BIKE: Bit Flipping Key Encapsulation

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Contents

1 Introduction

This document presents BIKE, a suite of algorithms for key encapsulation based on quasi-cyclic moderate density parity-check (QC-MDPC) codes that can be decoded using bit flipping decoding techniques. In particular, this document highlights the number of security, performance and simplicity advantages that make BIKE a compelling candidate for post-quantum key encapsulation standardization.

1.1 Notation and Preliminaries

Table 1 presents the used notation and is followed by preliminary concepts.

Notation	Description
$\mathbb{F}_2$:	Finite field of 2 elements.
$\mathcal{R}$:	The cyclic polynomial ring $\mathbb{F}_2[X]/\langle X^r - 1\rangle$.
$\lvert v\rvert$:	The Hamming weight of a binary polynomial v.
$u \xleftarrow{\$} U$:	Variable u is sampled uniformly at random from set U.
h_j:	The j-th column of a matrix H, as a row vector.
$\star$:	The component-wise product of vectors.

Table 1: Notation

Definition 1 (Linear codes)**.** *A* binary (n, k)-linear code $\mathcal{C}$ *of length* n *dimension* k *and co-dimension* $r = (n - k)$ *is a* k*-dimensional vector subspace of* $\mathbb{F}_2^n$*.*

Definition 2 (Generator and Parity-Check Matrices)**.** *A matrix* $G \in \mathbb{F}_2^{k\times n}$ *is called a* generator matrix *of a binary* (n, k)*-linear code* $\mathcal{C}$ *iff*

$$\mathcal{C} = \{mG \mid m \in \mathbb{F}_2^k\}.$$

A matrix $H \in \mathbb{F}_2^{(n-k)\times n}$ *is called a* parity-check matrix *of* $\mathcal{C}$ *iff*

$$\mathcal{C} = \{c \in \mathbb{F}_2^n \mid Hc^T = 0\}.$$

A *codeword* $c \in \mathcal{C}$ of a vector $m \in \mathbb{F}_2^{(n-r)}$ is computed as $c = mG$. A *syndrome* $s \in \mathbb{F}_2^r$ of a vector $e \in \mathbb{F}_2^n$ is computed as $s^T = He^T$.

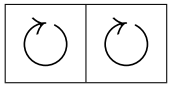
1.2 Quasi-Cyclic Codes

A binary circulant matrix is a square matrix where each row is the rotation one element to the right of the preceding row. It is completely defined by its first row. A block-circulant matrix is formed of circulant square blocks of identical size. The size of the circulant blocks is called the *order*. The *index* of a block-circulant matrix is the number of circulant blocks in a row.

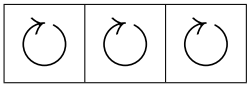
1.2.1 Definition

Definition 3 (Quasi-Cyclic Codes)**.** *A binary quasi-cyclic (QC) code of index n_0 and order r is a linear code which admits as generator matrix a block-circulant matrix of order r and index n_0. A (n_0, k_0)-QC code is a quasi-cyclic code of index n_0, length $n_0 r$ and dimension $k_0 r$.*

For instance:

$G =$ 

The rows of G span a $(2, 1)$-QC code

$G =$ 

The rows of G span a $(3, 1)$-QC code

1.2.2 Representation of QC Codes

Representation of Circulant Matrices. There exists a natural ring isomorphism, which we denote φ, between the binary $r \times r$ circulant matrices and the quotient polynomial ring $\mathcal{R} = \mathbb{F}_2[X]/(X^r - 1)$. The circulant matrix A whose first row is $(a_0, \ldots, a_{r-1})$ is mapped to the polynomial $\varphi(A) = a_0 + a_1 X + \cdots + a_{r-1} X^{r-1}$. This will allow us to view all matrix operations as polynomial operations.

Transposition. For any $a = a_0 + a_1 X + a_2 X^2 + \cdots + a_{r-1} X^{r-1}$ in $\mathcal{R}$, we define $a^T = a_0 + a_{r-1} X + \cdots + a_1 X^{r-1}$. This will ensure $\varphi(A^T) = \varphi(A)^T$.

Vector/Matrix Product. We may extend the mapping φ to any binary vector of $\mathbb{F}_2^r$. For all $\mathbf{v} = (v_0, v_1, \ldots, v_{r-1})$, we set $\varphi(\mathbf{v}) = v_0 + v_1 X + \cdots + v_{r-1} X^{r-1}$. To stay consistent with the transposition, the image of the column vector $\mathbf{v}^T$ must be $\varphi(\mathbf{v}^T) = \varphi(\mathbf{v})^T = v_0 + v_{r-1} X + \cdots + v_1 X^{r-1}$. It is easily checked that $\varphi(\mathbf{v}A) = \varphi(\mathbf{v})\varphi(A)$ and $\varphi(A\mathbf{v}^T) = \varphi(A)\varphi(\mathbf{v})^T$.

Representation of QC Codes as Codes over a Polynomial Ring. The generator matrix of an (n_0, k_0)-QC code can be represented as an $k_0 \times n_0$ matrix over $\mathcal{R}$. Similarly any parity check matrix can be viewed as an $(n_0 - k_0) \times n_0$ matrix over $\mathcal{R}$. Respectively

$$G = \begin{pmatrix} g_{0,0} & \cdots & g_{0,n_0-1} \\ \vdots & & \vdots \\ g_{k_0-1,0} & \cdots & g_{k_0-1,n_0-1} \end{pmatrix}, H = \begin{pmatrix} h_{0,0} & \cdots & h_{0,n_0-1} \\ \vdots & & \vdots \\ h_{n_0-k_0-1,0} & \cdots & h_{n_0-k_0-1,n_0-1} \end{pmatrix}$$

with all $g_{i,j}$ and $h_{i,j}$ in $\mathcal{R}$. In all respects, a binary (n_0, k_0)-QC code can be viewed as an $[n_0, k_0]$ code over the ring $\mathcal{R} = \mathbb{F}_2[X]/(X^r - 1)$.

1.3 QC-MDPC Codes

A binary MDPC (Moderate Density Parity Check) code is a binary linear code which admits a somewhat sparse parity check matrix, with a typical density of order $O(1/\sqrt{n})$. The existence of such a matrix allows the use of iterative decoders similar to those used for LDPC (Low Density Parity Check) codes [17], widely deployed for error correction in telecommunication.

