, Lemma 10] for $k = 3$ a list of $\binom{n}{3} - \binom{n}{2}$ words w_π were given such that with a specified ordering of the vertices of the graph (i.e. the triples), a matrix was produced in upper-triangular form with no zero elements on the diagonal. An analogous choice can be made for $k = 2$ to get a matrix in upper triangular form: see [8, Lemma 5.2.7]. Thus over any field the span of this matrix for $k = 2, 3$ has rank at least $\binom{n}{k} - \binom{n}{k-1}$, and thus we can assert:

Result 1 *For $n \geq 5$ for $k = 2$, and for $n \geq 7$ and $k = 3$, any prime p, $\dim(W_\Pi) \geq \binom{n}{k} - \binom{n}{k-1}$.*

This gives:

Lemma 10 *For $n \geq 7$, $k = 3$, and any prime p, $\dim(W_\Pi) = \binom{n}{3} - \binom{n}{2}$, and $W_2 = W_\Pi^\perp$. For $n \geq 5$, $k = 2$, and any prime p, $\dim(W_\Pi) = \binom{n}{2} - \binom{n}{1}$; if $p > 2$ then $W_\Pi = W_1^\perp$, and if $p = 2$, $W_\Pi \subset W_1^\perp$.*

Proof: For $k = 3$ and $p = 2, 3$ this is proved in [12,11]. For $p \geq 5$, from Lemma 6, we have $\dim(W_2) = \binom{n}{2}$. Now $W_2 \subseteq W_\Pi^\perp$, so $\dim(W_\Pi^\perp) \geq \binom{n}{2}$. Now from Result 1 we have $\dim(W_\Pi^\perp) \leq \binom{n}{2}$, and thus we have equality. A similar argument works for $k = 2$ by Lemma 2. ■

As in the case of $k = 3$ (see [12,11,14,15]), we have the following:

Proposition 1 *For $n \geq 2k+1$, any prime p, and any $k \geq 2$, if C is a code such that $W_\Pi \subseteq C \subseteq \mathbb{F}_p^V$, where $|V| = \binom{n}{k}$, then $C^\perp$ has minimum weight at least $n-k+1$.*

Proof: Let $\mathcal{B} = \{\mathrm{Supp}(w_\pi) \mid w_\pi \in W_\Pi\}$. Then $\mathcal{D} = (V, \mathcal{B})$ is a 1-$(\binom{n}{k}, 2^k, r)$ design where r is the number of blocks of $\mathcal{B}$ through a point. We have $r = \binom{n-k}{k}k!$, since for $x \in V$ there are $\binom{n-k}{k}$ ways of choosing a k-set to complete to a $2k$-set, and there are then $k!$ ways to choose the partition such that x is in the support.

The number of blocks of $\mathcal{B}$ that contain two distinct points x, y can have k values, depending on the size of $|x \cap y|$. For a fixed point x, for $0 \leq i \leq k-1$, if $y \neq x$, and $|x \cap y| = i$, let n_i denote the number of blocks of $\mathcal{B}$ that contain both x and y. Then counting yields that $n_i = \binom{n-2k+i}{i} i!(k-i)!$. For a fixed x we say a point y is of type-i if $|x \cap y| = i$.

Now let $S = \mathrm{Supp}(w)$ where $w \in C^{\perp}$, and $|S| = s$. Let $x \in S$. Let z_i, for $i = 0$ to 2^k be the number of blocks in $\mathcal{B}$ that contain x and meet S in i points. Then $z_0 = z_1 = 0$, and $\sum_{i=2}^{2^k} z_i = r$. For $0 \leq i \leq k-1$, suppose there are ℓ_i points in $S \setminus \{x\}$ of type-i. Then counting gives

$$\sum_{i=2}^{2^k} (i-1) z_i = \sum_{i=0}^{k-1} \ell_i n_i,$$

and $s - 1 = \sum_{i=0}^{k-1} \ell_i$.

For $n \geq 2k+1$, it is easy to prove directly that $n_{k-1} \geq n_i$ for $0 \leq i \leq k-1$. Thus we have

$$r \leq \sum_{i=2}^{2^k} (i-1) z_i = \sum_{i=0}^{k-1} \ell_i n_i \leq \sum_{i=0}^{k-1} \ell_i n_{k-1} = (s-1) n_{k-1}. \tag{5}$$

Since $r = (n-k) n_{k-1}$, this gives $n - k \leq s - 1$, i.e. $s \geq n - k + 1$. ■

Corollary 1 *For $n \geq 2k+1$, any $k \geq 2$, p prime, for $1 \leq s \leq k-1$, W_s has minimum weight at least $n-k+1$ and W_{k-1} has minimum weight $n-k+1$. For $n > 2k+1$, the minimum words of W_{k-1} are the scalar multiples of the w_{Λ} for $\Lambda \subset \Omega$ with $|\Lambda| = k-1$.*

Proof: Since $W_{\Pi} \subseteq W_s^{\perp}$ for $1 \leq s \leq k-1$, we have that W_s has minimum weight at least $n-k+1$ by Proposition 1. For $s = k-1 = |\Lambda|$, $\mathrm{wt}(w_{\Lambda}) = n-k+1$, so the minimum weight of W_{k-1} is $n-k+1$.

Let $w \in W_{k-1}$ have weight $n-k+1$ and support S. With the same notation as in the proposition, and writing $\ell_{k-1} = n - k - \sum_{i=0}^{k-2} \ell_i$, we have

$$r = (n-k) n_{k-1} \leq \sum_{i=0}^{k-2} n_i \ell_i + n_{k-1}(n - k - \sum_{i=0}^{k-2} \ell_i),$$

so that

$$r \leq \sum_{i=0}^{k-2} (n_i - n_{k-1}) \ell_i + r.$$

Now $n_{k-1} \geq n_i$ for $0 \leq i \leq k-2$, with possible equality only if $n = 2k+1$. Thus for $n > 2k+1$ we have $\ell_i = 0$ for $0 \leq i \leq k-2$. Thus $x \in S$ has $k-1$ elements in common with any other y in S, and this is true for all pairs of points.

Suppose $x = \{a_1, \dots, a_{k-1}, a_k\} \in S$. Let $y = \{a_1, \dots, a_{k-1}, a_{k+1}\} \in S$. If $\mathrm{Supp}(w) \neq \mathrm{Supp}(w_{\{a_1,\dots,a_{k-1}\}})$ and if $z \in S$ contains a further element a_{k+2} of Ω, but is not $\{a_1, \dots, a_{k-1}, a_{k+2}\}$, then z contains a_k, a_{k+1}, a_{k+2} and hence cannot satisfy that it intersects x, y in $k-1$ elements. Thus z contains only elements from $\{a_1, \dots, a_{k+1}\}$, and we can obtain at most $k+1$ such points. Since $n-k+1 \geq 2k+1-k+1 = k+2$, this is not the support of S. Thus it follows that $\mathrm{Supp}(w) = \mathrm{Supp}(w_{\{a_1,\dots,a_{k-1}\}})$.

Let $\Lambda = \{a_1, \dots, a_{k-1}\}$, and suppose $w = \sum_{a\in\Omega\setminus\Lambda} \alpha_a v^{\Lambda\cup\{a\}}$, where $\alpha_a \neq 0$. By scaling we can take $\alpha_a = 1$ for some a. Then $w - w_\Lambda = \sum_{b\in\Omega\setminus\Lambda}(\alpha_b - 1)v^{\Lambda\cup\{b\}}$ will have non-zero weight less than $n-k+1$ if not all the α_b are 1. Thus w is a scalar multiple of w_Λ. ■

5 Hulls

5.1 Hull(W_1)

Proposition 2 *For $n \geq 7$, $k \geq 2$, p prime, $\mathrm{Hull}(W_1)$ is one of $W_1, E(W_1), \langle \boldsymbol{\jmath} \rangle$ or $\{0\}$. Specifically, writing $n_1 = \binom{n-1}{k-1}$, $n_2 = \binom{n-2}{k-2}$, $H = \mathrm{Hull}(W_1)$,*

1. *if $n_1 \equiv n_2 \pmod p \equiv r \pmod p$, then if $r = 0$, $H = W_1$, and if $r \neq 0$, $H = E(W_1)$;*
2. *if $n_1 \not\equiv n_2 \pmod p$, then*
   - *(a) if $n_2 \equiv 0 \pmod p$ then $H = \{0\}$;*
   - *(b) if $n_1 \equiv 0 \pmod p$ then $H = \{0\}$ if $p|k$, and $H = \langle \boldsymbol{\jmath} \rangle$ if $p \nmid k$;*
   - *(c) if neither $n_1 \equiv 0 \pmod p$ nor $n_2 \equiv 0 \pmod p$, then $H = \{0\}$.*

If $p \geq n$ then $H = \{0\}$.

Proof: For any $a, b \in \Omega$, $(w_a, w_a) = \binom{n-1}{k-1} = n_1$, and $(w_a, w_b) = \binom{n-2}{k-2} = n_2$. Also, $(w_a, \boldsymbol{\jmath}) = n_1$, $\sum_a w_a = k\boldsymbol{\jmath}$, so $\boldsymbol{\jmath} \in W_1^\perp$ if $n_1 \equiv 0 \pmod p$, and $\boldsymbol{\jmath} \in H$ if also $p \nmid k$.

(1) Suppose that $n_1 \equiv n_2 \pmod p \equiv r \pmod p$. So $(w_a, w_b) \equiv r \pmod p$ for all $a, b \in \Omega$, so $w_a - w_b \in H$, and thus $E(W_1) \subseteq H$. If $r \equiv 0 \pmod p$ then $W_1 = H$, and if $r \not\equiv 0 \pmod p$, $E(W_1) = H$.

(2) Suppose that $n_1 \not\equiv n_2 \pmod p$. Let $w = \sum_{a\in\Omega} \alpha_a w_a \in H$. Then for any $b \in \Omega$, $(w, w_b) = \alpha_b n_1 + n_2 \sum_{a\neq b} \alpha_a = \alpha_b(n_1 - n_2) + n_2 \sum_{a\in\Omega} \alpha_a = 0$. Since $n_1 - n_2 \not\equiv 0 \pmod p$, this gives $\alpha_b = \frac{n_2}{n_2 - n_1} \sum_a \alpha_a$, i.e. $\alpha_b = \alpha$, a constant for all b, and $w = \alpha k \boldsymbol{\jmath}$.

(a) If $n_2 \equiv 0 \pmod p$ then $\alpha = 0$, so $H = \{0\}$.

(b) If $n_1 \equiv 0 \pmod p$ then $H = \{0\}$ if $p|k$, and $H = \langle \boldsymbol{\jmath} \rangle$ if $p \nmid k$.

(c) If neither $n_1 \equiv 0 \pmod p$ nor $n_2 \equiv 0 \pmod p$, then $(w, w_a) = \alpha k(w_a, \boldsymbol{\jmath}) = \alpha k n_1 = 0$, so $\alpha k = 0$ and $w = 0$.

