Golden codes: 4D hyperbolic regular quantum codes

Vivien Londe

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Outline

Quantum error correcting codes

- aim: protect k qubits of information.
  $\rightarrow$ quantum communication
  $\rightarrow$ quantum computing
- k logical qubits
  but n physical qubits $(n > k)$
  $\rightarrow$ codespace of dimension 2^k in a 2^n Hilbert space.

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  → quantum computing
- k logical qubits
  but n physical qubits (n > k)
  → codespace of dimension 2^k in a 2^n Hilbert space.
- minimal distance d of the code proportional to the maximal number of physical qubits which can be corrupted without corrupting the information carried by the logical qubits.
  → $[\![n,k,d]\!]$ quantum error correcting code

$[\![n,k,d]\!]$ hopes and results

n physical qubits

k logical qubits

d: minimal distance ≈ noise tolerance

- Chuck Norris' code $[\![n, \Theta(n), \Theta(n)]\!]$

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- ~~Chuck Norris'~~ dream code $[\![n, \Theta(n), \Theta(n)]\!]$
- toric code $[\![n, 2, \Theta(n^{0.5})]\!]$
  → encodes only 2 logical qubits.
- 4-dimensional hyperbolic code $[\![n, \Theta(n), \Omega(n^{0.2})]\!]$

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  → best known distance for a homological code with linear k.

Codespace defined by stabilizers

- codespace $\mathcal{C}$: space of logical qubits.
  $\rightarrow$ in the absence of noise, encoded states stay in the codespace $\mathcal{C}$.
- can be defined by a set S of commuting stabilizers:
  $\mathcal{C} = \{ |\Psi\rangle \in \mathbb{C}^{2^n} \quad | \quad \forall g \in S, \, g|\Psi\rangle = |\Psi\rangle \}$
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X-type stabilizer: $g_X = I \otimes X \otimes I ... \otimes I \otimes X ... \otimes I$
Z-type stabilizer: $g_Z = I \otimes ... \otimes Z \otimes I \otimes Z \otimes I ... \otimes I$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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- low number of X or Z factors in a stabilizer.
  $\rightarrow$ low number of interacting qubits.

Outline

Quantum error correction

Homological quantum codes

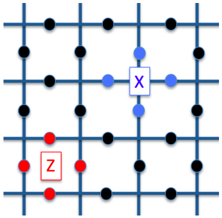
Surface codes

4-dimensional codes

Example of homological code: Toric code [Kitaev 2002]

- tessellation of the torus by squares.

- edge ↔ physical qubit
- square ↔ Z-type stabilizer
- vertex ↔ X-type stabilizer

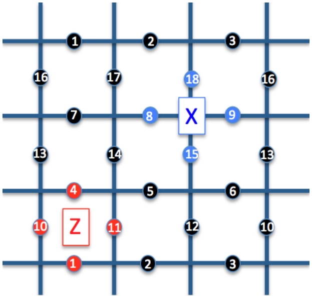


A torus is obtained by identifying left and right sides and identifying up and down sides of the square.

Stabilizers commute because each (face,vertex) pair shares an even number of edges.

Example of homological code: Toric code [Kitaev 2002]

- The Z-type stabilizer corresponding to the red face acts like Z on qubits 1, 10, 4 and 11.
- The X-type stabilizer corresponding to the blue vertex acts like X on qubits 8, 18, 9 and 15.



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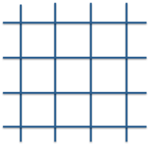
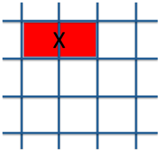
Detection of errors



represents a codestate $|\psi_0\rangle$.
All stabilizers have eigenvalue 1.



represents the state $X_{17}|\psi_0\rangle$.
red stabilizers have eigenvalue -1.