1.3.1 Definition

Definition 4 (QC-MDPC codes)**.** *An (n_0, k_0, r, w)-QC-MDPC code is an (n_0, k_0) quasi-cyclic code of length $n = n_0 r$, dimension $k = k_0 r$, order r (and thus index n_0) admitting a parity-check matrix with constant row weight $w = O(\sqrt{n})$.*

Remark 1. *Asymptotically, a QC-MDPC code could efficiently correct up to $t = O(\sqrt{n} \log n)$ errors. This is a corollary of Theorem 1 given in paragraph "Asymptotic Analysis for MDPC Codes" that follows. In this work, the parity-check row weight w and the error weight t will be chosen so that $wt = O(n)$. This is precisely the regime where the decryption failure rate is expected to decay exponentially in the codelength n (see Theorem 1).*

1.3.2 Decoding - The Bit Flipping Algorithm

The decoding of MDPC codes can be achieved by various iterative decoders. Among those, the *bit flipping algorithm* is particularly interesting because of its simplicity. In Algorithm 1 as it is given here the instruction to determine the threshold τ is unspecified. We will always consider regular codes, where all columns of h have the same weight d and we denote $T = \tau d$. There are several rules for computing the threshold T:

Algorithm 1 `Bit Flipping Algorithm`

Require: $H \in \mathbb{F}_2^{(n-k)\times n}$, $s \in \mathbb{F}_2^{n-k}$
Ensure: $eH^T = s$

1: $e \leftarrow 0$
2: $s' \leftarrow s$
3: **while** $s' \neq 0$ **do**
4: $\quad \tau \leftarrow$ threshold $\in [0, 1]$, found according to some predefined rule
5: $\quad$ **for** $j = 0, \ldots, n-1$ **do**
6: $\quad\quad$ **if** $|h_j \star s'| \geq \tau\, |h_j|$ **then**
7: $\quad\quad\quad e_j \leftarrow e_j + 1 \mod 2$
8: $\quad s' \leftarrow s - eH^T$
9: **return** e

h_j denotes the j-th column of H, as a row vector, $'\star'$ denotes the componentwise product of vectors, and $|h_j \star s|$ is the number of unchecked parity equations involving j.

- the maximal value of $|h_j \star s|$ minus some δ (typically $\delta = 5$), as in [32],
- precomputed values depending on the iteration depth, as in [12],
- variable, depending on the weight of the syndrome s', as in [11].

The algorithm takes as input a parity check matrix H and a word s and, if it stops, returns an error pattern e whose syndrome is s. If H is sparse enough and there exists an error e of small enough weight such that $s = eH^T$, then, with high probability, the algorithm stops and returns e.

Asymptotic Analysis for MDPC Codes For a fixed code rate k/n, let us denote w the weight of the rows of H and t the number of errors we are able to decode. Both w and t are functions of n. For LDPC codes, w is a constant and t will be a constant proportion of n, that is $wt = \Omega(n)$. For MDPC codes, we have $w = \Omega(\sqrt{n})$ and the amount of correctable errors will turn out to be a little bit higher than $t = \Omega(\sqrt{n})$.

To understand this point, let us first notice that experimental evidence seems to indicate that the decryption failure rate is dominated by the probability that the first round of the algorithm is unable to reduce significantly the number of initial errors. What we call here "round" of the decoding algorithm is an execution of for loop 5 in Algorithm 1. It also seems that at the first round of the decoding algorithm the individual bits of the syndrome bits s_i can be approximated by

independent random variables. This independence assumption can also be made for the vectors $h_j \star s = h_j \star s'$ at the first round. In other words, we make the following assumptions.

Assumption 1. *Let P_{err} be the probability that the bit flipping algorithm fails to decode. Let e^1 be the value of error-vector e after executing for loop 5 once in Algorithm 1 and let e^0 be the true error vector. Let $\Delta e = e^0 + e^1$ (addition is performed in $\mathbb{F}_2$) be the error vector that would remain if we applied the correction e^1 to the true error vector e^0.*

- *There exists a constant α in $(0, 1)$ such that*

$$P_{err} \leq \mathbb{P}(|\Delta e| \geq \alpha t).$$

- *The syndrome bits s_i are independent random variables.*
- *For $j = 0, \ldots, n-1$, the $h_j \star s$ are independent random variables.*

By making these assumptions we can prove that

Theorem 1. *Under assumption 1, the probability P_{err} that the bit flipping algorithm fails to decode with fixed threshold $\tau = \frac{1}{2}$ is upper-bounded by*

$$P_{err} \leq \frac{1}{\sqrt{\alpha \pi t}} e^{\frac{\alpha t w}{8} \ln\left(1-\varepsilon^2\right) + \frac{\alpha t}{8} \ln(n) + O(t)},$$

where $\varepsilon \overset{def}{=} e^{-\frac{2wt}{n}}$.

This theorem is proved in Section A of the appendix. This theorem shows that the decryption failure rate (DFR) decays exponentially in the codelength when $wt = O(n)$ and that the number of correctable errors is a little bit larger than $O(\sqrt{n})$ when $w = O(\sqrt{n})$: it can be as large as some constant $\beta\sqrt{n}\ln n$ as the upper-bound in this theorem is easily shown to converge to 0 for a small enough constant β.

Decoding with a Noisy Syndrome Noisy syndrome decoding is a variation of syndrome decoding in which, given H and s, we look for $e \in \mathbb{F}_2^n$ such that $s - eH^T$ and e are both of small weight. The bit flipping algorithm can be adapted to noisy syndromes. Two things must be modified. First the stopping condition: we do not require the quantity $s - eH^T$ to be null, only to have a small weight. Second, since we need to quantify the weight in this stopping condition, we need to specify a target weight u. For input (H, s, u) a pair e is returned such that $s = e' + eH^T$

Algorithm 2 `Extended Bit Flipping Algorithm`

Require: $H \in \mathbb{F}_2^{(n-k)\times n}$, $s \in \mathbb{F}_2^{n-k}$, integer $u \geq 0$
Ensure: $\left|s - eH^T\right| \leq u$

$e \leftarrow 0$
$s' \leftarrow s$
while $|s'| > u$ **do**
    $\tau \leftarrow$ threshold $\in [0, 1]$, found according to some predefined rule // *whatever that means*
    for $j = 0, \ldots, n-1$ **do**
        if $|h_j \star s'| \geq \tau\,|h_j|$ **then**
            $e_j \leftarrow e_j + 1 \mod 2$
    $s' \leftarrow s - eH^T$
return e

for some e' of weight at most u. If $u = 0$ we have the usual bit flipping algorithm. Again if H is sparse enough and there exists a solution the algorithm will stop with high probability. Note that if the algorithm stops, it returns a solution within the prescribed weight, but this solution might not be unique. In the case of MDPC codes, the column weight and the error weight are both of order $\sqrt{r}$ and the solution is unique with high probability.