Finally, if $p \geq n$, then (c) holds. ■

Corollary 2 *If $n \geq 7$, $k = 3$, $p \geq 5$, then, with $H = \mathrm{Hull}(W_1)$, if $n \equiv 1 \pmod p$ then $H = \langle \jmath \rangle$; if $n \equiv 2 \pmod p$ then $H = W_1$; if $n \not\equiv 1, 2 \pmod p$ then $H = \{0\}$.*

5.2 Hull(W_2)

For W_2 we have, for distinct $a, b, c, d \in \Omega$, $k \geq 3$:

$$(w_{a,b}, w_{a,b}) = \binom{n-2}{k-2} = n_2; (w_{a,b}, w_{a,c}) = \binom{n-3}{k-3} = n_3;$$

$$(w_{a,b}, w_{c,d}) = \binom{n-4}{k-4} = n_4, (w_{a,b}, w_a) = \binom{n-2}{k-2} = n_2;$$

$$(w_{a,b}, w_c) = \binom{n-3}{k-3} = n_3; \sum_{a,b \in \Omega} w_{a,b} = \binom{k}{2} \jmath.$$

Let $w = \sum_{a,b} \alpha_{a,b} w_{a,b} \in \mathrm{Hull}(W_2)$, $\sigma = \sum_{a,b} \alpha_{a,b}$, and $\beta_a = \sum_{b \neq a} \alpha_{a,b}$. Then $(\jmath, w) = n_2 \sigma$, and

$$(w, w_{a,b}) = n_2 \alpha_{a,b} + n_3 (\sum_{c \neq a,b} \alpha_{a,c} + \sum_{c \neq a,b} \alpha_{b,c}) + n_4 \sum_{c,d \in \Omega \setminus \{a,b\}} \alpha_{c,d} = 0,$$

$$(w, w_a) = n_2 \sum_{b \neq a} \alpha_{a,b} + n_3 \sum_{b,c \neq a} \alpha_{b,c}.$$

If $p \nmid (k-1)$ then $W_1 \subseteq W_2$, and we get, for $w \in \mathrm{Hull}(W_2)$:

1. $(w, w_{a,b}) = (n_2 - 2n_3 + n_4)\alpha_{a,b} + (n_3 - n_4)(\beta_a + \beta_b) + n_4 \sigma = 0$,
2. $(w, w_a) = (n_2 - n_3)\beta_a + n_3 \sigma = 0$.

Furthermore, $w = \sum_{x,y} \alpha_{x,y}(w_{x,y} - w_{a,b}) + \sigma w_{a,b}$, so if $\sigma = 0$ then $w \in E(W_2)$.

For $k = 3$ we can take $p \geq 5$, since $k = 2, 3$ was studied in [12,11,14,15]. With notation as given above we have $W_1 \subseteq W_2$, $n_2 = n - 2, n_3 = 1, n_4 = 0$.

Lemma 11 *For $k = 3$, $p \geq 5$, $H = \mathrm{Hull}(W_2)$, if $n \not\equiv 2, 3, 4 \pmod p$ then $H = \{0\}$; if $n \equiv 2 \pmod p$ then $H = \langle \jmath \rangle$; if $n \equiv 3 \pmod p$ then $E(W_1) \subseteq H$; if $n \equiv 4 \pmod p$ then for any 4-set $\{a, b, c, d\}$ of Ω, $w_{a,b} + w_{c,d} - w_{a,c} - w_{b,d} \in H$, and has weight $4(n-4)$. Furthermore, if $p \geq n-1$ then $H = \{0\}$.*

Proof: For $k = 3$ the two equations above for $w = \sum_{a,b} \alpha_{a,b} w_{a,b} \in H$ become

$$(n-4)\alpha_{a,b} + \beta_a + \beta_b = 0, \ \ (n-3)\beta_a + \sigma = 0.$$

If $n \not\equiv 3 \pmod p$, then $\beta_a = \alpha$, a constant, and $(n-4)\alpha_{a,b} = -2\alpha$. If $n \not\equiv 4 \pmod p$ then $\alpha_{a,b} = \beta$, a constant, so $w = \beta\binom{k}{2}\jmath = 3\beta\jmath$. Now $(\jmath, w_{a,b}) = n - 2$ so $\jmath \in W_2^{\perp}$ only if $n \equiv 2 \pmod p$. Thus if $n \not\equiv 2 \pmod p$ then $H = \{0\}$, proving the first assertion.

If $n \equiv 2 \pmod p$ then clearly $n \not\equiv 3,4 \pmod p$ (since $p \geq 5$), so from the above we have $H = \langle \jmath \rangle$.

If $n \equiv 3 \pmod p$ then $(w_{a,b}, w_a) = (w_{a,b}, w_c) \equiv 1 \pmod p$, so $E(W_1) \subseteq H$, since we already have $W_1 \subset W_2$.

If $n \equiv 4 \pmod p$ then it can be verified directly that for any 4-set $\{a,b,c,d\}$ of Ω, $w_{a,b} + w_{c,d} - w_{a,c} - w_{b,d} \in H$.

Finally, if $p \geq n-1$ then $p \nmid n_2, n_3, n_4$ and so we get $\beta_a = \frac{-n_3\sigma}{(n_2-n_3)} = \alpha$, a constant, so $\alpha_{a,b} = \beta$, a constant, and $w = \beta\binom{k}{2}\jmath$. If $\beta \neq 0$ then $\jmath \in H$. But $(w_{a,b}, \jmath) = n_2 \neq 0$, so $\beta = 0$ and $w = 0$. ■

6 Codes from $\Gamma(n,k,r)$ and $R\Gamma(n,k,r)$ and the W_i

Let $A_r^k, RA_r^k = A_r^k + I$ be adjacency matrices for $\Gamma(n,k,r)$ and $R\Gamma(n,k,r)$ respectively, where $0 \leq r \leq k-1$, $2 \leq k \leq n/2$. For any fixed prime p, let C_r, RC_r denote the row span over $\mathbb{F}_p$ of A_r^k, RA_r^k, respectively. Let $V = \Omega^{\{k\}}$.

Rows of A_r^k, RA_r^k for $r = 0,1,\ldots,k-1$ are denoted by r_x^r, s_x^r respectively, for $x \in V$, where we assume a fixed value of $k \geq 2$. Thus $s_x^r = r_x^r + v^x$. Also, we write

$$N_r(x) = \{y \in V \mid y \stackrel{r}{\sim} x\} = \{y \in V \mid r_x^r(y) = 1\}.$$

If C is any of these codes we will use the notation $E(C)$ to denote the span of the differences of the rows of the adjacency matrix $A_i, A_i + I$ over the relevant field. Thus, for example, $E(C_i) = \langle r_x^i - r_y^i \mid x,y \in V\rangle$, where $V = \Omega^{\{k\}}$ and r_x^i are the rows of A_i.

We can express the s_x^r, r_x^r in terms of the words w_Λ. For clarity we write $w_\Lambda^s = w_\Lambda$ where $|\Lambda| = s$. Then $w_\emptyset^0 = \jmath$ and $w_\Lambda^k = v^\Lambda$. Then counting gives the following relations, for $0 \leq r \leq k-1$, and any $x \in V$:

$$r_x^r = \sum_{j=0}^{k-r} (-1)^j \binom{j+r}{r} \sum_{\substack{y \in V \\ |x\cap y| = j+r}} w_{x\cap y}^{j+r}; \quad s_x^r = r_x^r + v^x. \tag{6}$$

For $\Gamma(n,k,r)$, the eigenvalues are found from the Eberlein polynomials in [18, Theorem 5.1] (see also [7,1,4,6] and [4, Theorem 4.6]) for $j = 0$ to k:

$$\varepsilon_j = \sum_{i=Max\{0,j-r\}}^{Min\{j,k-r\}} (-1)^i \binom{j}{i} \binom{k-j}{k-r-i} \binom{n-k-j}{k-r-i} \tag{7}$$

with multiplicity m_j as given above.

Note that in all cases $\varepsilon_0 = \binom{k}{k-r}\binom{n-k}{k-r}$, the valency of $\Gamma(n,k,r)$, and $\varepsilon_k = (-1)^{k-r}\binom{k}{k-r}$.

For $r = 1$, i.e. $\Gamma(n,k,1)$, this can be written more simply, for $j = 0$ to k:

$$\varepsilon_j = (-1)^{j-1} j \binom{n-k-j}{k-j} + (-1)^j (k-j) \binom{n-k-j}{k-j-1}. \tag{8}$$

6.1 Kneser and Johnson graphs

The Kneser and Johnson graphs have had special attention in the literature. The graph $\Gamma(n,k,0)$ is the Kneser graph $KG_{n,k}$. Its eigenvalues are, for $j=0$ to k:

$$\lambda_j = (-1)^j \binom{n-k-j}{k-j} \tag{9}$$

with multiplicity $m_j = \binom{n}{j} - \binom{n}{j-1}$ for $j>0$ and 1 for $j=0$.

The graph $\Gamma(n,k,k-1)$ is the Johnson graph $J(n,k)$. Its eigenvalues are (see [21] and the above formulal), for $j=0$ to k:

$$\theta_j = k(n-k) - j(n+1-j) \tag{10}$$

with multiplicity $m_j = \binom{n}{j} - \binom{n}{j-1}$ for $j>0$ and 1 for $j=0$.

We can use these eigenvalues to get information regarding the possible dimension of the codes C_0 and C_{k-1}. Since if λ is an eigenvalue for a matrix M then $\lambda+1$ is an eigenvalue for $M+I$, this will also give information about RC_0 and RC_{k-1}. If M is a $v\times v$ integral matrix with integral eigenvalues, then modulo p these will still be eigenvalues, but not necessarily all distinct. If none or at most one reduce to 0 modulo p then the p-rank of M will be v or $v-m_j$, respectively, where m_j is the multiplicity of the eigenvalue that is zero. In any case, the dimension of the zero eigenspace over $\mathbb{F}_p$ of the matrix A or $A+I$ is at most the sum m of the multiplicities of the eigenvalues that reduce to 0 modulo p, and thus the p-rank of A or $A+I$ is at least $\binom{n}{k} - m$. [1]

Note 2 Since $\lambda_k + 1 = (-1)^k + 1 = 0$ if k is odd, we see that $RC_0 \neq \mathbb{F}_p^{|V|}$ for k odd. In fact, notice that from Equation (6), for $x \in V = \Omega^{\{k\}}$:

$$s_x^0 = v^x + \sum_{j=0}^{k} (-1)^j \sum_{\substack{y\in V\\ |x\cap y|=j}} w^j_{x\cap y} = v^x + \jmath + \sum_{j=1}^{k-1} (-1)^j \sum_{\substack{y\in V\\ |x\cap y|=j}} w^j_{x\cap y} + (-1)^k v^x,$$

so if k is odd then $s_x^0 = \jmath + \sum_{i=1}^{k-1} u_i$, where $u_i \in W_i$. Thus if $\jmath, u_i \in W_{k-1}$, for $1 \le i \le k-1$, we have $RC_0 \subseteq W_{k-1}$. So if $p \nmid 2,3,\ldots,k$, this will hold.

From [9] (Propositions 3.1, 3.2, 3.3, 3.4) we can deduce the following connection with the binary codes from the Johnson graphs, $J(n,k)$, the uniform subset graph $\Gamma(n,k,k-1)$, and from [8, Theorem 6.2.7] or [10, Theorem 3.7], for the binary code from the odd graph $\mathcal{O}(k) = \Gamma(2k+1,k,0)$:

Result 2 *Let $k \ge 3$, $n \ge 2k+1$, $p=2$.*

1. *If $C = C_2(J(n,k))$, then*
   - *for n,k odd, $C = W_{k-1}^{\perp}$; for n odd, k even, $C = W_{k-1}$;*
   - *for n,k even, $C \subset W_{k-1}^{\perp}$; for n even, k odd, $C = \mathbb{F}_2^V$.*
2. *If $C = C_2(\mathcal{O}(k))$, then $C = W_{k-1}^{\perp}$.*

[1] The authors thank T.P. McDonough for this observation.

