The reach, metric distortion, geodesic convexity and the variation of tangent spaces

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Abstract

In this paper we discuss three results. The first two concern general sets of positive reach: We first characterize the reach by means of a bound on the metric distortion between the distance in the ambient Euclidean space and the set of positive reach. Secondly, we prove that the intersection of a ball with radius less than the reach with the set is geodesically convex, meaning that the shortest path between any two points in the intersection lies itself in the intersection. For our third result we focus on manifolds with positive reach and give a bound on the angle between tangent spaces at two different points in terms of the distance between the points and the reach.

1 Introduction

Metric distortion quantifies the maximum ratio between geodesic and Euclidean distances for pairs of points in a set $\mathcal{S}$. The reach of $\mathcal{S}$, defined by H. Federer [15], is the infimum of distances between points in $\mathcal{S}$ and points in its medial axis. Both reach and metric distortion are central concepts in manifold (re-)construction and have been used to characterize the size of topological features. Amenta and Bern [1] introduced a local version of the reach in order to give conditions for homeomorphic surface reconstruction and this criterion has been used in many works aiming at topologically faithful reconstruction. See the seminal paper of Niyogi, Smale and Weinberger [19] and Dey's book [12] for more context and references. A direct relation between the reach and the size of topological features is simply illustrated by the fact that the intersection of a set with reach $r > 0$ with a ball of radius less than r has reach at least r and is contractible [3]. In a certain way, metric distortion also characterizes the size of topological features. This is illustrated by the fact that a compact subset of $\mathbb{R}^n$ with metric distortion less than $\pi/2$ is simply connected (Section 1.14 in [16] , see also appendix A by P. Pansu where sets with a given metric distortion are called *quasi convex sets*).

In the first part of this paper, we provide tight bounds on metric distortion for sets of positive reach and, in a second part, we consider submanifolds of $\mathbb{R}^d$ and bound the angle between tangent spaces at different points. Whenever we mention manifolds we shall tacitly assume that it is embedded in Euclidean space. Previous versions of the metric distortion result, restricted to the manifold setting can be found in [19]. A significant amount of attention has gone to tangent space variation, see [4, 5, 7, 10, 12, 13, 19] to name but a few.

Our paper improves on these bounds, extend the results beyond the case of smooth manifolds and offers new insights and results. These results have immediate algorithmic consequences by, on one hand, improving the sampling conditions under which known reconstruction algorithms are valid and, on the other hand, allowing us to extend the algorithms to the class of manifolds of positive reach, which is much larger than the usually considered class of C^2 manifolds. Indeed, the metric distortion and tangent variation bounds for $C^{1,1}$ manifolds presented in this paper in fact suffice to extend the triangulation result of C^2 manifolds embedded in Euclidean space given in [6] to arbitrary manifolds with positive reach, albeit with slightly worse constants.

Overview of results For metric distortion, we extend and tighten the previously known results so much that our metric distortion result can be regarded as a completely new characterization of sets of positive reach. In particular, the standard manifold and smoothness assumptions are no longer necessary. Based on our new characterization of the reach by metric distortion, we can prove that the intersection of a set of positive reach with a ball with radius less than the reach is geodesically convex. This result is a far reaching extension of a result of [8] that has attracted significant attention, stating that, for smooth surfaces, the intersection is a pseudo-ball.

To study tangent variation along manifolds, we will consider two different settings, namely the C^2 setting, for which the bounds are tight, and the $C^{1,1}$ setting, where we achieve slightly weaker bounds.

The exposition for C^2 manifolds is based on differential geometry and is a consequence of combining the work of Niyogi, Smale, and Weinberger [19], and the two dimensional analysis of Attali, Edelsbrunner, and Mileyko [2] with some observations concerning the reach. We would like to stress that some effort went into simplifying the exposition, in particular the part of that article concerning the second fundamental form.

The second class of manifolds we consider consists of closed $C^{1,1}$ manifolds $\mathcal{M}$ embedded in $\mathbb{R}^d$. We restrict ourselves to $C^{1,1}$ manifolds because it is known that closed manifolds have positive reach if and only if they are $C^{1,1}$, see Federer [15, Remarks 4.20 and 4.21] and Scholtes [21] for a history of this result. Here we do not rely on differential geometry apart from simple concepts such as the tangent space. In fact most proofs can be understood in terms of simple Euclidean geometry. Moreover our proofs are very pictorial. Although the bounds we attain are slightly weaker than the ones we attain using differential geometry, we should note that we have sometimes simplified the exposition at the cost of weakening the bound.

We also prove that the intersection of a $C^{1,1}$ manifold with a ball of radius less than the reach of the manifold is a topological ball. This is a generalization of previous results. A sketch of a proof of the result in the C^2 case has been given by Boissonnat and Cazals [9]. Our result also extends a related result of Attali and Lieutier [3]. Is is furthermore related to the convexity result, but certainly not the same. This is because spaces can be geodesically convex without being topological disks, think for example of the equator of the sphere.

2 Metric distortion and convexity

For a closed set $\mathcal{S} \subset \mathbb{R}^d$, $d_{\mathcal{S}}$ denotes the geodesic distance in $\mathcal{S}$, i.e. $d_{\mathcal{S}}(a,b)$ is the infimum of lengths of paths in $\mathcal{S}$ between a and b. If there is at least one path between a and b with finite length, then it is known that a minimizing geodesic, i.e. a path with minimal length connecting a to b exists (see the second paragraph of part III, section 1: "Die Existenz geodätischer Bogen in metrischen Räumen" in [17]).

Next theorem can be read as an alternate definition of the reach, based on metric distortion. Observe that for fixed $|a-b|$, the function $r \mapsto 2r \arcsin \frac{|a-b|}{2r}$ is decreasing.

Theorem 2.1 *If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then*

$$\operatorname{rch} \mathcal{S} = \sup \left\{ r > 0,\, \forall a,b \in \mathcal{S},\, |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r} \right\},$$

where the sup *over the empty set is* 0.

Proof Lemma 2.5 states that if $r' < \operatorname{rch} \mathcal{S}$ then

$$\forall a,b \in \mathcal{S},\, |a-b| < 2r' \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r' \arcsin \frac{|a-b|}{2r'}.$$

This gives us

$$\sup \left\{ r > 0,\, \forall a,b \in \mathcal{S},\, |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r} \right\} \geq \operatorname{rch} \mathcal{S}.$$

If $\operatorname{rch} \mathcal{S} = \infty$, i.e. if $\mathcal{S}$ is convex, the theorem holds trivially. We assume now that the medial axis is non empty, i.e. $\operatorname{rch} \mathcal{S} < \infty$. Then by definition of the reach, if $r' > \operatorname{rch} \mathcal{S}$, there exists $x \in \mathbb{R}^d$ in the medial axis of $\mathcal{S}$ and $a, b \in \mathcal{S}, a \neq b$ such that $r' > r_x = d(x, \mathcal{S}) = d(x,a) = d(x,b)$. If for at least one of such pairs $\{a,b\}$ one has $d_{\mathcal{S}}(a,b) = \infty$ then $\|a-b\| \leq 2r_x < 2r'$ and:

$$\sup \left\{ r > 0,\, \forall a,b \in \mathcal{S},\, |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r} \right\} < r'$$

If not, consider a path γ in $\mathcal{S}$ between a and b: $\gamma(0) = a, \gamma(1) = b$. Because $\gamma([0,1])$ lies outside the open ball $B(x, r_x)^\circ$, its projection on the closed ball $B(x, r_x)$ cannot increase lengths. It follows that, for any $r \geq r'$:

$$d_{\mathcal{S}}(a,b) \geq 2r_x \arcsin \frac{|a-b|}{2r_x} > 2r \arcsin \frac{|a-b|}{2r}$$

which gives, for any $r' > \operatorname{rch} \mathcal{S}$,

$$\exists a, b \in \mathcal{S}, \forall r \geq r' \quad |a-b| < 2r \quad \text{and} \quad d_{\mathcal{S}}(a,b) > 2r \arcsin \frac{|a-b|}{2r},$$

and therefore

$$\sup \left\{ r > 0, \, \forall a, b \in \mathcal{S}, \, |a-b| < 2r \Rightarrow d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r} \right\} \leq r'.$$

□

Corollary 2.2 *Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with positive reach $r = \operatorname{rch} \mathcal{S} > 0$. Then, for any $r' < \operatorname{rch} \mathcal{S}$ and any $x \in \mathbb{R}^d$, if $B(x, r')$ is the closed ball centered at x with radius r', then $\mathcal{S} \cap B(x, r')$ is geodesically convex in $\mathcal{S}$.*

Proof First it follows from the theorem that if $a, b \in \mathcal{S} \cap B(x, r')$, then $d_{\mathcal{S}}(a,b) < \infty$ which means that there exists a path of finite length in $\mathcal{S}$ between a and b. From [17] there is at least one minimizing geodesic in $\mathcal{S}$ between a and b.

For a contradiction assume that a such geodesic γ goes outside $B(x, r')$. In other words there is at least one non empty open interval (t_1, t_2) such that $\gamma(t_1), \gamma(t_2) \in \partial B(x, r')$ and $\gamma((t_1, t_2)) \cap B(x, r') = \emptyset$. But then, since the projection on the ball $B(x, r')$ reduces lengths, one has:

$$d_{\mathcal{S}}(\gamma(t_1), \gamma(t_2)) > 2r' \arcsin \frac{|\gamma(t_1) - \gamma(t_2)|}{2r'},$$

a contradiction with the theorem. □

2.1 Projection of the middle point

For a closed set $\mathcal{S} \subset \mathbb{R}^d$ with positive reach $r = \operatorname{rch} \mathcal{S} > 0$ and a point $m \in \mathbb{R}^d$ with $d(m, \mathcal{S}) < r$, $\pi_{\mathcal{S}}(m)$ denotes the projection of m on $\mathcal{S}$ as depicted on Figure 1 on the left.