Noisy Syndrome *vs.* Normal Bit Flipping Interestingly, for MDPC codes, noisy syndromes affect only marginally the performance of the bit flipping algorithm. In fact, if e is the solution of $s = e' + eH^T$, then it is also the solution of $s = (e, 1)H'^T$ where H' is obtained by appending e' as $n+1$-th column. For MDPC codes, the error vector e' has a density which is similar to that of H and thus H' is sparse and its last column is not remarkably more or less sparse. Thus applying the bit flipping algorithm to (H', s) is going to produce e, except that we do not allow the last position to be tested in the loop and control is modified to stop the loop when the syndrome s' is equal to the last column of H'. Since we never test the last position we don't need to know the value of the last column of H' except for the stopping condition which can be replaced by a test on the weight. Thus we emulate (almost) the noisy syndrome bit flipping by running the bit flipping algorithm on a code of length $n + 1$ instead of n, to correct $|e| + 1$ errors instead of $|e|$.

QC-MDPC Decoding for Decryption Quasi-cyclicity does not change the decoding algorithm. The above algorithm will be used for $(2,1)$-QC MDPC codes. It allows us to define the procedure specified as follows. For any triple $(s, h_0, h_1) \in \mathcal{R}^3$ and any integer u

$\texttt{Decode}(s, h_0, h_1, u)$ returns $(e_0, e_1) \in \mathcal{R}^2$ with $|e_0 h_0 + e_1 h_1 + s| \leq u$.

The fourth argument u is an integer. If $u = 0$ the algorithm stops when $e_0 h_0 + e_1 h_1 = s$, that is the noiseless syndrome decoding, else it stops when $e_0 h_0 + e_1 h_1 = s + e$ from some e of weight at most u, that is the noisy syndrome decoding. In addition we will bound the running time (as a function of the block size r) and stop with a failure when this bound is exceeded.

1.4 Key Encapsulation Mechanisms

A key encapsulation mechanism (KEM) is composed by three algorithms: Gen which outputs a public encapsulation key pk and a private decapsulation key sk, Encaps which takes as input an encapsulation key pk and outputs a ciphertext c and a symmetric key K, and Decaps which takes as input a decapsulation key sk and a cryptogram c and outputs a symmetric key K or a decapsulation failure symbol $\perp$. For more details on KEM definitions, we refer the reader to [14].

2 Algorithm Specification (2.B.1)

BIKE relies purely on ephemeral keys, meaning that a new key pair is generated at each key exchange. In this way, the GJS attack [22], which depends on observing a large number of decoding failures for a same private key, is not applicable.

In the following we will present three variants of BIKE, which we will simply label BIKE-1, BIKE-2 and BIKE-3. All of the variants follow either the McEliece or the Niederreiter framework, but each one has some important differences, which we will discuss individually.

For a security level λ, let r be a prime such that $(X^r - 1)/(X - 1) \in \mathbb{F}_2[X]$ is irreducible, d_v be an odd integer and t be an integer such that decoding t errors with a uniformly chosen binary linear error-correcting code of length $n = 2r$ and dimension r, as well as recovering a base of column weight d_v given an arbitrary base of a code of the same length and dimension, both have a computational cost in $\Omega(\exp(\lambda))$. See Section 5 for a detailed discussion on parameters selection.

We denote by $\mathbf{K} : \{0,1\}^n \to \{0,1\}^{\ell_K}$ the hash function used by encapsulation and decapsulation, where ℓ_K is the desired symmetric key length (typically 256 bits).

2.1 BIKE-1

In this variant, we privilege a fast key generation by using a variation of McEliece. A preliminary version of this approach appears in [4].

First, in contrast to QC-MDPC McEliece [32] (and any QC McEliece variant), we do not compute the inversion of one of the private cyclic blocks and then multiply it by the whole private matrix to get systematic form. Instead, we hide the private code structure by simply multiplying its sparse private matrix by any random, dense cyclic block. The price to pay is the doubled size for the public key and the data since the public key will not feature an identity block anymore.

Secondly, we interpret McEliece encryption as having the message conveyed in the error vector, rather than the codeword. This technique is not new, following the lines of Micciancio's work in [31] and having already been used in a code-based scheme by Cayrel et al. in [9].

2.1.1 KeyGen

- Input: λ, the target quantum security level.
- Output: the sparse private key (h_0, h_1) and the dense public key (f_0, f_1).

0. Given λ, set the parameters r, w as described above.
1. Generate $h_0, h_1 \xleftarrow{\$} \mathcal{R}$ both of (odd) weight $|h_0| = |h_1| = w/2$.
2. Generate $g \xleftarrow{\$} \mathcal{R}$ of odd weight (so $|g| \approx r/2$).
3. Compute $(f_0, f_1) \leftarrow (gh_1, gh_0)$.

2.1.2 Encaps

- Input: the dense public key (f_0, f_1).
- Output: the encapsulated key K and the cryptogram c.

1. Sample $(e_0, e_1) \in \mathcal{R}^2$ such that $|e_0| + |e_1| = t$.
2. Generate $m \xleftarrow{\$} \mathcal{R}$.
3. Compute $c = (c_0, c_1) \leftarrow (mf_0 + e_0, mf_1 + e_1)$.
4. Compute $K \leftarrow \mathbf{K}(e_0, e_1)$.

2.1.3 Decaps

- Input: the sparse private key (h_0, h_1) and the cryptogram c.
- Output: the decapsulated key K or a failure symbol $\perp$.

1. Compute the syndrome $s \leftarrow c_0 h_0 + c_1 h_1$.
2. Try to decode s (noiseless) to recover an error vector (e'_0, e'_1).
3. If $|(e'_0, e'_1)| \neq t$ or decoding fails, output $\perp$ and halt.
4. Compute $K \leftarrow \mathbf{K}(e'_0, e'_1)$.

2.2 BIKE-2

In this variant, we follow Niederreiter's framework with a systematic parity check matrix. The main advantage is that this only requires a single block of length r for all the objects involved in the scheme, and thus yields a very compact formulation. On the other hand, this means that it is necessary to perform a polynomial inversion. In this regard, it is worth mentioning that an inversion-based key generation can be significantly slower than encryption (e.g., up to 21x as reported in [29]). A possible solution is to use a batch key generation as described in Section 3.4.

2.2.1 KeyGen

- Input: λ, the target quantum security level.
- Output: the sparse private key (h_0, h_1) and the dense public key h.

0. Given λ, set the parameters r, w as described above.
1. Generate $h_0, h_1 \xleftarrow{\$} \mathcal{R}$ both of (odd) weight $|h_0| = |h_1| = w/2$.
2. Compute $h \leftarrow h_1 h_0^{-1}$.

2.2.2 Encaps

- Input: the dense public key h.
- Output: the encapsulated key K and the cryptogram c.