From this result and the eigenvalues for $J(n,k)$ we can deduce the following for the binary codes for the Johnson graph and its associated reflexive graph:

Corollary 3 *For $k \geq 3$, $n \geq 2k+1$, $p=2$, let $C = C_2(J(n,k))$, $RC = C_2(RJ(n,k))$. Then*

1. *for n,k odd, $RC = W_{k-1} = C^{\perp}$, $\mathrm{Hull}(C) = \{0\}$;*
2. *for n odd, k even, $RC = W_{k-1}^{\perp} = C^{\perp}$, $\mathrm{Hull}(C) = \{0\}$;*
3. *for n,k even, $C \subseteq \mathrm{Hull}(W_{k-1})$, $RC = \mathbb{F}_2^V$;*
4. *for n even, k odd, $C = \mathbb{F}_2^V$, $RC \subseteq \mathrm{Hull}(W_{k-1})$.*

Proof: (1): By evaluating the inner products (r_x^{k-1}, s_y^{k-1}) we see it has the value $k(n-k)$ if $x=y$; 0 if $0 \leq |x \cap y| \leq k-3$; 4 if $|x \cap y| = k-2$; and $n-1$ if $|x \cap y| = k-1$. Thus $RC \subseteq C^{\perp}$ if n is odd, i.e. $RC \subseteq W_{k-1}$. Now from the eigenvalues for $J(n,k)$, we see that for n odd the null space of RC has dimension at most the sum of the multiplicities of the zero eigenvalues (for $A+I$), i.e. the sum of the m_i for i odd. This sum can be seen to be $\sum_{i=0}^{l}(-1)^{i+1}\binom{n}{i}$ where $l = k-1$ if k is even, $l = k$ if k is odd. Now for l odd it is easy to show that

$$\sum_{i=0}^{l}(-1)^{i+1}\binom{n}{i} = \binom{n-1}{l},$$

so for k odd the sum is $\binom{n-1}{k}$, and for k even the sum is $\binom{n-1}{k-1}$. Thus for n,k odd, $\dim(RC) \geq \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1} = \dim(W_{k-1})$ by Lemma 9. Thus $RC = W_{k-1}$.
(2): Since n is still odd, we have $RC \subseteq C^{\perp} = W_{k-1}^{\perp}$. The null space again has dimension at most the sum of the m_i for i odd, but now terminating in $k-1$. So $\dim(RC) \geq \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k} = \dim(W_{k-1}^{\perp})$. Thus $RC = W_{k-1}^{\perp}$.
(3): For n,k even, by Result 2, $C \subseteq W_{k-1}^{\perp}$. By Equation (6),

$$r_x^{k-1} = \sum_{j=0}^{1}(-1)^j\binom{j+k-1}{k-1}\sum_{\substack{y\in V\\ |x\cap y|=j+k-1}} w_{x\cap y}^{j+k-1}$$

$$= \sum_{\substack{y\in V\\ |x\cap y|=k-1}} w_{x\cap y}^{k-1} - \binom{k}{k-1}v^x = \sum_{\substack{y\in V\\ |x\cap y|=k-1}} w_{x\cap y}^{k-1}$$

since the last term is 0. Thus $C \subseteq \mathrm{Hull}(W_{k-1})$. Since all the eigenvalues for $A+I$ are non-zero modulo 2, $RC = \mathbb{F}_2^V$.
(4): For n even, k odd, the equation for r_x^{k-1} in terms of elements from W_{k-1} becomes $s_x^{k-1} = r_x^{k-1} + v^x = \sum_{\substack{y\in V\\ |x\cap y|=k-1}} w_{x\cap y}^{k-1}$, so $RC \subseteq W_{k-1}$. Checking then that $(s_x^{k-1}, w_z) \equiv 0 \pmod 2$ for all $x \in V$, $z \in \Omega^{\{k-2\}}$, we have $RC \subseteq \mathrm{Hull}(W_{k-1})$. The rest follows. ■

Note 3 (1) Computations indicate that the inequalities in (3) and (4) are equalities.
(2) From Identities 1(4), we have that if $k-s$ is odd and $p=2$, then $W_s \subseteq W_{s+1}$, for $0 \le s \le k-1$.

7 Codes from $\Gamma(n,3,i)$ and $R\Gamma(n,3,i)$ for $p \ge 5$

We examine the codes from the row span of adjacency matrices for $\Gamma = \Gamma(n,3,i)$ and its reflexive counterpart $R\Gamma = R\Gamma(n,3,i)$, for $i = 0,1,2$, and $p \ge 5$, $n \ge 7$. For $p = 2,3$ these codes are studied in [12,11,14,15]. Recall that $\Omega = \{1,\dots,n\}$. This section will be devoted to developing the various properties of the codes, and their links to W_1, W_2, W_π; the findings will be summarized in a more comprehensive manner in Section 8.

Let $V = \Omega^{\{3\}}$. Throughout this section, $n \ge 7$, $k = 3$ and $p \ge 5$. By Lemmas 5, 8, respectively, $W_1 \ne E(W_1)$, and $W_2 \ne E(W_2)$.

We first obtain some general relationships amongst the words of the codes from the graphs and the words from the codes W_1, W_2, W_Π.

7.1 General identities

With notation as before, for any $x \in V$,

$$\jmath = v^x + r_x^0 + r_x^1 + r_x^2 = -2v^x + s_x^0 + s_x^1 + s_x^2. \tag{11}$$

The valency ν_i for $\Gamma(n,3,i)$ for $i = 0,1,2$ is given by:

$$\nu_0 = \binom{n-3}{3}; \;\; \nu_1 = 3\binom{n-3}{2}; \;\; \nu_2 = 3(n-3), \tag{12}$$

and $\nu_i^* = \nu_i + 1$ is the valency of $R\Gamma$.

Note 4 1. From Identities 1 (4),(5), $\jmath \in W_1 \subseteq W_2$.
2. Since $\sum_x r_x^i = \nu_i \jmath$ and $\sum_x s_x^i = \nu_i^* \jmath$, $\jmath \in C_i$ if $\nu_i \not\equiv 0 \pmod p$ and $\jmath \in RC_i$ if $\nu_i^* \not\equiv 0 \pmod p$. Furthermore, $C_i \subseteq \langle \jmath \rangle^\perp$ if and only if $\nu_i \equiv 0 \pmod p$, and $RC_i \subseteq \langle \jmath \rangle^\perp$ if and only if $\nu_i^* \equiv 0 \pmod p$.

Definition 1 For $x \in \Omega^{\{3\}}$, $i = 0,1,2$, write

$$w_x^i = \sum_{y \in N_i(x) \cup \{x\}} s_y^i \text{ and } u_x^i = \sum_{y \in N_i(x)} r_y^i. \tag{13}$$

Then $w_x^i = v^x + 2r_x^i + u_x^i$, and, from counting arguments as detailed in [14,15,12,11]:

Identities 2 (1) $w_x^0 = (1 + \binom{n-3}{3})v^x + (\binom{n-6}{3} + 2)r_x^0 + \binom{n-5}{3} r_x^1 + \binom{n-4}{3} r_x^2$;
(2) $u_x^0 = \binom{n-3}{3} v^x + \binom{n-6}{3} r_x^0 + \binom{n-5}{3} r_x^1 + \binom{n-4}{3} r_x^2$;
(3) $w_x^1 = (60 - 10n)v^x + 9(n-6)s_x^0 + \frac{1}{2}(n^2 - 3n - 6)s_x^1 + (n-4)^2 s_x^2$;

(4) $u_x^1 = 3\binom{n-3}{2}v^x + 9(n-6)r_x^0 + \frac{(n-5)(n+2)}{2}r_x^1 + (n-4)^2 r_x^2$;
(5) $w_x^2 = 2(n-6)v^x + 4s_x^1 + ns_x^2$;
(6) $u_x^2 = 3(n-3)v^x + 4r_x^1 + (n-2)r_x^2$.

From Identities 1(4),(5), we have $\sum_{a\in\Omega} w_a = \sum_{a,b\in\Omega} w_{a,b} = 3\boldsymbol{\jmath}$, so, since $p \geq 5$, $\boldsymbol{\jmath} \in W_1, W_2$, and

$$2w_a = \sum_{b\neq a} w_{a,b} \implies W_1 \subseteq W_2, \text{ and } E(W_1) \subseteq E(W_2). \tag{14}$$

From Equation (6) with $k = 3$ we get the following:

Identities 3 *If $x = \{a, b, c\} \in V$,*

(1) $s_x^0 = \boldsymbol{\jmath} + (w_{a,b} + w_{a,c} + w_{b,c}) - (w_a + w_b + w_c)$;
(2) $r_x^0 = -v^x + \boldsymbol{\jmath} + (w_{a,b} + w_{a,c} + w_{b,c}) - (w_a + w_b + w_c)$;
(3) $s_x^1 = 4v^x - 2(w_{a,b} + w_{a,c} + w_{b,c}) + (w_a + w_b + w_c)$;
(4) $r_x^1 = 3v^x - 2(w_{a,b} + w_{a,c} + w_{b,c}) + (w_a + w_b + w_c)$;
(5) $s_x^2 = -2v^x + (w_{a,b} + w_{a,c} + w_{b,c})$;
(6) $r_x^2 = -3v^x + (w_{a,b} + w_{a,c} + w_{b,c})$.

Looking at sums of the s_x^i or r_x^i over various ranges gives:

Identities 4 – $i = 0$
(1) $\sum_{c\neq a,b} s^0_{\{a,b,c\}} = (n-4)w_{a,b} - (n-5)(w_a + w_b) + (n-5)\boldsymbol{\jmath}$;
(2) $\sum_{c\neq a,b} r^0_{\{a,b,c\}} = (n-5)w_{a,b} - (n-5)(w_a + w_b) + (n-5)\boldsymbol{\jmath}$;
(3) $\sum_{b,c\neq a} s^0_{\{a,b,c\}} = -\frac{1}{2}(n-3)(n-6)w_a + \binom{n-4}{2}\boldsymbol{\jmath}$;
(4) $\sum_{b,c\neq a} r^0_{\{a,b,c\}} = -\binom{n-4}{2}w_a + \binom{n-4}{2}\boldsymbol{\jmath}$;
– $i = 1$
(5) $\sum_{c\neq a,b} s^1_{\{a,b,c\}} = (-2n+12)w_{a,b} + (n-7)(w_a + w_b) + 3\boldsymbol{\jmath}$;
(6) $\sum_{c\neq a,b} r^1_{\{a,b,c\}} = (-2n+11)w_{a,b} + (n-7)(w_a + w_b) + 3\boldsymbol{\jmath}$;
(7) $\sum_{b,c\neq a} s^1_{\{a,b,c\}} = \frac{1}{2}(n^2 - 13n + 38)w_a + 3(n-4)\boldsymbol{\jmath}$;
(8) $\sum_{b,c\neq a} r^1_{\{a,b,c\}} = \frac{1}{2}(n-4)(n-9)w_a + 3(n-4)\boldsymbol{\jmath}$;
– $i = 2$
(9) $\sum_{c\neq a,b} s^2_{\{a,b,c\}} = (n-6)w_{a,b} + 2(w_a + w_b)$;
(10) $\sum_{c\neq a,b} r^2_{\{a,b,c\}} = (n-7)w_{a,b} + 2(w_a + w_b)$;
(11) $\sum_{b,c\neq a} s^2_{\{a,b,c\}} = (2n-8)w_a + 3\boldsymbol{\jmath}$;
(12) $\sum_{b,c\neq a} r^2_{\{a,b,c\}} = (2n-9)w_a + 3\boldsymbol{\jmath}$;