Lemma 2.3 *Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with reach $r = \operatorname{rch} \mathcal{S} > 0$. For $a, b \in \mathcal{S}$ such that $\delta = \frac{|a-b|}{2} < r$ and $m = \frac{a+b}{2}$ one has $|\pi_{\mathcal{S}}(m) - m| \leq \rho$, with $\rho = r - \sqrt{r^2 - \delta^2}$.*

The disk of center m and radius ρ appears in green on Figure 1 left and right.

Proof We shall now use a consequence Theorem 4.8 of [15]. In the following section we shall discuss this result for the manifold setting, where it generalizes the tubular neighbourhood results for C^2 manifolds from differential geometry and differential topology. For the moment we restrain ourselves to the following: If $\pi_{\mathcal{S}}(m) \neq m$ claim (12) in Theorem 4.8 of [15] gives us:

$$\forall \lambda \in [0, r), \, \pi_{\mathcal{S}} \left(\pi_{\mathcal{S}}(m) + \lambda \frac{m - \pi_{\mathcal{S}}(m)}{|m - \pi_{\mathcal{S}}(m)|} \right) = \pi_{\mathcal{S}}(m),$$

which means that for $\lambda \in [0, r)$:

$$y(\lambda) = \pi_{\mathcal{S}}(m) + \lambda \frac{m - \pi_{\mathcal{S}}(m)}{|m - \pi_{\mathcal{S}}(m)|}$$

is closer to $\pi_{\mathcal{S}}(m)$ than both to a and to b (see Figure 1).

Without loss of generality one assume that $|a - \pi_{\mathcal{S}}(m)| \geq |b - \pi_{\mathcal{S}}(m)|$. We denote $\mu = |\pi_{\mathcal{S}}(m) - m|$ and want to prove that $\mu \leq \rho$.

In the plane spanned by $a, b, \pi_{\mathcal{S}}(m)$ we consider the following frame $(m, \frac{a-m}{|a-m|}, \tau)$, where m denotes the origin, τ is a unit vector orthogonal to $a - m$ and such that $\langle \tau, \pi_{\mathcal{S}}(m) - m \rangle \leq 0$.

For some $\theta \in [0, \pi/2]$, the coordinates of $\pi_{\mathcal{S}}(m)$ in the frame are $(-\mu \sin\theta, -\mu \cos\theta)$. The coordinate of a are $(\delta, 0)$ and the coordinates of $y(\lambda)$ are, as shown in Figure 1, $((\lambda - \mu)\sin\theta, (\lambda - \mu)\cos\theta)$. Since $y(\lambda)$ is closer to $\pi_{\mathcal{S}}(m)$ than to a, one has

$$\forall \lambda \in [0, r), \quad (\delta - (\lambda - \mu)\sin\theta)^2 + (\lambda - \mu)^2 \cos^2\theta > \lambda^2.$$

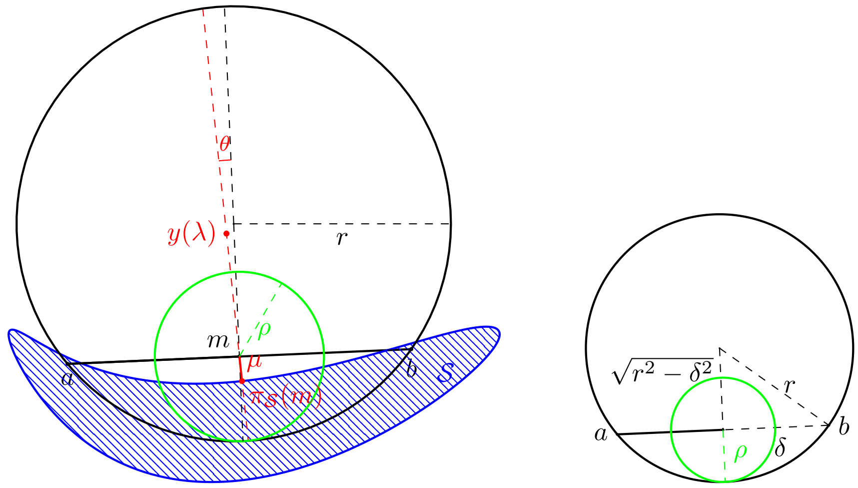

Figure 1: On the left the projection $\pi_{\mathcal{S}}(m)$ is contained in the disk of center m and radius ρ. The notation used in the proof of Lemma 2.3 is also added. From the right figure it is easy to deduce that $\rho = r - \sqrt{r^2 - \delta^2}$.

This is a degree 2 inequality in μ. One gets, for any $\lambda \in [0, r)$, if $\Delta \geq 0$,

$$\mu \notin \left[(\lambda - \delta \sin\theta) - \sqrt{\Delta},\ (\lambda - \delta\sin\theta) + \sqrt{\Delta}\right],$$

with $\Delta = (\lambda - \delta\sin\theta)^2 - (\delta^2 - 2\delta\lambda\sin\theta) = \lambda^2 - \delta^2 + (\delta\sin\theta)^2$. For $\lambda \geq \delta$ one has $\Delta \geq \lambda^2 - \delta^2$. Therefore: $(\lambda - \delta\sin\theta) - \sqrt{\Delta} \leq \lambda - \sqrt{\lambda^2 - \delta^2}$ and since $\lambda \mapsto \lambda - \sqrt{\lambda^2 - \delta^2}$ is continuous, one has:

$$\inf_{\lambda < r} \left\{(\lambda - \delta\sin\theta) - \sqrt{\Delta}\right\} \leq r - \sqrt{r^2 - \delta^2} = \rho,$$

also, when $\lambda \geq \delta$ one has $\sqrt{\Delta} \geq \delta\sin\theta$ and $(\lambda - \delta\sin\theta) + \sqrt{\Delta} \geq \delta$. Since $\mu \leq d(m, a) = \delta$, one finds that $\mu \leq \rho$. □

The following simple geometric Lemma is used in next section.

Lemma 2.4 *Consider a circle $\tilde{C}$ of radius r and two points $a, b \in \tilde{C}$ with $|a - b|/2 = \delta < r$. Define the middle point $m = \frac{a+b}{2}$ and consider a point p such that $|p - m| \leq \rho = r - \sqrt{r^2 - \delta^2}$. Denote $\tilde{C}_{a,b}$ the shortest of the arcs of circle in $\tilde{C}$ bounded by a and b. Define $\tilde{p} \in \tilde{C}_{a,b}$ as the unique point in $\tilde{C}_{a,b}$ such that $\frac{|a-\tilde{p}|}{|b-\tilde{p}|} = \frac{|a-p|}{|b-p|}$, then we have $|a - p| \leq |a - \tilde{p}|$ and $|b - p| \leq |b - \tilde{p}|$.*

The proof of this lemma is fairly straightforward and can be found in the appendix.

2.2 Upper bound on geodesic length

In this section we establish an upper bound on geodesic lengths through the iterative construction of a sequence of paths.

Lemma 2.5 *Let $\mathcal{S} \subset \mathbb{R}^d$ be a closed set with reach $r = \operatorname{rch}\mathcal{S} > 0$. For any $a, b \in \mathcal{S}$ such that $|a - b| < 2r$ one has $d_{\mathcal{S}}(a, b) \leq 2r \arcsin \frac{|a-b|}{2r}$.*

Proof We build two sequences of PL-functions (see Figure 2). For $i \in \mathbb{N}$, $\phi_i : [0, 1] \to \mathbb{R}^d$ and $\tilde{\phi}_i : [0, 1] \to \mathbb{R}^2$ are defined as follows.

First we define $\phi_0(t) = a + t(b - a)$. Denote $m = \frac{a+b}{2}$ the middle point of $[a, b]$. Since $d(m, \mathcal{S}) \leq d(m, a) = \delta < r$, the point $p = \pi_{\mathcal{S}}(m)$ is well defined. Secondly, we define

$$\phi_1(t) = \begin{cases} a + 2t(p - a) & \text{if} \quad t \leq 1/2 \\ p + (2t - 1)(b - p) & \text{if} \quad t \geq 1/2. \end{cases}$$

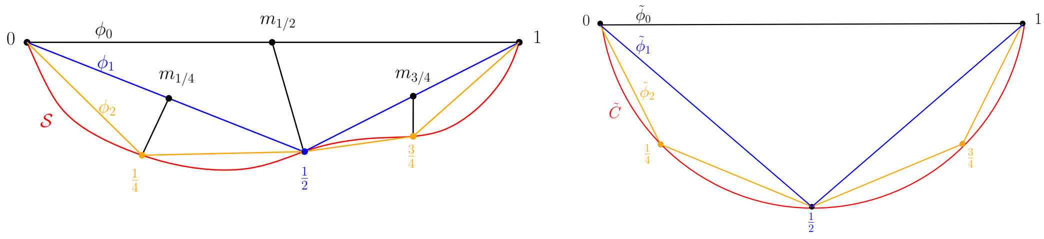

Figure 2: Left: ϕ_0, ϕ_1, ϕ_2 , Right: $\tilde{\phi}_0, \tilde{\phi}_1, \tilde{\phi}_2$.

as depicted in Figure 2 on the left.

