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Abstract This paper introduces an inductively defined tree notation for all the faces of polytopes arising from a simplex by truncations, that allows us to view inclusion of faces as the process of contracting tree edges. Our notation instantiates to the well-known notations for the faces of associahedra and permutohedra. Various authors have independently introduced combinatorial tools for describing such polytopes. We build on the particular approach developed by Došen and Petrić, who used the formalism of hypergraphs to describe the interval of polytopes from the simplex to the permutohedron. This interval was further stretched by Petrić to allow truncations of faces that are themselves obtained by truncations, and iteratively so. Our notation applies to all these polytopes. We illustrate this by showing that it instantiates to a notation for the faces of the permutohedron-based associahedra, that consists of parenthesised words with holes. The paper also explores links between polytopes and categorified operads. Došen and Petrić have exhibited some families of hypergraph polytopes (associahedra, permutohedra, and hemiassociahedra) describing the coherences, and the coherences between coherences etc., arising by weakening sequential and parallel associativity of operadic composition. We complement their work with a criterion allowing us to recover the information whether edges of these "operadic polytopes" come from sequential, or from parallel associativity. We also give alternative proofs for some of the original results of Došen and Petrić.

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1 Introduction

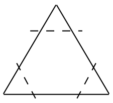
Classically, a (convex) polytope is defined as a bounded intersection of a finite set of half-spaces. More precisely, a polytope P is specified as the set of solutions to a system $Ax \geq b$ of linear inequalities, where A is an $m \times n$ matrix, x is an $n \times 1$ column vector of variables, and b is an $m \times 1$ column vector of constants. Here, n is the dimension of the ambient space containing P, and m is the number of half-spaces defining P. The actual dimension of P is the maximum dimension of an open ball contained in P.

A *face* of P is any intersection of P with one of its bounding hyperplanes (such a hyperplane intersects P and bounds a closed half-space containing P). Following usual terminology, the 0-dimensional (resp. 1-dimensional) faces of a polytope are called *vertices* (resp. *edges*), and if P is n-dimensional, we call its $(n-1)$-dimensional faces *facets*. If the definition of a face is extended to allow the empty set to be considered as a face, then the faces of a convex polytope form a bounded lattice called its *face lattice*, the partial ordering being the set containment of faces. The whole polytope (resp. the empty set) is the maximum (resp. minimum) element of the lattice.

As opposed to the classically (or geometrically) defined polytopes, an *abstract polytope* is a structure that captures only the combinatorial properties of the face lattice of a polytope, ignoring some of its other properties, particularly measurable ones, such as angles, edge lengths, etc. An abstract polytope is given as a set of faces, together with an order relation that satisfies certain axioms reflecting the incidence properties of polytopes in the classical sense.

In [4], Došen and Petrić investigate a family of polytopes that may be obtained by truncating the vertices, edges and other faces of simplices of any finite dimension. The permutohedra are limit cases in that family, where all possible truncations have been made. The limit cases at the other end, where no truncation has been made, are simplices. (Alternatively, one may choose the permutohedron as starting point, and reach the simplex by successive contractions, see e.g. [17]). Other independent (and even predating) approaches have been developed for describing polytopes in this family [6,7,2,3,13,14]. While the combinatorial description in all these works is essentially the same (with a different terminology), the ways of describing the geometric realisation are quite diverse, as we shall point out.

An easy example of a transition from a simplex to a permutohedron is obtained by truncating all the vertices of a 2-dimensional simplex (i.e., a triangle) to get a 2-dimensional permutohedron (i.e., a hexagon):



In higher dimensions, the number of possible truncations increases with the number of faces of different dimensions. For example, at dimension 3 we can truncate not only the vertices of a tetrahedron, but also its edges. The connected subsets of a hypergraph $\boldsymbol{H}$ with n vertices act as truncating instructions to be applied to the simplex of dimension $n-1$. The polytopes obtained in this manner are called *hypergraph polytopes.*

In [4], the faces of hypergraph polytopes are named by combinatorial objects called *constructs*, for which we develop here a new approach. While they were originally defined in [4] as certain sets of connected subsets of a hypergraph, we define them as decorated trees obtained in an algorithmic manner; this dynamic point of view extends to the definition of the partial order on constructs (Section 2). We show the equivalence with the original definition of Došen and Petrić in Section 3, where we also provide an alternative proof for the main theorem of [4], stating the order-isomorphism between the poset of constructs and the poset of faces in the geometric realisation. Unlike the original proof, our proof builds the isomorphism explicitly.

Došen and Petrić developed hypergraph polytopes in connection to their work on the categorification of operads [5]. The coherences arising in this setting display themselves as faces of some hypergraph polytopes. We complement their work with a criterion for recognising whether edges in these polytopes arise from sequential or parallel associativity isomorphisms (Section 4).

Finally, in Section 5, we show how to extend our tree notation for constructs to cover iterated truncations, i.e., truncations of faces themselves obtained after (possibly iterated) truncations, as captured combinatorially in [12], and we illustrate it for the case of the permutohedron-based associahedron (underlying the coherences of symmetric monoidal categories). We present an ad hoc notation for the faces of this polytope (in any finite dimension), based on words with holes and directly suggested by our construct notation.

We have tried to give, as much as possible, a self-contained exposition of the material presented.

Terminological warning: Throughout the paper, there will be trees (all rooted), graphs, hypergraphs, and polytopes, sometimes discussed next to each other. When speaking about "vertices" or "edges", it should always be clear to which of these structures we are referring.

We shall use two notions of subtree. By a *subtree of a construct* T we shall mean a tree obtained by picking a node of T and taking all its descendants. But in the context of *operadic trees* $\mathcal{T}$ (Section 4), we shall call subtree *any connected subset of* $\mathcal{T}$.

2 Hypergraph polytopes and constructs

In this section, we recall the definition of a hypergraph and some basic related notions. Then we give our own definition of constructs and of the partial ordering between them, postponing to Section 3 the proof that these coincide up to isomorphism with the definitions given in [4].

2.1 Hypergraphs

A hypergraph is given by a set H of vertices (the carrier), and a subset $\boldsymbol{H} \subseteq \mathcal{P}(H)\backslash\emptyset$ such that $\bigcup \boldsymbol{H} = H$. The elements of $\boldsymbol{H}$ are called the *hyperedges* of $\boldsymbol{H}$. We always assume that $\boldsymbol{H}$ is *atomic*, by which we mean that $\{x\} \in \boldsymbol{H}$, for all $x \in H$. Identifying x with $\{x\}$, H can be seen as the set of hyperedges of cardinality 1, also called *vertices*. We shall always use the convention to give the same name to the hypergraph and to its carrier, the former being the bold version of the latter. A hyperedge of cardinality 2 is called an *edge*. Note that any ordinary graph (V, E) can be viewed as the atomic hypergraph $\{\{v\} \,|\, v \in V\} \cup \{e \,|\, e \in E\}$ (with no hyperedge of cardinality ≥ 3).

If $\boldsymbol{H}$ is a hypergraph, and if $X \subseteq H$, we set

$$\boldsymbol{H}_X = \{Z \,|\, Z \in \boldsymbol{H} \text{ and } Z \subseteq X\}.$$

We say that $\boldsymbol{H}$ is *connected* if there is no non-trivial partition $H = X_1 \cup X_2$ such that $\boldsymbol{H} = \boldsymbol{H}_{X_1} \cup \boldsymbol{H}_{X_2}$. All our hypergraphs will be finite. It is easily seen that for each finite hypergraph there exists a partition $H = X_1 \cup \ldots \cup X_m$ such that each $\boldsymbol{H}_{X_i}$ is connected and $\boldsymbol{H} = \bigcup(\boldsymbol{H}_{X_i})$. The $\boldsymbol{H}_{X_i}$'s are called the *connected components* of $\boldsymbol{H}$. We shall also use the following notation:

$$\boldsymbol{H}\backslash X = \boldsymbol{H}_{H\backslash X}.$$

As a (standard) abuse of notation, we call a non-empty subset X of vertices connected (resp. a connected component) whenever $\boldsymbol{H}_X$ is connected (resp. a connected component). We define the *saturation* of $\boldsymbol{H}$ as the hypergraph

$$Sat(\boldsymbol{H}) = \{X \,|\, \emptyset \subsetneq X \subseteq H \text{ and } \boldsymbol{H}_X \text{ is connected}\}.$$

A hypergraph is called saturated when $\boldsymbol{H} = Sat(\boldsymbol{H})$. Atomic and saturated hypergraphs are called *building sets* in the works of Postnikov et al. [13,14], and are generalised, with the same name, from the present setting of $\mathcal{P}(H)$ to that of arbitrary finite lattices in the works of Feichtner et al. [6,7].

The notation

$$\boldsymbol{H}, X \rightsquigarrow H_1, \ldots, H_n \quad \text{(resp. } \boldsymbol{H}, X \rightsquigarrow \{H_i \,|\, i \in I\})$$

will mean that $H_1, \ldots, H_n \subseteq H\backslash X$ are the (resp. $\{H_i \mid i \in I\}$ is the set of) connected components of $\boldsymbol{H}\backslash X$. We shall write $\boldsymbol{H}_i$ for $\boldsymbol{H}_{H_i}$.

We call a *quasi-partition of a set* X a collection of disjoint (possibly empty) subsets whose union is X. We shall need the following (standard) property.

Lemma 1 *Let $\boldsymbol{H}$ be a connected hypergraph, and let $Y \subseteq X \subseteq H$. Let $\boldsymbol{H}, Y \rightsquigarrow \{K_j \,|\, j \in J\}$ and $\boldsymbol{H}, X \rightsquigarrow \{H_i \,|\, i \in I\}$. Then the following two claims hold:*

1. *If $H_i \cap K_j \neq \emptyset$, then $H_i \subseteq K_j$.*
2. *There exists a quasi-partition $\{I_j \mid j \in J\}$ of I, such that, for each $j \in J$, $K_j\backslash X = \bigcup_{i \in I_j} H_i$. Consequently, we have $\boldsymbol{K}_j, X \rightsquigarrow \{H_i \mid i \in I_j\}$.*

2.2 Constructs and constructions

A connected hypergraph $\boldsymbol{H}$ gives rise to a partial order of *constructs*, which we define below inductively.

Definition 1 Let $\boldsymbol{H}$ be a connected hypergraph and Y be an arbitrary non-empty subset of H:

- If $Y = H$, then the one-node tree decorated with H, written H, is a construct of $\boldsymbol{H}$.
- Otherwise, if $\boldsymbol{H}, Y \leadsto H_1, \ldots, H_n$, and if $T_1, \ldots, T_n$ are constructs of $\boldsymbol{H}_1, \ldots, \boldsymbol{H}_n$, respectively, then the tree whose root is decorated by Y, with n outgoing edges on which the respective T_i's are grafted, written $Y(T_1, \ldots, T_n)$, is a construct.

A *construction* is a construct whose nodes are all decorated with singletons. We shall often use the letter V to denote a construction (since constructions denote vertices in the geometric realisation, see Section 3.3).

In $Y(T_1, \ldots, T_n)$, the order of the constructs $T_1, ..., T_n$ is irrelevant. We shall write $Y\{T_i \mid i \in I\}$ when the constructs T_i are indexed over some finite set I. When $I = \emptyset$, we get that $Y\{T_i \mid i \in I\}$ stands for Y, corresponding to the base case in Definition 1 (note that the only hypergraph with an empty set of connected components is the empty one). It is also convenient to allow ourselves to write $\emptyset\{T_i \,|\, i \in I\}$, with $I = \{i_0\}$, as a stuttering form of T_{i_0} (if $Y = \emptyset$, we are left with building a construct of the original hypergraph).

The intuition behind this definition is algorithmic: *a construct is built by picking a non-empty subset Y of H and then branching to the connected components of $\boldsymbol{H}\backslash Y$, and continuing recursively in all the branches.*

The labels of the nodes of a construct of $\boldsymbol{H}$ form a partition of H. We shall freely confuse the nodes with their labels, since they are a fortiori all distinct. For every node Y of T, we denote by $\uparrow_T(Y)$ (or simply $\uparrow(Y)$) the union of the labels of the descendants of Y in T (all the way to the leaves), including Y. For every construct T of $\boldsymbol{H}$ and every node Z of T, the subtree of S rooted at Z is a construct of $\boldsymbol{H}_{\uparrow_T(Z)}$.

The notation $T : \boldsymbol{H}$ will mean that T is a construct of $\boldsymbol{H}$. The following formal system summarises our definition of constructs:

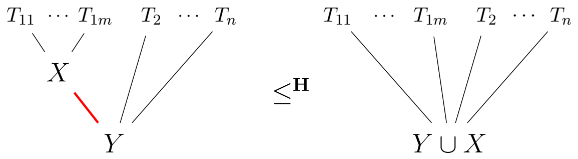
$$\boxed{\dfrac{}{H : \boldsymbol{H}} \qquad \dfrac{\boldsymbol{H}, X \leadsto H_1, \ldots, H_n \qquad T_1 : \boldsymbol{H}_1, \ldots, T_n : \boldsymbol{H}_n}{X(T_1, \ldots, T_n) : \boldsymbol{H}}}$$

We note that while the inductively-defined constructions in tree form appear in [4][Section 3] and in [13][Proposition 8.5] exactly like in Definition 1, these authors did not notice or exploit the fact that the tree notation could be extended to all constructs simply by replacing singletons with arbitrary subsets. As we shall see, this simple observation gives additional insights. In particular, it allows us to formulate various equivalent and useful characterisations of the partial order between constructs.

The tree notation for all constructs appears in [7][Proposition 3.17], but without an inductive characterisation.

2.3 Ordering constructs

We next define a partial order between constructs. The algorithmic intuition is that, given S, one can get a larger construct by contracting an edge of S, and then merging the decorations of the two nodes related by that edge, as illustrated in the following picture:


T_{11} ⋯ T_{1m} T_2 ⋯ T_n
X
Y
$\leq^{\mathbf{H}}$
T_{11} ⋯ T_{1m} T_2 ⋯ T_n
$Y \cup X$

Formally, the partial order $\leq^{\boldsymbol{H}}$ (or simply $\leq$, when $\boldsymbol{H}$ is understood) is defined as the smallest partial order generated by the following rules:

$$\frac{\begin{array}{c}\boldsymbol{H}, Y \rightsquigarrow K_1,\ldots,K_n \qquad \boldsymbol{K}_1, X \rightsquigarrow H_{11},\ldots,H_{1m}\\ T_{11}:\boldsymbol{H}_{11},\ldots,T_{1m}:\boldsymbol{H}_{1m} \qquad T_2:\boldsymbol{K}_2,\ldots,T_n:\boldsymbol{K}_n\end{array}}{Y(X(T_{11},\ldots,T_{1m}),T_2,\ldots,T_n) \leq^{\boldsymbol{H}} (Y\cup X)(T_{11},\ldots,T_{1m},T_2,\ldots,T_n)}$$

$$\frac{\boldsymbol{H}, Y \rightsquigarrow H_1,\ldots,H_n \qquad T_2:\boldsymbol{H}_2,\ldots,T_n:\boldsymbol{H}_n \qquad T_1 \leq^{\boldsymbol{H}_1} T_1'}{Y(T_1,T_2,\ldots,T_n) \leq^{\boldsymbol{H}} Y(T_1',T_2,\ldots,T_n)}$$

This definition is well-formed, in the sense that, if $S : \boldsymbol{H}$ and if $S \leq T$ is inferred, then $T : \boldsymbol{H}$ can be inferred. The one-node construct H is maximum, while the constructions are the minimal elements (there is no $X \cup Y$ to split).

The partial order $\leq$ admits two other equivalent definitions, for which we shall provisionally write $\leq_2^{\boldsymbol{H}}$ and $\leq_3^{\boldsymbol{H}}$ (shortly $\leq_2$ and $\leq_3$, respectively) before we prove that they define the same relation as $\leq^{\boldsymbol{H}}$. The formulation $\leq_2$ will allow us to prove the equivalence of our definitions with the original ones of [4], while, the formulation $\leq_3$ underlies an algorithm for enumerating all the vertices inferior to a given construct (see Section 2.5).

The definition of $\leq_2$ is given by two clauses (guided by Lemma 1):

$$\frac{\qquad}{H \leq_2^{\boldsymbol{H}} H}$$

$$\frac{\begin{array}{c}Y \subseteq X \qquad \boldsymbol{H}, Y \rightsquigarrow K_1,\ldots,K_m \qquad \boldsymbol{H}, X \rightsquigarrow H_1,\ldots,H_n\\ S_1:\boldsymbol{K}_1,\ldots,S_m:\boldsymbol{K}_m \qquad T_1:\boldsymbol{H}_1,\ldots,T_n:\boldsymbol{H}_n\\ S_j \leq_2^{\boldsymbol{K}_j} (K_j \cap X)\{T_i \,|\, H_i \subseteq K_j\} \text{ for all } j\end{array}}{Y(S_1,\ldots,S_m) \leq_2^{\boldsymbol{H}} X(T_1,\ldots,T_n)}$$

The relation $\leq_2$ formalises the following intuition. As in the definition of $\leq$, given a construct S, we want to know which constructs T (of the same hypergraph) lie above S in the partial order. If $S = H$, then S is the maximum construct of $\boldsymbol{H}$, hence $H \leq T$ boils down to $T = H$. Otherwise, the root of S

must be a subset of the root of T, and the task of showing $S \leq T$ is reduced to that of verifying that each S_j lies lower than an appropriate term.