To consider the possibilities of W_1, W_2 to be inside the dual codes of the graph, we note first the following:

Identities 5 *For $a, b \in \Omega$, $x \in V$:*

$$(w_a, s_x^0) = \begin{cases} 1 & a \in x \\ \binom{n-4}{2} & a \notin x \end{cases}, (w_a, r_x^0) = \begin{cases} 0 & a \in x \\ \binom{n-4}{2} & a \notin x \end{cases},$$

$$(w_{a,b}, s^0_x) = \begin{cases} 1 & a, b \in x \\ n-5 & a \notin x, b \notin x \\ 0 & a \text{ or } b \in x \end{cases}, (w_{a,b}, r^0_x) = \begin{cases} 0 & a, b \in x \\ n-5 & a \notin x, b \notin x \\ 0 & a \text{ or } b \in x \end{cases},$$

$$(w_a, s^1_x) = \begin{cases} 1 + \binom{n-3}{2} & a \in x \\ 3(n-4) & a \notin x \end{cases}, (w_a, r^1_x) = \begin{cases} \binom{n-3}{2} & a \in x \\ 3(n-4) & a \notin x \end{cases},$$

$$(w_{a,b}, s^1_x) = \begin{cases} 1 & a, b \in x \\ 3 & a \notin x, b \notin x \\ n-4 & a \text{ or } b \in x \end{cases}, (w_{a,b}, r^1_x) = \begin{cases} 0 & a, b \in x \\ 3 & a \notin x, b \notin x \\ n-4 & a \text{ or } b \in x \end{cases},$$

$$(w_a, s^2_x) = \begin{cases} 1 + 2(n-3) & a \in x \\ 3 & a \notin x \end{cases}, (w_a, r^2_x) = \begin{cases} 2(n-3) & a \in x \\ 3 & a \notin x \end{cases},$$

$$(w_{a,b}, s^2_x) = \begin{cases} n-2 & a, b \in x \\ 0 & a \notin x, b \notin x \\ 2 & a \text{ or } b \in x \end{cases}, (w_{a,b}, r^2_x) = \begin{cases} n-3 & a, b \in x \\ 0 & a \notin x, b \notin x \\ 2 & a \text{ or } b \in x \end{cases}.$$

For W_1, W_2:

$$(w_{a,b}, w_{c,d}) = \begin{cases} n-2 & \{a,b\} = \{c,d\} \\ 1 & \{c,d\} \ni a \text{ or } b \\ 0 & \{a,b\} \cap \{c,d\} = \emptyset \end{cases}, (w_{a,b}, w_c) = \begin{cases} n-2 & c \in \{a,b\} \\ 1 & c \notin \{a,b\} \end{cases},$$

$$(w_a, w_b) = \begin{cases} \binom{n-1}{2} & a = b \\ n-2 & a \neq b \end{cases}.$$

7.2 The codes

We use the identities established in the previous subsection to obtain relationships amongst the codes. Again, throughout, $n \geq 7$, $k = 3$, and $p \geq 5$.

From Identities 3 and since $W_1 \subseteq W_2$, we have

Lemma 12 *For $C = C_i$ or RC_i, for $i = 1, 2$, if $W_2 \subseteq C$ then $C = \mathbb{F}_p^V$; if $E(W_2) \subseteq C$ then $\langle \boldsymbol{\jmath} \rangle^\perp \subseteq C$. If $\boldsymbol{\jmath} \in C_0$ then if $W_2 \subseteq C_0$ then $C_0 = \mathbb{F}_p^V$; if $E(W_2) \subseteq C_0$ then $\langle \boldsymbol{\jmath} \rangle^\perp \subseteq C_0$.*

If $\nu \in \{\nu_i, \nu_i^* \mid 0 \leq i \leq 2\}$ and if $\nu \equiv 0 \pmod p$ then the corresponding code $C \subseteq \langle \boldsymbol{\jmath} \rangle^\perp$.

Proof: Follows immediately from Identities 3. ■

Also we have

Lemma 13 *1. $RC_0 \subseteq W_2$;*
2. if $n \equiv 5 \pmod p$, then $W_2 = RC_0$;
3. if $n \equiv 4 \pmod p$, then $W_1 \subseteq RC_0$, and $W_1 \subseteq RC_1$.

Proof: (1) From Identities 3, since $\jmath \in W_2$, and $W_1 \subseteq W_2$, we have $RC_0 \subseteq W_2$ for all n, giving the first assertion.

(2) For $n \equiv 5 \pmod{p}$, from Identities 4(1) we get $W_2 \subseteq RC_0$, and hence $W_2 = RC_0$.

(3) For $n \equiv 4 \pmod{p}$, from Identities 4(3) we get $W_1 \subseteq RC_0$, and from Identities 4(7) we get $W_1 \subseteq RC_1$. ■

Lemma 14

$$(1)\ W_\Pi \subseteq C_0, C_1, C_2, RC_0^\perp, RC_1, RC_2;$$

$$(2)\ W_\Pi \not\subseteq C_0^\perp, C_1^\perp, C_2^\perp, RC_0, RC_1^\perp, RC_2^\perp.$$

Proof: Let w_π be as defined in Equation (2). Then it can be shown directly that $\sum_{x\in X} r_x^i - \sum_{x\in X^c} r_x^i = \alpha_i w_\pi$, for $i = 0, 1, 2$, where $\alpha_0 = -1, \alpha_1 = 3, \alpha_2 = -3$, and $\sum_{x\in X} s_x^i - \sum_{x\in X^c} s_x^i = \beta_i w_\pi$ for $i = 1, 2$, where $\beta_1 = 4, \beta_2 = -2$.

That $w_\pi \in RC_0^\perp$, but not in $C_0^\perp, C_1^\perp, C_2^\perp, RC_1^\perp, RC_2^\perp$, can be verified directly.

To show $W_\Pi \not\subseteq RC_0$, since $RC_0 \subseteq W_2 = W_\Pi^\perp$ by Lemma 13 and Lemma 10, if $w_\pi \in RC_0$ then $w_\pi \in W_\Pi^\perp$, so $(w_\pi, w_\pi) = 0$. But $(w_\pi, w_\pi) = 8 \neq 0$ for p odd, so we have a contradiction. ■

From Identities 5:

Lemma 15 *1.* $i = 0$
$W_1, W_2, E(W_2) \not\subseteq RC_0^\perp$ *for all* n*;* $E(W_1) \subseteq RC_0^\perp$ *if* $n \equiv 3, 6 \pmod{p}$*;*
$W_1 \subseteq C_0^\perp$ *for* $n \equiv 4, 5 \pmod{p}$*;* $W_2 \subseteq C_0^\perp$ *for* $n \equiv 5 \pmod{p}$*;*

2. $i = 1$
$W_1, W_2, E(W_2) \not\subseteq RC_1^\perp$ *for all* n*;* $E(W_1) \subseteq RC_1^\perp$ *if* $n^2 - 13n + 38 \equiv 0 \pmod{p}$*;*
$W_2, E(W_2) \not\subseteq C_1^\perp$ *for all* n*;* $W_1 \subseteq C_1^\perp$ *for* $n \equiv 4 \pmod{p}$*;* $E(W_1) \subseteq C_1^\perp$ *for* $n \equiv 4, 9 \pmod{p}$*;*

3. $i = 2$
$W_1, W_2, E(W_2) \not\subseteq RC_2^\perp$ *for all* n*;* $E(W_1) \subseteq RC_2^\perp$ *if* $n \equiv 4 \pmod{p}$*;*
$W_1, W_2, E(W_2) \not\subseteq C_2^\perp$ *for all* n*;* $E(W_1) \subseteq C_2^\perp$ *if* $2n \equiv 9 \pmod{p}$*.*

Result 3 *For the Kneser and Johnson graphs with* $k = 3$*, the eigenvalues and multiplicities are as follows, using Equations (9), (10), for* A *and* $A + I$*:*

- *Kneser* $K_{n,3}$, $\nu_0 = \binom{n-3}{3}$, $n \geq 7$*;*
  1. $\lambda_0 = \binom{n-3}{3}$, $\lambda_0 + 1 = \frac{1}{6}(n-2)(n^2 - 10n + 27)$, $m_0 = 1$*;*
  2. $\lambda_1 = -\binom{n-4}{2}$, $\lambda_1 + 1 = -\frac{1}{2}(n-3)(n-6)$, $m_1 = n - 1$*;*
  3. $\lambda_2 = n - 5$, $\lambda_2 + 1 = n - 4$, $m_2 = \binom{n}{2} - n$*;*
  4. $\lambda_3 = -1$, $\lambda_3 + 1 = 0$, $m_3 = \binom{n}{3} - \binom{n}{2}$*.*
- *Johnson* $J(n, 3)$, $\nu_2 = 3(n-3)$, $n \geq 7$*;*
  1. $\theta_0 = 3(n-3)$, $\theta_0 + 1 = 3n - 8$, $m_0 = 1$*;*
  2. $\theta_1 = 2n - 9$, $\theta_1 + 1 = 2n - 8$, $m_1 = n - 1$*;*
  3. $\theta_2 = n - 7$, $\theta_2 + 1 = n - 6$, $m_2 = \binom{n}{2} - n$*;*
  4. $\theta_3 = -3$, $\theta_3 + 1 = -2$, $m_3 = \binom{n}{3} - \binom{n}{2}$*.*

Note 5 1. Peeters in [21, p. 142] gives a formula for the p-rank of the Johnson graph, but there is an error in his excluded cases as for $p = 5$ and $n \equiv 2 \pmod 5$ and for $n \equiv 1 \pmod 5$ the formulas for A and $A + I$, respectively, are incorrect. Thus we will only give a value when we have all or all but one of the eigenvalues are non-zero.
2. Since $\lambda_3 + 1 = 0$, $RC_0 \neq \mathbb{F}_p^{|V|}$ for all p.

Lemma 16 *For $n \geq 7$, $p \geq 5$, notation as before, for $J(n, 3)$ we have:*

1. *if $n \not\equiv 3, 7, 9/2 \pmod p$ then $C_2 = \mathbb{F}_p^V$;*
2. *if $n \equiv 3 \pmod p$ then $C_2 = \langle \jmath \rangle^\perp$;*
3. *if $n \equiv 7 \pmod p$ and $p \neq 5$, then $\dim(C_2) = \binom{n}{3} - \binom{n}{2} + n$;*
4. *if $n \equiv 9/2 \pmod p$ and $p \neq 5$, then $\dim(C_2) = \binom{n}{3} - n + 1$;*
5. *if $n \not\equiv 4, 6, 8/3 \pmod p$ then $RC_2 = \mathbb{F}_p^V$;*
6. *if $n \equiv 8/3 \pmod p), p > 5$ then $RC_2 = \langle \jmath \rangle^\perp$;*
7. *if $n \equiv 4 \pmod p$ then $RC_2^\perp = E(W_1)$;*
8. *if $n \equiv 6 \pmod p$ and $p \neq 5$, then $\dim(RC_2) = \binom{n}{3} - \binom{n}{2} + n$. If $p = 5$ then $\dim(RC_2) \geq \binom{n}{3} - \binom{n}{2} + n - 1$.*

Proof: The proof follows by determining when the number of zero eigenvalues is zero or 1, as described in Section 6.1. All cases are covered apart from when $p = 5$ where the dimension of C_2 if $n \equiv 2 \pmod 5$ and RC_2 if $n \equiv 1 \pmod 5$ are not determined. ∎

Lemma 17 *For $n \geq 7$, $p \geq 5$, notation as before, for $KG_{n,3}$ we have:*

1. *if $n \not\equiv 3, 4, 5 \pmod p$ then $C_0 = \mathbb{F}_p^V$; if $n \equiv 3, 4, 5 \pmod p$ then $C_0 \subseteq \langle \jmath \rangle^\perp$.*
2. *if $n \equiv 3 \pmod p$ then $C_0 = \langle \jmath \rangle^\perp$;*
3. *if $n \equiv 0, 5 \pmod p$ then $RC_0 = W_2$;*
4. *if $n \equiv 1, 7 \pmod p$, $p \neq 5$ then $RC_0 = W_2$.*

Proof: As in Lemma 16, and using Lemma 13. ∎

Note 6 For the reflexive case of the Kneser graphs, see also Lemma 13.