From Lemma 2.3, one has $|p-m| \leq \rho = r - \sqrt{r^2 - \delta^2} < r$ and thus

$$\min(|a-p|, |b-p|) \geq \delta - \rho > 0 \qquad\qquad \max(|a-p|, |b-p|) \leq \delta + \rho.$$

We also fix a circle $\tilde{C}$ in $\mathbb{R}^2$ with radius r and we consider $\tilde{a}, \tilde{b} \in R^2$ such that $\tilde{a}, \tilde{b} \in \tilde{C}$ and $|\tilde{a} - \tilde{b}| = |a - b|$ and we define $\tilde{\phi}_0(t) = \tilde{a} + t(\tilde{b} - \tilde{a})$. Denote by $\tilde{C}_{\tilde{a},\tilde{b}}$ the shortest of the two arcs of $\tilde{C}$ bounded by $\tilde{a}, \tilde{b}$ and $\tilde{p}$ as constructed in Lemma 2.4 i.e. $\tilde{p} \in \tilde{C}_{\tilde{a},\tilde{b}}$ such that $\frac{|\tilde{p}-\tilde{a}|}{|\tilde{p}-\tilde{b}|} = \frac{|p-a|}{|p-b|}$, as shown in Figure 2 on the right, and define

$$\tilde{\phi}_1(t) = \begin{cases} \tilde{a} + 2t(\tilde{p} - \tilde{a}) & \text{if} \quad t \leq 1/2 \\ \tilde{p} + (2t-1)(\tilde{b} - \tilde{p}) & \text{if} \quad t \geq 1/2 \end{cases}$$

Applying Lemma 2.4 we get $|a - p| \leq |\tilde{a} - \tilde{p}|$, $|b - p| \leq |\tilde{b} - \tilde{p}|$, and

$$\text{length}(\phi_1) = |a-p| + |b-p| \leq |\tilde{a} - \tilde{p}| + |\tilde{b} - \tilde{p}| = \text{length}(\tilde{\phi}_1).$$

For $i \geq 2$, ϕ_i and $\tilde{\phi}_i$ are PL functions with 2^i intervals. For $k \in \mathbb{N}$, $0 \leq k \leq 2^i$, $\phi_i(k/2^i) \in \mathcal{S}$, $\tilde{\phi}_i(k/2^i) \in \tilde{C}_{\tilde{a},\tilde{b}}$ are defined by applying to each of the 2^{i-1} segments of $\phi_{i-1}([0,1])$ and $\tilde{\phi}_{i-1}([0,1])$ the same subdivision process used when defining ϕ_1 and $\tilde{\phi}_1$.

If k is even we set $\phi_i(k/2^i) = \phi_{i-1}(k/2^i)$ and $\tilde{\phi}_i(k/2^i) = \tilde{\phi}_{i-1}(k/2^i)$.

If k is odd define:

$$m_{k/2^i} = \frac{\phi_i((k-1)/2^i) + \phi_i((k+1)/2^i)}{2} \quad \text{and} \quad \phi_i(k/2^i) = \pi_{\mathcal{S}}\left(m_{k/2^i}\right)$$

Note that $m_{1/2}$ corresponds to m defined above.

Let $\tilde{\phi}_i(k/2^i) \in \tilde{C}_{\tilde{\phi}_{i-1}((k-1)/2^i), \tilde{\phi}_{i-1}((k+1)/2^i)} \subset \tilde{C}_{\tilde{a},\tilde{b}}$ be such that:

$$\frac{|\tilde{\phi}_i(k/2^i) - \tilde{\phi}_{i-1}((k-1)/2^i)|}{|\tilde{\phi}_i(k/2^i) - \tilde{\phi}_{i-1}((k+1)/2^i)|} = \frac{|\phi_i(k/2^i) - \phi_{i-1}((k-1)/2^i)|}{|\phi_i(k/2^i) - \phi_{i-1}((k+1)/2^i)|}.$$

Figure 2 left shows the curves ϕ_1 and ϕ_2 in blue and yellow respectively.

Applying Lemma 2.4, since by induction,

$$\left|\phi_{i-1}((k+1)/2^{i-1}) - \phi_{i-1}(k/2^{i-1})\right| \leq \left|\tilde{\phi}_{i-1}((k+1)/2^{i-1}) - \tilde{\phi}_{i-1}(k/2^{i-1})\right|$$

we get that for $i \in \mathbb{N}$ and $p = 0, \ldots, 2^i - 1$:

$$|\phi_i((k+1)/2^i) - \phi_i(k/2^i)| \leq |\tilde{\phi}_i((k+1)/2^i) - \tilde{\phi}_i(k/2^i)|,$$

and therefore:

$$\begin{aligned}\text{length}(\phi_i) &= \sum_{k=0}^{2^i-1} |\phi_i((k+1)/2^i) - \phi_i(k/2^i)| \\ &\leq \sum_{k=0}^{2^i-1} |\tilde{\phi}_i((k+1)/2^i) - \tilde{\phi}_i(k/2^i)| \\ &= \text{length}(\tilde{\phi}_i) \leq \text{length}(\tilde{C}_{\tilde{a},\tilde{b}}) = 2r \arcsin \frac{|a-b|}{2r}. \end{aligned} \tag{1}$$

We study now the behavior of the sequence $\phi_i, i \in \mathbb{N}$. Define $\delta_0 = \delta$ and $\rho_0 = \rho$. Further define δ_i as

$$\delta_i = \frac{1}{2} \max_{0 \leq k \leq 2^i - 1} |\phi_i((k+1)/2^i) - \phi_i(k/2^i)|.$$

i.e. half the max of lengths of all segments of $\phi_i([0,1])$ and $\rho_i = r - \sqrt{r^2 - \delta_i^2}$. We make the following assertion:

Claim 2.6

$$\lim_{i \to \infty} \delta_i = 0. \tag{2}$$

The proof of this claim is given in the appendix.

Since for any $i \geq 0$ and $t \in [0,1]$, $d(\phi(t), \mathcal{S}) \leq \delta_i$ and $\delta_i < \text{rch}\,\mathcal{S}$ the curves $\pi_{\mathcal{S}} \circ \phi_i$, projections of ϕ_i on $\mathcal{S}$ are well defined, with $\pi_{\mathcal{S}} \circ \phi_i : [0,1] \to \mathcal{S}$, $\pi_{\mathcal{S}} \circ \phi_i(0) = a$ and $\pi_{\mathcal{S}} \circ \phi_i(1) = b$.

Claim (8) in Theorem 4.8 of [15] states that for $\mu < r = \text{rch}\mathcal{S}$ the restriction of $\pi_{\mathcal{S}}$ to the μ-tubular neighbourhood $\mathcal{S}^\mu$ is $\frac{\text{rch}\mathcal{S}}{\text{rch}\mathcal{S} - \mu}$-Lipschitz. This together with (1) above gives us an upper bound on the lengths of curves $\pi_{\mathcal{S}} \circ \phi_i$:

$$\text{length}(\pi_{\mathcal{S}} \circ \phi_i) \leq \frac{\text{rch}\,\mathcal{S}}{\text{rch}\,\mathcal{S} - \delta_i} \text{length}(\phi_i) \leq \frac{\text{rch}\,\mathcal{S}}{\text{rch}\,\mathcal{S} - \delta_i} 2r \arcsin \frac{|a-b|}{2r}$$

This together with (2) yields $d_{\mathcal{S}}(a,b) \leq 2r \arcsin \frac{|a-b|}{2r}$.

□

3 Variation of tangent spaces

In this section we shall first discuss the the bound on the variation of tangent spaces in the C^2 setting, and then generalize to the $C^{1,1}$ setting. For this generalization we need a topological result, which will be presented in Section 3.2.

3.1 Bounds for $\mathbf{C^2}$ submanifolds

We shall be using the following result, Theorem 4.8(12) of [15]:

Theorem 3.1 (Federer's tubular neighbourhoods) *Let $B_{N_p\mathcal{M}}(r)$, be the ball of radius r centred at p in the normal space $N_p\mathcal{M} \subset \mathbb{R}^d$ of a $C^{1,1}$ manifold $\mathcal{M}$ with reach $\text{rch}(\mathcal{M})$, where $r < \text{rch}(\mathcal{M})$. For every point $x \in B_{N_p\mathcal{M}}(r)$, $\pi_{\mathcal{M}}(x) = p$.*

The fact that such a tubular neighbourhood exists is non-trivial, even for a neighbourhood of size ϵ. From Theorem 3.1 we immediately see that:

Corollary 3.2 *Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^d$ and $p \in \mathcal{M}$. Any open ball $B(c,r)$ that is tangent to $\mathcal{M}$ at p and whose radius r satisfies $r \leq \text{rch}(\mathcal{M})$ does not intersect $\mathcal{M}$.*

Proof Let $r < \text{rch}(\mathcal{M})$. Suppose that the intersection of $\mathcal{M}$ and the open ball is not empty, then the $\pi_{\mathcal{M}}(c) \neq p$ contradicting Federer's tubular neighbourhood theorem. The result for $r = \text{rch}(\mathcal{M})$ now follows by taking the limit. □

Here we prove the main result for C^2 manifolds. Our exposition is the result of straightforwardly combining the work of Niyogi, Smale, and Weinberger [19], and the two dimensional analysis of Attali, Edelsbrunner, and Mileyko [2] with some observations concerning the reach.