For the definition of $\leq_3$, we need to introduce a variation of the notion of construct. We define the *partial constructs* of a connected hypergraph $\boldsymbol{H}$ by adding one clause to the inductive definition of constructs:

– The single-node tree decorated with Ω_H is a partial construct of $\boldsymbol{H}$.

(and by replacing "construct" with "partial construct" in the original clauses).

To distinguish partial constructs from constructs, we use the font $\mathbb{S}, \mathbb{T}, \ldots$ for the former. We summarise the definition of partial constructs as follows:

$$\boxed{\frac{}{\Omega_H : \boldsymbol{H}} \qquad \frac{}{H : \boldsymbol{H}} \qquad \frac{\boldsymbol{H}, X \rightsquigarrow H_1, \ldots, H_n \qquad \mathbb{T}_1 : \boldsymbol{H}_1, \ldots, \mathbb{T}_n : \boldsymbol{H}_n}{X(\mathbb{T}_1, \ldots, \mathbb{T}_n) : \boldsymbol{H}}}$$

We define a partial construction to be a partial construct in which all the non-Ω nodes are labelled by singletons.

Note the difference between decorations X and Ω_X in a partial construct: the latter stands for "undefined", in the spirit of Scott domain theory. We shall write $\mathbb{T}[\Omega_X \leftarrow \mathbb{S}]$ for the partial construct obtained from $\mathbb{T}$ by replacing Ω_X with $\mathbb{S} : \boldsymbol{H}_X$.

We shall use the notation $\mathbb{T} \blacktriangleright^{\boldsymbol{H}} X$ to indicate that X is the union of all non-Ω-decorations of $\mathbb{T}$, and we shall say that $\mathbb{T}$ *spans* X. Formally, this predicate is inductively defined as follows:

$$\boxed{\frac{}{\Omega_H \blacktriangleright^{\boldsymbol{H}} \emptyset} \qquad \frac{}{H \blacktriangleright^{\boldsymbol{H}} H} \qquad \frac{\boldsymbol{H}, X \rightsquigarrow H_1, \ldots, H_n \qquad \mathbb{T}_1 \blacktriangleright^{\boldsymbol{H}_1} X_1, \ldots, \mathbb{T}_n \blacktriangleright^{\boldsymbol{H}_n} X_n}{X(\mathbb{T}_1, \ldots, \mathbb{T}_n) \blacktriangleright^{\boldsymbol{H}} X \cup X_1 \cup \ldots \cup X_n}}$$

Lemma 2 *If $\mathbb{T} \blacktriangleright^{\boldsymbol{H}} X$, with $\boldsymbol{H}, X \rightsquigarrow H_1, \ldots, H_n$, then, for each $i \in \{1, \ldots, n\}$, there exists exactly one occurrence of Ω_{H_i} in $\mathbb{T}$, and these are all the occurrences of an Ω in $\mathbb{T}$.*

Proof. The proof is by structural induction on the proof of well-formedness of $\mathbb{T}$. The case $\mathbb{T} = Y(\mathbb{T}_1, \ldots, \mathbb{T}_n)$ is settled by appealing to Lemma 1. □

It follows from this lemma that the partial constructs (resp. constructions) that span the whole carrier H of a hypergraph $\boldsymbol{H}$ are exactly the constructs (resp. constructions) of $\boldsymbol{H}$.

Lemma 3 *With the notations of Lemma 2, if $S_1, \ldots, S_n$ are constructs of $\boldsymbol{H}_1, \ldots, \boldsymbol{H}_n$, respectively, then, for all $1 \leq i \leq n$, $\mathbb{T}[\ldots, \Omega_{H_i} \leftarrow S_i, \ldots]$ is a construct of $\boldsymbol{H}$, and $\mathbb{T}[\ldots, \Omega_{H_i} \leftarrow S_i, \ldots] \leq X(S_1, \ldots, S_n)$.*

Proof. By structural induction on $\mathbb{T}$. We treat the case $\mathbb{T} = Y(\mathbb{T}_1, \ldots, \mathbb{T}_m)$, with $\mathbb{T}_j$ spanning Y_j for all j. Setting $I_j = \{i \mid \Omega_{H_i} \text{ occurs in } \mathbb{T}_j\}$ and T'_j to be the result of replacing each Ω_{H_i} by S_i in $\mathbb{T}_j$ (i ranging over I_j), we get by induction that $T'_j \leq Y_j\{S_i \mid i \in I_j\}$, and we conclude as follows:

$$\begin{aligned}
\mathbb{T}[\ldots, \Omega_{H_i} \leftarrow S_i, \ldots] &= Y(T'_1, \ldots, T'_m) \\
&\leq Y(Y_1\{S_i \mid i \in I_1\}, \ldots, Y_m\{S_i \mid i \in I_m\}) \\
&\leq (Y \cup \{Y_1, \ldots, Y_m\})(S_1, \ldots, S_n) \;=\; X(S_1, \ldots, S_n)\,. \quad \square
\end{aligned}$$

We have now all the prerequisites for our third presentation of the partial order. We define $\leq_3^{\boldsymbol{H}}$ by the following two clauses:

$$\boxed{\begin{array}{c} \dfrac{S : \boldsymbol{H}}{S \leq_3^{\boldsymbol{H}} H} \\[2em] \dfrac{\begin{array}{c} \mathbb{T} \blacktriangleright^{\boldsymbol{H}} X \qquad \boldsymbol{H}, X \rightsquigarrow H_1, \ldots, H_n \\ S_1 : \boldsymbol{H}_1, \ldots, S_n : \boldsymbol{H}_n \quad T_1 : \boldsymbol{H}_1, \ldots, T_n : \boldsymbol{H}_n \qquad S_i \leq_3^{\boldsymbol{H}_i} T_i \text{ for all } i \end{array}}{\mathbb{T}[\ldots, \Omega_{H_i} \leftarrow S_i, \ldots] \leq_3^{\boldsymbol{H}} X(T_1, \ldots, T_n)} \end{array}}$$

Unlike for $\leq$ and $\leq_2$, the algorithmic reading of $S \leq_3 T$ answers the question of when S lies lower than some fixed T. If $T = H$, then any construct of $\boldsymbol{H}$ lies lower than T. Otherwise, S "starts by spanning X" (and recursively so).

Proposition 1 *The relations $\leq$, $\leq_2$, and $\leq_3$ coincide.*

Proof. That $S \leq T$ implies $S \leq_2 T$ is proved by showing that $\leq_2$ is closed under the rules that define $\leq$, including reflexivity and transitivity. Let us look at transitivity. Suppose that $\boldsymbol{H}, Y \rightsquigarrow K_1, \ldots, K_m$, $\boldsymbol{H}, X \rightsquigarrow H_1, \ldots, H_n$, $\boldsymbol{H}, Z \rightsquigarrow G_1, \ldots, G_k$, and

$$Y(S_1, \ldots, S_m) \leq_2 X(T_1, \ldots, T_n) \leq_2 Z(U_1, \ldots, U_k)\,.$$

We discuss only the case where $m, n, k \geq 1$. We have to show that the two conditions allowing to deduce $Y(S_1, \ldots, S_m) \leq_2 Z(U_1, \ldots, U_k)$ hold. Collecting the first conditions in clause 2 of $\leq_2$, relative to our present two assumptions, we have that $Y \subseteq X$ and $X \subseteq Z$, and hence $Y \subseteq Z$. We now show that the second condition holds. Let us fix $j \in \{1, \ldots, m\}$ and let

$$Y_j = K_j \cap Z, \quad Y'_j = K_j \cap X \quad \text{and} \quad X_i = H_i \cap Z.$$

We have to prove that

$$S_j \leq_2 Y_j(\{U_l \,|\, l \in L_j\}), \text{ where } L_j = \{l \in \{1, \ldots, k\} \,|\, G_l \subseteq K_j\}\,.$$

The second condition for our first assumption gives us that, for all j:

$$S_j \leq_2 Y'_j(\{T_i \,|\, i \in I_j\}), \text{ where } I_j = \{i \in \{1, \ldots, n\} \,|\, H_i \subseteq K_j\}\,.$$

Now, for each T_i, where $i \in I_j$, the second condition for the second assumption gives us that

$$T_i \leq_2 X_i(\{U_m \,|\, m \in M_i\}), \tag{1}$$

where $M_i = \{m \in \{1, \ldots, k\} \,|\, G_m \subseteq H_i\}$. Next, we have that

$$\begin{aligned} Y_j &= (K_j \cap X) \cup (K_j \cap (Z \backslash X)) \\ &= Y'_j \cup ((K_j \backslash X) \cap Z) \\ &= Y'_j \cup ((\textstyle\bigcup_{i \in I_j} H_i) \cap Z) \\ &= Y'_j \cup \textstyle\bigcup_{i \in I_j} (H_i \cap Z) \\ &= Y'_j \cup \textstyle\bigcup_{i \in I_j} X_i. \end{aligned} \tag{2}$$

And, lastly, since $K_j \backslash Z = (\bigcup_{i \in I_j} H_i)\backslash Z$, we have that

$$\begin{aligned} L_j &= \{l \in \{1,\ldots,k\} \,|\, G_l \subseteq K_j\} \\ &= \{l \in \{1,\ldots,k\} \,|\, G_l \subseteq (\textstyle\bigcup_{i \in I_j} H_i)\} \\ &= \textstyle\bigcup_{i \in I_j} \{l \in \{1,\ldots,k\} \,|\, G_l \subseteq H_i\} \\ &= \textstyle\bigcup_{i \in I_j} M_i. \end{aligned} \tag{3}$$

Finally, (1), (2), (3) and (4), together with the rules from the definition of $\leq$, give us that

$$\begin{array}{lr} S_j \leq_2 Y_j'(\{T_i \,|\, i \in I_j\}) & (1) \\ \quad \leq_2 Y_j'(\{X_i(\{U_m \,|\, m \in M_i\}) \,|\, i \in I_j\}) & (2), \text{congruence} \\ \quad \leq_2 (Y_j' \cup \bigcup_{i \in I_j} X_i)(\{U_m \,|\, m \in M_i \text{ and } i \in I_j\}) & \text{axiom of } \leq \\ \quad = Y_j(\{U_m \,|\, m \in \bigcup_{i \in I_j} M_i\}) & (3) \\ \quad = Y_j(\{U_l \,|\, l \in L_j\}). & (4) \end{array}$$

Note that this proof is valid provided one has shown beforehand that $\leq_2$ is closed under the other defining clauses of $\leq$.

That $S \leq_2 T$ implies $S \leq_3 T$ (resp. $S \leq_3 T$ implies $S \leq T$) is proved by induction on the proof of $S \leq_2 T$ (resp. of $S \leq_3 T$). □

2.4 Examples of hypergraphs and constructs

In this section, we provide a few examples of hypergraphs and their constructs, conveying an intuitive understanding of their geometric realisation. We shall freely write x instead of $\{x\}$ etc. for the labels of singleton nodes of constructs.

As our first example, we describe the n–1-dimensional simplex:

$$\boldsymbol{H} = \{\{x_1\},\ldots,\{x_n\},\{x_1,\ldots,x_n\}\}.$$

All of its constructs have the form $X(y_1,\ldots,y_p)$, where $X \subseteq \{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_p\} = \{x_1,\ldots,x_n\}\backslash X$. Note that $\boldsymbol{H}$ is indeed a hypergraph, and not just the discrete graph with n vertices, because we insist that the hyperedge $\{x_1,\ldots,x_n\}$ is included.

– At dimension 2 and writing x, y, z instead of x_1, x_2, x_3, we have 3 vertices, 3 edges or facets and the maximum face:

$$\begin{array}{rlll} \text{vertices} & x(y,z) & y(x,z) & z(x,y) \\ \text{facets} & \{x,y\}(z) & \{y,z\}(x) & \{x,z\}(y) \\ \text{whole polytope} & \{x,y,z\}. & & \end{array}$$

Note that $x(y,z) \leq \{x,y\}(z)$ and $y(x,z) \leq \{x,y\}(z)$, which says combinatorially that the edge $\{x,y\}(z)$ connects the vertices $x(y,z)$ and $y(x,z)$.

– At dimension 3, we get 4 vertices, 6 edges and 4 facets.

We illustrate now how the hypergraph structure allows us to make truncations. The desired effects of truncation will be obtained by adding hyperedges to the bare “simplex hypergraph”.

- Truncation of a vertex, say $x(y,z)$, of the 2-dimensional simplex (cf. Section 1). We add the hyperedge $\{y,z\}$ to the simplex hypergraph:

  $$\boldsymbol{H} = \{\{x\},\{y\},\{z\},\{y,z\},\{x,y,z\}\} \ .$$

  Then $x(y,z)$ is not a construction anymore, since $\boldsymbol{H}_{\{y,z\}}$ is now connected. Instead, we have 3 new constructs (encoding two vertices and one edge):

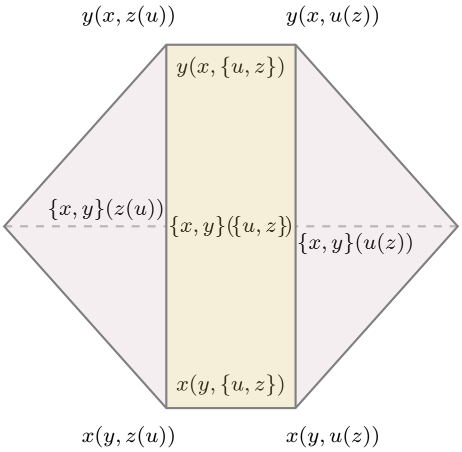
  $$x(y(z)) \quad x(z(y)) \quad x(\{y,z\}) \ .$$

- Truncation of an edge, say $\{x,y\}(u,z)$, of the 3-dimensional simplex. Similarly, we add the hyperedge $\{u,z\}$ to the simplex hypergraph:

  $$\boldsymbol{H} = \{\{x\},\{y\},\{z\},\{u\},\{u,z\},\{x,y,z,u\}\} \ .$$

  The edge $\{x,y\}(u,z)$ and its end vertices are now replaced by a rectangular face (9 new constructs):

  $$\begin{array}{llll} x(y,u(z)) & x(y,z(u)) & y(x,u(z)) & y(x,z(u)) \\ x(y,\{u,z\}) & y(x,\{u,z\}) & \{x,y\}(u(z)) & \{x,y\}(z(u)) \\ \{x,y\}(\{u,z\}) \ . & & & \end{array}$$

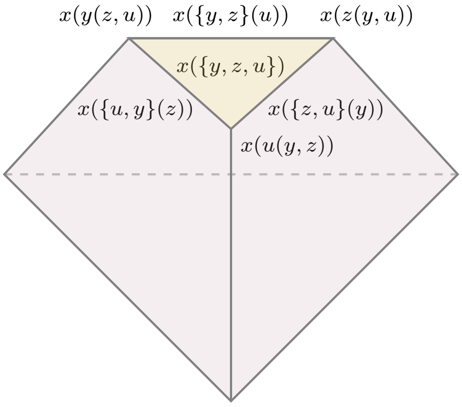


- Truncation of a vertex, say $x(y,z,u)$, of the 3-dimensional simplex. We achieve this by adding the hyperedge $\{y,z,u\}$ to the simplex hypergraph:

  $$\boldsymbol{H} = \{\{x\},\{y\},\{z\},\{u\},\{y,z,u\},\{x,y,z,u\}\} \ .$$

  This hypergraph disallows the construction $x(y,z,u)$ since $\boldsymbol{H}\backslash\{x\}$ is now connected, and replaces it by 3 vertices, 3 edges, and a facet:

  $$\begin{array}{lll} x(y(z,u)) & x(z(y,u)) & x(u(y,z)) \\ x(\{y,z\}(u)) & x(\{z,u\}(y)) & x(\{u,y\}(z)) \\ x(\{y,z,u\}) \ . & & \end{array}$$



Our next example is the family of associahedra. One of the standard labellings of the faces of the n-dimensional associahedron is by all the (partial or total) parenthesisations of a word of $n+2$ letters. Here, the idea is to focus, not on the letters (or the leaves of the corresponding tree), but on the $n+1$ "compositions of these letters" involved. These compositions are next to each other, as suggested in the following picture for dimension 3 (where $a,b,c,\dots$ are the letters and $x,y,\dots$ are the compositions):

$$\begin{matrix} a & & b & & c & & d & & e \\ & x & & y & & z & & u & \end{matrix} \tag{4}$$

- At dimension 2, this suggests to take the following graph (in hypergraph form), expressing "x is next to y which is next to z":

  $$\boldsymbol{H} = \{\{x\},\{y\},\{z\},\{x,y\},\{y,z\}\}\ .$$

  (Note that the hyperedge $\{x,y,z\}$ is no longer necessary to ensure that $\boldsymbol{H}$ is connected.) The edges $\{x,y\}$ and $\{y,z\}$ are prescriptions for truncating two vertices of a triangle, yielding a pentagon. The 5 vertices are

  $$x(y(z)) \quad x(z(y)) \quad y(x,z) \quad z(x(y)) \quad z(y(x))\ .$$

- At dimension 3, we take

  $$\boldsymbol{H} = \{\{x\},\{y\},\{z\},\{u\},\{x,y\},\{y,z\},\{z,u\}\}\ ,$$

  which seems like a prescription for truncating (only) three edges of the simplex. But look at what has become of the vertex $u(x,y,z)$. It has been also truncated! Indeed, it has been split into 5 constructions (with corresponding edges and face):

  $$u(x(y(z))) \quad u(x(z(y))) \quad u(y(x,z)) \quad u(z(x(y)) \quad u(z(y(x)))\ .$$

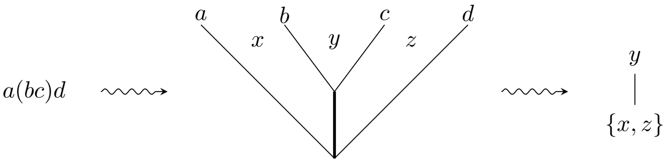
  To build these constructions, we have used that $\boldsymbol{H}_{\{x,y,z\}}$ is connected. In fact, the truncation prescriptions are all hyperedges of $Sat(\boldsymbol{H})$.