Proposition 3 *If $n \equiv 6 \pmod p$, then $RC_1 = RC_2 = W_1 \oplus W_2^\perp$.*

Proof: If $n \equiv 6 \pmod p$ then $\nu_1^* \equiv 10 \pmod p$, so $\jmath \in RC_1$. From Identities 4(7), $\sum_{b,c \neq a} s^1_{\{a,b,c\}} = \frac{1}{2}(n^2 - 13n + 38)w_a + 3(n-4)\jmath = -2w_a + 6\jmath$, so $W_1 \subseteq RC_1$. From Lemma 14, $W_\Pi = W_2^\perp \subseteq RC_1$, so $W_1 + W_2^\perp \subseteq RC_1$. Now $W_2^\perp \subseteq W_1^\perp$, so $W_1 \cap W_2^\perp \subseteq \mathrm{Hull}(W_1) = \{0\}$, by Proposition 2, since, in the notation of that proposition, $n_1 \equiv 10 \pmod p$ and $n_2 \equiv 4 \pmod p$. Thus $W_1 + W_2^\perp = W_1 \oplus W_2^\perp$, and $W_1 \oplus W_2^\perp \subseteq RC_1$.

To show that $RC_1 = RC_2$, from Identities 2(3),(5), $w_x^1 = 6s_x^1 + 4s_x^2$ and $w_x^2 = 4s_x^1 + 6s_x^2$, so $RC_1 = RC_2$.

Finally, we need to show that $RC_1 \subseteq W_1 \oplus W_2^\perp$, i.e. that $s_x^1 \in W_1 \oplus W_2^\perp$ for all x, so we need $u \in W_1$ such that $s_x^1 - u \in W_2^\perp$. Let $x = \{a, b, c\}$ and $u =$

$\sum_d \alpha_d w_d$. Let $N = \sum_d \alpha_d$. Recall that $(w_{a,b}, w_a) = n - 2 \equiv 4 \pmod p$, and $(w_{a,b}, w_c) = 1$ for $c \neq a, b$. Now we use Identities 5 and put $(s_x^1 - u, w_{d,e}) = 0$ for all $d, e \in \Omega$ to obtain:

$$(s_x^1 - \sum_d \alpha_d w_d, w_{a,b}) = 0 \Rightarrow 1 - \sum_{d \neq a,b} \alpha_d - 4(\alpha_a + \alpha_b) = 0 \Rightarrow 1 - N = 3(\alpha_a + \alpha_b),$$

and similarly for $d, e \notin x$,

$$2 - N = 3(\alpha_a + \alpha_d), \ \ 3 - N = 3(\alpha_d + \alpha_e).$$

This implies that $\alpha_a = \alpha_b = \alpha_c = \alpha$ and $\alpha_d = \beta$ for $d \notin x$, and so $N = 3\alpha + (n-3)\beta$, $1 = 3(\beta - \alpha)$, so $\alpha = 0$ and $\beta = 3^{-1}$. Thus $s_x^1 - 3^{-1} \sum_{d \notin x} w_d \in W_2^\perp$. ■

Recall that from Lemma 14, $W_2^\perp \subseteq C_1, RC_1$ for all p.

Proposition 4 *1. If $n \equiv 3, 4 \pmod p$ then $C_1 \subseteq \langle \jmath \rangle^\perp$, and*
- *(a) if $n \equiv 3 \pmod p$ then $E(W_1) \subseteq W_2^\perp \subseteq C_1$, and if $p > 5$, then $C_1 = \langle \jmath \rangle^\perp$;*
- *(b) if $n \equiv 4 \pmod p$ then $C_1 = C_0$.*

2. If $n \not\equiv 3, 4 \pmod p$ then $\jmath \in C_1$ and
- *(a) if $n \not\equiv 9 \pmod p$ then $W_1 + W_2^\perp \subseteq C_1$ and $\dim(C_1) \geq \binom{n}{3} - \binom{n}{2} + n - 1$; if also $2n \not\equiv 11 \pmod p$ then $C_1 = \mathbb{F}_p^V$;*
- *(b) if $n \equiv 9 \pmod p$ then $p \geq 7$, and if $p = 7$ then $W_1 \subseteq C_1$; if $p > 7$, then $C_1 \subseteq W_1^\perp + \langle \jmath \rangle$.*

3. For $n \not\equiv 6 \pmod p$:
- *(a) If $\nu_1^* \equiv 0 \pmod p$, then $RC_1 \subseteq \langle \jmath \rangle^\perp$, and if $n^2 - 13n + 38 \not\equiv 0 \pmod p$ then $RC_1 = \langle \jmath \rangle^\perp$. If $n^2 - 13n + 38 \equiv 0 \pmod p$ then $p = 19$ and $n \equiv 0 \pmod{19}$, $\jmath \in RC_1$, and $RC_1 \subseteq E(W_1)^\perp$.*
- *(b) If $\nu_1^* \not\equiv 0 \pmod p$, and $(n^2 - 13n + 38) \not\equiv 0 \pmod p$, then $RC_1 = \mathbb{F}_p^V$.*
- *(c) If $\nu_1^* \not\equiv 0 \pmod p$, $(n^2 - 13n + 38) \equiv 0 \pmod p$, then $W_2^\perp \subseteq RC_1 \subseteq E(W_1)^\perp$, and $\jmath \in RC_1$.*

Proof: (1): If $n \equiv 3 \pmod p$ and $p \neq 5$, then by the Identities 4(6),(8), $5w_{a,b} - 4(w_a + w_b) + 3\jmath$ and $3w_a - 3\jmath$ are in C_1 for any $a, b \in \Omega$. The second equation gives $E(W_1) \subseteq C_1$, and then subtracting two equations of the first type gives $E(W_2) \subseteq C_1$. By Lemma 12, $C_1 = \langle \jmath \rangle^\perp$. This does not work for $p = 5$ although we still have $E(W_1) \subseteq C_1$. For $n \equiv 3 \pmod p$, $E(W_1) \subseteq \mathrm{Hull}(W_2)$ by Lemma 11.

If $n \equiv 4 \pmod p$ then by Identities 2(2),(4), $C_0 = C_1$.

(2): For $n \not\equiv 3, 4 \pmod p$, if $n \not\equiv 9 \pmod p$ then by Identities 4 (8) $W_1 \subseteq C_1$, and $\dim(C_1) \geq \dim(W_1 + W_2^\perp) \geq \binom{n}{3} - \binom{n}{2} + n - 1$ since $\mathrm{Hull}(W_2) \subseteq \langle \jmath \rangle$, by Lemma 11. If $n \not\equiv 9 \pmod p$ and $2n \not\equiv 11 \pmod p$, then similarly using also Identities 4 (8) and Lemma 12.

If $n \equiv 9 \pmod p$, then $p \geq 7$ and by Identities 5, $(w_a, r_x^1) = 15 \neq 0$, a constant for all a, x, so $E(W_1) \subseteq C_1^\perp$. If $p = 7$ then $2n \equiv 11 \pmod p$ so $W_1 \subseteq C_1$ by Identities 4(6). If $p > 7$ then $(w_a, \jmath) = 28 \neq 0$, so $(w_a, r_x^1 - \frac{15}{28}\jmath) = 0$ for all a, x, so $r_x^1 - \frac{15}{28}\jmath \in W_1^\perp$.

(3): For $n \not\equiv 6 \pmod p$, if $\nu_1^* \equiv 0 \pmod p$ then, since $(s_x^1, \jmath) \equiv 0 \pmod p$, we have $RC_1 \subseteq \langle \jmath \rangle^\perp$. If $(n^2 - 13n + 38) \not\equiv 0 \pmod p$ then from Identities 4 (7), $E(W_1) \subseteq RC_1$. Then Identities 4 (5) gives $E(W_2) \subseteq RC_1$, and hence by Lemma 12, $RC_1 = \langle \jmath \rangle^\perp$. If $(n^2 - 13n + 38) \equiv 0 \pmod p$, then from Identities 5, $(w_a, s_x^1) = \alpha$, a non-zero constant for all a, x, and $RC_1 \subseteq E(W_1)^\perp$. But $\nu_1^* \equiv 0 \pmod p$ and $(n^2 - 13n + 38) \equiv 0 \pmod p$ implies $n \equiv 0 \pmod p$ or $n \equiv 4 \pmod p$. Only the first is possible with $p = 19$. From Identities 4 (7), since $n \not\equiv 4 \pmod p$ in this case, it follows that $\jmath \in RC_1$.

If $\nu_1^* \not\equiv 0 \pmod p$ then $\jmath \in RC_1$, and from Identities 4 (7), if $(n^2 - 13n + 38) \not\equiv 0 \pmod p$ then $W_1 \subseteq RC_1$. From Identities 4(5), if $n \not\equiv 6 \pmod p$ then $W_2 \subseteq RC_1$, and hence from Lemma 12, $RC_1 = \mathbb{F}_p^V$.

If $\nu_1^* \not\equiv 0 \pmod p$ and $(n^2 - 13n + 38) \equiv 0 \pmod p$, then from Identities 5, $(w_a, s_x^1) = \alpha$, a non-zero constant for all a, x. Thus $RC_1 \subseteq E(W_1)^\perp$. From Identities 4(7), since $n \not\equiv 4 \pmod p$ in this case, it follows that $\jmath \in RC_1$. ■

Proposition 5 *If* $n \equiv 6 \pmod p$ *then* $W_2 = RC_0 + W_1$, $RC_0 \cap W_1 = \langle \jmath \rangle$, *and* $\dim(RC_0) = \binom{n-1}{2}$.