We start with the following simple observation:

Lemma 3.3 *Let $\gamma(t)$ be a geodesic parametrized according to arc length on $\mathcal{M} \subset \mathbb{R}^d$, then $|\ddot{\gamma}| \leq 1/\mathrm{rch}(\mathcal{M})$, where we use Newton's notation, that is we write $\ddot{\gamma}$ for the second derivative of γ with respect to t.*

Proof Because $\gamma(t)$ is a geodesic, $\ddot{\gamma}(t)$ is normal to $\mathcal{M}$ at $\gamma(t)$. Now consider the sphere of radius $\mathrm{rch}(\mathcal{M})$ tangent to $\mathcal{M}$ at $\gamma(t)$, whose centre lies on the line $\{\gamma(t) + \lambda\ddot{\gamma} \mid \lambda \in \mathbb{R}\}$. If now $|\ddot{\gamma}|$ were larger than $1/\mathrm{rch}(\mathcal{M})$, the geodesic γ would enter the tangent sphere, which would contradict Corollary 3.2. □

Note that $|\ddot{\gamma}|$ is the normal curvature, because γ is a geodesic. Using the terminology of [19, Section 6], Lemma 3.3 can also be formulated as follows: $1/\mathrm{rch}(\mathcal{M})$ bounds the principal curvatures in the normal direction ν, for any unit normal vector $\nu \in N_p\mathcal{M}$. In particular, $1/\mathrm{rch}(\mathcal{M})$ also bounds the principal curvatures if $\mathcal{M}$ has codimension 1.

We now have the following, which is a straightforward extension of an observation in [2] to general dimension:

Lemma 3.4 *Let $\gamma(t)$ be a geodesic parametrized according to arc length, with $t \in [0, \ell]$ on $\mathcal{M} \subset \mathbb{R}^d$, then:*

$$\angle\dot{\gamma}(0)\dot{\gamma}(\ell) \leq \frac{d_{\mathcal{M}}(\gamma(0), \gamma(\ell))}{\mathrm{rch}(\mathcal{M})}.$$

Proof Because γ is parametrized according to arc length $|\dot{\gamma}| = 1$ and $\dot{\gamma}(t)$ can be seen as a curve on the sphere $\mathbb{S}^{d-1}$. Moreover $\ddot{\gamma}$ can be seen as tangent to this sphere. The angle between two tangent vectors $\dot{\gamma}(0)$ and $\dot{\gamma}(\ell)$ equals the geodesic distance on the sphere. The geodesic distance between any two points is smaller or equal to the length of any curve connecting these points, and $\{\dot{\gamma}(t) \mid t \in [0, \ell]\}$ is such a curve. We therefore have

$$\angle\dot{\gamma}(0)\dot{\gamma}(\ell) \leq \int_0^\ell \left|\frac{d}{dt}\dot{\gamma}\right| \mathrm{d}t = \int_0^\ell |\ddot{\gamma}| \mathrm{d}t \leq \frac{\ell}{\mathrm{rch}(\mathcal{M})} \leq \frac{d_{\mathcal{M}}(\gamma(0), \gamma(\ell))}{\mathrm{rch}(\mathcal{M})}, \tag{3}$$

where we used Lemma 3.3. □

We can now turn our attention to the variation of tangent spaces. Here we mainly follow Niyogi, Smale, and Weinberger [19], but use one useful observation of [2]. We shall be using the second fundamental form, which we assume the reader to be familiar with. We refer to [14] as a standard reference.

The second fundamental form $\mathrm{II}_p(u, v)$ has the geometric interpretation of the normal part of the covariant derivative, where we assume now that u, v are vector fields. In particular $\mathrm{II}(u, v) = \bar{\nabla}_u v - \nabla_u v$, where $\bar{\nabla}$ is the connection in the ambient space, in this case Euclidean space, and ∇ the connection with respect to the induced metric on the manifold $\mathcal{M}$. $\mathrm{II}_p : T_p\mathcal{M} \times T_p\mathcal{M} \to N_p\mathcal{M}$ is a symmetric bi-linear form, see for example Section 6.2 of [14] for a proof. This means that we only need to consider vectors in the tangent space and not vector fields, when we consider $\mathrm{II}_p(u, v)$.

We can now restrict our attention to u, v lying on the unit sphere $\mathbb{S}^{n-1}_{T_p\mathcal{M}}$ in the tangent space and ask for which of these vectors $|\mathrm{II}_p(u, v)|$ is maximized. Let us assume that the $\mathrm{II}_p(u, v)$ for which the maximum[1] is attained lies in the direction of $\eta \in N_p\mathcal{M}$ where η is assumed to have unit length.

We can now identify $\langle \mathrm{II}_p(\cdot, \cdot), \eta\rangle$, with a symmetric matrix. Because of this $\langle \mathrm{II}_p(u, v), \eta\rangle$, with $u, v \in \mathbb{S}^{n-1}_{T_p\mathcal{M}}$, attains its maximum for u, v both lying in the direction of the unit eigenvector w of $\langle \mathrm{II}_p(\cdot, \cdot), \eta\rangle$ with the largest[2] eigenvalue. In other words the maximum is assumed for $u = v = w$. Let us now consider a geodesic γ_w on $\mathcal{M}$ parametrized by arclength such that $\gamma_w(0) = p$ and $\dot{\gamma}_w(0) = w$. Now, because γ_w is a geodesic and the ambient space is Euclidean,

$$\mathrm{II}_p(w, w) = \mathrm{II}_p(\dot{\gamma}_w, \dot{\gamma}_w) = \bar{\nabla}_{\dot{\gamma}_w}\dot{\gamma}_w - \nabla_{\dot{\gamma}_w}\dot{\gamma}_w = \bar{\nabla}_{\dot{\gamma}_w}\dot{\gamma}_w - 0 = \ddot{\gamma}_w.$$

Due to Lemma 3.3 and by definition of the maximum, we now see that $|\mathrm{II}_p(u, v)| \leq |\mathrm{II}_p(w, w)| \leq 1/\mathrm{rch}\mathcal{M}$, for all u, v of length one.

Having discussed the second fundamental form, we can give the following lemma:

[1] If there is more than one direction we simply pick one.

[2] We can assume positivity without loss of generality, and, again, if there is more than one direction, we pick one.

Lemma 3.5 *Let $p, q \in \mathcal{M}$, then*

$$\angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \frac{d_{\mathcal{M}}(p,q)}{\text{rch}(\mathcal{M})}.$$

Proof Let γ be a geodesic connecting p and q, parametrized by arc length. We consider an arbitrary unit vector u and parallel transport this unit vector along γ, getting the unit vectors $u(t)$ in the tangent spaces $T_{\gamma(t)}\mathcal{M}$. The maximal angle between $u(0)$ and $u(\ell)$, for all u bounds the angle between $T_p\mathcal{M}$ and $T_q\mathcal{M}$. Now

$$\frac{du}{dt} = \bar{\nabla}_{\dot{\gamma}} u(t) = \mathrm{II}_p(\dot{\gamma}, u(t)) + \nabla_{\dot{\gamma}} u(t) = \mathrm{II}_p(\dot{\gamma}, u(t)) + 0,$$

where we used that $u(t)$ is parallel and thus by definition $\nabla_{\dot{\gamma}} u(t) = 0$. So using our discussion above $|\frac{du}{dt}| \leq 1/\text{rch}(\mathcal{M})$. Now we note that, similarly to what we have seen in the proof of Lemma 3.4, $u(t)$ can be seen as a curve on the sphere and thus $\angle(u(0), u(\ell)) \leq \int_0^\ell |\frac{du}{dt}| \text{d}t \leq \ell/\text{rch}(\mathcal{M})$. □

This bound is tight as it is attained for a sphere.

Combining Theorems 2.1 and 3.5 we find that

Corollary 3.6

$$\sin\left(\frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2}\right) \leq \frac{|p-q|}{2\text{rch}(\mathcal{M})}.$$

The proof is almost immediate, but has been added to the appendix for completeness.

With the bound on the angles between the tangent spaces it is not difficult to prove that the projection map onto the tangent space is locally a diffeomorphism, as has been done in [19]. Although the results were given in terms of the (global) reach to simplify the exposition, the results can be easily formulated in term of the local feature size.

3.2 A topological result

We shall now give a full proof of a statement by Boissonnat and Cazals [9, Proposition 12] in the more general $C^{1,1}$ setting:

Proposition 3.7 *Let B be a closed ball that intersects a $C^{1,1}$ manifold $\mathcal{M}$. If B does not contain a point of the medial axis ($\text{ax}(\mathcal{M})$) of $\mathcal{M}$ then $B \cap \mathcal{M}$ is a topological ball.*

The proof uses some result from topology, namely a variation of [18, Theorem 3.2]:

Lemma 3.8 *Let $d_c|_{\mathcal{M}}$ be the $C^{1,1}$ function on $\mathcal{M}$ defined, as in Lemma A.1, as the restriction to $\mathcal{M}$ of $d_c : \mathbb{R}^d \to \mathbb{R}, d_c(x) = |x-c|$. Assume that y is a global isolated minimum of $d_c|_{\mathcal{M}}$ and let r_c be the second critical value of $d_c|_{\mathcal{M}}$. Then for all $0 < \eta < r_c - |c-y|$, $\mathcal{M}^{r_c - \eta}$ is a topological ball.*

The proof of this lemma can be found in the appendix.

Proof Proof of Proposition 3.7 Write r for the radius of B and c for its center. The result is trivial if c belongs to the medial axis of $\mathcal{M}$. Therefore assume that $c \notin \text{axis}(\mathcal{M})$.

Let y be the (unique) point of $\mathcal{M}$ closest to c. We denote by B_y the closed ball centered at c with radius $|c-y|$ (see Figure 3). By Corollary 3.2, the interior of B_y does not intersect $\mathcal{M}$ and $B_y \cap \mathcal{M} = \{y\}$. This means that the conditions of Lemma 3.8 are satisfied and $B(c, r_c - \eta) \cap \mathcal{M}$ is a topological ball for all $0 < \eta < r_c - |c-y|$, where r_c is the second critical value of the distance function to c restricted to $\mathcal{M}$. In other words r_c is the radius for which the ball centred on c is tangent to $\mathcal{M}$ for the second time.