– At dimension n, we take

$$\boldsymbol{H} = \{\{x_1\},\ldots,\{x_{n+1}\},\{x_1,x_2\},\ldots,\{x_n,x_{n+1}\}\}\ .$$

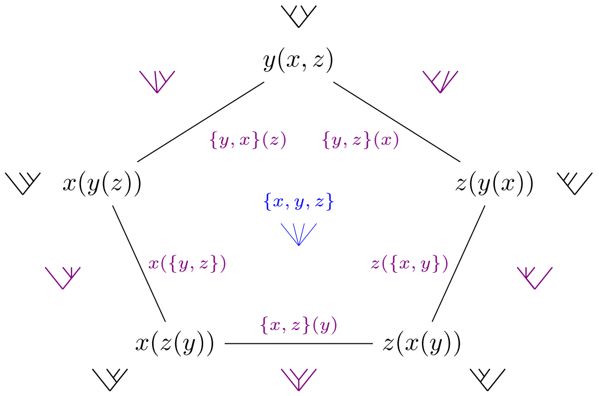
Here is the recipe showing how to move between three equivalent presentations of the faces of the associahedra: partially parenthesised words, rooted (undecorated) planar trees (see e.g. [10]), and constructs.

- From rooted planar trees to constructs. Label all the intervals beween the leaves of a tree with $n+2$ leaves by $x_1,\ldots,x_{n+1}$ (from left to right). Consider the x_i's as balls and let them fall. Label each node of the tree by the set of balls which fall to that node. Finally, remove all the leaves. For example:



- From constructs to parenthesisations. Read a construct from the leaves to the root, and each node as an instruction for building a parenthesis. If the label is, for example, $\{x_i,x_{i+2}\}$, then the instruction is to do an unbiased composition of three partially parenthesised words w_1,w_2,w_3 “above” x_i and x_{i+2} (for example, $a,(bc),d$ are above x and z in (4)), in one shot, resulting in $(w_1w_2w_3)$.
- From parenthesisations to trees. This is standard.

For the 2-dimensional associahedron, the representation with planar rooted trees / constructs is given on the next picture:

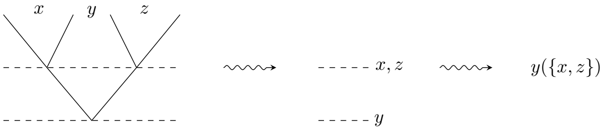


Our final example is the family of permutohedra. Here we take the complete graph on the set of vertices as the hypergraph. We discuss directly the general case at dimension n:

$$\boldsymbol{H} = \{\{x_1\},\ldots,\{x_{n+1}\}\} \cup \{\{x_i,x_j\}\,|\,i,j\in\{1,\ldots,n+1\} \text{ and } i\neq j\}.$$

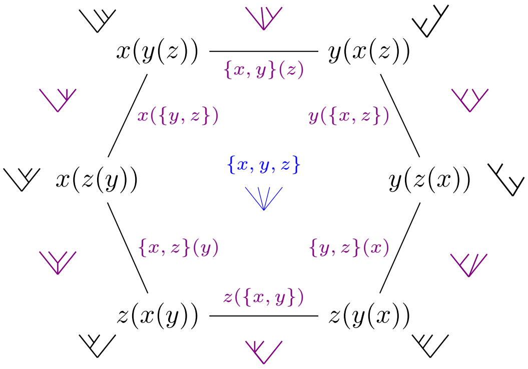
Note that all the constructs of the permutohedra are filiform, i.e., are trees reduced to a branch. The faces of the permutohedra have been described in the literature as surjections, and also as planar rooted trees with levels. The three representations are related as follows:

- From trees with levels to constructs. Consider again the x_i's as balls being thrown in the successive intervals between the leaves, and let them fall. Then we form the construct $Y_1(Y_2(\ldots(Y_m)\ldots))$, where Y_i is the collection of balls that fall to level i (counting levels from the root). For example:



- A construct $Y_1(Y_2(\ldots(Y_m)\ldots))$ defines a surjection from $\{x_1, \ldots, x_{n+1}\}$ to $\{1, \ldots, m\}$ mapping each x to i, where i is such that $x \in Y_i$.
- From surjections to trees with levels. We refer to [9].

For the 2-dimensional permutohedron, the representation with planar rooted trees with levels / constructs is given on the next picture:



2.5 Vertices of faces

As a preparation for the following section, given a hypergraph $\boldsymbol{H}$ and a construct $T : \boldsymbol{H}$, we give a device for finding all constructions V such that $V \leq^{\boldsymbol{H}} T$. If T is a construction, then this set is reduced to T itself. We shall use the notation $V \lessdot^{\boldsymbol{H}} T$ for "$V \leq^{\boldsymbol{H}} T$ and V is a construction".

First, we notice that, by a straightforward tuning of the definition of $\leq_3$, the predicate $\lessdot$ is defined by the following clauses:

$$\boxed{\begin{array}{c}\dfrac{V \text{ is a construction of } \boldsymbol{H}}{V \lessdot^{\boldsymbol{H}} H}\\[4ex] \dfrac{\begin{array}{c}\boldsymbol{H}, X \leadsto H_1,\ldots,H_n\\ V_1 \lessdot^{\boldsymbol{H}_1} T_1 \quad \ldots \quad V_n \lessdot^{\boldsymbol{H}_n} T_n \qquad \mathbb{V}_0 \blacktriangleright^{\boldsymbol{H}} X\end{array}}{\mathbb{V}_0[\varOmega_{H_1} \leftarrow V_1,\ldots,\varOmega_{H_n} \leftarrow V_n] \lessdot^{\boldsymbol{H}} X(T_1,\ldots,T_n)}\end{array}}$$

where, in the second clause, $\mathbb{V}_0$ is a partial construction.

This suggests an algorithm. For every node X of T, we should "zoom in" and replace it with a partial construction spanning X. Here is a formal device for searching all the partial constructions of $\boldsymbol{H}$ spanning a given fixed set $X \subseteq H$. One starts from $\varOmega_H$, and one performs rewriting (non-deterministically), until exhaustion of X, as follows:

$$\boxed{\dfrac{\mathbb{V} \blacktriangleright^{\boldsymbol{H}} Y \qquad Y \subsetneq X \quad x \in X\backslash Y}{\mathbb{V} \longrightarrow_X \mathbb{V}[\varOmega_K \leftarrow x(\varOmega_{K_1},\ldots,\varOmega_{K_p})]}}$$

where K is the connected component of $\boldsymbol{H}\backslash Y$ to which x belongs and where $\boldsymbol{K}, \{x\} \leadsto K_1,\ldots,K_p$.

We write ${\longrightarrow_X}^*$ for the reflexive and transitive closure of $\longrightarrow_X$. We shall say that a partial construction $\mathbb{V}$ is *accepted* if $\varOmega_H \;{\longrightarrow_X}^*\; \mathbb{V}$ and there exists no $\mathbb{V}'$ such that $\mathbb{V} \longrightarrow_X \mathbb{V}'$. As immediate observations, we have:

1. If $\mathbb{V} \longrightarrow_X \mathbb{V}' = V[\varOmega_K \leftarrow x(\varOmega_{K_1},\ldots,\varOmega_{K_p})]$, then $\mathbb{V}' \blacktriangleright^{\boldsymbol{H}} Y \cup \{x\}$, i.e., $\mathbb{V}'$ is a partial construction spaning $Y \cup \{x\}$.
2. The rewriting system $\longrightarrow_X$ is terminating, since the cardinality of the spanned subset increases by 1 at each step, while remaining a subset of X.

Lemma 4 *(1) The accepted partial constructions are precisely the partial constructions spanning X. (2) For every element $x \in X$, there exists a partial construction spanning X whose root is decorated by x.*

PROOF. If $\mathbb{V}$ is accepted, then $\mathbb{V}$ spans X by definition. Conversely, we proceed by induction on the cardinality of X. We can write $\mathbb{V}$ as $\mathbb{V}'[\varOmega_K \leftarrow x(\varOmega_{K_1},\ldots,\varOmega_{K_p})]$, since every tree has a node all of whose outgoing edges are leaves, and then apply induction to $\mathbb{V}'$, which spans $X\backslash\{x\}$.

As for the second claim, given $x \in X$, we can start the rewriting sequence with $\varOmega_H \longrightarrow_X x(\varOmega_{K_1},\ldots,\varOmega_{K_p})$. Then any continuation of this sequence leads to a partial construction spanning X, and has x as a root. □

Returning to our goal of finding all constructions V such that $V \lessdot T$ (for fixed T), we transform our definition of $\lessdot$ into an algorithmic one by replacing

$$\mathbb{V}_0 \blacktriangleright^{\boldsymbol{H}} X \quad \text{with} \quad \varOmega_H \longrightarrow_X^* \mathbb{V}_0 \blacktriangleright^{\boldsymbol{H}} X$$

in the second clause: we apply the device repetitively at all nodes of T.

Corollary 1 *For each construct $X(T_1,\ldots,T_n)$ and each $x \in X$, there exists at least one construction of the form $x(S_1,\ldots,S_m)$, such that $x(S_1,\ldots,S_m) \lessdot X(T_1,\ldots,T_n)$.*

3 Constructs as geometric faces

In this section, we recall the geometric realisation of hypergraph polytopes, following Došen and Petrić, and we provide a new proof of their theorem stating that the poset of constructs is isomorphic to the poset of geometric faces. The original proof in [4] relies on Birkhoff's representation theorem, without providing an explicit description of the isomorphism. Our proof is constructive, in that it exhibits the isomorphism.

We first prove the equivalence between our notion of constructs and theirs. Then we recall the geometric realisation of hypergraph polytopes. Finally, we translate both formalisations in the language of simplicial complexes, which provides the environment for exhibiting the desired isomorphism.

3.1 Non-inductive characterisation of constructs

Let $\boldsymbol{H}$ be a finite, atomic and connected hypergraph. We can define a map ψ from the set of constructs of $\boldsymbol{H}$ to $\mathcal{P}(\mathcal{P}(H)\backslash\emptyset)\backslash\emptyset$, as follows (with notation $\uparrow$ from Section 2.2):

$$\psi(T) = \{\uparrow(Y) \,|\, Y \text{ is a (label of a) node of } T\} \ .$$

We note that the Hasse diagram of $(\psi(T), \supseteq)$ is the same tree as T, replacing everywhere Y by $\uparrow(Y)$. We also observe that the old decoration can be recovered from the new one by noticing that

$$Y = \uparrow(Y) \backslash \bigcup \{\uparrow(Z) \,|\, Z \text{ is a child of } Y \text{ in } T\} \ .$$

From these observations, one can easily conclude that ψ is injective.

Lemma 5 *The map ψ is (contravariantly) monotonic and order-reflecting.*

Proof. Monotonicity is easy, following the inductive definition of $\leq$. For the second part of the statement, we show that if $\psi(T') \subseteq \psi(T)$, then $T \leq_2 T'$, by induction on the size of T. In what follows, for an arbitrary construct T, we will denote with $\rho(T)$ the root of T.

If $T = H$, then $\psi(T) = \{H\}$ and $\{H\} = \uparrow(\rho(T')) \subseteq \psi(T') \subseteq \psi(T)$ implies that also $\psi(T') = \{H\}$, and hence $T' = H$, and we conclude by clause 1 of the definition of $\leq_2$.

If $T = Y(S_1, \ldots, S_m)$ $(m \geq 1)$, let $T' = X(T_1, \ldots, T_n)$ $(n \geq 0)$. Since $\psi(T') \subseteq \psi(T)$, we get

$$\{\uparrow(\rho(T_i)) \,|\, 1 \leq i \leq n\} \subseteq \{\uparrow(Z) \,|\, Z \text{ is a node of } T\}.$$

Denote with $X_1, \ldots, X_n$ the nodes of T for which we have $\uparrow(\rho(T_i)) = \uparrow(X_i)$, and let, for each $1 \leq i \leq n$, U_i be the subtree of T rooted at X_i. Note that all X_i's must be different from Y. Indeed, if we had that $X_i = Y$ for some

$1 \leq i \leq n$, i.e., that $\uparrow(\rho(T_i)) = \uparrow(Y) = H$, this would imply that $X \subseteq \uparrow(\rho(T_i))$, which is not possible. We now have

$$Y = H \backslash \bigcup_{j=1}^{m} \uparrow(\rho(S_j)) \subseteq H \backslash \bigcup_{i=1}^{n} \uparrow(X_i) = H \backslash \bigcup_{i=1}^{n} \uparrow(\rho(T_i)) = X.$$

Therefore, the first condition in the second clause defining $\leq_2$ holds for T and T'. For the second condition, it is enough to establish (for all j)

$$\psi((K_j \cap X)\{T_i \mid H_i \subseteq K_j\}) \subseteq \psi(S_j) \ ,$$

which amounts to proving $\psi(T_i) \subseteq \psi(S_j)$, for every i such that $H_i \subseteq K_j$. We have, on one hand, $\psi(T_i) \subseteq \psi(T') \subseteq \psi(T)$, and, on the other hand, for each element Z of $\psi(T_i)$, $Z \subseteq H_i \subseteq K_j$, from which $\psi(T_i) \subseteq \psi(S_j)$ follows, since every non-root node Z' of S, other than a node appearing in S_j, appears in some other $S_{j'}$, hence is included in $K_{j'}$, and not in K_j. □

We now describe the image of ψ. We shall characterise the constructs among all possible trees decorated with disjoint subsets of H, in a non-inductive way. We note that the definition of $\uparrow$ makes sense for any such tree.

Recall that an antichain in a poset is a subset of pairwise uncomparable elements. We say that an antichain is proper if its cardinality is at least 2.

Lemma 6 *Any of the following properties characterises constructs among trees T decorated with subsets of H:*

1. *At every non-leaf node of T, $\uparrow(Y_1), \ldots, \uparrow(Y_m)$ are the connected components of $\boldsymbol{H}_{\uparrow(Y)\backslash Y}$, where Y is the label of the node, and $Y_1, \ldots, Y_m$ are the labels of its child nodes.*
2. *The following three conditions hold:*
   A *All labels of the nodes of T are pairwise disjoint and their union is H.*
   B *At each node X, $\uparrow(X)$ is such that $\boldsymbol{H}_{\uparrow(X)}$ is connected.*
   C' *At every non-leaf node Y whose child nodes are $Y_1, \ldots, Y_m$, and any subset I of $\{1, \ldots, m\}$ of cardinality at least 2, $\boldsymbol{H}_{\bigcup\{\uparrow(Y_i) \mid i \in I\}}$ is not connected.*
3. *Conditions (A) and (B) hold, together with:*
   C *For each set $\{X_1, \ldots, X_m\}$ of labels of T such that $\{\uparrow(X_1), \ldots, \uparrow(X_m)\}$ is a proper antichain, $\boldsymbol{H}_{\uparrow(X_1)\cup\ldots\cup\uparrow(X_m)}$ is not connected.*

Proof. (1) is a paraphrase of our inductive definition of construct. We have that (3) obviously implies (2), since (C) a fortiori implies (C').