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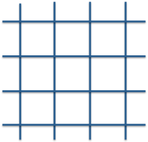

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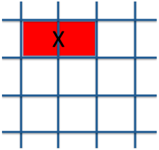
$$g_Z|\psi_0 >= |\psi_0\rangle$$



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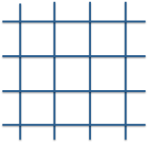
$$g_Z X_{17}|\psi_0\rangle$$
$$= -X_{17} g_Z|\psi_0\rangle$$
$$= -X_{17}|\psi_0\rangle$$

Detection of errors



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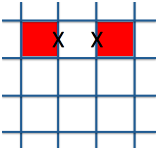
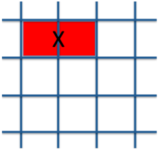


state $X_{17}X_{18}|\psi_0\rangle$.
red stabilizers have eigenvalue -1.



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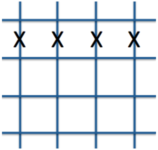
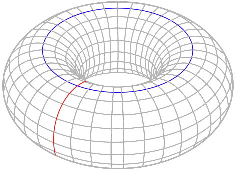


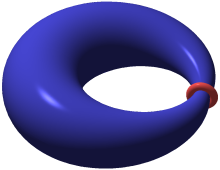
state $X_{16}X_{17}X_{18}X_{19}|\psi_0\rangle$:
codestate different from $|\psi_0\rangle$.

Geometric interpretation of n and d

- number of physical qubits n proportional to the area of the manifold.
- minimal distance d proportional to the systole of the manifold.
- the systole of a torus is the length of its shortest non contractible loop.



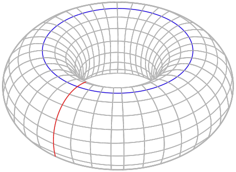
$d = \textit{systole} = \#(\text{red edges})$
$n = 2 \times \textit{area} = \#(\text{edges})$



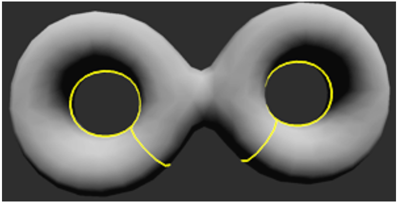
torus leading to
low minimal distance d

Geometric interpretation of k

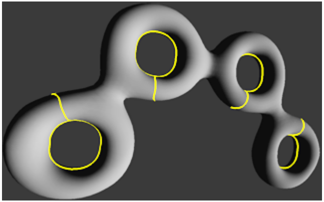
- The number of logical qubits is the rank of H_1, the first homology group of the manifold.
- Informally, it is the number of different loops of the manifold.



$k = 2$



$k = 4$



$k = 8$

Outline

Quantum error correction

Homological quantum codes

Surface codes

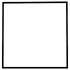
4-dimensional codes

Schläfli symbols

Schläfli symbols
$\rightarrow$ regularly tessellated surfaces and higher dimensional manifolds.

Schläfli symbols are defined recursively:

- $\{4\}$ is a square.



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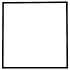
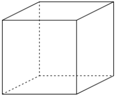


- $\{4{,}3\}$ is the regular polyhedron such that each vertex is incident to 3 squares ($\{4\}$): the cube.



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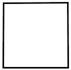

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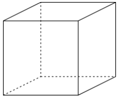
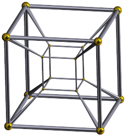
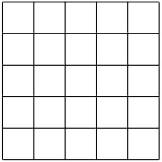


- $\{4,3,3\}$ is the regular polytope such that each edge is incident to 3 cubes ($\{4,3\}$): the hypercube.



Regular tessellation

- {4,4} is the regular tessellation of Euclidean plane such that each vertex is incident to 4 squares.
  It is the grid of the toric code.
  $\rightarrow [\![n, 2, \Theta(\sqrt{n})]\!]$



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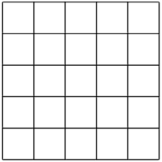
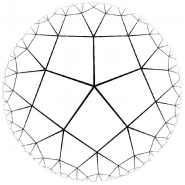
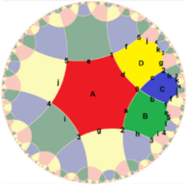


- {4,5} is the regular tessellation such that each vertex is incident to 5 squares. It is a tessellation of the hyperbolic plane.
  $\rightarrow [\![n, \Theta(n), \Theta(\log n)]\!]$



Finite codes

- Identifications on the boundary of a finite number of polygons yields a closed surface with the local structure of the regular tessellation.