Proof: $RC_0 \subseteq W_2$ for all $p \geq 5$ by Lemma 13. Identities 4(1) then gives $\sum_{c \neq a,b} s^0_{\{a,b,c\}} = 2w_{a,b} - (w_a + w_b) + \jmath$ and (3) gives $\sum_{b,c \neq a} s^0_{\{a,b,c\}} = \jmath$, and so $W_2 \subseteq RC_0 + W_1$. Since also $RC_0, W_1 \subseteq W_2$, we have $W_2 = RC_0 + W_1$. Also by Lemma 15 (1), for $n \equiv 6 \pmod p$ $E(W_1) \subseteq RC_0^\perp$, so $RC_0 \subseteq E(W_1)^\perp$. Clearly $RC_0 \cap W_1 \ni \jmath$. Suppose $v \in RC_0 \cap W_1$, where $v = \sum_i \alpha_i w_i \in RC_0$. So $(v, w_a - w_b) = 0$ for all $a, b \in \Omega$. Now $(w_a, w_a) = \binom{n-1}{2} = 10$ and $(w_a, w_b) = n-2 = 4$. So $(v, w_a - w_b) = \sum_i \alpha_i (w_i, w_a) - \sum_i \alpha_i (w_i, w_b) = 10\alpha_a + 4\sum_{i \neq a} \alpha_i - 10\alpha_b - 4\sum_{i \neq b} \alpha_i = 10\alpha_a + 4(\sum_i \alpha_i - \alpha_a) - 10\alpha_b - 4(\sum_i \alpha_i - \alpha_b) = 6\alpha_a - 6\alpha_b = 0$, so $\alpha_a = \alpha_b = \alpha$ for all $a, b \in \Omega$, and $v = 3\alpha\jmath$. Finally by the dimension formula we get $\dim(RC_0) = \binom{n}{2} - n + 1 = \binom{n-1}{2}$. ■

Proposition 6 *If* $n \not\equiv 6 \pmod p$ *then*

1. $E(W_2) \subseteq RC_0$;
2. *if* $\jmath \in RC_0$, *then* $RC_0 = W_2$, *and in particular if* $\nu_0^* \not\equiv 0 \pmod p$;
3. *if* $\nu_0^* = \frac{1}{6}(n-2)(n^2 - 10n + 27) \equiv 0 \pmod p$, *then* $n \equiv 2 \pmod p$ *or* $n \equiv 5 \pm \sqrt{-2} \pmod p$. *If* $n \not\equiv 2 \pmod p$ *then* $p \equiv 1, 3 \pmod 8$ *and* $RC_0 = E(W_2)$.

Proof: (1). First show that $RC_0 \supseteq E(W_2)$, and first take $n \not\equiv 3, 4 \pmod p$. Identities 4(1) gives $u_1 = (n-4)w_{a,b} - (n-5)(w_a + w_b) + (n-5)\jmath \in RC_0$, and $u_2 = (n-4)w_{a,c} - (n-5)(w_a + w_c) + (n-5)\jmath \in RC_0$. We take $n \not\equiv 5 \pmod p$ since we know $W_2 = RC_0$ in this case. Thus $u_1 - u_2 = (n-4)(w_{a,b} - w_{a,c}) - (n-5)(w_b - w_c) \in RC_0$. From 4(3), $u_3 = -\frac{1}{2}(n-3)(n-6)w_b + \binom{n-4}{2}\jmath \in RC_0$ and $u_4 = -\frac{1}{2}(n-3)(n-6)w_c + \binom{n-4}{2}\jmath \in RC_0$. Thus $u_3 - u_4 = -\frac{1}{2}(n-3)(n-6)(w_b - w_c) \in RC_0$, and since also $n \not\equiv 3 \pmod p$,we have $E(W_1) \subseteq RC_0$ and then from the above that $E(W_2) \subseteq RC_0$ for $n \not\equiv 3, 4, 6 \pmod p$.

Now suppose $n \equiv 3 \pmod p$. Then $\nu_0^* \neq 0$, so $\jmath \in RC_0$. From Identities 4(1), we have $-w_{a,b} + 2(w_a + w_b), -w_{a,c} + 2(w_a + w_c), -w_{b,c} + 2(w_b + w_c) \in$

RC_0, and summing gives $-(w_{a,b}+w_{a,c}+w_{b,c})+4(w_a+w_b+w_c) \in RC_0$. From Identities 3, we see that $4(w_{a,b}+w_{a,c}+w_{b,c})-4(w_a+w_b+w_c) \in RC_0$, and summing with the previous element gives $(w_{a,b}+w_{a,c}+w_{b,c}) \in RC_0$ and $(w_a+w_b+w_c) \in RC_0$. Now $RC_0 \ni -w_{a,b}+2(w_a+w_b) = -w_{a,b}+2(w_a+w_b+w_c)-2w_c$, and hence $w_{a,b}+2w_c \in RC_0$ for all a,b,c and subtracting shows that $E(W_2) \subseteq RC_0$.

Now suppose $n \equiv 4 \pmod{p}$. Then $\nu_0^* \neq 0$, so $\boldsymbol{\jmath} \in RC_0$, and $W_1 \subseteq RC_0$ by Identities 4(3). From Identities 3, we see that $(w_{a,b}+w_{a,c}+w_{b,c}) \in RC_0$ for all $a,b,c \in \Omega$. In particular we have $(w_{a,b}+w_{a,d}+w_{b,d}), (w_{a,d}+w_{a,c}+w_{d,c}) \in RC_0$, and subtracting gives $(w_{a,b}-w_{a,c})+(w_{d,b}-w_{d,c}) \in RC_0$. By symmetry we also have $(w_{c,a}-w_{c,d})+(w_{b,a}-w_{b,d}) \in RC_0$. Summing these two last elements gives $2(w_{a,b}-w_{c,d}) \in RC_0$ and hence $E(W_2) \subseteq RC_0$.

(2). Suppose that $\boldsymbol{\jmath} \in RC_0$. Since $3\boldsymbol{\jmath} = \sum_{a,b} w_{a,b} = \sum_{a,b}(w_{a,b}-w_{c,d})+\binom{n}{2}w_{c,d}$, if $n \not\equiv 0,1 \pmod{p}$, $w_{c,d} \in RC_0$. However, for $n \equiv 0,1 \pmod{p}$ we have $RC_0 = W_2$ by Lemma 17, unless $p=5$ and $n \equiv 1 \pmod{5}$. But we have excluded this case because $1 \equiv 6 \pmod{5}$. Thus the result follows since $\nu_0^*\boldsymbol{\jmath} = \sum_{x\in V} s_x^0$.

(3). If $\nu_0^* \equiv 0 \pmod{p}$ and $n \not\equiv 2 \pmod{p}$ then $n \equiv 5 \pm \sqrt{-2} \pmod{p}$. Now it can be proved that -2 is a square in $\mathbb{F}_p$ for prime $p \geq 5$ if and only if $p \equiv 1,3 \pmod{8}$: see [2, Lemma 2.10.1] where the proof can be modified easily to the case of -2. If $\nu_0^* \equiv 0 \pmod{p}$ then $RC_0 \subset \langle \boldsymbol{\jmath} \rangle^\perp$, so if $RC_0 = W_2$ then $W_2 \subset \langle \boldsymbol{\jmath} \rangle^\perp$, so $n \equiv 2 \pmod{p}$. Thus if $n \not\equiv 2 \pmod{p}$ and $\nu_0^* \equiv 0 \pmod{p}$ then $RC_0 = E(W_2)$. ■

For design theory notation see [2, Chapter 1], and for a discussion of Steiner triple systems see [2, Chapter 8].

Proposition 7 *For all primes p, if $n \geq 7$ and $n \equiv 1,3 \pmod{6}$ then $\frac{1}{6}(n^2 - 10n+27)\boldsymbol{\jmath} \in RC_0$. If also $n \equiv 2 \pmod{p}$ and $p \geq 5$, then $RC_0 = W_2$ unless, possibly, $p = 11$.*

Proof: By [17,20], a Steiner triple system on n points, i.e. a 2-$(n,3,1)$ design, exists if and only if $n \equiv 1,3 \pmod{6}$. Let $\mathcal{S}$ be a Steiner triple system on n points with block set $\mathcal{B}$. Then if $\Omega = \{1,\ldots,n\}$, $\mathcal{B} \subset \Omega^{\{3\}} = V$, and the number r of blocks through a point is $r = \frac{1}{2}(n-1)$. Thus $b = |\mathcal{B}| = \frac{1}{6}n(n-1)$.

Let $w = \sum_{x\in\mathcal{B}} s_x^0$. For $x \in \mathcal{B}$, x meets $3(r-1) = \frac{3}{2}(n-3)$ other blocks, so x is disjoint from, or equal to, $b-3(r-1) = b-\frac{3}{2}(n-3) = \frac{1}{6}(n^2-10n+27)$ blocks. Thus for $x \in \mathcal{B}$, $w(x) = \frac{1}{6}(n^2-10n+27)$.

For $x \in \Omega^{\{3\}} \setminus \mathcal{B}$, we have $x = \{a,b,c\}$ is not a block of $\mathcal{S}$, and blocks in $\mathcal{B}$ can meet x in only one element, so the number of blocks meeting x is $3r-3$, since the blocks ab, ac and bc of $\mathcal{B}$ are counted twice. Thus the number not meeting x is $b-(3r-3) = \frac{1}{6}(n^2-10n+27)$, as above, so for $x \notin \mathcal{B}$, we have $w(x) = \frac{1}{6}(n^2-10n+27)$, and so $w = \frac{1}{6}(n^2-10n+27)\boldsymbol{\jmath}$, as asserted.

Suppose $n = 2+kp$. Then $\frac{1}{6}(n^2-10n+27) = \frac{1}{6}((2+kp)^2-10(2+kp)+27) = \frac{1}{6}((kp)^2-6(kp)+11) = \frac{1}{6}11$, which is non-zero if $p \neq 11$. ■

Lemma 18 *If $p = 5$, $n \geq 7$, $n \equiv 2 \pmod{5}$, then $E(C_2) = W_1 + W_2^\perp$, and $\dim(C_2) = \binom{n}{3} - \binom{n}{2} + n$.*

Proof: By Identities 4 (10), $w_a + w_b \in C_2$ for all $a, b \in \Omega$, and hence $W_1 \subseteq C_2$. By Lemmas 10, 14, $W_\Pi = W_2^\perp \subseteq C_2$. It is quite direct to show that $W_1 \cap W_2^\perp = \langle \boldsymbol{\jmath} \rangle$, and thus $\dim(W_1 + W_2^\perp) = \binom{n}{3} - \binom{n}{2} + n - 1$. We show that $E(C_2) = W_1 + W_2^\perp$ by showing that for each pair $x, y \in V$, there is an element $w = \sum_{i=1}^n \alpha_i w_i$ such that $r_x^2 - r_y^2 - w \in W_2^\perp$. There are three cases for x, y to be considered and they are covered by the following: (1) $r^2_{\{1,2,3\}} - r^2_{\{1,2,4\}} + 2(w_3 - w_4)$; (2) $r^2_{\{1,2,3\}} - r^2_{\{1,4,5\}} - 3(w_2 + w_3) - 2(w_4 + w_5)$; (3) $r^2_{\{1,2,3\}} - r^2_{\{4,5,6\}} + 2(w_1 + w_2 + w_3) - 2(w_4 + w_5 + w_6)$.

Thus $E(C_2) \subseteq W_1 + W_2^\perp \subseteq C_2$. Since $W_1 + W_2^\perp \subseteq \langle \boldsymbol{\jmath} \rangle^\perp$, but $C_2 \not\subseteq \langle \boldsymbol{\jmath} \rangle^\perp$, we have $W_1 + W_2^\perp = E(C_2)$, and thus $\dim(C_2) = \binom{n}{3} - \binom{n}{2} + n$. ■

8 Summary for codes from $\Gamma(n,3,r)$ and $R\Gamma(n,3,r)$, $n \geq 7$, $p \geq 5$

We summarize the results for the codes C_i and RC_i from $\Gamma(n,3,r)$ and $R\Gamma(n,3,r)$, respectively, for $p \geq 5$ that we have established in Section 7. Note that there are open questions remaining. Take $n \geq 7$, $p \geq 5$ in all the following.