1. Sample $(e_0, e_1) \in \mathcal{R}^2$ such that $|e_0| + |e_1| = t$.
2. Compute $c \leftarrow e_0 + e_1 h$.
3. Compute $K \leftarrow \mathbf{K}(e_0, e_1)$.

2.2.3 Decaps

- Input: the sparse private key (h_0, h_1) and the cryptogram c.
- Output: the decapsulated key K or a failure symbol $\perp$.

1. Compute the syndrome $s \leftarrow ch_0$.
2. Try to decode s (noiseless) to recover an error vector (e'_0, e'_1).
3. If $|(e'_0, e'_1)| \neq t$ or decoding fails, output $\perp$ and halt.
4. Compute $K \leftarrow \mathbf{K}(e'_0, e'_1)$.

2.3 BIKE-3

This variant follows the work of Ouroboros [15]. Looking at the algorithms description, the variant resembles BIKE-1, featuring fast, inversion-less key generation and two blocks for public key and data. The main difference is that the decapsulation invokes the decoding algorithm on a "noisy" syndrome. This also means that BIKE-3 is fundamentally distinct from BIKE-1 and BIKE-2, mainly in terms of security and security-related aspects like choice of parameters. We will discuss this in the appropriate section.

2.3.1 KeyGen

- Input: λ, the target quantum security level.
- Output: the sparse private key (h_0, h_1) and the dense public key (f_0, f_1).

0. Given λ, set the parameters r, w as described above.
1. Generate $h_0, h_1 \xleftarrow{\$} \mathcal{R}$ both of (odd) weight $|h_0| = |h_1| = w/2$.
2. Generate $g \xleftarrow{\$} \mathcal{R}$ of odd weight (so $|g| \approx r/2$).
3. Compute $(f_0, f_1) \leftarrow (h_1 + gh_0, g)$.

2.3.2 Encaps

- Input: the dense public key (f_0, f_1).
- Output: the encapsulated key K and the cryptogram c.

1. Sample $(e, e_0, e_1) \in \mathcal{R}^3$ with $|e| = t/2$ and $|e_0| + |e_1| = t$.
2. Compute $c = (c_0, c_1) \leftarrow (e + e_1 f_0, e_0 + e_1 f_1)$.
3. Compute $K \leftarrow \mathbf{K}(e_0, e_1)$.

2.3.3 Decaps

- Input: the sparse private key (h_0, h_1) and the cryptogram c.
- Output: the decapsulated key K or a failure symbol $\perp$.

1. Compute the syndrome $s \leftarrow c_0 + c_1 h_0$.
2. Try to decode s (with noise at most $t/2$) to recover error vector (e'_0, e'_1).
3. If $|(e'_0, e'_1)| \neq t$ or decoding fails, output $\perp$ and halt.
4. Compute $K \leftarrow \mathbf{K}(e'_0, e'_1)$.

Comparison between BIKE versions. For ease of comparison, we provide a summary of the three schemes in Table 2 below.

	BIKE-1	BIKE-2	BIKE-3
SK	(h_0, h_1) with $\|h_0\| = \|h_1\| = w/2$		
PK	$(f_0, f_1) \leftarrow (gh_1, gh_0)$	$(f_0, f_1) \leftarrow (1, h_1 h_0^{-1})$	$(f_0, f_1) \leftarrow (h_1 + gh_0, g)$
Enc	$(c_0, c_1) \leftarrow (mf_0 + e_0, mf_1 + e_1)$	$c \leftarrow e_0 + e_1 f_1$	$(c_0, c_1) \leftarrow (e + e_1 f_0, e_0 + e_1 f_1)$
	$K \leftarrow \mathbf{K}(e_0, e_1)$		
Dec	$s \leftarrow c_0 h_0 + c_1 h_1$; $u \leftarrow 0$	$s \leftarrow c h_0$; $u \leftarrow 0$	$s \leftarrow c_0 + c_1 h_0$; $u \leftarrow t/2$
	$(e'_0, e'_1) \leftarrow \texttt{Decode}(s, h_0, h_1, u)$		
	$K \leftarrow \mathbf{K}(e'_0, e'_1)$		

Table 2: Algorithm Comparison

We remark that e can be represented with only $\lceil \log_2 \binom{n}{t} \rceil$ bits and such a compact representation can be used if memory is the preferred metric of optimization (the hash function $\mathbf{K}$ would need to be changed as well to receive $\lceil \log_2 \binom{n}{t} \rceil$ bits instead of n).

2.4 Suggested Parameters

The parameters suggested in this section refer to the security levels indicated by NIST's call for papers, which relate to the hardness of a key search attack on a block cipher, like AES. More precisely, we indicate parameters for Levels 1, 3 and 5, corresponding to the security of AES-128, AES-192 and AES-256 respectively.

In addition, the block size r is chosen so that the MDPC decoder described in Section 2.5 has a failure rate not exceeding 10^{-7} (validated through exhaustive simulation). Table 3 summarizes these three parameter suggestions.

	BIKE-1 and BIKE-2				BIKE-3			
Security	n	r	w	t	n	r	w	t
Level 1	20,326	10,163	142	134	22,054	11,027	134	154
Level 3	39,706	19,853	206	199	43,366	21,683	198	226
Level 5	65,498	32,749	274	264	72,262	36,131	266	300

Table 3: Suggested Parameters.

2.5 Decoding

In all variants of BIKE, we will consider the decoding as a black box running in bounded time and which either returns a valid error pattern or fails. It takes as arguments a (sparse) parity check matrix $H \in \mathbb{F}_2^{(n-k)\times n}$, a syndrome $s \in \mathbb{F}_2^{n-k}$, and an integer $u \geq 0$. Any returned value e is such that the Hamming distance between eH^T and s is smaller than u.

For given BIKE parameters r, w, t and variant, the key features are going to be the decoding time and the DFR (Decoding Failure Rate). Let $\mathcal{R} = \mathbb{F}_2[X]/(X^r - 1)$, the decoder input (H, s, u) is such that:

- the matrix H is block-circulant of index 2, that is a $H = (h_0^T \;\; h_1^T) \in \mathcal{R}^{1\times 2}$ such that $|h_0| = |h_1| = w/2$
- the integer u is either 0 (noiseless syndrome decoding, BIKE-1 and BIKE-2) or $t/2$ (noisy syndrome decoding, BIKE-3).
- the syndrome s is equal to $e' + e_0h_0 + e_1h_1$ for some triple $(e', e_0, e_1) \in \mathcal{R}^3$ such that $|e'| = u$ and $|e_0| + |e_1| = t$.