We now prove that (1) implies (3). (A) and (B) are obvious through the equivalence of (1) with our definition of inductively defined constructs. We notice that (C) is vacuously true if T is reduced to one node. So let $T = Y(T_1, \ldots, T_p)$, with $\boldsymbol{H}, Y \rightsquigarrow H_1, \ldots, H_p$. Let $S = \{X_1, \ldots, X_m\}$ be as specified in the statement, and suppose that $\boldsymbol{H}_{\uparrow(X_1)\cup\ldots\cup\uparrow(X_m)}$ is connected. Then it is included in one of the H_i's (note that $(\uparrow(X_1) \cup \ldots \cup \uparrow(X_m)) \cap Y = \emptyset$). But then induction applies and we have a contradiction.

Finally, we prove that (2) implies (1). By induction, it suffices to check the property (1) at the root of $T = X(T_1, \ldots, T_q)$. By (B), we have that every $\uparrow(X_i)$ (X_i root of T_i, $i \in \{1, \ldots, q\}$) is included in some H_j. By (A), we have in fact that each H_j is a union of some $\uparrow(X_i)$'s. Formally, there exists a non-empty set I_j such that $H_j = \bigcup\{\uparrow(X_i) \mid i \in I_j\}$. But, by (C'), I_j must have cardinality 1 (for every j). Hence, up to permutation, we have $p = q$ and it follows that (1) holds at the root of T. □

Proposition 2 *The map ψ is a (contravariant) order-isomorphism between the set of constructs-as-decorated-trees and the collections of sets M of connected (non-empty) subsets of H, containing H, and satisfying the following property:*

C For each proper antichain $S = \{X_1, \ldots, X_m\} \subseteq M$, $\boldsymbol{H}_{X_1 \cup \ldots \cup X_m}$ is not connected.

Proof. By Lemma 6, we have that, for any T, $\psi(T)$ satisfies (C) (which we did not even care to rename!). Conversely, we first show that the Hasse diagram of a set M satisfying the conditions of the statement, ordered by reverse inclusion, is a tree. We note that if X, Y are in M and neither $X \subseteq Y$ nor $Y \subseteq X$, and thus $\{X, Y\}$ is an antichain, then, by (C), $\boldsymbol{H}\backslash X \cup Y$ is not connected. This entails in particular that $X \cap Y$ is empty, as otherwise, since $\boldsymbol{H}\backslash X$ and $\boldsymbol{H}\backslash Y$ are connected by assumption, $\boldsymbol{H}\backslash X \cup Y$ would be connected. It follows that there cannot be a Z above X and Y in the Hasse diagram, as this would imply $Z \subseteq X \cap Y$, but all elements of M, and Z in particular, are non-empty: contradiction. Hence this Hasse diagram is a tree, with root H. Then, as remarked above, it is easy to find a decoration of the same tree where at each node the decoration X is such that $\uparrow(X)$ is the corresponding element in the Hasse diagram. Finally, we know from Lemma 6 that this tree is indeed a construct T, and we have $\psi(T) = M$ by construction. □

Remark 1 1. Sets as in Proposition 2 are called *nested sets* in [7,13]. Proposition 2 thus states that constructs as inductively defined trees are in order-isomorphic correspondence with nested sets. In their work, Došen and Petrić adopt an intermediate viewpoint: they define constructions inductively, and they define constructs as subsets of $\psi(V)$ containing H, for some construction V. They prove in [4][Proposition 6.13] that their definition is equivalent to that of nested set.

2. When $\boldsymbol{H}$ is a graph (or has its set of connected subsets unchanged if restricted to hyperedges of cardinality ≤ 2), the assumption (C) can be further relaxed to:

   C_g For each antichain $S = \{X_1, X_2\} \subseteq M$ (with respect to inclusion) of cardinality 2, we have that $\boldsymbol{H}_{X_1 \cup X_2}$ is not connected.

   Indeeed, if (referring to (C)) $\boldsymbol{H}_{X_1 \cup \ldots \cup X_m}$ were connected, then since connectedness is path-connectedness in a graph, we would have that $\boldsymbol{H}_{X_i \cup X_j}$ is connected, for every pair of distinct $i, j \in \{1, \ldots, m\}$ (actually, picking just one such pair is enough for proving that (C_g) implies (C)).

Here is an example of why the stronger condition (C) is needed for general hypergraphs. Consider

$$\boldsymbol{H} = \{\{x\}, \{y\}, \{z\}, \{x, y, z\}\} \ .$$

Then (C$_g$) holds, but $\{\{x\}, \{y\}, \{z\}\}$ is a witness that (C) does not hold.

3. Going back to graph polytopes, condition (C$_g$) is equivalent to the conditions (1) and (2) below:
   1 If $X_1, X_2 \in M$ are such that $X_1 \cap X_2 \neq \emptyset$, then $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$.
   2 If $X_1, X_2 \in M$ are such that $X_1 \cap X_2 = \emptyset$, then $\boldsymbol{H}_{X_1 \cup X_2}$ is not connected.

   That (1) and (2) together imply (C$_g$) is obvious. Conversely, we get (the contraposite of) (1) by arguing as in the proof above, and since the implication $((X_1 \subseteq X_2$ or $X_2 \subseteq X_1) \Rightarrow X_1 \cap X_2 \neq \emptyset)$ holds obviously, (1) is actually an equivalence, through which (2) can be rephrased as (C$_g$). Conditions (1) and (2) are those given for *tubings* in [2].
4. We summarise the terminologies used in the literature in the following table (see also Section 3.2):

Combinatorial	Hypergraphs constructs	Graphs tubings	Building sets nested sets
Geometrical	Hypergraph polytopes	Graph associahedra	Nestohedra

3.2 Geometric realisation

Following Došen and Petrić, given a hypergraph $\boldsymbol{H}$, we show how to associate

- actual half-spaces and hyperplanes to the connected subsets of $\boldsymbol{H}$ (i.e., to the hyperedges of $Sat(\boldsymbol{H})$),
- an actual polytope $\mathcal{G}(\boldsymbol{H})$ to the whole hypergraph and
- actual faces of $\mathcal{G}(\boldsymbol{H})$ to constructs of $\boldsymbol{H}$.

Let $H = \{x_1, \ldots, x_n\}$. For every (non-empty) $A \subseteq \{1, \ldots, n\}$, we define two subsets of $\mathbb{R}^n$, as follows:

$$\pi_A^+ = \{(x_1, \ldots, x_n) \,|\, \textstyle\sum_{i \in A} x_i \geq 3^{|A|}\} \qquad \pi_A = \{(x_1, \ldots, x_n) \,|\, \textstyle\sum_{i \in A} x_i = 3^{|A|}\} \ .$$

where $|A|$ is the cardinality of A. Then the polytope associated with $\boldsymbol{H}$ is defined as follows:

$$\mathcal{G}(\boldsymbol{H}) = \bigcap \{\pi_Y^+ \,|\, Y \in Sat(\boldsymbol{H}) \backslash \{H\}\} \ \cap \ \pi_H \ .$$

For an arbitrary $M \subseteq Sat(\boldsymbol{H})$, we define

$$\varPi(M) = \bigcap \{\pi_Y \,|\, Y \in M\} \cap \mathcal{G}(\boldsymbol{H}) \ .$$

The definition of $\mathcal{G}(\boldsymbol{H})$ implements the truncation instructions encoded by the hypergraph $\boldsymbol{H}$.

This construction extends the realisation of the associahedra and of the cyclohedra originally proposed in [15,16]. In [2], graph-associahedra are also realised by means of truncations, although the concrete implementation of truncations is not described (interestingly, Devadoss gives a more precise realisation in terms of convex hulls in [3], that is also based on powers of 3).

In the setting of building sets (cf. Remark 1), a realisation that associates a linear inequality to every element of the building set, like in [4], can be found in [18]. On the other hand, Feichtner et al. use the elements of a building set as instructions for performing successive stellar subdivisions, starting from the simplex, while Postnikov et al. realise a building set by associating (via a fixed coordinate system) a simplex with each of its elements, and then by taking the Minkowski sum of these simplices. They call the resulting polytopes nestohedra.

3.3 Isomorphism between combinatorial and geometric faces

In this section, we exhibit an isomorphism between combinatorial and geometric faces of $\mathcal{G}(\boldsymbol{H})$, which exploits the fact that $\mathcal{G}(\boldsymbol{H})$ is a simple polytope.

We first give an alternative definition of a geometric face. We defined a face of a polytope as the intersection of the polytope with a single hyperplane. But by allowing the intersection with several hyperplanes, the choice of those hyperplanes can be restricted, as stated in the following proposition, which is often taken as an alternative definition of geometric face.

Proposition 3 *Each non-empty face of a polytope P presented by a collection $\mathcal{S}$ of half-spaces is defined as the intersection of P with some of the hyperplanes bounding the half-spaces in $\mathcal{S}$.*

We next introduce some notation. Given a polytope P, we let the letters F, G (resp. Φ) range over the geometric faces (resp. the facets) of P. We define a map ϕ from faces to sets of facets as follows:

$$\phi(F) = \{\Phi \mid F \subseteq \Phi\} .$$

We shall use the following equivalent characterisations of the notion of simple polytope (which are the item (iii) and a sharpened version of the item (v) of Proposition 2.16 of [19]):

S1 Each vertex of the polytope belongs to exactly n facets of the polytope, where n is the dimension of the polytope.

S2 For every face F, the restriction of ϕ to $\{G \mid F \subseteq G\}$ is an order-isomorphism onto $\mathcal{P}(\phi(F))$.

We shall also use the following properties, which are consequences of Lemmas 9.2, 9.4 and 9.5 of [4]:

H1 For every $M \subseteq \boldsymbol{H}$, if $\Pi(M)$ is non-empty, then M satisfies condition (C) of Proposition 2.

H2 For every construction V, $\Pi(\psi(V))$ is a vertex $\{v\}$ of $\mathcal{G}(\boldsymbol{H})$, and for every $Y \in \boldsymbol{H}\backslash\psi(V)$, we have $v \notin \pi_Y$. Conversely, every vertex of $\mathcal{G}(\boldsymbol{H})$ is obtained as $\Pi(\psi(V))$ for some construction V.

We take three steps in order to come up with the desired isomorphism.

A) The poset of (non-empty) faces of a simple polytope is isomorphic to an abstract simplicial complex.

This is well-known, but since we want to express our isomorphisms explicitly, we briefly review here how this goes. We start by some observations on polytopes (not necessarily simple). In any polytope, we have (cf. [19][Propositions 2.3 and 2.2]):

– Every face of a polytope is the convex hull of its vertices.

We shall exploit two consequences of this property.

P1 The map which associates with a face the set of all vertices that it contains is monotonic and order-reflecting, and by polarity (cf. [19][Section 2.3]), it follows that the map ϕ defined above is (contravariantly) monotonic and order-reflecting.

P2 Every non-empty face contains a vertex.

Let P be a polytope, with vertices $\{v_1\},\ldots,\{v_n\}$. By P2, the lattice of faces (minus the empty face) can be written as $\mathbb{L} = \mathbb{L}_1 \cup \ldots \cup \mathbb{L}_n$, where $\mathbb{L}_i$ is the set of faces containing v_i. If the polytope is simple, we know moreover by S2 that ϕ restricts to an order-isomorphism between $\mathbb{L}_i$ and $\mathcal{P}(\phi(\{v_i\}))$. Let us now define an abstract simplicial complex $\mathbb{N}$ associated with P. Recall that a finite abstract simplicial complex (abreviated here as simplicial complex) is given by specifying a set X, called the support, and subsets $X_1,\ldots,X_n \subseteq X$, called the bases, that are pairwise incomparable (w.r.t. inclusion), and are such that $X = X_1 \cup \ldots \cup X_n$. The simplicial complex associated to these data is by definition the set $\mathcal{P}(X_1) \cup \ldots \cup \mathcal{P}(X_n)$, ordered by inclusion. The complex $\mathbb{N}$ is defined as follows:

– the support X of $\mathbb{N}$ is the set of facets of P;
– we take as bases the sets $\phi(\{v\})$, for all vertices of P.

Since the local isomorphisms between the $\mathbb{L}_i$'s and $\mathcal{P}(\phi(\{v_i\}))$'s are restrictions of the same function ϕ, it follows that ϕ is an isomorphism from $\mathbb{L}$ to $\mathbb{N}$.

B) An isomorphism of simplicial complexes.

First, we remark that the partial order $\mathcal{A}(\boldsymbol{H})$ of constructs of $\boldsymbol{H}$ can itself be organised as a simplicial complex, up to the isomorphism identifying each construct T with $\psi(T)\backslash\{H\}$. Under these glasses, $\mathcal{A}(\boldsymbol{H})$ is isomorphic to the simplicial complex $\mathbb{M}$

– whose support is $\boldsymbol{H}\backslash\{H\}$,
– and whose bases are the sets $\psi(V)\backslash\{H\}$, where V ranges over the constructions of $\boldsymbol{H}$.

This follows from noting that any subset of a set satisfying condition (C) of Proposition 2 also satisfies that condition.

Our goal is to define an isomorphism from $\mathbb{M}$ to $\mathbb{N}$.

Lemma 7 *A set N of facets belongs to $\mathbb{N}$ if and only if $\bigcap N$ is non-empty.*

Proof. If $N \in \mathbb{N}$, then $N \subseteq \phi(\{v\})$, for some v, by definition of $\mathbb{N}$. It follows that $\bigcap \phi(\{v\}) \subseteq \bigcap N$. But $v \in \bigcap \phi(\{v\})$, by definition of ϕ, hence $\bigcap N$ is not empty. Conversely, if $\bigcap N$ is not empty, then, by (H1), $\{X \mid \Pi(\{X\}) \in N\} = \psi(T)$, for some construct T. By Corollary 1, we can choose a construction V, such that $V \leq T$. Moreover, by (H2), $\Pi(\psi(V))$ is a vertex $\{v\}$ of $\mathcal{G}(\boldsymbol{H})$. Let now $\chi(X)$ be a facet in N. Then $X \in \psi(V)$ since $\psi(T) \subseteq \psi(V)$, and therefore $\{v\} = \Pi(\psi(V)) \subseteq \chi(X)$, and hence $N \subseteq \phi(\{v\})$. □

Lemma 8 *If T is a construct of $\boldsymbol{H}$, and if $X \in \boldsymbol{H}\backslash\psi(T)$, then there exists a construction $V \lessdot T$, such that $X \in \boldsymbol{H}\backslash\psi(V)$.*

Proof. By induction on T. Let $T = Y(T_1, \ldots, T_n)$ (possibly with $n = 0$). We distinguish two cases:

1. $X \cap Y = \emptyset$. Then, $n \geq 1$, and for each $1 \leq i \leq n$, since $\psi(T_i) \subseteq \psi(T)$ for each i, we can apply induction to T_i's, and get $V_i \lessdot T_i$ satisfying the statement relative to T_i. Let then S' be an arbitrary partial construction spanning Y. By grafting the V_i's on the corresponding occurrences of Ω of S', we get a construction $V \lessdot T$, which satisfies the statement: this is clear for all nodes x coming from the V_i's, while all nodes coming from S', being elements of Y, are such that $\uparrow(x) \cap Y \neq \emptyset$, which implies $\uparrow(x) \neq X$.
2. $X \cap Y \neq \emptyset$. Let $y \in X \cap Y$. By Corollary 1, we can choose a construction $V \lessdot T$ whose root is decorated by y. Then $\psi(V)\backslash\{H\}$ consists only of sets that do not contain y, hence none of them can be X. □

Lemma 9 *The elements of $\boldsymbol{H}\backslash\{H\}$ are in one-to-one correspondence with the facets of $\mathcal{G}(\boldsymbol{H})$, through the map χ defined by $\chi(X) = \Pi(\{X\})$.*

Proof. We need to show that χ is both bijective and well-defined, in the sense that $\Pi(\{X\})$ is actually a facet. We take the following steps.

1. For all $X, Y \in \boldsymbol{H}\backslash\{H\}$, if $X \neq Y$, then $\chi(X)$ is not included in $\chi(Y)$ (this a fortiori implies that χ is injective). Since $X \neq Y$, we have $Y \notin \psi((H\backslash X)(X))$. Then, by Lemma 8, there exists a construction V such that $V \leq (H\backslash X)(X)$ and $Y \notin \psi(V)$. By (H2), we have $\Pi(\psi(V)) = \{v\}$ for some v such that $v \notin \pi_Y$, and therefore $v \notin \chi(Y)$. On the other hand, $V \leq (H\backslash X)(X)$ implies $v \in \chi(X)$, which proves the claim.
2. $\chi(X)$ is a facet, for all X. Suppose that $\chi(X) \subsetneq F$ for some face F of $\mathcal{G}(\boldsymbol{H})$. It follows from Proposition 3 that every face is included in some $\chi(Y)$ (just pick one of the hyperplanes in the statement). So we have $F \subseteq \chi(Y)$ for some Y, and a fortiori $\chi(X) \subseteq \chi(Y)$, from which we deduce $X = Y$ by (1). But this forces $\chi(X) = F$, contradicting our assumption.
3. χ is surjective. We already observed that every face is included in some $\chi(Y)$, from which surjectivity follows. □

Then the claimed isomorphism from $\mathbb{M}$ to $\mathbb{N}$ is defined through the map χ of Lemma 9, using the following easy fact.