8.1 C_0

$\nu_0 = \binom{n-3}{3}$; $\forall p$, $W_2^\perp \subseteq C_0$, $W_2 \not\supseteq C_0$ (Lemma 14)

1. $n \not\equiv 3, 4, 5 \pmod p \implies C_0 = \mathbb{F}_p^V$ (Lemma 17 (1))
2. $n \equiv 3 \pmod p \implies C_0 = \langle \boldsymbol{\jmath} \rangle^\perp$ (Lemma 17 (2))
3. $n \equiv 4 \pmod p \implies C_0 = W_1^\perp$ (Lemma 15 (1) gives $C_0 \subseteq W_1^\perp$ so $\dim(C_0) \leq \binom{n}{3} - n$. Eigenvalues for Kneser gives $\lambda_0 = \lambda_1 = 0$, and the other two non-zero, so $\dim(C_0) \geq \binom{n}{3} - (m_0 + m_1) = \binom{n}{3} - n$. Thus $\dim(C_0) = \binom{n}{3} - n$ and $C_0 = W_1^\perp$)
4. $n \equiv 5 \pmod p \implies C_0 = W_2^\perp$ (Lemma 14 (1) gives $W_2^\perp \subseteq C_0$, Lemma 15 (1) gives $C_0 \subseteq W_2^\perp$, hence equality)

n cong	$\not\equiv 3,4,5$	$\equiv 3$	$\equiv 4$	$\equiv 5$
$p \geq 5$				
ν	$\not\equiv 0$	$\equiv 0$	$\equiv 0$	$\equiv 0$
C_0	$\mathbb{F}_p^V$	$\langle \boldsymbol{\jmath} \rangle^\perp$	$W_1^\perp$	$W_2^\perp$

Table 1 C_0, $p \geq 5$, congruences modulo p

8.2 RC_0

$\nu_0^* = \nu_0 + 1 = \binom{n-3}{3} + 1 = \frac{1}{6}(n-2)(n^2-10n+27)$; $\forall p$, $RC_0 \subseteq W_2$, $W_2^\perp \not\subseteq RC_0$ (Lemma 14)

1. $n \equiv 6 \pmod p \implies W_2 = RC_0 + W_1$, $RC_0 \cap W_1 = \langle \boldsymbol{\jmath} \rangle$, $\dim(RC_0) = \binom{n-1}{2}$ (Proposition 5)
2. $n \not\equiv 6 \pmod p \implies E(W_2) \subseteq RC_0 \subseteq W_2$ (Proposition 6 and Lemma 13)
   (a) $\nu_0^* \not\equiv 0 \pmod p \implies RC_0 = W_2$ (Proposition 6)
   (b) $\nu_0^* \equiv 0 \pmod p), n \not\equiv 2 \pmod p \implies n \equiv 5 \pm \sqrt{-2} \pmod p$, for $p \equiv 1, 3 \pmod 8 \implies RC_0 = E(W_2)$ (Proposition 6)
   (c) $n \equiv 2 \pmod p), n \equiv 1, 3 \pmod 6), p \neq 11 \implies RC_0 = W_2$ (Proposition 7)

n cong	$\equiv 6$	$\equiv 2$, $\equiv 1, 3 \pmod 6$	$\not\equiv 6$	$\not\equiv 2, 6$, $\equiv 5 \pm \sqrt{-2}$	$\equiv 2$, $\not\equiv 1, 3 \pmod 6$
$p \geq 5$		$\neq 11$			
ν^*	$\not\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\equiv 0$	$\equiv 0$
RC_0	$W_2 = RC_0 + W_1$	W_2	W_2	$E(W_2)$	$E(W_2) \subseteq RC_0 \subseteq W_2$

Table 2 RC_0, $p \geq 5$, congruences modulo p

8.3 C_1

$\nu_1 = 3\binom{n-3}{2}$; $\forall p$, $W_2^\perp \subseteq C_1$, $W_2 \not\supseteq C_1$ (Lemma 14)

1. $n \equiv 3 \pmod p \implies E(W_1) \subseteq C_1 \subseteq \langle \boldsymbol{\jmath} \rangle^\perp$ and for $p > 5$, $C_1 = \langle \boldsymbol{\jmath} \rangle^\perp$ (Proposition 4)
2. $n \equiv 4 \pmod p \implies C_1 = W_1^\perp$ (Identities 2 gives $C_1 = C_0$, and $C_0 = W_1^\perp$ by Subsection 8.1)
3. if $n \equiv 9 \pmod p$ then $p \geq 7$, and if $p = 7$ then $W_1 \subseteq C_1$; if $p > 7$, then $C_1 \subseteq W_1^\perp + \langle \boldsymbol{\jmath} \rangle$. (Proposition 4)
4. $n \not\equiv 3, 4, 9 \pmod p \implies W_1 \subseteq C_1, \dim(C_1) \geq \dim(W_2^\perp) + n - 1$ (Proposition 4)
5. $n \not\equiv 3, 4, 9 \pmod p), 2n \not\equiv 11 \pmod p \implies C_1 = \mathbb{F}_p^V$ (Proposition 4)

8.4 RC_1

$\nu_1^* = \nu_1 + 1 = 3\binom{n-3}{2} + 1 = \frac{1}{2}(3n^2 - 21n + 38)$; $\forall p$, $W_2^\perp \subseteq RC_1$, $W_2 \not\supseteq RC_1$ (Lemma 14)

1. $n \equiv 6 \pmod p \implies RC_1 = W_1 \oplus W_2^\perp$ ($RC_1 = RC_2$ by Identities 2 (3),(5), so see 11.6 below)

n cong	$\equiv 3$	$\equiv 3$	$\equiv 4$	$\equiv 9$	$\equiv 9$	$\not\equiv 3,4,9$	$\not\equiv 3,4,$ $9,\frac{11}{2}$
$p \geq 5$	5	>5		7	>7		
ν	$\equiv 0$	$\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
$C_1 \supseteq W_2^\perp$	$E(W_1) \subseteq C_1$	$\langle \jmath \rangle^\perp$	$W_1^\perp$	$\supseteq W_1 + W_2^\perp$	$\subseteq \langle W_1^\perp, \jmath \rangle$	$\supseteq W_1 + W_2^\perp$	$\mathbb{F}_p^V$

Table 3 C_1, $p \geq 5$, congruences modulo p

2. $n \not\equiv 6 \pmod p, \nu_1^* \equiv 0 \pmod p \implies RC_1 \subseteq \langle \jmath \rangle^\perp$, $(n^2-13n+38) \not\equiv 0 \pmod p \implies RC_1 = \langle \jmath \rangle^\perp$; $(n^2-13n+38) \equiv 0 \pmod p \implies n \equiv 0 \pmod{19}, p = 19 \implies RC_1 \subseteq E(W_1)^\perp$ (Proposition 4)
3. $n \not\equiv 6 \pmod p, \nu_1^* \not\equiv 0 \pmod p, (n^2-13n+38) \not\equiv 0 \pmod p \implies RC_1 = \mathbb{F}_p^V$ (Proposition 4)
4. $n \not\equiv 6 \pmod p, \nu_1^* \not\equiv 0 \pmod p, (n^2-13n+38) \equiv 0 \pmod p \implies W_2^\perp \subseteq RC_1 \subseteq E(W_1)^\perp$ (Proposition 4).

n cong	$\equiv 6$	$\not\equiv 6$; $h(n) \not\equiv 0$	$\not\equiv 6$; $h(n) \equiv 0$	$\not\equiv 6$; $h(n) \not\equiv 0$	$\not\equiv 6$; $h(n) \equiv 0$
$p \geq 5$	$5,\ >5$		19		
ν^*	$\equiv 0, \not\equiv 0$	$\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
$RC_1 \supseteq W_2^\perp$	$W_1 \oplus W_2^\perp$	$\langle \jmath \rangle^\perp$	$\subseteq E(W_1)^\perp$	$\mathbb{F}_p^V$	$\subseteq E(W_1)^\perp$

Table 4 RC_1, $p \geq 5$, congruences modulo p, $h(n) = n^2 - 13n + 38$

8.5 C_2

$\nu_2 = 3(n-3)$; $\forall p$, $W_2^\perp \subseteq C_2$, $W_2 \not\supseteq C_2$ (Lemma 14)

1. $n \not\equiv 3, 7, 9/2 \pmod p \implies C_2 = \mathbb{F}_p^V$ (Lemma 16)
2. $n \equiv 3 \pmod p \implies C_2 = \langle \jmath \rangle^\perp$ (Lemma 16)
3. $n \equiv 7 \pmod p, p \neq 5 \implies \dim(C_2) = \binom{n}{3} - \binom{n}{2} + n, C_2 = W_1 \oplus W_2^\perp$ (Lemma 16 and $\jmath \in C_2 \implies W_1 \subseteq C_2$ by Identities 4 (12), $W_1 \cap W_2^\perp \subseteq \mathrm{Hull}(W_2) = \{0\}$ by Lemma 11)
4. $n \equiv 9/2 \pmod p, p \neq 5 \implies \dim(C_2) = \binom{n}{3} - n + 1, C_2 = E(W_1)^\perp$ (Lemma 16, and by Identities 5, $C_2 \subseteq E(W_1)^\perp$, and $\dim(E(W_1)^\perp) = \binom{n}{3} - n + 1$)
5. $n \equiv 2 \pmod p, p = 5 \implies E(C_2) = W_1 + W_2^\perp, \dim(C_2) = \binom{n}{3} - \binom{n}{2} + n$ (Lemma 18)

8.6 RC_2

$\nu_2^* = \nu_2 + 1 = 3(n-3) + 1$; $\forall p$, $W_2^\perp \subseteq RC_2$, $W_2 \not\supseteq RC_2$ (Lemma 14)

n cong	$\not\equiv 3, 7, \frac{9}{2}$	$\equiv 3$	$\equiv 7$	$\equiv \frac{9}{2}$	$\equiv 2$
$p \geq 5$			> 5	> 5	5
ν	$\not\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\not\equiv 0$
C_2	$\mathbb{F}_p^V$	$\langle \jmath \rangle^\perp$	$W_1 \oplus W_2^\perp$	$E(W_1)^\perp$	$E(C_2) = W_1 + W_2^\perp$

Table 5 C_2, $p \geq 5$, congruences modulo p

1. if $n \not\equiv 4, 6, 8/3 \pmod p \implies RC_2 = \mathbb{F}_p^V$ (Lemma 16)
2. if $n \equiv 8/3 \pmod p), p > 5 \implies RC_2 = \langle \jmath \rangle^\perp$ (Lemma 16)
3. if $n \equiv 4 \pmod p \implies RC_2 = E(W_1)^\perp$ (Lemma 16)
4. if $n \equiv 6 \pmod p \implies \dim(RC_2) = \binom{n}{3} - \binom{n}{2} + n, RC_2 = W_1 \oplus W_2^\perp$ (for $p \neq 5$: Lemma 16, and $\jmath \in RC_2 \implies W_1 \subseteq RC_2$ by Identities 4 (11), $W_1 \cap W_2^\perp \subseteq \mathrm{Hull}(W_2) = \{0\}$ by Lemma 11. For $p = 5$: by Lemma 16, $\dim(RC_2) \geq \binom{n}{3} - m_0 - m_2 = \binom{n}{3} - \binom{n}{2} + n - 1$; by Identities 4 (9), $W_1 \subseteq RC_2$, so $W_1 \oplus W_2^\perp \subseteq RC_2$. Since $s^2_{\{a,b,c\}} - 2\jmath + w_a + w_b + w_c \in W_2^\perp$, equality follows)

n cong	$\not\equiv 4, 6, \frac{8}{3}$	$\equiv \frac{8}{3}$	$\equiv 4$	$\equiv 6$	$\equiv 6$
$p \geq 5$		> 5		> 5	5
ν^*	$\not\equiv 0$	$\equiv 0$	$\not\equiv 0$	$\not\equiv 0$	$\equiv 0$
RC_2	$\mathbb{F}_p^V$	$\langle \jmath \rangle^\perp$	$E(W_1)^\perp$	$W_1 \oplus W_2^\perp$	$W_1 \oplus W_2^\perp$

Table 6 RC_2, $p \geq 5$, congruences modulo p

According to Magma, the unsolved cases for $k = 3, p \geq 5$:
8.2 (2)(c): $RC_0 = W_2$ unless $p = 11$ when $RC_0 = E(W_2)$; 8.3 (1): for $p = 5$, $\dim(C_1) = \dim(W_2^\perp) + n - 1$; 8.3 (3): for $p = 7$, $\dim(C_1) = \dim(W_2^\perp) + n$; $p > 7$, $C_1 = W_1^\perp + \langle \jmath \rangle$; 8.3.(4): $\dim(C_1) = \dim(W_2^\perp) + n - 1$; 8.4 (2): if $p = 19, n \equiv 0 \pmod{19}$ then $RC_1 = E(W_1)^\perp$; 8.4 (4): $RC_1 = E(W_1)^\perp$.