For each parameter set and each BIKE variant, the decoder input is entirely defined by h_0, h_1, e', e_0, e_1. The DFR is defined as the probability for the decoder to fail when the input (h_0, h_1, e', e_0, e_1) is distributed uniformly such that $|h_0| = |h_1| = w/2$, $|e'| = u$, and $|e_0| + |e_1| = t$.

2.5.1 One-Round Decoding

We will use the decoder defined in Algorithm 3. As it is defined, this algorithm returns a valid error pattern when it stops but it may not stop. In practice, A maximum running time is set, when this maximum is reached the algorithm stops with a failure. For given BIKE parameters r, w, and t, we have $n = 2r$ and $k = r$. In addition, we must (1) set values for S and δ and (2) provide a rule for computing the threshold (instruction 1).

Algorithm 3 `One-Round Bit Flipping Algorithm`

Require: $H \in \mathbb{F}_2^{(n-k)\times n}$, $s \in \mathbb{F}_2^{n-k}$, integer $u \geq 0$
Ensure: $\left|s - eH^T\right| \leq u$

$T \leftarrow \texttt{threshold}(|s|)$
for $j = 0, \ldots, n-1$ **do**
  $\ell \leftarrow \texttt{min}(\texttt{ctr}(H, s, j), T)$
  $J_\ell \leftarrow J_\ell \cup \{j\}$ *// all J_ℓ empty initially*
$e \leftarrow J_T$ (**)
$s' \leftarrow s - eH^T$
while $|s'| > S$ **do** (***)
  for $\ell = 0, \ldots, \delta$ **do** (***)
    $e' \leftarrow \texttt{check}(H, s', J_{T-\ell}, d/2)$
    $(e, s') \leftarrow (e + e', s' - e'H^T)$ *// update error and syndrome*
$e' \leftarrow \texttt{check}(H, s', e, d/2)$ (**)
$(e, s') \leftarrow (e + e', s' - e'H^T)$ *// update error and syndrome*
while $|s'| > u$ **do**
  $j \leftarrow \texttt{guess\_error\_pos}(H, s', d/2)$
  $(e_j, s') \leftarrow (e_j + 1, s' + h_j)$ (*)
return e

`check`(H, s, J, T)
  $e \leftarrow 0$
  for $j \in J$ **do**
    if `ctr`$(H, s, j) \geq T$ **then**
      $e_j \leftarrow 1$
  return e

`guess_error_pos`(H, s, T)
  loop *// until success*
    $i \xleftarrow{\$} s$ (**)
    for $j \in eq_i$ **do** (*),(**)
      if `ctr`$(H, s, j) \geq T$ **then**
        return j

`ctr`(H, s, j)
  return $|h_j \cap s|$ (*),(**)

`threshold`(S)
  return function of r, w, t, and S

(*) *h_j the j-th column of H (as a row vector), eq_i the i-th row of H*
(**) *we identify binary vectors with the set of their non zero positions*
(***) *the algorithm uses two parameters S and δ which depend of r, w, and t*

Threshold Selection Rule. This rule derives from [10]. We use the notation of the algorithm, $s = eH^T$ is the input syndrome and e the corresponding (unknown) error. We denote $d = w/2$ the column weight of H. Let

$$\pi_1 = \frac{|s| + X}{td} \text{ and } \pi_0 = \frac{w\,|s| - X}{(n-t)d} \text{ where } X = \sum_{\ell \text{ odd}} (\ell - 1) \frac{r\binom{w}{\ell}\binom{n-w}{t-\ell}}{\binom{n}{t}}.$$

The counter value $|h_j \cap d|$ follows a distribution very close to a binomial distribution[1] $B(d, \pi_1)$ and $B(d, \pi_0)$ respectively if $e_j = 1$ or $e_j = 0$. From that it follows that the best threshold is the smallest integer T such that

$$t\binom{d}{T}\pi_1^T(1-\pi_1)^{d-T} \geq (n-t)\binom{d}{T}\pi_0^T(1-\pi_0)^{d-T},$$

that is (note that $\pi_1 \geq \pi_0$)

$$T = \left\lceil \frac{\log \frac{n-t}{t} + d \log \frac{1-\pi_0}{1-\pi_1}}{\log \frac{\pi_1}{\pi_0} + \log \frac{1-\pi_0}{1-\pi_1}} \right\rceil. \tag{1}$$

This value depends only of $n = 2r$, $w = 2d$, $t = |e|$ the error weight, and $|s|$ the syndrome weight. Details can be found in [10]. For any set of parameters thresholds can be precomputed.

In practice for a given set of parameters the formula (1) is very accurately approximated, in the relevant range for the syndrome weight, by an affine function:

- for BIKE-1 and BIKE-2
  - security level 1: $T = \lceil 13.530 + 0.0069722\ |s| \rceil$,
  - security level 3: $T = \lceil 15.932 + 0.0052936\ |s| \rceil$,
  - security level 5: $T = \lceil 17.489 + 0.0043536\ |s| \rceil$,
- for BIKE-3
  - security level 1: $T = \lceil 13.209 + 0.0060515\ |s| \rceil$,
  - security level 3: $T = \lceil 15.561 + 0.0046692\ |s| \rceil$,
  - security level 5: $T = \lceil 17.061 + 0.0038459\ |s| \rceil$.

2.6 Auxiliary Functions

Possible realizations of the auxiliary functions required by BIKE are described next. Other techniques can be used as long as they meet the target security level.

[1] $B(n, p)$ the number of success out of n Bernouilli trials of probability p

2.6.1 Pseudorandom Random Generators

Three types of pseudorandom bits stream generation are considered: no constraints on the output weight (Alg. 4), odd weight (Alg. 5), and specific weight w (Alg. 6). The common building block for them is AES-CTR-PRF based on AES-256, in CTR mode (following NIST SP800-90A guidelines [3]). For typical BIKE parameters the number of calls to AES with a given key is way below the restrictions on using AES in CTR mode. We remark that such AES-CTR-PRF generator is very efficient on modern processors equipped with dedicated AES instructions (e.g., AES-NI).