- If χ is a bijection from the support of $\mathbb{M}$ to the support of $\mathbb{N}$ whose extension to subsets (notation $\chi[M] = \{\chi(X) \,|\, X \in M\}$) is such that, for all subsets M, N of the respective supports, we have $M \in \mathbb{M} \Leftrightarrow \chi[M] \in \mathbb{N}$, then it defines an order-isomorphism between $\mathbb{M}$ and $\mathbb{N}$.

C) $\mathcal{G}(\boldsymbol{H})$ is simple.

First, we establish the dimension of $\mathcal{G}(\boldsymbol{H})$.

Lemma 10 *If H has cardinality $n+1$, then $\mathcal{G}(\boldsymbol{H})$ has dimension n.*

Proof. It is enough to prove the statement in the case of the permutohedron, since $\mathcal{G}(\boldsymbol{H})$ contains the permutohedron defined by the complete graph on H. Simple calculations prove that the point $(3^{n+1}/n+1, \ldots, 3^{n+1}/n+1)$ lies in π_H and in the interior of π_Y^+ for all non-empty $Y \subsetneq H$, from which one concludes easily. □

We prove simplicity via condition S1, as follows. First, the dimension of $\mathcal{G}(\boldsymbol{H})$ is $|H| - 1$, by Lemma 10. Second, we note that a construction V has always exactly $|H|$ nodes, hence $\psi(V)$ has exactly $|H| - 1$ elements different from H. Since, by (H2), every vertex can be written as $\{v\} = \Pi(\psi(V))$, for some construction V, we conclude by observing that $\Pi(\psi(V))$ is included by definition in all of the $|H| - 1$ facets $\chi(X)$, for X ranging over $\psi(V) \backslash \{H\}$, and in no other facet, by (H2) and Lemma 9.

Thus we can combine steps (B) and (A).

Theorem 1 *The map $\Pi \circ \psi$, where ψ and Π are defined in Sections 3.1 and 3.2, is an order-isomorphism.*

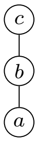
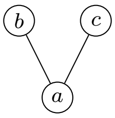
Proof. Our analysis gives us the isomorphism $\phi^{-1} \circ \chi$. It can be shown that "taking the intersection" is inverse to ϕ, which allows us to reformulate the isomorphism as follows $\phi^{-1}(\chi[\psi(T) \backslash \{H\}]) = \Pi(\psi(T))$. □

4 Operadic coherences

In [5], Došen and Petrić have used hypergraph polytopes in the study of coherences arising when categorifying the notion of operad [10], i.e., when the axioms of sequential and parallel associativity are turned into coherent isomorphisms β and θ, the coherence conditions being naturally associated with suitable polytopes. We shall not need the precise definition of an operad, and shall rely instead on simple graphical intuitions.

4.1 Weak Cat-operads

In monoidal categories, a coherence condition is imposed on the associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, ensuring that all the diagrams made of instances of α (possibly whiskered by identites), and their inverses, commute. This condition is expressed by the commutation of Mac Lane's pentagon (see diagram (2) on the next page). In an operad, the role of the objects A, B of a monoidal category is now played by operations labelling the nodes of a rooted (non-planar) tree. We call such a tree, acting as a pasting scheme, an *operadic tree*. Any two neigbouring operations in an operadic tree may be composed (imagine that the edge connecting them is contracted in the process), and then composed with a neigbouring operation, etc. The axioms of operads guarantee that the overall composition of the operations in the tree does not depend on the order of compositions. Consider the two trees with three nodes:

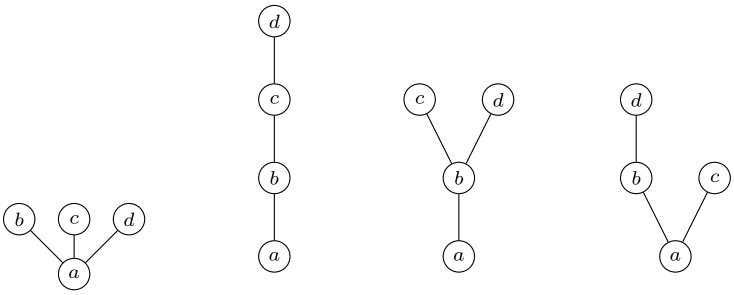
 and 

The axiom of sequential (resp. parallel) associativity says that the two ways to build the tree on the left (resp. on the right) by means of grafting and to perform compositions accordingly, yield the same operation: first compose a with b, and then compose with (or *insert*) c, or first compose b with c and then insert a (resp. first compose a with b, or first compose a with c). In a weak Cat-operad, these identifications are turned into isomorphisms

$$\beta : (ab)c \to a(bc) \quad \text{and} \quad \theta : (ab)c \to (ac)b$$

(writing composition as juxtaposition).

To synthesise the coherence conditions that β and θ have to satisfy, we need to consider the four possible shapes of operadic trees with four nodes:



(1) (2) (3) (4)

Each of these trees guides the interpretation of parenthesised words such as $((ab)c)d$ as sequences of insertions, and each of the diagrams below (one for each tree) features the resolution of the critical pair (or overlapping)

$$(\underbrace{(ab)c})d \qquad \underbrace{((ab)c)d}\,,$$

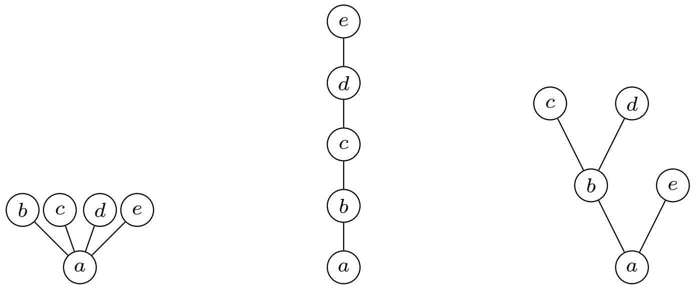
interpreted diversely according to whether the associativities are parallel or sequential, as prescribed by the respective trees.

$$\begin{tikzcd}
& ((ab)c)d \arrow[r, "\theta"] \arrow[dl, "\theta"'] & ((ac)b)d \arrow[dr, "\theta"] & \\
((ab)d)c \arrow[dr, "\theta"'] & (1) & & ((ac)d)b \arrow[dl, "\theta"] \\
& ((ad)b)c \arrow[r, "\theta"] & ((ad)c)b &
\end{tikzcd}
\qquad
\begin{tikzcd}
& ((ab)c)d \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\
(a(bc))d \arrow[d, "\beta"'] & (2) & (ab)(cd) \arrow[d, "\beta"] \\
a((bc)d) \arrow[rr, "\beta"] & & a(b(cd))
\end{tikzcd}$$

$$\begin{tikzcd}
& ((ab)c)d \arrow[r, "\theta"] \arrow[dl, "\beta"'] & ((ab)d)c \arrow[dr, "\beta"] & \\
(a(bc))d \arrow[dr, "\beta"'] & (3) & & (a(bd))c \arrow[dl, "\beta"] \\
& a((bc)d) \arrow[r, "\theta"] & a((bd)c) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& ((ab)c)d \arrow[dl, "\theta"'] \arrow[dr, "\theta"] & \\
((ac)b)d \arrow[d, "\beta"'] & (4) & ((ab)d)c \arrow[d, "\beta"] \\
(ac)(bd) \arrow[rr, "\theta"] & & (a(bd))c
\end{tikzcd}$$

Remark 2 Before asking the question of distinguishing β and θ edges in these "operadic polytopes" (and, in general, in operadic polytopes of arbitrary dimension), one must be able to systematically assign them labels. In Section 2.4, we have seen various ways to label all the faces of the pentagon and of the hexagon, that would work here for the pentagon made of β-arrows only and the hexagon made of θ-arrows only. However, it is a priori not clear how we could do this for the other two mixed β/θ-diagrams. This question will be addressed in Section 4.2.

By "lifting" the methodology of coherence chasing to the 3-dimensional setting, i.e., by considering trees with 5 nodes, we find 9 possible configurations. We shall draw only three of them:



The expression $(((ab)c)d)e$ is now subject to a three-fold overlapping,

$$((\underbrace{(ab)c})d)e \;\; (\underbrace{((ab)c)d})e \;\; \underbrace{(((ab)c)d)e}$$

which is resolved differently for each of the 9 trees, leading to 9 "coherence conditions between coherences" (in a framework where the coherence equations would not hold up to equality), each described by a suitable 3-dimensional polytope.

For the first two trees above, we get the 3-dimensional permutohedron and associahedron, respectively, whose edges all stand for θ-arrows in the first case, and β-arrows in the second. For the third one, we get a polytope called the *hemiassociahedron*, which, as we shall see, also belongs to the familly of hypergraph polytopes. In Figure 1, we labelled some of the vertices of this polytope, matching them with decompositions of our example tree (this matching will be spelled out in Proposition 4).

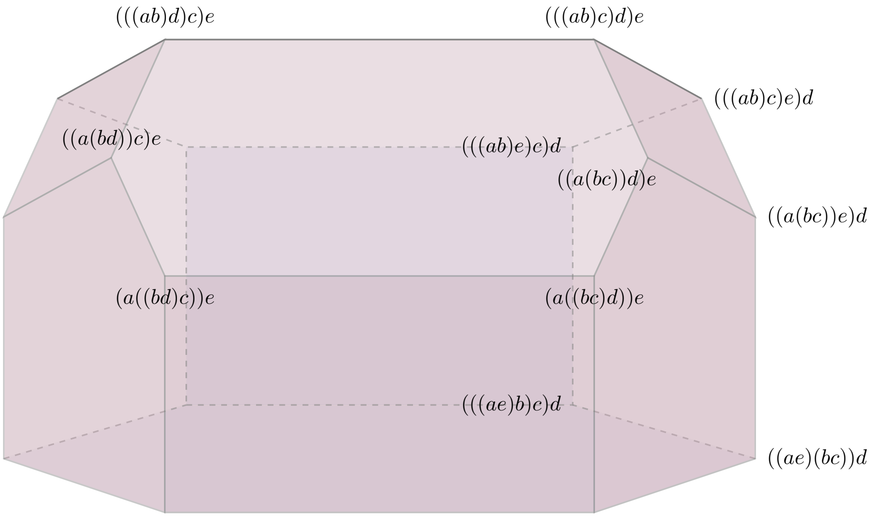


Fig. 1 The hemiassociahedron

4.2 Graphs associated with operadic trees

To every rooted tree $\mathcal{T}$ representing a pasting scheme for operadic operations, Došen and Petrić associate a graph $\mathbb{G}(\mathcal{T})$, obtained as follows. Its vertices are the edges of $\mathcal{T}$, and two vertices are connected whenever as edges of $\mathcal{T}$ they share a common vertex.

It is clear that one can identify the edges of $\mathcal{T}$ with the non-root nodes of $\mathcal{T}$ (for example, in Figure 2, there is a bijection mapping x to c, y to d, z to b, and u to e). By this identification, seeing now the nodes of $\mathbb{G}(\mathcal{T})$ as the non-root nodes of $\mathcal{T}$, all edges of $\mathcal{T}$, apart from those stemming from the root, are in $\mathbb{G}(\mathcal{T})$. All the other edges of $\mathbb{G}(\mathcal{T})$ are edges witnessing that two edges of $\mathcal{T}$ are siblings. We record the latter (resp. the former) by representing them with a dashed (resp. solid) line.

The graph $\mathbb{G}(\mathcal{T})$ is connected and can be represented itself as a tree with some horizontal dashed edges such that, by construction, each dashed horizontal zone is a complete graph all of whose nodes are connected to their father

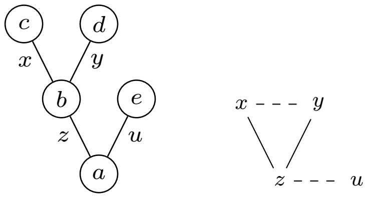


Fig. 2 The $\mathbb{G}(\mathcal{T})$ construction

node (if it exists) by solid edges. The nodes of $\mathbb{G}(\mathcal{T})$ are thus organised in levels. We say that $\mathbb{G}(\mathcal{T})$ has a root when there is no horizontal dashed layer at the bottom of $\mathbb{G}(\mathcal{T})$.

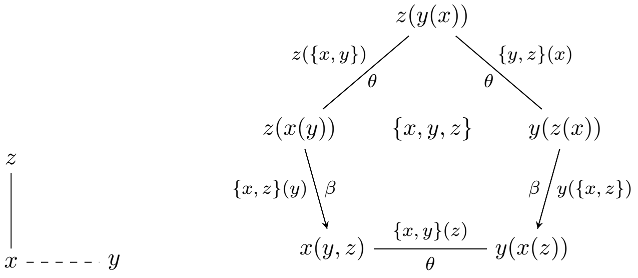
Figure 2 shows the graph associated to the third tree considered at the end of Section 4.1 (z, u are at level 1, and x, y are at level 2).

We insist that the dashed/solid informations on the edges of $\mathbb{G}(\mathcal{T})$ are not part of the graph structure $\mathbb{G}(\mathcal{T})$: they are additional data that we shall use to derive both the type (β or θ) and (in the case of β) the orientation of all edges of the corresponding polytope (as dictated by $\mathcal{T}$).

Recall that, in the language of constructs, vertices are trees whose nodes are all labelled with singletons. An edge E is a tree whose nodes are all singletons, *except one*, which is a two-element set $\{u_E, v_E\}$. We will show that $\mathbb{G}(\mathcal{T})$, together with its bipartition of dashed and solid edges, determines the type (and the orientation) of E. Let us call a min-path of a graph a path of minimum length between two vertices (we will show that in $\mathbb{G}(\mathcal{T})$ min-paths are always unique). Our criterion is the following:

† If the min-path between u_E and v_E in $\mathbb{G}(\mathcal{T})$ is made only of solid edges, E corresponds to a β-arrow, oriented towards the vertex of E in which the label u_E appears below the label v_E if and only if the level of u_E is inferior to the level of v_E in $\mathbb{G}(\mathcal{T})$. Otherwise, E witnesses a θ-arrow.

As an example, let us derive the edge information for the mixed pentagon (4), out of the associated graph:



According to the criterion, the orientation of, say, the β edge connecting $z(x(y))$ and $x(y, z)$ is dictated by the fact that x is below z in $\mathbb{G}(\mathcal{T})$. The orientation of the θ edges is then determined after choosing a starting vertex (one of the three upper vertices).

We now embark on the proof of soundness and completeness of this criterion. We shall formulate the criterion in different ways, and we shall exhibit the relationship between the connectedness properties of $\mathcal{T}$ and of $\mathbb{G}(\mathcal{T})$.

We first observe that for any two distinct vertices u, v of $\mathbb{G}(\mathcal{T})$, exactly one of the following two situations occurs (referring to u, v as edges of $\mathcal{T}$):

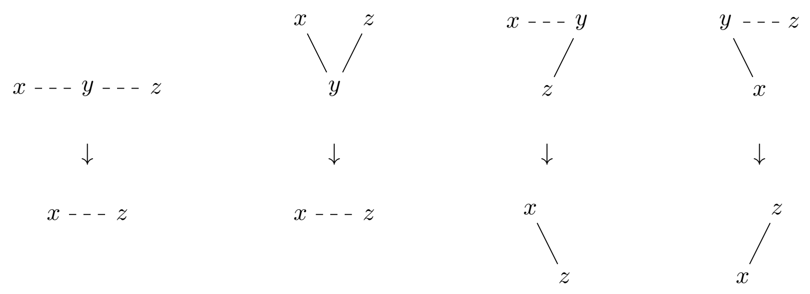
- Type I: u is above v or conversely.
- Type II: u and v are situated in disjoint branches of a subtree of $\mathcal{T}$. We will denote by $meet(u, v)$ the node of $\mathcal{T}$ at which the two branches diverge.

We can reformulate these two situations in $\mathbb{G}(\mathcal{T})$, without reference to $\mathcal{T}$:

- Type I: There is a descending path of solid edges (i.e., the level decreases by 1 at each node in the path) from u to v or from v to u (such a path will be called of type I);
- Type II: There exists a path $p = p_1, u', v', p_2$ from u to v whose parts p_1, u' and v', p_2 are descending and ascending, respectively (and therefore are made of solid edges only) and which is such that (u', v') is a dashed edge (such a path will be called of type II).

That this indeed is a reformulation is obvious for type I, while for type II, the desired path in $\mathbb{G}(\mathcal{T})$ is obtained by going down in $\mathcal{T}$ from (the child vertex of) u all the way down to u' whose father node is $meet(u, v)$, then through a dashed arrow to the branch carrying v', and then all the way up to (the child vertex of) v. Conversely, transcribing the path p_1, u', v', p_2 in the language of $\mathcal{T}$, we find a configuration of type II there.