8.7 Summary for codes from $\Gamma(n, 3, r)$ for $p = 2, 3, 5, 7$

Tables 7, 8 summarize the results from [14,15,12,11] for $\Gamma(n, 3, r)$ and $R\Gamma(n, 3, r)$ for $p = 2, 3$, respectively. Tables 9, 10 summarize what we have shown in Tables 1 to 6 for $p = 5, 7$, respectively. In the tables, $H(C) = \mathrm{Hull}(C)$.

9 Conclusion

It can be shown by similar methods that for $\Gamma(n, 2, r)$ and $R\Gamma(n, 2, r)$, $r = 0, 1$, C_r, RC_r are always one of

$$\langle \jmath \rangle^\perp, \mathbb{F}_p^V, W_1, E(W_1), W_1^\perp, E(W_1)^\perp, \langle \jmath, W_1 \rangle, \langle \jmath, E(W_1) \rangle, W_\Pi, W_\Pi^\perp.$$

n cong	C_0	RC_0	C_1	RC_1	C_2	RC_2
0	$W_1^\perp$	$W_1 + H(W_2)$	$W_1^\perp$	W_1	$\mathbb{F}_2^V$	$H(W_2)$
1	$W_1^\perp \cap W_2^\perp$	$W_1 + W_2$	$E(W_1)^\perp$	W_1	$W_2^\perp$	W_2
2	$\mathbb{F}_2^V$	$\subset W_1 + H(W_2)$	$\mathbb{F}_2^V$	W_1	$\mathbb{F}_2^V$	$H(W_2)$
3	$W_2^\perp$	$W_1 + W_2$	$\langle \jmath \rangle^\perp$	W_1	$W_2^\perp$	W_2

Table 7 $\Gamma(n,3,r)$, $R\Gamma(n,3,r)$, $p = 2$, congruences modulo 4

n cong	C_0	RC_0	C_1	RC_1	C_2	RC_2
0,0	$\mathbb{F}_3^V$	$\subset W_2 + \langle \jmath \rangle$	$E(W_2)$	$RC_0^\perp + W_2^\perp$	$E(W_2)$	RC_1
0,3	$\langle \jmath \rangle^\perp$	$\subset W_2 + \langle \jmath \rangle$	$E(W_2)$	$RC_0^\perp + W_2^\perp$	$E(W_2)$	RC_1
0,6	$\mathbb{F}_3^V$	$\subset W_2 + \langle \jmath \rangle$	$E(W_2)$	$RC_0^\perp + W_2^\perp$	$E(W_2)$	RC_1
1,1	$W_1^\perp$	$E(W_2) + \langle \jmath \rangle$	$H(W_2)$	$\mathbb{F}_3^V$	$E(W_2)$	$W_1^\perp$
1,4	$W_1^\perp = C_0 + \langle \jmath \rangle$	$E(W_2) + \langle \jmath \rangle$	$H(W_2)$	$\mathbb{F}_3^V$	$E(W_2)$	$W_1^\perp$
1,7	$W_1^\perp$	$E(W_2) + \langle \jmath \rangle$	$H(W_2)$	$\mathbb{F}_3^V$	$E(W_2)$	$W_1^\perp$
2,2	$W_2^\perp$	$W_2 + \langle \jmath \rangle$	W_2	$\mathbb{F}_3^V$	W_2	$\mathbb{F}_3^V$
2,5	$(W_2 + \langle \jmath \rangle)^\perp$	$W_2 + \langle \jmath \rangle$	W_2	$\mathbb{F}_3^V$	W_2	$\mathbb{F}_3^V$
2,8	$W_2^\perp$	$W_2 + \langle \jmath \rangle$	W_2	$\mathbb{F}_3^V$	W_2	$\mathbb{F}_3^V$

Table 8 $\Gamma(n,3,r)$, $R\Gamma(n,3,r)$, $p = 3$, congruences modulo 3,9

n cong	C_0	RC_0	C_1	RC_1	C_2	RC_2
0	$W_2^\perp$	W_2	$\mathbb{F}_5^V$	$\mathbb{F}_5^V$	$\mathbb{F}_5^V$	$\mathbb{F}_5^V$
1	$\mathbb{F}_5^V$	$W_2 = RC_0 + W_1$	$\mathbb{F}_5^V$	$W_1 \oplus W_2^\perp$	$\mathbb{F}_5^V$	$W_1 \oplus W_2^\perp$
2	$\mathbb{F}_5^V$	$E(W_2) \subseteq RC_0 \subseteq W_2$	$\mathbb{F}_5^V$	$\mathbb{F}_5^V$	$E(C_2) = W_2^\perp + W_1$	$\mathbb{F}_5^V$
3	$\langle \jmath \rangle^\perp$	W_2	$\supseteq W_2^\perp$	$\mathbb{F}_5^V$	$\langle \jmath \rangle^\perp$	$\mathbb{F}_5^V$
4	$W_1^\perp$	W_2	$W_1^\perp$	$\mathbb{F}_5^V$	$\mathbb{F}_5^V$	$E(W_1)^\perp$

Table 9 $\Gamma(n,3,r)$, $R\Gamma(n,3,r)$, $p = 5$, congruences modulo 5

n cong	C_0	RC_0	C_1	RC_1	C_2	RC_2
0	$\mathbb{F}_7^V$	W_2	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$	$W_1 \oplus W_2^\perp$	$\mathbb{F}_7^V$
1	$\mathbb{F}_7^V$	W_2	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$	$W_1^\perp$	$\mathbb{F}_7^V$
2	$\mathbb{F}_7^V$	$E(W_2) \subseteq RC_0 \subseteq W_2$	$\supseteq W_1 + W_2^\perp$	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$
3	$\langle \jmath \rangle^\perp$	W_2	$W_1^\perp$	$\mathbb{F}_7^V$	$\langle \jmath \rangle^\perp$	$\mathbb{F}_7^V$
4	$W_1^\perp$	W_2	$W_1^\perp$	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$	$E(W_1)^\perp$
5	$W_2^\perp$	W_2	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$	$\mathbb{F}_7^V$	$\langle \jmath \rangle^\perp$
6	$\mathbb{F}_7^V$	$W_2 = RC_0 + W_1$	$\mathbb{F}_7^V$	$W_1 \oplus W_2^\perp$	$\mathbb{F}_7^V$	$W_1 \oplus W_2^\perp$

Table 10 $\Gamma(n,3,r)$, $R\Gamma(n,3,r)$, $p = 7$, congruences modulo 7

Note that $\Gamma(n,2,0) = KG_{n,2}$, and $\Gamma(n,2,1) = J(n,2)$, the triangular graph. We summarize the results for $\Gamma(n,2,r)$ and $R\Gamma(n,2,r)$ for $p = 2,3$ in Tables 11 and 12, where $|V| = \binom{n}{2}$. Some of the proofs can be found in [14,8], and those that are not there can be deduced easily in a way similar to the proofs here for $k = 3$.

n cong	C_0	RC_0	C_1	RC_1
0	$\mathbb{F}_2^V$	$\langle E(W_1), \jmath\rangle$	$E(W_1)$	$\mathbb{F}_2^V$
1	$W_1^\perp$	$\langle W_1, \jmath\rangle$	W_1	$W_1^\perp$
2	$\langle \jmath\rangle^\perp$	$\langle E(W_1), \jmath\rangle$	$E(W_1)$	$\mathbb{F}_2^V$
3	W_Π	$W_\Pi^\perp$	W_1	$W_1^\perp$

Table 11 $\Gamma(n, 2, r)$, $R\Gamma(n, 2, r)$, $p = 2$, congruences modulo 4

n cong	C_0	RC_0	C_1	RC_1
0	W_Π	$\mathbb{F}_3^V$	$\mathbb{F}_3^V$	$\langle W_1^\perp, \jmath\rangle$
1	$\mathbb{F}_3^V$	$E(W_1)^\perp$	$E(W_1)^\perp$	$\mathbb{F}_3^V$
2	$\langle \jmath\rangle^\perp$	$\mathbb{F}_3^V$	$\langle \jmath\rangle^\perp$	$\mathbb{F}_3^V$

Table 12 $\Gamma(n, 2, r)$, $R\Gamma(n, 2, r)$, $p = 3$, congruences modulo 3

Thus, what was found for $k = 2, 3$, all p, and for other values of k (for example for Johnson and odd graphs for all k and $p = 2$, from Fish [8]): the codes of $\Gamma(n, k, r)$ and $R\Gamma(n, k, r)$ are all some combination of the codes C, where C is W_s, $E(W_s)$, W_Π, $\langle \jmath\rangle$, $\mathbb{F}_p^V$, along with their duals and hulls. The tables do show some inconclusive results, but computational examination with Magma of these cases all indicate that the above statement holds in general.

We leave a more complete study of the codes W_i, W_Π, in the general case, to a future paper. We have also not included the codes from the complementary graphs in our discussion; these are uniform subset graphs only if $k = 2$. Since our aim was to describe, as far as possible, the codes from $\Gamma(n, k, r)$ in terms of the W_i, W_Π, we have not listed the minimum weights in the tables, nor the hulls. Some of these can be deduced from Sections 3, 4, 5 and 7. In addition some of the codes are LCD, i.e. have zero hull, and this happens in particular when $C_i = RC_i^\perp$, since $C_i + RC_i = \mathbb{F}_p^V$. For example if $n \equiv 5 \pmod p$, $p \geq 5$, then from Tables 1 and 2, $C_0^\perp = RC_0 = W_2$. An example of a zero hull when $C \neq RC^\perp$ is from Tables 3 and 4, where $C_1 = W_1^\perp$ for $n \equiv 4 \pmod p$, but $RC_1 \neq W_1$, and from Corollary 2, $\mathrm{Hull}(W_1) = \mathrm{Hull}(W_1^\perp) = \{0\}$. See also [16] for further results on LCD codes from graphs.

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