Algorithm 4 GenPseudoRand(seed, len)

Require: seed (32 bytes)
Ensure: $\bar{z}$ (len pseudo-random bits z embedded in array of bytes).
1: s = AES-CTR-INIT(seed, $0, 2^{32}-1$)
2: z = $\text{truncate}_{\text{len}}$ (AES-CTR-PRF (s, len))
3: **return** $\bar{z}$

Algorithm 5 GenPseudoRandOddWeight(seed, len)

Require: seed (32 bytes), len
Ensure: $\bar{z}$ (len pseudorandom bits z of odd weight, in a byte array).
1: z = GenPseudoRand(seed, len)
2: **if** $weight(z)$ is even **then** z[0] = z[0] $\oplus$1
3: **return** $\bar{z}$

Algorithm 6 WAES-CTR-PRF(s, wt, len)

Require: s (AES-CTR-PRF state), wt (32 bits), len
Ensure: A list (wlist) of wt bit-positions in $[0, \ldots, \text{len}-1]$, updated s.
1: wlist= ϕ; valid_ctr = 0
2: **while** valid_ctr < wt **do**
3: (pos, s) = AES-CTR-PRF(s, 4)
4: **if** ((pos < len) AND (pos $\notin$ wlist)) **then**
5: wlist = wlist $\cup$ {pos}; valid_ctr = valid_ctr + 1
6: **return** wlist, s

2.6.2 Efficient Hashing

In this section, we describe a parallelized hash technique (see [18, 19, 21]) that can be used to accelerate the hashing process. We stress that a simple hash (e.g., SHA2 or SHA3 hash family) call can be used instead if (for interoperability reasons, for instance) a standard hash function is preferred. Let hash be a hash function with digest length of ld bytes that uses a compression function compress which consumes a block of hbs bytes. The ParallelizedHash, with s slices, and pre-padding length srem, is described in Alg. 7. In our accompanying implementations, we instantiated hash with SHA-384.

Algorithm 7 ParallelizedHash

Require: an array of la bytes array[la − 1 : 0], such that la ≥ s > 0
Ensure: digest (ld bytes)

1: **procedure** ComputeSliceLen(la)
2: tmp := floor $\left(\frac{la}{s}\right)$ − slicerem
3: α := floor $\left(\frac{tmp}{hbs}\right)$
4: **return** $\alpha \times$ hbs + slicerem
5: **procedure** ParallelizedHash(array, la)
6: ls := ComputeSliceLen(la)
7: lrem := la - (ls × s)
8: **for** i := 0 to (s -1) **do**
9: slice[i] = array[$(i + 1) \times$ ls − 1 : $i \times$ ls]
10: $X[i]$ = hash(slice[i])
11: Y = array[la − 1: ls × s]
12: YX= Y ‖ X[s − 1] ‖ X[s − 2] . . . ‖ X[0]
13: **return** hash(YX)

3 Performance Analysis (2.B.2)

In this section, we discuss the performance of BIKE with respect to both latency and communication bandwidth. The performance numbers presented in sections 3.1, 3.2 and 3.3 refer to our reference code implementation, while section 3.4 refers to optimizations and their corresponding latency gains.

The platform used in the experiments was equipped with an Intel® Core™ i5-6260U CPU running at 1.80GHz. This platform has 32 GB RAM, 32K L1d and L1i cache, 256K L2 cache, and 4,096K L3 cache. Intel® Turbo Boost and Intel®

Hyper-Threading technologies were all disabled. For each benchmark, the process was executed 25 times to warm-up the caches, followed by 100 iterations that were clocked (using the RDTSC instruction) and averaged. To minimize the effect of background tasks running on the system, each such experiment was repeated 10 times, and averaged. Our code was compiled using gcc/g++ 5.4.0 (build 20160609) with OpenSSL library (v1.0.2g, 1 Mar 2016) and NTL library (v6.2.1-1).

Regarding memory requirements, we remark that BIKE private keys are composed by $(h_0, h_1) \in \mathcal{R}$ with $|h_0| = |h_1| = w/2$. Each element can either be represented by (r) bits or, in a more compact way, by the $w/2$ non-zero positions, yielding a $(\frac{w}{2} \cdot \lceil \log_2(r) \rceil)$-bits representation.

3.1 Performance of BIKE-1

3.1.1 Memory Cost

Table 4 summarizes the memory required for each quantity.

Quantity	Size	Level 1	Level 3	Level 5
Private key	$w \cdot \lceil \log_2(r) \rceil$	$2,130$	$2,296$	$4,384$
Public key	n	$20,326$	$43,786$	$65,498$
Ciphertext	n	$20,326$	$43,786$	$65,498$

Table 4: Private Key, Public Key and Ciphertext Size in Bits.

3.1.2 Communication Bandwidth

Table 5 shows the bandwidth cost per message.

Message Flow	Message	Size	Level 1	Level 3	Level 5
Init. $\rightarrow$ Resp.	(f_0, f_1)	n	$20,326$	$43,786$	$65,498$
Resp. $\rightarrow$ Init.	(c_0, c_1)	n	$20,326$	$43,786$	$65,498$

Table 5: Communication Bandwidth in Bits.

3.1.3 Latency

Operation	Level 1	Level 3	Level 5
Key Generation	$730,025$	$1,709,921$	$2,986,647$
Encapsulation	$689,193$	$1,850,425$	$3,023,816$
Decapsulation	$2,901,203$	$7,666,855$	$17,483,906$

Table 6: Latency Performance in Number of Cycles.

3.2 Performance of BIKE-2

3.2.1 Memory Cost

Table 7 summarizes the memory required for each quantity.

Quantity	Size	Level 1	Level 3	Level 5
Private key	$w \cdot \lceil \log_2(r) \rceil$	$2,130$	$3,296$	$4,384$
Public key	r	$10,163$	$21,893$	$32,749$
Ciphertext	r	$10,163$	$21,893$	$32,749$

Table 7: Private Key, Public Key and Ciphertext Size in Bits.

3.2.2 Communication Bandwidth

Table 8 shows the bandwidth cost per message.

Message Flow	Message	Size	Level 1	Level 3	Level 5
Init. $\rightarrow$ Resp.	f_1	r	$10,163$	$21,893$	$32,749$
Resp. $\rightarrow$ Init.	c	r	$10,163$	$21,893$	$32,749$

Table 8: Communication Bandwidth in Bits.

3.2.3 Latency

Operation	Level 1	Level 3	Level 5
Key Generation	6,383,408	22,205,901	58,806,046
Encapsulation	281,755	710,970	1,201,161
Decapsulation	2,674,115	7,114,241	16,385,956

Table 9: Latency Performance in Number of Cycles.

3.3 Performance of BIKE-3

3.3.1 Memory Cost

Table 10 summarizes the memory required for each quantity.

Quantity	Size	Level 1	Level 3	Level 5
Private key	$w \cdot \lceil \log_2(r) \rceil$	2,010	3,168	4,522
Public key	n	22,054	43,366	72,262
Ciphertext	n	22,054	43,366	72,262

Table 10: Private Key, Public Key and Ciphertext Size in Bits.

3.3.2 Communication Bandwidth

Table 11 shows the bandwidth cost per message.

Message Flow	Message	Size	Level 1	Level 3	Level 5
Init. $\rightarrow$ Resp.	(f_0, f_1)	n	22,054	43,366	72,262
Resp. $\rightarrow$ Init.	(c_0, c_1)	n	22,054	43,366	72,262

Table 11: Communication Bandwidth in Bits.