Let us now assume that there exists a point $z \neq y$ of $\mathcal{M}$ such that $r_c = |c-z| > |c-y|$ where the ball $B(c, r_c)$ is tangent to $\mathcal{M}$. We consider the set $\mathcal{B}_z$ of closed balls that are tangent to $\mathcal{M}$ at z and are centred on the line segment $[zc]$. The balls in $\mathcal{B}_z$ can be ordered according to their radius. Note that $B(c, r_c)$ is the ball of $\mathcal{B}_z$ centered at c. Since the interior of $B(c, r_c)$ contains y and therefore intersects $\mathcal{M}$, there must exist a largest ball $B_z \in \mathcal{B}_z$, whose interior does not intersect $\mathcal{M}$. The center of B_z belongs to both $\text{ax}(\mathcal{M})$ and B since $B_z \subset B(c, r_c) \subset B$. □

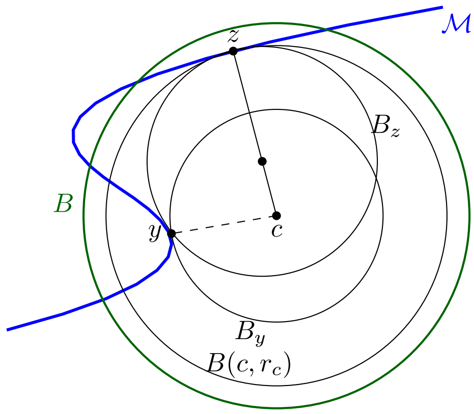

Figure 3: For the proof of Proposition 3.7.

3.3 Bounds for $C^{1,1}$ submanifolds

We shall now give an elementary exposition for the result of the previous section.

3.3.1 From manifold to tangent space and back

We start with the following lemma, which is due to Federer. It bounds the distance of a point $q \in \mathcal{M}$ to the tangent space of a point that is not too far away.

Lemma 3.9 (Distance to tangent space, Theorem 4.8(7) of [15]) *Let $p, q \in \mathcal{M} \subset \mathbb{R}^d$ such that $|p-q| < \operatorname{rch}(M)$. We have*

$$\sin \angle([pq], T_p\mathcal{M}) \leq \frac{|p-q|}{2\operatorname{rch}(\mathcal{M})}, \tag{4}$$

and

$$d_{\mathbb{E}}(q, T_p\mathcal{M}) \leq \frac{|p-q|^2}{2\operatorname{rch}(\mathcal{M})}. \tag{5}$$

We also have the converse statement of the distance bounds in Lemma 3.9. The following lemma is an improved version of Lemma B.2 in [5]. This result too can be traced back to Federer [15], in a slightly different guise. Before we give the lemma we first introduce the following notation. Let $C(T_p\mathcal{M}, r_1, r_2)$ denote the 'filled cylinder' given by all points that project orthogonally onto a ball of radius r_1 in $T_p\mathcal{M}$ and whose distance to this ball is less than r_2.

In the following lemma we prove for all points $v \in T_p\mathcal{M}$, such that $|v-p|$ is not too large, that a pre-image on $\mathcal{M}$, if it exists, under the projection to $T_p\mathcal{M}$ cannot be too far from $T_p\mathcal{M}$. The existence of such a point on $\mathcal{M}$ is proven below.

Lemma 3.10 (Distance to Manifold) *Suppose that $v \in T_p\mathcal{M}$ and $|v-p| < \operatorname{rch}(\mathcal{M})$. Let $q = \pi^{-1}_{(\mathcal{M}\to T_p\mathcal{M})}(v)$ be the inverse of the (restricted) projection $\pi_{T_p\mathcal{M}}$ from $\mathcal{M} \cap C(T_p\mathcal{M}, \operatorname{rch}(\mathcal{M}), \operatorname{rch}(\mathcal{M}))$ to $T_p\mathcal{M}$ of v, if it exists. Then*

$$|q-v| \leq \left(1 - \sqrt{1 - \left(\frac{|v-p|}{\operatorname{rch}(\mathcal{M})}\right)^2}\right)\operatorname{rch}(\mathcal{M}) \leq \frac{1}{2}\frac{|v-p|^2}{\operatorname{rch}(\mathcal{M})} + \frac{1}{2}\frac{|v-p|^4}{\operatorname{rch}(\mathcal{M})^3}.$$

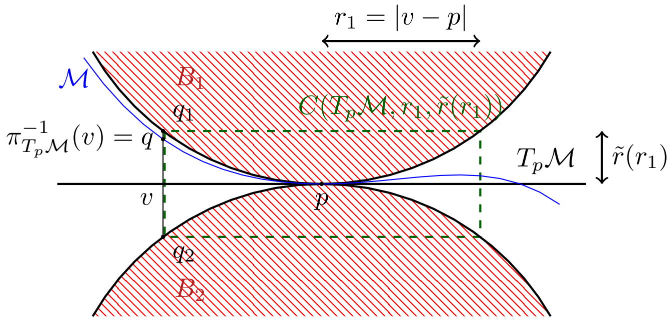

Figure 4: The set of all tangent balls to the tangent space of radius $\text{rch}(\mathcal{M})$ bounds the region in which $\mathcal{M}$ can lie. Here we depict the 2 dimensional analogue.

Remark 3.11 *It follows immediately that* $\mathcal{M} \cap C(T_p\mathcal{M}, r_1, \text{rch}(\mathcal{M})) \subset C(T_p\mathcal{M}, r_1, \tilde{r}(r_1))$*, with*

$$\tilde{r}(r_1) = \left(1 - \sqrt{1 - \left(\frac{r_1}{\text{rch}(\mathcal{M})}\right)^2}\right) \text{rch}(\mathcal{M}). \tag{6}$$

This cylinder is indicated in green in Figure 4. Let $C_{top/bottom}(T_p\mathcal{M}, r_1, \tilde{r}(r_1))$ *denote the subset of* $C(T_p\mathcal{M}, r_1, \tilde{r}(r_1))$ *that projects orthogonally onto the open ball of radius* r_1 *in* $T_p\mathcal{M}$ *and lies a distance* $\tilde{r}(r_1)$ *away. We also see that* $\mathcal{M} \cap C_{top/bottom}(T_p\mathcal{M}, r_1, \tilde{r}(r_1)) = \emptyset$ *and that* $\mathcal{M} \cap C(T_p\mathcal{M}, r_1, \text{rch}(\mathcal{M})) \cap N_p\mathcal{M} = \{p\}$*. We write*

$$C_{side\ rim}(T_p\mathcal{M}, r_1, \tilde{r}(r_1)) = \partial C(T_p\mathcal{M}, r_1, \tilde{r}(r_1)) \setminus C_{top/bottom}(T_p\mathcal{M}, r_1, \tilde{r}(r_1)).$$

3.3.2 The result

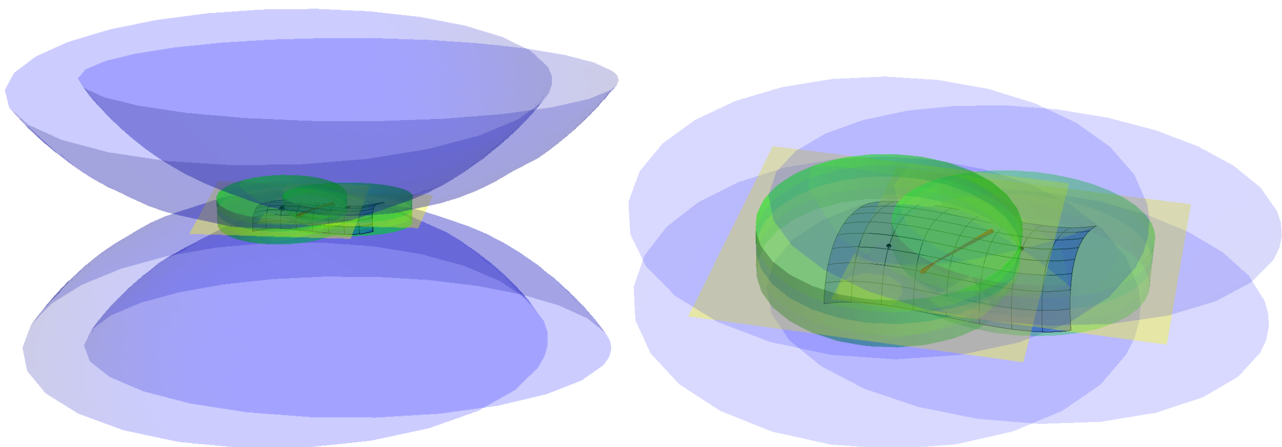

Figure 5: The tangent spaces $T_p\mathcal{M}$ and $T_q\mathcal{M}$ are drawn in yellow. The cylinders $C(T_p\mathcal{M}, r_1, \tilde{r})$ and $C(T_q\mathcal{M}, r_1, \tilde{r})$ are indicated in green. The red line segment lies in both cylinders and therefore its angle with both $T_p\mathcal{M}$ and $T_q\mathcal{M}$ is small.

This section revolves around the following observation: If r_1 roughly the distance between p and q, there is a significant part of $\mathcal{M}$ that is contained in the intersection $C(T_p\mathcal{M}, r_1, \tilde{r}) \cap C(T_q\mathcal{M}, r_1, \tilde{r})$. In particular any line segment, whose length is denoted by ℓ, connecting two points in $\mathcal{M} \cap C(T_p\mathcal{M}, r_1, \tilde{r}) \cap C(T_q\mathcal{M}, r_1, \tilde{r})$ is contained in both $C(T_p\mathcal{M}, r_1, \tilde{r})$ and $C(T_q\mathcal{M}, r_1, \tilde{r})$. If this line segment is long, the angle with both $T_p\mathcal{M}$ and $T_q\mathcal{M}$ is small. This bounds the angle between $T_p\mathcal{M}$ and $T_q\mathcal{M}$, see Figure 5.