In the next lemma, we show how to transform any path into a path of type I or II with the same end nodes. The transformations are specified by the following picture:



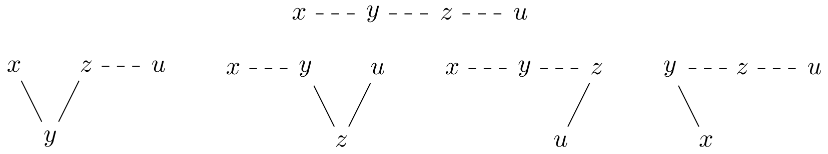
This specification is then used to define a rewriting system:

$$p_1, x, y, z, p_2 \longrightarrow p_1, x, z, p_2$$

when x, y, z are in one of the four configurations at the top of the picture.

Lemma 11 *This rewriting system is confluent and terminating. It is complete in the sense that any two paths between the same pair of end points are provably equal by a zigzag of such rewritings, and sound in the sense that any such zigzag always relates two paths with the same endpoints. The normal forms of the rewriting system are the paths of type I or II, and are the min-paths.*

Proof. Termination is obvious, since the length decreases by 1 at each step. As for confluence, we list the critical pairs, which all admit immediate solutions (note that the sequence (x, y) solid, (y, z) dashed, (z, u) solid is excluded since one would then have $x = u$, which contradicts the definition of a path):



That the paths of type I and II are in normal form is also immediate (there is no matching for the left hand sides of our rewriting rules). It remains to check that all normal forms are indeed of one of these two shapes. We proceed by induction on the length of the normal form p. Every path of length 1 is indeed of type I or II. Let now $p = u, v, p_1$. We can assume by induction that p_1 is of type I or II. There are three cases:

- (u, v) is solid with v one level up from u. Then p_1 cannot start with a solid edge going down, because then p would visit u twice, nor with a dashed edge, because p would then not be a normal form. Hence p_1 is of type I, and morevoer goes up (again because otherwise p would not be a path). Then adding (u, v) in front still results in a path of type I.
- (u, v) is solid with v one down from u. Then p_1 cannot start with a solid edge going up, since p would not be in normal form. Hence prefixing p_1 with (u, v) yields a path of type I (resp. II) if p_1 was of type I (resp. II).
- (u, v) is dashed. Then p_1 cannot start with a dashed edge nor a solid edge going down, as p would then not be in normal form. Hence p_1 has to be of type I, going up, which makes p a path of type II.

We now prove completeness. We have already observed the uniqueness of the paths of type I or II. Since we have established that the normal forms are the paths of type I or II, it follows that all paths in normal form from u to v coincide (notation $\equiv$), and we have, for any two paths p_1, p_2 from u to v, and writing $nf(p)$ for the normal form of a path p

$$p_1 \longrightarrow^* nf(p_1) \equiv nf(p_2)^* \longleftarrow p_2$$

Conversely, the rewriting system leaves the endpoints of the path unchanged at each step, and hence any zigzag maintains this inviariant, which establishes soundness.

That every minpath is normal is clear, since any rewriting step decreases the length of a path. For the converse, we use completeness. Suppose that p is normal, but is not a min-path, and let p_1 be a min-path with the same endpoints as p. By completeness, there exists a zigzag between p and p_1, or equivalently, by confluence, p and p_1 have the same normal form. But $nf(p_1)$ has a fortiori a length strictly smaller than $p = nf(p)$: contradiction. □

Summing up, the following are characterisations of "being of type I or II", for two distinct vertices u, v of $\mathbb{G}(\mathcal{T})$ (or equivalently, two edges u, v of $\mathcal{T}$):

Type I	Type II
u, v are one above the other in $\mathcal{T}$	u, v are on disjoint branches of a subtree of $\mathcal{T}$
u, v connected by a path of type I	u, v connected by a path of type II
min-path between u, v is of type I	min-path between u, v is of type II
min-path between u, v contains only solid edges	min-path between u, v contains at least one dashed edge

Indeed, by Lemma 11, we know that the min-paths are exactly the paths of type I or II, and crossing or not a dashed edge is what distinguishes among min-paths those that are of type II or I, respectively.

Lemma 12 *There is a one-to-one correspondence between the subtrees of $\mathcal{T}$ and the connected subsets of $\mathbb{G}(\mathcal{T})$.*

Proof. The connected subset of $\mathbb{G}(\mathcal{T})$ corresponding to a subtree $\mathcal{T}'$ of $\mathcal{T}$ is precisely $\mathbb{G}(\mathcal{T}')$. In the other direction, let K be a connected subset of $\mathbb{G}(\mathcal{T})$. By connectedness, for any u, v in K there exists a path p from u to v that is included in K. By Lemma 11, we know that $nf(p)$ is also included in K. By this observation, through the transcription in $\mathcal{T}$ of paths of type I or II of $\mathbb{G}(\mathcal{T})$, we conclude that the subgraph of $\mathcal{T}$ whose edges are precisely the vertices of K is a subtree. □

In what follows, in the context of operadic trees, we shall say that a tree is *non-Empty* if it contains at least one edge (whence the capital "E"). Clearly, all operadic trees relevant for describing operadic laws are non-Empty.

Lemma 13 *If $x_1, \ldots, x_n$ are arbitrary distinct edges of $\mathcal{T}$, then the following claims hold.*

1. *By removing $x_1, \ldots, x_n$ from $\mathcal{T}$, we obtain exactly $n+1$ subtrees of $\mathcal{T}$.*
2. *The number k of non-Empty subtrees of $\mathcal{T}$ obtained in this way is equal to the number of connected components of $\mathbb{G}(\mathcal{T})$ obtained by removing the vertices $x_1, \ldots, x_n$, and $k \in \{0, \ldots, n+1\}$.*
3. *Let $\mathcal{T}'$ be one of the non-Empty subtrees of $\mathcal{T}$ obtained by removing $x_1, \ldots, x_n$, and let K be the connected subset of $\mathbb{G}(\mathcal{T})$ associated with $\mathcal{T}$ by (2). Then, if y is an edge of $\mathcal{T}'$ and a vertex of K, we have that $\mathbb{G}(\mathcal{T}') = K$.*

Proof. We consider only the case $n = 1$ (the general case follows easily by induction), and we write x for x_1. Let a and b be the vertices adjacent to x, with a being the child vertex for b.

The first claim is standard: the subtrees obtained after the removal of x are the subtree $\mathcal{T}_1$ rooted at a and containing all descendants of a, and the subtree $\mathcal{T}_2$ obtained from $\mathcal{T}$ by removing all of $\mathcal{T}_1$. Note that b is a leaf of $\mathcal{T}_2$.

We prove the other two claims in parallel, by looking at the possible configurations of $\mathcal{T}$. If x is the only edge of $\mathcal{T}$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are the vertex a

and the vertex b, respectively, and, therefore, $k = 0$. Suppose that x is not the only edge of $\mathcal{T}$. Then, if x is on the highest level in $\mathcal{T}$, $\mathcal{T}_1$ is just the vertex a, while $\mathcal{T}_2$ is clearly non-Empty, and, hence, $k = 1$. We also get $k = 1$ when x is the unique edge on the first level of $\mathcal{T}$, in which case $\mathcal{T}_1$ is non-Empty and $\mathcal{T}_2$ is just the vertex b. In all other situations, we have $k = 2$.

Let us now prove that k is also the number of connected components of $\mathbb{G}(\mathcal{T})$ obtained by removing the vertex x. We examine only the case $k = 2$. Let $K_1 = \mathbb{G}(\mathcal{T}_1)$ and $K_2 = \mathbb{G}(\mathcal{T}_2)$. Since K_1 and K_2 are connected and disjoint and $\mathbb{G}(\mathcal{T})\backslash\{x\} = K_1 \cup K_2$, we only have to show that the set of edges of $\mathbb{G}(\mathcal{T})\backslash\{x\}$ is also the (disjoint) union of sets of edges of K_1 and K_2. For this, we use the fact that the removal of x from $\mathbb{G}(\mathcal{T})$ involves the removal of all edges of $\mathbb{G}(\mathcal{T})$ that have x as one of its adjacent vertices. Let e be an edge of $\mathbb{G}(\mathcal{T})$, with y and z being its adjacent vertices.

Suppose first that e is an edge of $\mathbb{G}(\mathcal{T})\backslash\{x\}$. We then know that both y and z are different from x, and share a common vertex v when considered as edges of $\mathcal{T}$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ form a partition of the set of vertices of $\mathcal{T}$, let us assume, say, that v is a vertex of $\mathcal{T}_1$. If $v \neq a$, we can immediately conclude that y and z are edges of $\mathcal{T}_1$, and, if $v = a$, then, since both y and z are different from x, it must be the case that v is the parent vertex for both y and z, which also implies that y and z are edges of $\mathcal{T}_1$. Therefore, y and z are both vertices of K_1, and, hence, e is an edge of K_1.

Conversely, if e is an edge of K_1, then y and z are edges of $\mathcal{T}_1$, and therefore must both be different from x. Since they share a common vertex in $\mathcal{T}_1$, and, hence, in $\mathcal{T}$, we conclude that e is an edge of $\mathbb{G}(\mathcal{T})\backslash\{x\}$. □

The following proposition is only implicit in [5].

Proposition 4 *For every operadic tree $\mathcal{T}$, the constructions of $\mathbb{G}(\mathcal{T})$ (considered as hypergraph) are in one-to-one correspondence with the (fully) parenthesised words that denote decompositions of $\mathcal{T}$.*

Proof. To every decomposition/parenthesisation of $\mathcal{T}$, one can associate a tree each of whose nodes is decorated by an edge of $\mathcal{T}$: one proceeds from the most internal parentheses to the most external ones, recording each insertion on the way.

Formally, the fullly parenthesised words are declared by the syntax $w :: a \mid (ww)$, where a ranges over the nodes of $\mathcal{T}$ (all named with different letters).

Not all words correspond to decompositions of $\mathcal{T}$. When this is the case, we say that w is admissible for $\mathcal{T}$ (the precise definition of admissibility can be easily reconstructed from the inductive construction below).

Since we deal with non-Empty trees, our base case is that of a word (ab) corresponding to a single edge operadic tree connecting a and b. Then there is only one decomposition and one construction, hence the statement holds.

Otherwise, we have a word $w = (w_1 w_2)$, where at least one of the words w_1 or w_2 is not reduced to a letter. We proceed by structural induction on w, providing both the decorated tree and the proof that is indeed a construction. Let us call $\mathcal{T}_1$, $\mathcal{T}_2$ the trees decomposed by w_1, w_2, respectively (cf. Lemma

13). Let x be the edge on which $\mathcal{T}_1$ is grafted on the tree $\mathcal{T}_2$. We distinguish three cases.

1. If neither w_1 nor w_2 are reduced to a letter, then $\mathbb{G}(\mathcal{T}_1)$ and $\mathbb{G}(\mathcal{T}_2)$ are both non-empty, and are the connected components of $\mathbb{G}(\mathcal{T})\backslash\{x\}$. We can thus apply induction: if V_1 and V_2 are the constructions associated with w_1 and w_2, then we associate $x(V_1, V_2)$ with (w_1w_2), which is a construction.
2. If $w_2 = a$ is a reduced to a letter and w_1 is not reduced to a letter, then $\mathbb{G}(\mathcal{T}_2)$ is empty, and x is a leaf of $\mathbb{G}(\mathcal{T})$. We conclude by induction that the tree $x(V_1)$ associated with $((w_1)a)$ is a construction.
3. If $w_1 = a$ is reduced to a letter, then $\mathcal{T}$ is of the form $a(\mathcal{T}_2)$, i.e. a is the root and has only one child which is the root of $\mathcal{T}_2$. We conclude by induction that the tree $x(V_2)$ associated with $(a(w_2))$ is a construction.

Note that case 1 (resp. cases 2 and 3) correspond to the situation where $k = 2$ (resp. $k = 1$), while the base case is the case where $k = 0$ (in the terminology of Lemma 13).

The converse mapping is defined much in the same way. We observe that, for any $\mathcal{T}$ (with at least 3 nodes), constructions of $\mathbb{G}(\mathcal{T})$ can only be of the form $x(V)$ or $x(V_1, V_2)$. They are of the first (resp. second) form when the node x is either a leaf or the root of $\mathbb{G}(\mathcal{T})$, (resp. when x is any other node). We can deploy induction on the number of nodes of T and map constructions back to parenthesised words. By induction too, we can show that these are inverse transformations. □

As an illustration, referring to Figure 2, $(ae)((bd)c)$ is mapped to $z(x(y), u)$, obtained as follows: the leaf u records ae, while in parallel the leaf y records bd and then $x(y)$ encodes $(bd)c$ and, finally, the last insertion is along z. (Note that, following common practice, in examples, we do not write the most external parentheses.)

We are now in a position to conclude.

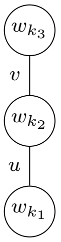
Theorem 2 *The criterion* (†) *is sound and complete.*

Proof. We write u, v for u_E, v_E. Let E' be the subtree of E whose root is $\{u, v\}$ and let K be the connected subset of $\mathbb{G}(\mathcal{T})$ out of which E' arises as a construct. Let $\mathcal{T}'$ be the subtree of $\mathcal{T}$ that corresponds to K by Lemma 12.

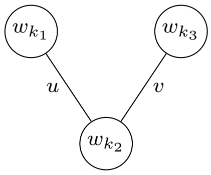
The number of constructions grafted to $\{u, v\}$ in E' is the number of connected components of $\boldsymbol{K}\backslash\{u, v\}$. By Lemma 13, it is also the number of non-Empty subtrees of $\mathcal{T}'$ obtained by removing the edges u and v. Moreover, there can be at most 3 such subtrees. Let us now introduce some names.

Let $\mathcal{T}_1', \mathcal{T}_2'$ and $\mathcal{T}_3'$ be the subtrees of $\mathcal{T}'$ obtained by removing u and v. Let $I \subseteq \{1, 2, 3\}$ be such that $i \in I$ if and only if $\mathcal{T}_i'$ is non-Empty, and let $J = \{1, 2, 3\}\backslash I$. Let, for all $i \in I$, K_i be the connected component of $\boldsymbol{K}\backslash\{u, v\}$ corresponding to $\mathcal{T}_i'$, V_i be the construction of K_i that is grafted to $\{u, v\}$ in E', and w_i be the decomposition of $\mathcal{T}_i'$ corresponding to V_i according to Proposition 4. On the other hand, for all $j \in J$, $\mathcal{T}_j'$ is a vertex a_j, and let w_j be precisely a_j.

By analysing the case $n = 2$ of Lemma 13, we get that E' determines an (incomplete) decomposition W of $\mathcal{T}'$, in which the insertions of u and v are the only ones not yet performed, and which has one of the following two shapes:



and

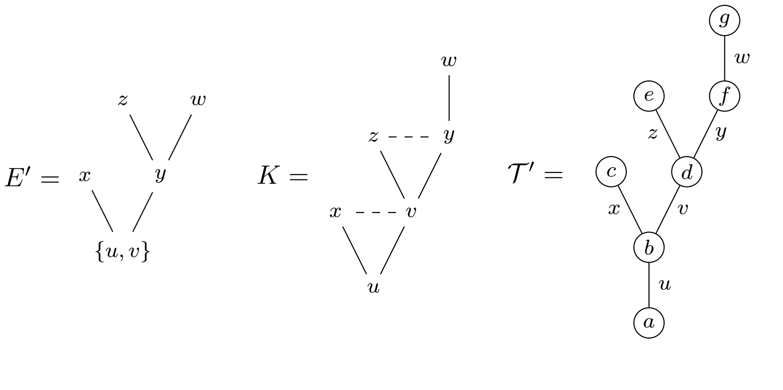


where $\{k_1, k_2, k_3\} = I \cup J$, and where the words w_{k_i} are as defined above. The shape on the left arises in the case when there exists a sequence $u = x_0, \ldots, x_n = v$ of edges in $\mathcal{T}'$ such that the child vertex of x_{i-1} is a parent vertex of x_i, for all $1 \leq i \leq n$, and the one on the right when there exists a subtree of $\mathcal{T}'$ that has u and v on different branches. We observe that $\{u, v\}$ is of type I (resp. of type II) in $\mathcal{T}'$ (and hence in $\mathcal{T}$) if W has the shape on the left (resp. on the right).

Now, if V_1 and V_2 are the vertices of $P_{\mathcal{T}}$ adjacent to E, then, in order to get complete decompositions of $\mathcal{T}'$ corresponding to V_1 and V_2, it remains to add u and v (in a way dictated by V_1 and V_2, respectively) in the sequence of insertions obtained previously from E'. More precisely, if we assume that u is the child of v (resp. v is the child of u) in V_1 (resp. V_2), then in the decomposition of $\mathcal{T}'$ guided by V_1 (resp. V_2), the insertion of v (resp. u) will be applied last. We then conclude by examining the two possible shapes of W.