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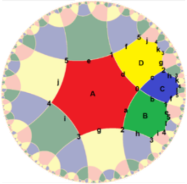

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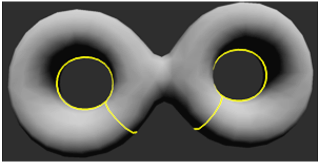

topology of the $\{6,4\}_3$ closed surface

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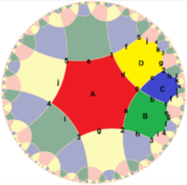

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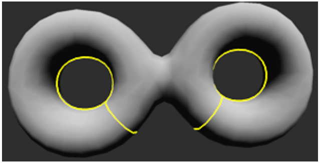

topology of the $\{6,4\}_3$ closed surface

- To this tessellated closed surface corresponds a $[\![12, 4, 2]\!]$ quantum code.

Limits of surface codes

- Tradeoffs for reliable quantum information storage in surface codes and color codes [Delfosse 2013]:
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– hyperbolic codes family: $[\![n, k, d]\!] = [\![n, \Theta(n), \Theta(\log n)]\!]$
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Codes based on 4-manifolds can overcome this bound!

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4-dimensional codes

- Tessellation of a 4-dimensional manifold by polytopes.
  - 2-face (polygon) $\leftrightarrow$ qubit
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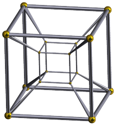
- Logical qubits correspond to 2-dimensional holes.
  $\rightarrow$ the number k of logical qubits is the rank of the second homology group of the manifold.
- Minimal distance d is proportional to the 2-systole of the manifold.
  $\rightarrow$ it is the smallest area of a surface surrounding a 2-dimensional hole.

4-dimensional tessellations



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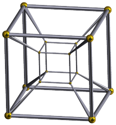

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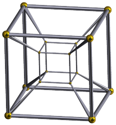

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- {4,3,3,5} is a regular tessellation of hyperbolic 4-space $\mathbb{H}^4$.
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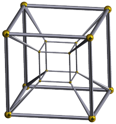

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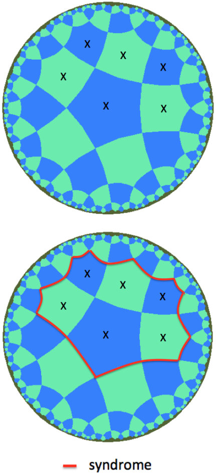
  $[\![n, \Theta(n), \Omega(n^{0.2})]\!] \rightarrow$ violates the bound on surface codes!

Beyond ⟦n, k, d⟧

Other advantages of 4D hyperbolic regular codes:

- Syndromes are cycles of edges.
  $\rightarrow$ better complexity for decoding algorithms.



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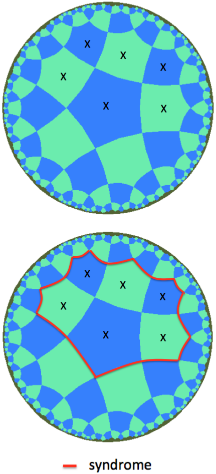

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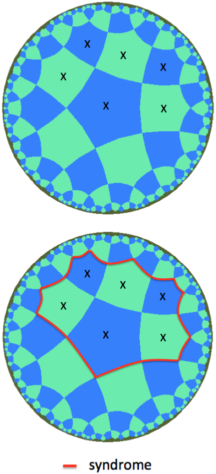
- Regularity of the tessellation leads to explicit decoding algorithms.



Summary and Outlook

Summary:

- 4-dimensional hyperbolic codes $[\![n, \Theta(n), \Omega(n^{0.2})]\!]$ are the best known homological codes in terms of $[\![n, k, d]\!]$.
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Thank you for your attention!