For the existence of the line segment that is contained in both $C(T_p\mathcal{M}, r_1, \tilde{r})$ and $C(T_q\mathcal{M}, r_1, \tilde{r})$ we need the following corollary of Proposition 3.7:

Corollary 3.12 *For each $v \in T_pM$ such that $|v - p| < \frac{\sqrt{3}}{2}\mathrm{rch}(\mathcal{M})$ there exists at least one original $\pi^{-1}_{T_pM}(v)$.*

The proofs of this statement can be found in the appendix.

Theorem 3.13 *Let $|p - q| \leq \mathrm{rch}(\mathcal{M})/3$, then the angle φ between $T_p\mathcal{M}$ and $T_q\mathcal{M}$ is bounded by*

$$\sin\frac{\varphi}{2} \leq \frac{\left(1 - \sqrt{1-\alpha^2}\right)}{\sqrt{\frac{\alpha^2}{4} - (\frac{\alpha^2}{2} + 1 - \sqrt{1-\alpha^2})^2}} \simeq \alpha + 9\alpha^3/4,$$

where $\alpha = |p - q|/\mathrm{rch}(\mathcal{M})$.

The proof of this result follows the lines as sketched in the overview, and can by found in full in the appendix.

Remark 3.14 *The bound we presented above can be tightened by further geometric analysis, in particular by splitting $T_p\mathcal{M}$ into the span of $\pi_{T_p\mathcal{M}}(q) - p$ and its orthocomplement. However we chose to preserve the elementary character of the argument.*

With the bound on the angles between the tangent spaces it is not difficult to prove that the projection map is locally a diffeomorphism, as has been done in [19].

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A Proofs

Proof Proof of Lemma 2.4 Since $\rho < r$, one has $|b-p| \geq r - \rho > 0$ and the quotient is well defined. Because $|p-m| \leq \rho$, c belongs to both disks with radius r with a and b on their boundary. This can be expressed through angles comparison as $\psi = \angle apb \geq \angle a\tilde{p}b = \tilde{\psi} \geq \pi/2$. If we denote $\tau = \frac{|a-p|}{|b-p|}$ one has

$$\begin{aligned}(a-b)^2 &= ((a-p)+(p-b))^2 \\ &= |a-p|^2 + |b-p|^2 - 2|a-p||b-p|\cos\psi \\ &= |b-p|^2\left(1+\tau^2-2\tau\cos\psi\right).\end{aligned}$$

Similarly,

$$(a-b)^2 = |b-\tilde{p}|^2\left(1+\tau^2-2\tau\cos\tilde{\psi}\right),$$

so that

$$= |b-\tilde{p}|^2\left(1+\tau^2-2\tau\cos\tilde{\psi}\right) = |b-p|^2\left(1+\tau^2-2\tau\cos\psi\right).$$

But $\psi \geq \tilde{\psi} \geq \pi/2$ gives

$$1 + \tau^2 - 2\tau \cos \tilde{\psi} \leq 1 + \tau^2 - 2\tau \cos \psi,$$

and we get $|b - \tilde{p}| \geq |b - p|$ and $|a - \tilde{p}| = \frac{|b-\tilde{p}|}{|b-p|}|a - p| \geq |a - p|$. $\square$

Proof Proof of Claim 2.6 Thanks to the definitions of δ_i and ρ_i, one has for $i \geq 1$

$$\delta_i \leq \frac{1}{2}(\delta_{i-1} + \rho_{i-1}) = \frac{1}{2}\left(1 + \frac{\rho_{i-1}}{\delta_{i-1}}\right)\delta_{i-1}. \tag{7}$$

Moreover for any $i \in \mathbb{N}$,

$$\frac{\rho_i}{\delta_i} = \frac{r}{\delta_i} - \sqrt{\left(\frac{r}{\delta_i}\right)^2 - 1} = \frac{1}{\frac{r}{\delta_i} + \sqrt{\left(\frac{r}{\delta_i}\right)^2 - 1}} \leq \frac{\delta_i}{r}. \tag{8}$$

Equations (7) and (8) give:

$$\delta_i \leq \frac{1}{2}\left(1 + \frac{\delta_{i-1}}{r}\right)\delta_{i-1}. \tag{9}$$

Since

$$\frac{\delta_0}{r} = \frac{\delta}{r} < 1,$$

(9) allows the induction

$$\frac{\delta_i}{r} < 1 \Rightarrow \frac{\delta_{i+1}}{r} < \frac{\delta_i}{r} < 1.$$

We get that the sequence $(\delta_i)_{i\in\mathbb{N}}$ is decreasing and $\frac{\delta_i}{r} \leq \frac{\delta}{r}$. Replacing and iterating in (9) gives

$$\delta_i \leq \left(\frac{1}{2}\left(1 + \frac{\delta_0}{r}\right)\right)^i \delta_0.$$

Since $\frac{1}{2}\left(1 + \frac{\delta_0}{r}\right) < 1$ we see that δ_i decreases faster than a geometric sequence, in particular:

$$\lim_{i\to\infty} \delta_i = 0. \tag{10}$$

$\square$

Proof Proof of Corollary 3.6 Lemma 3.5 gives

$$\sin\left(\frac{\angle(T_p\mathcal{M}, T_q\mathcal{M})}{2}\right) \leq \sin\left(\frac{d_{\mathcal{M}}(p,q)}{2\text{rch}(\mathcal{M})}\right)$$

and Theorems 2.1 yields

$$\sin\left(\frac{d_{\mathcal{M}}(p,q)}{2\text{rch}(\mathcal{M})}\right) \leq \frac{|p-q|}{2\text{rch}(\mathcal{M})}.$$

The result now follows. $\square$

Before we can proof Lemma 3.8, we need a result from Morse theory [18, Theorem 3.1] with a modification:

Lemma A.1 *Consider the distance function from c: $d_c : \mathbb{R}^d \to \mathbb{R}, d_c(x) = |x - c|$ restricted to $\mathcal{M}$. Let $a = d_c(x')$ and $b = r$ and suppose that the set $d_c^{-1}[a, b]$, consisting of all $p \in \mathcal{M}$ with $a \leq d_c(p) \leq b$, contains no critical points of d_c (that is, no point q of $\mathcal{M}$ where $B(c, q)$ is tangent to $\mathcal{M}$). Then $\mathcal{M}^a = \{x \in \mathcal{M}, d_c \leq a\} = \mathcal{M} \cap B(c, a)$ is homeomorphic (if d_c is $C^{1,1}$) to $\mathcal{M}^b = \{x \in \mathcal{M}, d_c \leq b\}$. Furthermore M^a is a deformation retract of M^b.*

Proof Proof of Lemma A.1 The key change compared to original statement by Milnor, which is in the C^2 setting, is the passing from a diffeomorphism to a homeomorphism. This lemma is true because of the following: The proof of Theorem 3.1 of [18] mentions the assumption that the function (in this case d_c) is smooth, however in the proof relies on using gradient flow, that is solving a differential equation. Thanks to Picard-Lindelöf theorem, see [11, Theorem 3.1], we know that the initial value problem $x' = g(x)$ has a unique continuous solution if g is Lipschitz. In the proof presented by Milnor, g is the gradient of a (Morse) function (in this case the distance function). This implies that it suffices that the gradient of the distance function is Lipschitz, or equivalently that the function itself is $C^{1,1}$. Because the gradient flow is only continuous in this Lipschitz setting we find a homeomorphism in the $C^{1,1}$ setting, instead of the diffeomorphism as in the C^2 case. □

Proof Proof of Lemma 3.8 Due to Lemma A.1, in particular the deformation retract, we have that $\mathcal{M}^{r_c-\eta} \setminus \{y\}$ is homeomorphic to $(0, 1] \times (d_c|_{\mathcal{M}})^{-1}(r_c-\eta)$, for all $0 < \eta < r_c - |c-y|$. This gives that $\mathcal{M}^{r_c-\eta}$ is homeomorphic to the cone of $(d_c|_{\mathcal{M}})^{-1}(r_c - \eta)$ with the point y as its tip. Because $\mathcal{M}^{r_c-\eta}$ is a $C^{1,1}$ manifold with boundary and y does not lie on its boundary we have the following: Firstly, $(d_c|_{\mathcal{M}})^{-1}(r_c - \eta)$ is a C^1 manifold and can be triangulated, see [20, section 7] and [22] respectively, giving a triangulation of the cone. We can now use the following definition and result from topology [23, Chapter 3]:

Definition A.2 *A complex K is called a combinatorial n-manifold if the link (the boundary of the star) of each vertex is an $(n-1)$-sphere or an $(n-1)$-ball.*

Lemma A.3 ([23, Lemma 9 of Chapter 3]) *Suppose that $|K| = \mathcal{M}$. Then K is a combinatorial manifold if and only if $\mathcal{M}$ is a manifold.*

Because $(d_c|_{\mathcal{M}})^{-1}(r_c - \eta)$ is the link of y, $(d_c|_{\mathcal{M}})^{-1}(r_c - \eta)$ is a sphere and $\mathcal{M}^{r_c-\eta}$ a ball. □