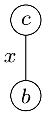
- In the type I case, V_1 and V_2 differ only by the subwords $(w_{k_1}w_{k_2})w_{k_3}$ and $w_{k_1}(w_{k_2}w_{k_3})$, respectively. Hence E' features a β-arrow. Moreover, the orientation prescribed in the statement of the criterion tells us that the edge should be oriented from V_1 to V_2, given our (arbitrary) choice of placing u under v in our drawing on the left.
- In the type II case, V_1 and V_2 differ only by the subwords $(w_{k_1}w_{k_2})w_{k_3}$ and $(w_{k_1}w_{k_3})w_{k_2}$, respectively. Hence E' features a θ-arrow. □

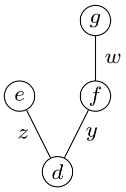
We illustrate the constructions of the proof below, with $E', K, \mathcal{T}'$ as follows:



The subtrees we get after removing u and v from $\mathcal{T}'$ are

$\mathcal{T}'_1 =$

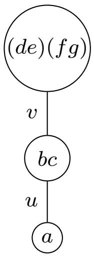
$\mathcal{T}'_2 =$ 

$\mathcal{T}'_3 =$ 

and the corresponding decompositions are

$$w_1 = a \qquad\qquad w_2 = bc \qquad\qquad w_3 = (de)(fg)$$

Hence, E' corresponds to the following decomposition of $\mathcal{T}'$:

$W =$ 

For this example, the vertices V_1 and V_2 adjacent to E induce decompositions $(a(bc))((de)(fg))$ and $a((bc)((de)(fg)))$, respectively, and E features a β-arrow from V_1 to V_2.

5 Iterated truncations

In this section, we recast the iterated truncations of [12] in our setting. Hypergraph polytopes allow us to describe all the polytopes in the interval between the simplex and the permutohedron. The hypergraph specifies at once all truncations to be made to reach a particular polytope in this interval. But what about truncating a new face that was not present in the original simplex, i.e. a face already obtained as a result of a truncation? We shall build a whole "tree" of polytopes, each polytope in the tree giving rise to a whole interval of truncations which are all its child nodes in the tree. A polytope at distance n of the root will be obtained through n runs of truncations. The root is occupied by the simplex. Our tree notation for constructs extends to this setting.

5.1 Successive rounds of truncations

Let $\mathcal{X}$ be a set (whose elements stand for the facets of the initial simplex). All the work will be carried out within $\mathcal{M}^f(\mathcal{X})$, the set of finite multisets of elements of $\mathcal{X}$, which gives rise to a monad. At each round of truncation, we are given

- a non-empty set $H \subseteq \mathcal{M}^f(\mathcal{X})$ (whose elements stand for the facets of the polytope that is to be truncated);

- a hypergraph $\boldsymbol{H}^v$ (whose hyperedges stand for the ***v***ertices of the same polytope);
- an atomic and connected hypergraph $\boldsymbol{H}^{\boldsymbol{t}}$ (whose connected subsets give instructions for the ***t***runcations to be performed at this round).

We require that $(\cup \boldsymbol{H}^v) = H = (\cup \boldsymbol{H}^t)$. We also require $\boldsymbol{H}^v$ to satisfy the following property::

$(P) \quad \forall x \in H, \, \exists V \in \boldsymbol{H}^v, \, x \in V$.

The intuition is that $\boldsymbol{H}^v$ serves to tame the constructs of $\boldsymbol{H}^t$. Indeed, the polytopes that we are building in this way are simple (this is a consequence of [12][Proposition 9.3]), so from every vertex at round n, the local view is that of a simplex – a property that is often taken as the definition of simplicity –, which makes those polytopes liable to the machinery of hypergraph polytopes.

We modify the definition of construct (and construction) as follows. Constructs are defined exactly as in Section 2, except for the root, for which one has to pick, not an arbitrary non-empty subset Y of H, but one which contains the complement of some hyperedge V of $\boldsymbol{H}^v$. Such a construct will be called a *construct of* $\boldsymbol{H}^t$ *rel to* $\boldsymbol{H}^v$, and we shall say that it is *tamed* by V. Constructs in the "old sense" will be called plain constructs. Here is the full definition. Pick an arbitrary subset $Y \subseteq H$ such that $(H \backslash V) \subseteq Y$ for some $V \in \boldsymbol{H}^v$.

- If $Y = H$, then the one node tree decorated with Y, and written Y, is a construct of $\boldsymbol{H}^t$ rel to $\boldsymbol{H}^v$.
- Otherwise, if $\boldsymbol{H}^t, Y \rightsquigarrow H_1, \ldots, H_n$, and if $T_1, \ldots, T_n$ are plain constructs of $\boldsymbol{H}_1, \ldots, \boldsymbol{H}_n$, respectively, then $Y(T_1, \ldots, T_n)$, is a construct of $\boldsymbol{H}^t$ rel to $\boldsymbol{H}^v$.

Note that the taming is only performed at the root. We denote by $\mathcal{A}_{\boldsymbol{H}^v}(\boldsymbol{H}^t)$ the set of constructs rel to $\boldsymbol{H}^v$.

The definition of construction is also slightly modified: it is a tree where all non-root nodes are decorated by singletons while the root is decorated exactly by the complement of some hyperedge of $\boldsymbol{H}^v$.

The initial round of truncations is along the simplex-permutohedron interval. We take:

- $H_1 = \mathcal{X}$ (identifying an element x of $\mathcal{X}$ with the associated one element multiset);
- $\boldsymbol{H}_1^v = \{\mathcal{X} \backslash \{x\} \,|\, x \in \mathcal{X}\}$;
- $\boldsymbol{H}_1^t$ is any atomic connected hypergraph on H_1.

Note that the constructs of $\boldsymbol{H}_1^t$ rel to $\boldsymbol{H}_1^v$ are all the constructs of $\boldsymbol{H}_1^t$ (no taming yet).

We explain now how round $n+1$ is prepared from round n. From $H_n \subseteq \mathcal{M}^f(\mathcal{X})$, $\boldsymbol{H}_n^v$, $\boldsymbol{H}_n^t$, we generate $\mathcal{A}_{\boldsymbol{H}_n^v}(\boldsymbol{H}_n^t)$, which induces a set $H_{n+1} \subseteq \mathcal{M}^f(\mathcal{X})$ and a hypergraph $\boldsymbol{H}_{n+1}^v$ on H_{n+1}, as follows:

- The maximal elements of $\mathcal{A}_{\boldsymbol{H}_n^v}(\boldsymbol{H}_n^t) \backslash \{H_n\}$, which we shall call *constrs*, are all of the form $X(Y)$, where $X \cup Y = H_n$ (by definition of constructs), and hence are entirely characterized by Y. We set

$$H_{n+1} = \{(\mu_{\mathcal{X}} \circ \sigma_{\mathcal{M}^f(\mathcal{X})})(Y) \mid (H\backslash Y)(Y) \text{ is a constr of } \boldsymbol{H}_n^t \text{ rel to } \boldsymbol{H}_n^v\},$$

where σ turns a set into the formal sum of its elements, and where μ is the multiplication of the monad $\mathcal{M}^f$.

- $\boldsymbol{H}_{n+1}^v$ is in bijection with the set of constructions of $\boldsymbol{H}_n^t$ rel to $\boldsymbol{H}_n^v$:

$$\begin{array}{l}\boldsymbol{H}_{n+1}^v = \{\mathcal{P}(\mu_{\mathcal{X}} \circ \sigma_{\mathcal{M}^f(\mathcal{X})})(\psi(T)\backslash\{H_n\}) \mid \\ \qquad\qquad T \text{ is a construction of } \boldsymbol{H}_n^t \text{ rel to } \boldsymbol{H}_n^v\}\ .\end{array}$$

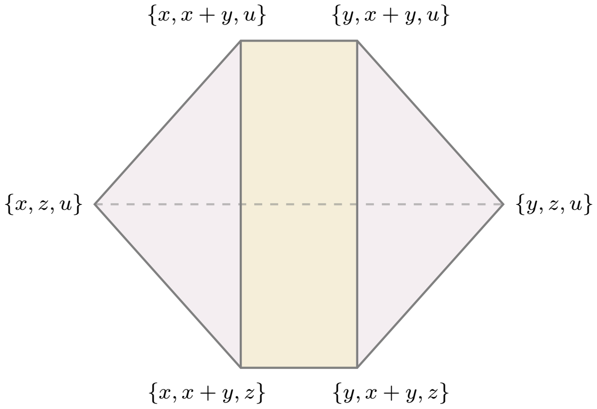
Proposition 5 *1. $\boldsymbol{H}_{n+1}^v$ is indeed a subset of $\mathcal{P}(H_{n+1})$, and $\mu_{\mathcal{X}} \circ \sigma_{\mathcal{M}^f(\mathcal{X})}$ is bijective on the subsets to which it is applied.*
2. At every round, $\boldsymbol{H}_n^v$ satisfies property (P).
3. At every round, we have $H_n \subseteq H_{n+1}$.

Proof. For (1), we refer (mutatis mutandis) to [12][section 7]. We prove (3) first. In terms of constructs, we have to show that for each $y \in H_n$, $(H_n\backslash\{y\})(y)$ is a construct rel to $\boldsymbol{H}_n^v$. Unrolling what it means, we see that this is the case if there exists V such that $(H_n\backslash\{y\}) \supseteq (H\backslash V)$ which is statement (2) (at round n). We now prove (2) at round $n+1$. Let Y be such that $(H_{n+1}\backslash Y)(Y)$ is a constr of $\boldsymbol{H}_n^t$ rel to $\boldsymbol{H}_n^v$. By an easy adaptation of the devices described in Section 2.5, we can obtain at least one construction V of $\boldsymbol{H}_n^t$ rel to $\boldsymbol{H}_n^v$ such that $V \leq (H\backslash Y)(Y)$, which entails $Y \in \psi(V)$. □

As an illustration, here is how to recast the example of [12][p. 11]. We take $\mathcal{X} = \{x, y, z, u\}$. We set:

$$\begin{array}{l}H_1 = \{x, y, z, u\} \quad \text{(considered as a subset of } \mathcal{M}^f(\mathcal{X}))\\ \boldsymbol{H}_1^v = \{\{x, y, z\}, \{y, z, u\}, \{z, u, x\}, \{u, x, y\}\}\\ \boldsymbol{H}_1^t = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, y, z, u\}\}\end{array}$$

resulting in the following truncation of the 3-dimensional simplex (decorating the vertices as in $\boldsymbol{H}_2^v$, by anticipation):



For example, $\{x, x+y, u\}$ is obtained from the construction $V = z(y(x), u)$ as prescribed by the specification of $\boldsymbol{H}_2^v$: slowly, we have

$$\psi(V)\backslash H_1 = \{\{x\}, \{x, y\}, \{u\}\}\ ,$$

from which we get $\{x, x+y, u\}$ by applying σ elementwise (no μ to perform here).

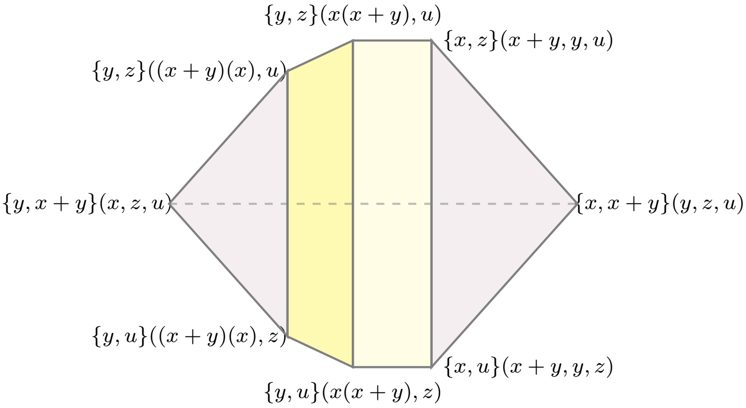
The first round induces

$$\begin{aligned} H_2 &= \{x, y, z, u, x+y\} \\ \boldsymbol{H}_2^v &= \{\{y, z, u\}, \{x, z, u\}, \{y, x+y, z\}, \{x, x+y, z\}, \{y, x+y, u\}, \{x, x+y, u\}\}\ , \end{aligned}$$

and let the second round be instructed by

$$\boldsymbol{H}_2^t = \{\{u\}, \{x\}, \{y\}, \{z\}, \{x+y\}, \{x, x+y\}, \{u, x, y, z, x+y\}\}.$$

resulting in the following polytope:



in which the new edge between x and $x+y$ (created after the first round) has been itself truncated. This induces

$$\begin{aligned} H_3 &= \{x, y, z, u, x+y, 2x+y\} \\ \boldsymbol{H}_3^v &= \{\{x, z, u\}, \{y, z, u\}, \{x+y, y, z\}, \{x+y, y, u\}, \{x+y, 2x+y, z\}, \\ &\quad \{x, 2x+y, z\}, \{x+y, 2x+y, u\}, \{x, 2x+y, u\}\} \end{aligned}$$

Here, say, $\{x+y, 2x+y, u\}$ corresponds to $\{y, z\}(x(x+y), u)$ (note the use of μ on $x+(x+y)$).

One could go on on this example: we could truncate the new edge between the faces x and $2x+y$, and create the new face $3x+y$, etc. It can be shown (see [12][Section 6]) that the flattening from $x+(x+y)$ to $2x+y$ (or of $x+(2x+y)$ to $3x+y$) incurs no loss of information, provided the traces of the rounds (i.e., the successive pairs of hypergraphs) are recorded. In this way, an untractable combinatorial explosion in the description of iterated truncations is avoided.

5.2 The permutohedron-based associahedron

As a more sophisticated example of iterated truncations, we now describe the combinatorics of the family of permutohedron-based associahedra, which are polytopes describing the coherences of symmetric monoidal categories (see Figures 3 and 4). They were introduced in [12], and further studied in [1]. These polytopes are different from the permutoassociahedra, which were introduced for the same purpose in [8], and which are not simple polytopes. The reason for this diversity is that different choices of generating isomorphisms lead to different combinatorial / geometrical interpretations of the same coherence theorem.

We take $\mathcal{X} = \{x_1, \dots, x_{n+1}\}$. The first round of truncation is that producing the permutohedron, with as truncation hypergraph the complete graph over $\mathcal{X}$:

$$\begin{array}{l} H_1 = \mathcal{X} \\ \boldsymbol{H}_1^v = \{\mathcal{X} \backslash \{x\} \,|\, x \in \mathcal{X}\} \\ \boldsymbol{H}_1^t = \{\{x_i\} \,|\, i \in [1, n+1]\} \cup \{\{x_i, x_j\} \,|\, i, j \in [1, n+1],\ i \neq j\} \ . \end{array}$$

Recall from Section 2.4 that all constructs of the permuohedra are filiform. Thus constrs are in bijection with proper subsets of $\mathcal{X}$ (different from $\mathcal{X}$), and the set of constructions is in bijection with the symmetric group S_{n+1}. More precisely, a construction $x_{\sigma(n+1)}(\dots(x_{\sigma(2)}(x_{\sigma(1)}))\dots)$ is encoded as

$$x_\sigma \overset{\Delta}{=} \{x_{\sigma(1)} \,,\, x_{\sigma(1)} + x_{\sigma(2)} \,,\, \dots \,,\, x_{\sigma(1)} + \dots + x_{\sigma(n)}\} \ ,$$

where $\sigma \in S_{n+1}$. This leads us to

$$\begin{array}{l} H_2 = \{\{\Sigma_{i \in I} x_i \,|\, i \in I\} \,|\, \emptyset \neq I \subsetneq \mathcal{X}\} \\ \boldsymbol{H}_2^v = \{x_\sigma \,|\, \sigma \in S_{n+1}\} \ . \end{array}$$

We shall write $x_I = \Sigma_{i \in I} x_i$. The next round of truncations is defined by

$$\boldsymbol{H}_2^t = \{\{x_I\} \,|\, \emptyset \neq I \subsetneq \mathcal{X}\} \cup \{\{x_I, x_J\} \,|\, I \subseteq J \text{ and } J \backslash I \text{ is a singleton}\} \ .$$

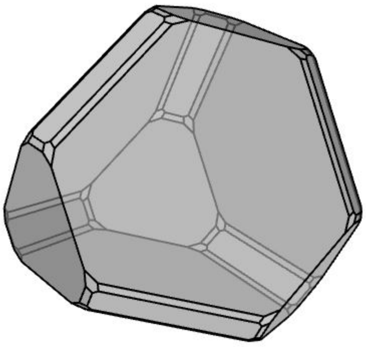


Fig. 3 The 3-dimensional permutohedron-based associahedron

We note that, for each $V \in \boldsymbol{H}_2^v$, $(\boldsymbol{H}_2^t)_V$ is the hypergraph specifying the associahedron. Now, recall from Section 2.4 that in the setting of associahedra we have a bijective correspondence between (fully) parenthesized words and constructions. In the present case, it is guided by the following picture:

$$\begin{array}{ccccccccc} x_{\sigma(1)} & & x_{\sigma(2)} & & x_{\sigma(3)} & & x_{\sigma(n)} & & x_{\sigma(n+1)} \\ & x_{\sigma(1)} & \rule[0.5ex]{3em}{0.4pt} & x_{\sigma(1)}+x_{\sigma(2)} & \rule[0.5ex]{3em}{0.4pt} & \cdots & \rule[0.5ex]{3em}{0.4pt} & x_{\sigma(1)}+\ldots,+x_{\sigma(n)} & \end{array}$$

We get that the set of constructs of $\boldsymbol{H}_2^t$ tamed by x_σ is in one-to-one correspondence with the set of parenthesized words over $\mathcal{X}$, in which the order of the letters from left to right is the one that we adopted in the definition of x_σ. This takes care of all 24 pentagons (corresponding to all possible permutations σ) of Figure 3.. We next show how to name the remaining edges and faces.