3.3.3 Latency

Operation	Level 1	Level 3	Level 5
Key Generation	433,258	1,100,372	2,300,332
Encapsulation	575,237	1,460,866	3,257,675
Decapsulation	3,437,956	7,732,167	18,047,493

Table 12: Latency Performance in Number of Cycles.

3.4 Optimizations and Performance Gains

Optimizations for BIKE and corresponding performance gains are discussed next.

3.4.1 BIKE-2 Batch Key Generation

BIKE-2 key generation needs to compute a (costly) polynomial inversion, as described in Section 2.2. To reduce the impact of this costly operation and still benefit from the lower communication bandwidth offered by BIKE-2, we propose a *batch* version of BIKE-2 key generation. The main benefit of this approach is that only one polynomial inversion is computed for every N key generations, assuming a predefined $N \in \mathbb{N}$, instead of one inversion per key generation.

This technique is based on Montgomery's trick [33] and assumes that multiplication is fairly less expensive than inversion. As a toy example, suppose that one needs to invert two polynomials $x, y \in \mathcal{R}$. Instead of computing the inverse of each one separately, it is possible to compute them with one inversion and three multiplications: set $tmp = x \cdot y$, $inv = tmp^{-1}$ and then recover $x^{-1} = y \cdot inv$ and $y^{-1} = x \cdot inv$. This can be easily generalized to $N > 2$ polynomials: in this case, $2N$ multiplications are needed and inverses need to be recovered one at a time and in order. Because of this, our implementation requires the maintenance of a global variable $0 \leq \texttt{keyindex} < N$ that must be accessible only to the legitimate party willing to generate BIKE-2 keys and increased after each key generation. Algorithm 8 describes this optimization. Most of the work is done in the first key generation ($\texttt{keyindex} = \texttt{0}$). In this way, the amortized cost of BIKE-2 key generation is reduced significantly as illustrated in Table 13.

Algorithm 8 `BIKE-2 Batch Key Generation`

Require: `keyindex`, $N \in \mathbb{N}$, code parameters (n, k, w)
Ensure: $(h_{0,0}, \ldots, h_{0,N-1}, h_1) \in \mathcal{R}^{N+1}$, $|h_{0,i}|_{0 \leq i < N} = |h_1| = w$

1: Sample $h_1 \xleftarrow{\$} \mathcal{R}$ such that $|h_1| = w$
2: **if** $\texttt{keyindex} = 0$ **then**
3: Sample $h_{0,i} \xleftarrow{\$} \mathcal{R}$ such that $|h_{0,i}| = w$ for $0 < i < N$
4: $\texttt{prod}_{0,0} = \texttt{h}_{0,0}$
5: $\texttt{prod}_{0,\texttt{i}} = \texttt{prod}_{0,\texttt{i}-1} \cdot \texttt{h}_{0,\texttt{i}}$, for $1 \leq i < N$
6: $\texttt{prod}_{1,\texttt{N}-1} = \texttt{prod}_{0,\texttt{N}-1}^{-1}$
7: $\texttt{prod}_{1,\texttt{i}} = \texttt{prod}_{1,\texttt{i}+1} \cdot \texttt{h}_{0,\texttt{i}+1}$, for $N - 2 \geq i > 0$
8: $inv = \texttt{prod}_{1,1} \cdot \texttt{h}_{0,1}$
9: **else**
10: $inv = \texttt{prod}_{1,\texttt{keyindex}} \cdot \texttt{prod}_{0,\texttt{keyindex}-1}$
11: $h \leftarrow h_1 \cdot inv$
12: $\texttt{keyindex} \leftarrow \texttt{keyindex} + 1$
13: **return** $(h_{0,\texttt{keyindex}}, h_1, h)$

Operation	Reference	Batch	Gain (%)
Level 1	6,383,408	1,647,843	74.18%
Level 3	22,205,901	4,590,452	79.32%
Level 5	58,806,046	9,296,144	84.19%

Table 13: Reference Versus Batch Key Generation (in cycles, for $N = 100$).

We stress that an implementer interested in the benefits offered by BIKE-2 batch key generation will need to meet the additional security requirements of protecting from adversaries and securely updating the variables `keyindex`, $prod_0$ and $prod_1$. It is also important to stress that the keys generated through this batch process are not related to each other. Finally, we remark that the use (or not) of the batch optimization does not impact on the encapsulation and decapsulation processes described in Section 2.2.

3.5 Additional Implementation

To illustrate the potential performance that BIKE code may achieve when running on modern platforms, we report some results of an additional implementation.

These preliminary BIKE-1 and BIKE-2 results can be expected to be further improved.

The performance is reported in processor cycles (lower is better), reflecting the performance per a *single core*. The results were obtained with the same measurement methodology declared in Section 3. The results are reported in Tables 14, 15, and 16 for BIKE-1, and in Tables 17, 18, and 19 for BIKE-2.

The additional implementation code. The core functionality was written in x86 assembly, and wrapped by assisting C code. The implementations use the PCLMULQDQ, AES−NI and the AVX2 and AVX512 architecture extensions. The code was compiled with gcc (version 5.4.0) in 64-bit mode, using the "O3" Optimization level, and run on a Linux (Ubuntu 16.04.3 LTS) OS. Details on the implementation and optimized components are provided in [16], and the underlying primitives are available in [20].

The benchmarking platform. The experiments were carried out on a platform equipped with the latest 8^{th} Generation Intel® Core^{TM} processor ("Kaby Lake") - Intel® Xeon® Platinum 8124M CPU at 3.00 GHz Core® i5 − 750. The platform has 70 GB RAM, 32K L1d and L1i cache, 1,024K L2 cache, and 25,344K L3 cache. It was configured to disable the Intel® Turbo Boost Technology, and the Enhanced Intel Speedstep® Technology.

	—			Constant time implementation		
	KeyGen	Encaps	Decaps	KeyGen	Encaps	Decaps
AVX2	0.09	0.11	1.13	0.20	0.15	5.30
AVX512	0.09	0.11	1.02	0.19	0.13	4.86

Table 14: Performance (in millions of cycles) of BIKE-1 Level 1.

	—			Constant time implementation		
	KeyGen	Encaps	Decaps	KeyGen	Encaps	Decaps
AVX2	11.99	0.27	2.70	12.45	0.39	10.74
AVX512	11.99	0.25	2.14	12.34	0.34	8.93

Table 19: Performance (in millions of cycles) of BIKE-2 Level 5.