Proof Proof of Lemma 3.10 Consider the plane H in which v, q and p lie. Let in addition B_1, B_2 be the two disks in H that are tangent to $\mathcal{M}$ at p and thus to $T_p\mathcal{M}$ with radius $\mathrm{rch}(\mathcal{M})$. Due to Lemma 3.2 q can not lie inside the interior of B_1 or B_2. Let us now extend the line $[vq]$ and call the first intersection of this line with B_1, q_1 and with B_2, q_2. We call the centres of B_1 and B_2, c_1 and c_2, and the angles $\angle([q_1c_1], [c_1p]) = \angle([q_2c_2], [c_2p]) = \theta$. We find that $|v - p| = \mathrm{rch}(\mathcal{M}) \sin\theta$, while

$$|q - v| \leq |v - q_1| = |v - q_2| = (1 - \cos\theta)\mathrm{rch}(\mathcal{M}).$$

This gives us

$$|q - v| \leq \left(1 - \sqrt{1 - \left(\frac{|v - p|}{\mathrm{rch}(\mathcal{M})}\right)^2}\right) \mathrm{rch}(\mathcal{M}) \leq \frac{1}{2}\frac{|v - p|^2}{\mathrm{rch}(\mathcal{M})} + \frac{1}{2}\frac{|v - p|^4}{\mathrm{rch}(\mathcal{M})^3},$$

using Taylor's theorem. □

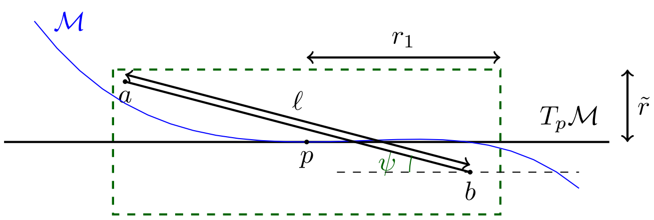

Figure 6: An illustration of the notation used in Remark A.

Remark A.4 *Let $[ab]$ be a line segment with length ℓ that is contained in $C(T_p\mathcal{M}, r_1, \tilde{r})$. Then the angle ψ between $[ab]$ and $T_p\mathcal{M}$ is bounded by $\sin(\psi) \leq \frac{2\tilde{r}}{\ell}$.*

We now need the following corollary of Proposition 3.7:

Corollary A.5 *We have:*

1. *For any ball $B(p,r)$ of radius $r < \mathrm{rch}(\mathcal{M})$ centred at $p \in \mathcal{M}$, $B(p,r) \cap M$ is a topological ball.*
2. *For every $0 < r < \mathrm{rch}(\mathcal{M})$, $\partial(B(p,r) \cap M)$ is contained in a set homeomorphic to $C_{side\ rim}(T_p\mathcal{M}, r, \tilde{r}(r))$, this homeomorphism is a projection, which is denoted by h_r and indicated in Figure 7 in green.*
3. *There exists an isotopy from the image of $\partial(B(p,r) \cap M)$ under the homeomorphism to $C_{side\ rim}(T_p\mathcal{M}, r, \tilde{r}(r))$ to the sphere that is the boundary of the open ball of radius r in $T_p\mathcal{M}$.*

Proof Proof of Corollary A.4 The first observation is a straightforward consequence of Proposition 3.7 and the definition of reach.

$B(p,r) \subset C(T_p\mathcal{M}, r, \mathrm{rch}(\mathcal{M}))$, so thanks to Remark 3.3.1 we see that

$$B(p,r) \cap \mathcal{M} \cap N_p\mathcal{M} = \mathcal{M} \cap C(T_p\mathcal{M}, r_1, \mathrm{rch}(\mathcal{M})) \cap N_p\mathcal{M} = \{p\}.$$

Because $\mathcal{M}$ does not have a boundary $\partial(B(p,r) \cap \mathcal{M}) \subset \partial(B(p,r)) \setminus N_p\mathcal{M}$ whose closure is homeomorphic to $C_{\text{side rim}}(T_p\mathcal{M}, r, \tilde{r}(r))$. This gives us the second observation.

The third observation is obviously true for sufficiently small $r = \epsilon$, because the tangent space is the first order approximation of the manifold. Because the second observation holds for any $r' \leq \mathrm{rch}(\mathcal{M})$, the third observation follows where the isotopy can be found by following $\partial(B(p,r') \cap \mathcal{M})$ from $r' = r$ to the limit of r' going to zero. The isotopy can be understood in the following steps: Thanks to the Proposition 3.7, $\mathcal{M} \cap \partial B(p,r)$, is a topological sphere. For each $0 < r' < \mathrm{rch}(\mathcal{M})$, $h_{r'}(\partial(B(p,r') \cap M))$ lies on $C_{\text{side rim}}(T_p\mathcal{M}, r', \tilde{r}(r'))$. In turn $C_{\text{side rim}}(T_p\mathcal{M}, r', \tilde{r}(r'))$ can rescaled in the radial direction such that the image is contained in $C_{\text{side rim}}(T_p\mathcal{M}, r, \tilde{r}(r))$. This rescaling is denoted by the map $R_{r' \to r}$. The map $R_{r' \to r}(h_{r'}(\partial(B(p,r') \cap M)))$ now gives the isotopy, because the limit $\lim_{r' \to 0} R_{r' \to r}(h_{r'}(\partial(B(p,r') \cap M)))$ is in fact the sphere in the tangent space. □

Proof Proof of Corollary 3.11 The proof, by contradiction, is completely pictorial in nature, see Figure 7.

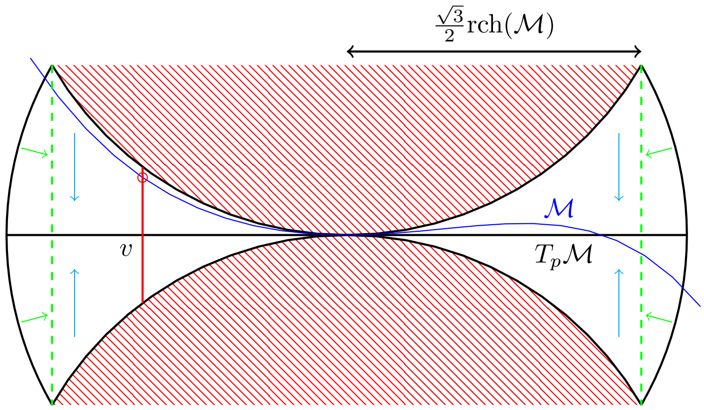

Figure 7: The manifold $\mathcal{M}$ in a neighbourhood of the point p lies in region bounded by all tangent balls of $T_p\mathcal{M}$ at p, indicated by the red balls. The projection on the boundary of $C(T_p\mathcal{M}, \frac{\sqrt{3}}{2}\mathrm{rch}(\mathcal{M}), \frac{1}{2}\mathrm{rch}(\mathcal{M}))$ is indicated in green. The projection onto the tangent page is indicated in cyan.

So let us suppose that there exists a $v \in T_pM$ with $|v - p| < \frac{\sqrt{3}}{2}\mathrm{rch}(\mathcal{M})$ such that there does not exist a $\pi^{-1}_{T_pM}(v)$. Consider the ball $B(p, \mathrm{rch}(\mathcal{M}))$. $\mathcal{M} \cap B(p, \mathrm{rch}(\mathcal{M}))$ is a topological ball, by Corollary A.4. We now map (radially) the part of this ball outside the cylinder $C(T_p\mathcal{M}, \frac{\sqrt{3}}{2}\mathrm{rch}(\mathcal{M}), \frac{1}{2}\mathrm{rch}(\mathcal{M}))$ onto the boundary of $C(T_p\mathcal{M}, \frac{\sqrt{3}}{2}\mathrm{rch}(\mathcal{M}), \frac{1}{2}\mathrm{rch}(\mathcal{M}))$, as indicated in Figure 7. We then project everything onto $T_p\mathcal{M}$. By Corollary A.4 one has that the result is the image of a topological ball whose boundary coincides with the boundary of $B_{T_p\mathcal{M}}(\frac{\sqrt{3}}{2}\mathrm{rch}(\mathcal{M}))$. However because we assumed that there did not exist a $\pi^{-1}_{T_pM}(v)$, this image of the topological ball is topologically non-trivial, which yields a clear contradiction, because if there is a puncture the boundary would no longer be homologically trivial. □

Proof Proof of Theorem 3.12 The idea of the proof is pictorial, as we have seen in the overview in Figure 5 and below. We shall now give the details.

We consider the balls of radius $|p-q|$ centred at p and q respectively. The ball of radius $\frac{|p-q|}{2}$ centred at the midpoint $m = \frac{p+q}{2}$ is clearly contained in both larger balls, being $B(p, |p-q|)$ and $B(q, |p-q|)$, as indicated in Figure 8.

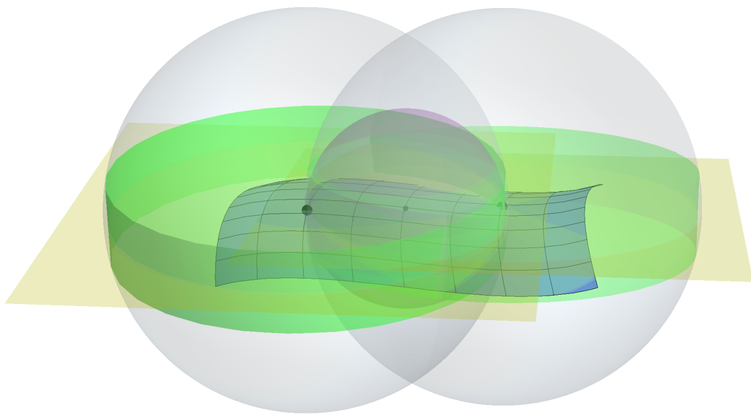

Figure 8: The ball $B(m, \frac{|p-q|}{2})$ lies in both $B(p, |p-q|)$ and $B(q, |p-q|)$.