- Let us set $a = x_1, b = x_1 + x_2, c = x_1 + x_2 + x_3, d = x_1 + x_3$. Then

  $$\left.\begin{array}{l} (H_2\backslash\{a,b,c\})(b(a,c)) \\ (H_2\backslash\{a,c,d\})(d(a,c)) \end{array}\right\} \text{ correspond to } \left\{\begin{array}{l} (x_1x_2)(x_3x_4) \\ (x_1x_3)(x_2x_4) \end{array}\right.$$

  There is an edge between these two vertices, named by $(H_2\backslash\{a,c\})(a,c)$. Here is a way to name it in the style of parenthesized words:

  $$((x_1\bullet_1)(\bullet_1x_4)\ ,\ (\bullet_1 \mapsto \{x_2,x_3\}))$$

  The notation here is a way to formalise the surjection that maps x_1 to x_1, x_2, x_3 to $\bullet_1$, and x_4 to x_4. After all, we are seeking a mix of the notation for associahedra and permutohedra, hence a mix of parenthisations and surjections!

  In this way, we account for all single edges relating two pentagons.
- We now account for parallel edges between two pentagons, and the corresponding rectangular faces:

  $$\left.\begin{array}{l} (\bullet_1(\bullet_1(x_3x_4)),(\bullet_1{\mapsto}\{x_1{,}x_2\})) \\ (\bullet_1((\bullet_1x_3)x_4),(\bullet_1{\mapsto}\{x_1{,}x_2\})) \end{array}\right\} \text{ for } \left\{\begin{array}{l} x_1(x_2(x_3x_4)) - x_2(x_1(x_3x_4)) \\ x_1((x_2x_3)x_4) - x_2((x_1x_3)x_4) \end{array}\right.$$

  $$\bullet_1(\bullet_1x_3x_4) \ \text{ for } \ (H_2\backslash\{b,c\})(\{b,c\})$$

  and

  $$\left.\begin{array}{l} ((x_1(x_2\bullet_1))\bullet_1,(\bullet_1{\mapsto}\{x_3{,}x_4\})) \\ (((x_1x_2)\bullet_1)\bullet_1,(\bullet_1{\mapsto}\{x_3{,}x_4\})) \end{array}\right\} \text{ for } \left\{\begin{array}{l} (x_1(x_2x_3))x_4 - (x_1(x_2x_4))x_3 \\ ((x_1x_2)x_3)x_4 - ((x_1x_2)x_4)x_3 \end{array}\right.$$

  $$(x_1x_2\bullet_1)\bullet_1 \ \text{ for } \ (H_2\backslash\{a,b\})(\{a,b\})$$

- We are left with the remaining faces. The eight dodecagons are named by

  $$((x_i\bullet_1)\bullet_1\bullet_1,(\bullet_1 \mapsto (H_2\backslash\{x_i\})) \ \text{ and } \ (\bullet_1\bullet_1(\bullet_1x_i),(\bullet_1 \mapsto (H_2\backslash\{x_i\}))$$

  standing for $(H_2\backslash\{x_i\})(x_i)$ and $(H_2\backslash\{\sum_{j\neq i} x_j\})(\sum_{j\neq i} x_j)$, respectively, and the 6 octagons by, say:

  $$(\bullet_1(\bullet_1\bullet_2)\bullet_2,(\bullet_1 \mapsto \{x_1,x_2\}\ ,\ \bullet_2 \mapsto \{x_3,x_4\})).$$

Indeed, this octagon should contain the following four edges (which are sides of four pentagons), for each of which we give the corresponding construct:

$$\begin{array}{ll}
x_1(x_2x_3)x_4 & (H_2\backslash\{a,b,c\})(\{a,c\})(b)) \\
x_1(x_2x_4)x_3 & (H_2\backslash\{a,b,f\})(\{a,f\})(b)) \\
x_2(x_1x_3)x_4 & (H_2\backslash\{e,b,c\})(\{e,c\})(b)) \\
x_2(x_1x_4)x_3 & (H_2\backslash\{e,b,f\})(\{e,f\})(b))
\end{array}$$

where $a = x_1$, $b = x_1 + x_2$, $c = x_1 + x_2 + x_3$, $e = x_2$, $f = x_1 + x_2 + x_4$. The least upper bound of these constructs is $(H_2\backslash\{b\})(b)$, and all what this construct specifies is that we should do the operation b as innermost operation. It is a "Mastermind" kind of partial information:

$$\begin{array}{ccccccc}
\bullet_1 & & \bullet_1 & & \bullet_2 & & \bullet_2 \\
 & ?_1 & \text{——————} & x_1 + x_2 & \text{——————} & ?_2 &
\end{array}$$

Note that b, being the sum of two letters, has to be the central node, and that it being the sum of x_1 and x_2 entails that $?_1$ is x_1 or x_2, and $?_2$ is $x_1 + x_2 + x_3$ or $x_1 + x_2 + x_4$. The same information is carried out by our encoding.

The notation can be systematised in any finite dimension. We shall describe an algorithm transforming any construct $T = (H_2\backslash Y)(T_1, \ldots, T_p)$ of $\boldsymbol{H}_2^t$ rel

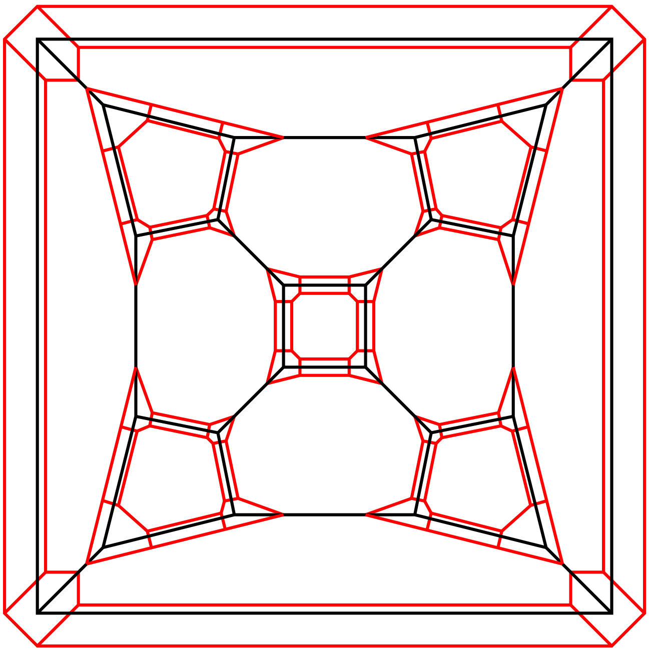


Fig. 4 Plane projection of the permutohedron-based associahedron of dimension 3

to $\boldsymbol{H}_2^v$ into a pair (W,π) of the kind discovered above. We can write

$$Y=\{\sum_{i\in I_1}x_i,\ldots,\sum_{i\in I_k}x_i\}\qquad\text{with } I_1\subsetneq I_2\subsetneq\ldots\subsetneq I_k\ .$$

We can encode the information provided by Y through the following map $\tilde{\pi}_Y$:

$$\begin{array}{ll}\tilde{\pi}_Y(i1)=\{x_i\,|\,i\in I_1\}, & \tilde{\pi}_Y(2)=\{x_i\,|\,i\in I_2\backslash I_1\},\ldots,\\ \tilde{\pi}_Y(k)=\{x_i\,|\,i\in I_k\backslash I_{k-1}\} & \tilde{\pi}_Y(k+1)=\{x_i\,|\,i\in[1,n+1]\backslash I_k\}\ .\end{array}$$

We associate with $\tilde{\pi}_Y$ the word $\tilde{w}_Y$ starting with $|I_1|$ occurrences of the letter $\bullet_1$, followed by $|I_2\backslash I_1|$ occurrences of the letter $\bullet_2$,..., ending with $|[1,n+1]\backslash I_k|$ occurrences of $\bullet_{k+1}$. We then do a bit of "making up": we replace in $\tilde{w}_Y$ all letters $\bullet_i$ occurring only once by the unique element of $\tilde{\pi}_Y(i)$; we also renumber the remaining letters $\bullet_j$, and we reindex $\tilde{\pi}_Y$ accordingly. We denote the new word by w_Y and the new map by π_Y We call the $\cdot_j$'s and the x_i's holes and determined letters, respectively. For example, if

$$\begin{array}{lll}\tilde{\pi}_Y(1)=\{x_9\} & \tilde{\pi}_Y(3)=\{x_3\} & \tilde{\pi}_Y(5)=\{x_6\}\\ \tilde{\pi}_Y(2)=\{x_2,x_4,x_8\} & \tilde{\pi}_Y(4)=\{x_1,x_7\} & \tilde{\pi}_Y(6)=\{x_5,x_7\}\ ,\end{array}$$

then $\bullet_1\bullet_2\bullet_2\bullet_2\bullet_3\bullet_4\bullet_4\bullet_5\bullet_6\bullet_6$ becomes

$$x_9\bullet_1\bullet_1\bullet_1x_3\bullet_2\bullet_2x_6\bullet_3\bullet_3\ ,$$

and we have

$$\pi_Y(1)=\{x_2,x_4,x_8\}\quad\pi_Y(2)\{x_1,x_7\}\quad\pi_Y(3)=\{x_5,x_7\}\ .$$

We complete the making up by placing parentheses in w_Y:

- around every subword $\bullet_jx_{i_1}\ldots x_{i_l}\bullet_{j+1}$ (l may be 0), and, if this applies,
- around the prefix of w_Y of the form $x_{i_1}\ldots x_{i_l}\bullet_1$ ($l>0$),
- and around the suffix of w_Y of the form $\bullet_kx_{i_1}\ldots x_{i_l}$ ($l>0$).

(with all letters in the ...'s determined). We denote the obtained parenthesised word by $(w_Y)^{st}$ (for *st*andardisartion). We use square brackets for writing the parentheses in $(w_Y)^{st}$, to distinguish them (visually only) from further parentheses that will be induced by $T_1,\ldots,T_p$. For our example, we get:

$$[x_9\bullet_1]\bullet_1[\bullet_1x_3\bullet_2][\bullet_2x_6\bullet_3]\bullet_3$$

We now examine how to encode the information provided by $T_1,\ldots,T_p$. The square brackets in $(w_Y)^{st}$ delimit the zones of w_Y that correspond to the connected components of $(\boldsymbol{H}_2^t)_Y$, which have the form

$$\{\sum_{i\in I_m}x_i,\sum_{i\in I_{m+1}}x_i,\ldots,\sum_{i\in I_{m+q}}x_i\}\text{ with}\left\{\begin{array}{l}I_m\subseteq I_{m+1}\subseteq\ldots\subseteq I_{m+q}\\ |I_{m+1}\backslash I_m|=\ldots=|I_{m+q}\backslash I_{m+q-1}|=1.\end{array}\right.$$

We examine first the two degenerate cases:

- $Y = \emptyset$. Then $k = 0$, and we set by convention $I_0 = \emptyset$, so that $\pi(1) = \{1, \ldots, n+1\} \backslash I_0 = \{1, \ldots, n+1\}$. Then $(w_Y)^{st} = w_Y = \bullet_1 \ldots \bullet_1$ (with length $n+1$), which encodes the maximum face, i.e., the entire polytope.
- $Y = x_\sigma$ for some σ. Then all sets I_1, $I_2 \backslash I_1, \ldots$, $\{1, \ldots, n+1\} \backslash I_k$ are singletons, and the construct T is of the form $(H \backslash x_\sigma)(S)$, where S is a construct of the associahedron generated by the hypergraph

  $$\boldsymbol{H}_\sigma \overset{\Delta}{=} \{\{x_{\sigma(1)}\}, \ldots, \{x_{\sigma(n+1)}\}, \{x_{\sigma(1)}, x_{\sigma(2)}\}, \ldots, \{x_{\sigma(n)}, x_{\sigma(n+1)}\}\}.$$

  It follows that $(w_{x_\sigma})^{st} = w_{x_\sigma} = x_{\sigma(1)}\, x_{\sigma(2)}\, \ldots\, x_{\sigma(n+1)}$. Then S determines a parenthesisation of this word.

In the non-degenerate cases, if we fix a permutation σ such that $\emptyset \subsetneq Y \subsetneq x_\sigma$, we can show that $T_\sigma = (x_\sigma \backslash Y)(T_1, \ldots, T_p)$ is a construct of $\boldsymbol{H}_{x_\sigma}$, hence the data $T_1, \ldots$, T_p amount to giving parenthesisations in the p zones delimited by the square parentheses, resulting in a parenthesised word which we denote by W_T. It can be shown easily that the synthesis of W_T does not depend on the choice of σ such that $Y \subsetneq x_\sigma$.

Our analysis also identifies the target of the translation associating W_T and π_Y to $T = (H_2 \backslash Y)(T_1, \ldots, T_p)$. It consists of all pairs (W, π), where

- π is a map from $\{1, \ldots, q\}$ to the set of subsets of $\mathcal{X}$ of cardinality at least 2, for some q,
- W is a parenthesised word over $\mathcal{X} \cup \{\bullet_1, \ldots, \bullet_q\}$ such that, writing $\overline{W}$ for the word obtained by removing all parentheses from W:
  - each letter x_i appears at most once in $\overline{W}$;
  - for all $i \in \{1, \ldots, q\}$, all the occurrences of $\bullet_i$ appear as a block of length $|\pi(i)|$ in $\overline{W}$, and before any occurrence of $\bullet_{i+1}$ (if $i < q$).
  - the sets $\pi(r)$ (for r ranging over $\{1, \ldots, q\}$) and the singletons $\{x_i\}$ such that x_i appears in $\overline{W}$ form a partition of $\mathcal{X}$;
  - all the parentheses of W are within the scope of some parentheses of $(\overline{W})^{st}$ (as defined above), and W carries all the parentheses of $(\overline{W})^{st}$.

  As an example, in reference to the avove example of standardisation,

  $$[x_9\bullet_1]\bullet_1[\bullet_1(x_3\bullet_2)][\bullet_2 x_6\bullet_3]\bullet_3$$

  respects the scoping condition. As a prototypical counter-example, the word $(x_1 x_2)(\bullet_1\bullet_1)$ is not accepted, since $(x_1 x_2 \bullet_1 \bullet_1)^{st} = [x_1 x_2 \bullet_1]\bullet_1$.

We leave to the reader the proof that the translation is well defined and bijective, and that the following description of a partial order makes it actually an isomorphism:

- set $(W, \pi) \leq (W', \pi)$ if W' has one pair of parentheses (other than the standard ones) less than W;
- set $(W, \pi) \leq (W', \pi')$, if W inherits the parentheses of W', and if π is an elementary refinement of π', i.e., $\pi(1) = \pi'(1), \ldots, \pi(i-1) = \pi'(i-1)$, $\pi(i) \cup \pi(i+1) = \pi'(i)$, $\pi(i+2) = \pi'(i+1), \ldots$, up to "making up";
- close by reflexivity and transitivity.

For example, we have:

$$\begin{array}{lll}
\bullet_1((\bullet_1 x_3)x_4) < \bullet_1(\bullet_1 x_3 x_4) & (\pi(1) = \{x_1, x_2\}) & \text{by the first rule} \\
\bullet_1((\bullet_1 x_3)x_4) < \bullet_1(\bullet_1 \bullet_2)\bullet_2 & (\pi(1) = \{x_1, x_2\}) & \text{by the second rule} \\
(x_1 \bullet_1)\bullet_1 \bullet_1 < \bullet_1 \bullet_1 \bullet_1 \bullet_1 & (\pi(1) = \{x_2, x_3, x_4\}) & \text{by the second rule .}
\end{array}$$

We detail the derivation of $\bullet_1((\bullet_1 x_3)x_4) < \bullet_1(\bullet_1 \bullet_2)\bullet_2$: the refinement splits $\pi'(2) = \{x_3, x_4\}$, yielding the standardised word $\bullet_1(\bullet_1 x_3 x_4)$, and because parentheses are inherited, we indeed get $\bullet_1((\bullet_1 x_3)x_4)$ as a predecessor.

6 Directions for future work

We plan to apply hypergraph polytopes to study other coherence problems. In recent work, the first two authors have identified the coherence conditions for categorified cyclic operads, but it is not yet clear what the relevant polytopes are in this setting. The third author is working on giving precise geometric realisations of the polytopes obtained by iterated truncations. The case of the permutohedron-based associahedron has already been settled in [1].

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