We now note that $\mathcal{M} \cap B(m, \frac{|p-q|}{2})$ is contained in both the cylinders $C(T_p\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$ and $C(T_q\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$. Moreover, there exists an n-dimensional ball $B_{T_p\mathcal{M}}(\ell)$ of diameter ℓ in $T_p\mathcal{M}$ (the dark disk in Figure 9) such that $\pi^{-1}_{T_p\mathcal{M}}(x) \in B(m, \frac{|p-q|}{2})$ for all $x \in B_{T_p\mathcal{M}}(\ell)$. Determining ℓ is the only part of this proof for which we have to do some calculations, which we postpone until the end of the proof.

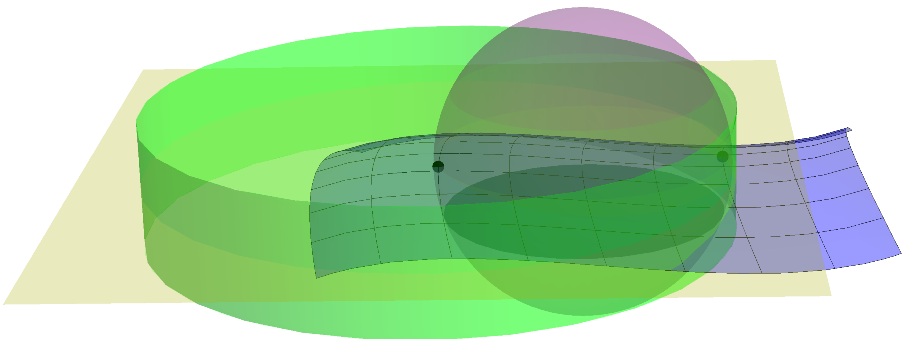

Figure 9: $B_{T_p\mathcal{M}}(\ell)$ is the dark disks that lies the sphere.

For each direction in $T_p\mathcal{M}$ we can consider the line segment connecting two antipodal point y_1, y_2 on the sphere $\partial B_{T_p\mathcal{M}}(\ell)$ and the line segment connecting $\pi^{-1}_{T_p\mathcal{M}}(y_1)$ and $\pi^{-1}_{T_p\mathcal{M}}(y_2)$, see Figure 10. These two points exist because of Corollary 3.11. This line segment has at least length ℓ. Moreover it lies in both $C(T_p\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$, $C(T_q\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$, with $\tilde{r}$ as in (6).

We now have a line segment for each direction in $T_p\mathcal{M}$ that is close to that direction in $T_p\mathcal{M}$, because it lies in $C(T_p\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$, and is close to $T_q\mathcal{M}$, because the line segment lies in $C(T_q\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$.

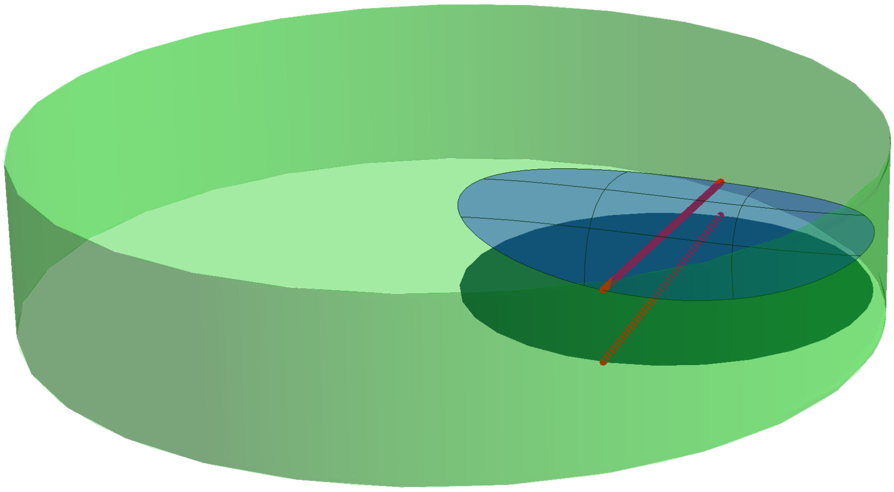

Figure 10: The line segment connecting two antipodal point y_1, y_2 on the sphere $\partial B_{T_p\mathcal{M}}(\ell)$ is indicated as a dotted red line and the line segment connecting $\pi^{-1}_{T_p\mathcal{M}}(y_1)$ and $\pi^{-1}_{T_p\mathcal{M}}(y_2)$ is indicated in red.

If this line segment is not too short compared to $\tilde{r}(|p-q|)$, Lemma 3.9, Lemma 3.10 and Remark A give us that the angle between $T_p\mathcal{M}$ and $T_q\mathcal{M}$ is small.

The only thing which is left is to give a lower bound ℓ. For this we shall use Figure 11. We shall denote the orthogonal translation of $T_p\mathcal{M}$ that goes through a point x by $\text{Trans}_x(T_p\mathcal{M})$. Let $\text{Trans}_{\max}(T_p\mathcal{M})$ be the orthogonal translation of $T_p\mathcal{M}$ to the furthest possible affine subspace from q, such that the intersection of $\text{Trans}(T_p\mathcal{M})$ and $C(T_p\mathcal{M}, |p-q|, \tilde{r}(|p-q|))$ is nonempty. $\text{Trans}_{\max}(T_p\mathcal{M})$ is indicated by a thick dashed line in Figure 11. The radius of the intersection of $\text{Trans}_{\max}(T_p\mathcal{M})$ with $B(m, \frac{|p-q|}{2})$ gives us $\ell/2$.

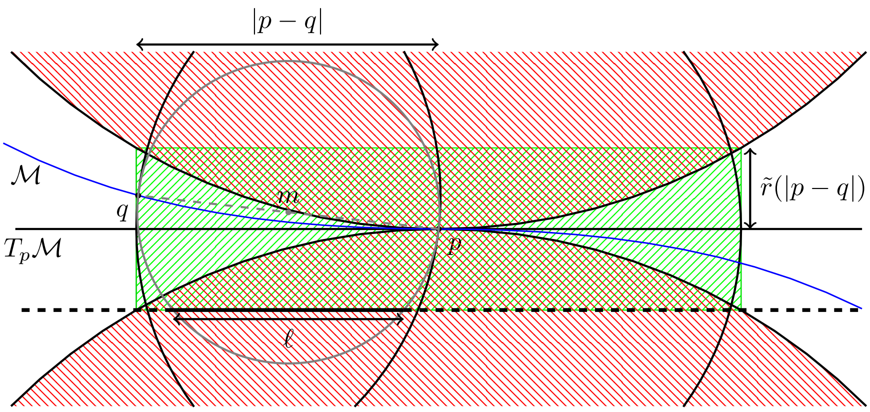

Figure 11: The intersection region of the balls centred at p and q with radius $|p-q|$.

Because Lemma 3.9 gives us that m lies at most $\frac{|p-q|^2}{2\text{rch}(\mathcal{M})}$ from $T_p\mathcal{M}$ and the distance between $\text{Trans}_{\max}(T_p\mathcal{M})$ and $T_p\mathcal{M}$ is $\tilde{r}(|p-q|)$ we have, by Pythagoras,

$$(\ell/2)^2 = \left(\frac{|p-q|}{2}\right)^2 - \left(\frac{|p-q|^2}{2\text{rch}(\mathcal{M})} + \tilde{r}(|p-q|)\right)^2.$$

Using Remark A, we see that

$$\sin\frac{\varphi}{2} \leq \frac{2\tilde{r}(|p-q|)}{\ell},$$

where the factor 2 on the left hand side is due to the fact that we apply the bound twice, once for each cylinder. To be precise we have used

$$\phi = \angle(T_p\mathcal{M}, T_q\mathcal{M}) \leq \sup_{y_1, y_2 \in \partial B_{T_p\mathcal{M}}(\ell)} \angle(T_p\mathcal{M}, \pi^{-1}_{T_p\mathcal{M}}(y_1) - \pi^{-1}_{T_p\mathcal{M}}(y_2)) + \angle(T_q\mathcal{M}, \pi^{-1}_{T_p\mathcal{M}}(y_1) - \pi^{-1}_{T_p\mathcal{M}}(y_2)),$$

where we understand that the supremum is taken over antipodal points y_1 and y_2 in $\partial B_{T_p\mathcal{M}}(\ell)$ and $\sin(a+b) \leq \sin(a) + \sin(b)$.

Combining the results yields

$$\begin{aligned}
\sin\frac{\varphi}{2} &\leq \frac{\tilde{r}(|p-q|)}{\sqrt{\left(\frac{|p-q|}{2}\right)^2 - \left(\frac{|p-q|^2}{2\mathrm{rch}(\mathcal{M})} + \tilde{r}(|p-q|)\right)^2}}, \\
&= \frac{\left(1 - \sqrt{1 - \left(\frac{|p-q|}{\mathrm{rch}(\mathcal{M})}\right)^2}\right)\mathrm{rch}(\mathcal{M})}{\sqrt{\left(\frac{|p-q|}{2}\right)^2 - \left(\frac{|p-q|^2}{2\mathrm{rch}(\mathcal{M})} + \left(1 - \sqrt{1 - \left(\frac{|p-q|}{\mathrm{rch}(\mathcal{M})}\right)^2}\right)\mathrm{rch}(\mathcal{M})\right)^2}} \\
&= \frac{\left(1 - \sqrt{1-\alpha^2}\right)}{\sqrt{\frac{\alpha^2}{4} - (\frac{\alpha^2}{2} + 1 - \sqrt{1-\alpha^2})^2}} \\
&\simeq \alpha + 9\alpha^3/4,
\end{aligned}$$

where $\alpha = |p-q|/\mathrm{rch}(\mathcal{M})$. □