	—			Constant time implementation		
	KeyGen	Encaps	Decaps	KeyGen	Encaps	Decaps
AVX2	0.25	0.28	3.57	0.45	0.36	16.74
AVX512	0.25	0.27	2.99	0.45	0.33	15.26

Table 15: Performance (in millions of cycles) of BIKE-1 Level 3.

	—			Constant time implementation		
	KeyGen	Encaps	Decaps	KeyGen	Encaps	Decaps
AVX2	0.25	0.29	2.75	0.67	0.42	9.84
AVX512	0.25	0.27	2.24	0.69	0.36	8.27

Table 16: Performance (in millions of cycles) of BIKE-1 Level 5.

4 Known Answer Values – KAT (2.B.3)

4.1 KAT for BIKE-1

The KAT files of BIKE-1 are available in:

- req file: `KAT/PQCkemKAT_BIKE1-Level1_2542.req`
- rsp file: `KAT/PQCkemKAT_BIKE1-Level1_2542.rsp`
- req file: `KAT/PQCkemKAT_BIKE1-Level3_4964.req`
- rsp file: `KAT/PQCkemKAT_BIKE1-Level3_4964.rsp`
- req file: `KAT/PQCkemKAT_BIKE1-Level5_8188.req`
- rsp file: `KAT/PQCkemKAT_BIKE1-Level5_8188.rsp`

4.2 KAT for BIKE-2

The KAT files of BIKE-2 are available in:

- req file: `KAT/PQCkemKAT_BIKE2-Level1_2542.req`

	—			Constant time implementation		
	KeyGen	Encaps	Decaps	KeyGen	Encaps	Decaps
AVX2	4.38	0.09	1.12	4.46	0.12	5.55
AVX512	4.38	0.08	0.86	4.45	0.11	5.12

Table 17: Performance (in millions of cycles) of BIKE-2 Level 1.

	—			Constant time implementation		
	KeyGen	Encaps	Decaps	KeyGen	Encaps	Decaps
AVX2	7.77	0.17	2.88	8.04	0.27	17.36
AVX512	7.79	0.18	3.48	8.05	0.23	15.63

Table 18: Performance (in millions of cycles) of BIKE-2 Level 3.

- rsp file: `KAT/PQCkemKAT_BIKE2-Level1_2542.rsp`
- req file: `KAT/PQCkemKAT_BIKE2-Level3_4964.req`
- rsp file: `KAT/PQCkemKAT_BIKE2-Level3_4964.rsp`
- req file: `KAT/PQCkemKAT_BIKE2-Level5_8188.req`
- rsp file: `KAT/PQCkemKAT_BIKE2-Level5_8188.rsp`

4.3 KAT for BIKE-3

The KAT files of BIKE-3 are available in:

- req file: `KAT/PQCkemKAT_BIKE3-Level1_2758.req`
- rsp file: `KAT/PQCkemKAT_BIKE3-Level1_2758.rsp`
- req file: `KAT/PQCkemKAT_BIKE3-Level3_5422.req`
- rsp file: `KAT/PQCkemKAT_BIKE3-Level3_5422.rsp`
- req file: `KAT/PQCkemKAT_BIKE3-Level5_9034.req`
- rsp file: `KAT/PQCkemKAT_BIKE3-Level5_9034.rsp`

5 Known Attacks (2.B.5)

This section discusses the practical security aspects of our proposal.

5.1 Hard Problems and Security Reduction

In the generic (*i.e.* non quasi-cyclic) case, the two following problems were proven NP-complete in [6].

Problem 1 (Syndrome Decoding – SD)**.**
Instance: $H \in \mathbb{F}_2^{(n-k)\times n}$, $s \in \mathbb{F}_2^{n-k}$, *an integer* $t > 0$.
Property: *There exists* $e \in \mathbb{F}_2^n$ *such that* $|e| \leq t$ *and* $eH^T = s$.

Problem 2 (Codeword Finding – CF)**.**
Instance: $H \in \mathbb{F}_2^{(n-k)\times n}$, *an integer* $t > 0$.
Property: *There exists* $c \in \mathbb{F}_2^n$ *such that* $|c| = t$ *and* $cH^T = 0$.

In both problems the matrix H is the parity check matrix of a binary linear $[n, k]$ code. Problem 1 corresponds to the decoding of an error of weight t and Problem 2 to the existence of a codeword of weight t. Both are also conjectured to be hard on average. This is argued in [1], together with results which indicate that the above problems remain hard even when the weight is very small, i.e. $t = n^{\varepsilon}$, for any $\varepsilon > 0$. Note that all known solvers for one of the two problems also solve the other and have a cost exponential in t.

5.1.1 Hardness for QC codes.

Coding problems (SD and CF) in a QC-code are NP-complete, but the result does not hold for when the index is fixed. In particular, for $(2, 1)$-QC codes or $(3, 1)$-QC codes, which are of interest to us, we do not know whether or not SD and CF are NP-complete.

Nevertheless, they are believed to be hard on average (when r grows) and the best solvers in the quasi-cyclic case have the same cost as in the generic case up to a small factor which never exceeds the order r of quasi-cyclicity. The problems below are written in the QC setting, moreover we assume that the parity check matrix H is in systematic form, that is the first $(n_0 - k_0) \times (n_0 - k_0)$ block of H is the identity matrix. For instance, for $(2, 1)$-QC and $(3, 1)$-QC codes codes, the parity check matrix (over $\mathcal{R}$) respectively have the form

$$\begin{pmatrix} 1 & h \end{pmatrix} \text{ with } h \in \mathcal{R}, \text{ and } \begin{pmatrix} 1 & 0 & h_0 \\ 0 & 1 & h_1 \end{pmatrix} \text{ with } h_0, h_1 \in \mathcal{R}.$$

In our case, we are interested only by those two types of QC codes and to the three related hard problems below:

Problem 3 ($(2,1)$-QC Syndrome Decoding – $(2,1)$-QCSD)**.**
Instance: s, h *in* $\mathcal{R}$*, an integer* $t > 0$*.*
Property: *There exists* e_0, e_1 *in* $\mathcal{R}$ *such that* $|e_0| + |e_1| \leq t$ *and* $e_0 + e_1 h = s$*.*

Problem 4 ($(2,1)$-QC Codeword Finding – $(2,1)$-QCCF)**.**
Instance: h *in* $\mathcal{R}$*, an integer* $t > 0$*.*
Property: *There exists* c_0, c_1 *in* $\mathcal{R}$ *such that* $|c_0| + |c_1| = t$ *and* $c_0 + c_1 h = 0$*.*

Problem 5 ($(3,1)$-QC Syndrome Decoding – $(3,1)$-QCSD)**.**
Instance: s_0, s_1, h_0, h_1 *in* $\mathcal{R}$*, an integer* $t > 0$*.*
Property: *There exists* e_0, e_1, e_2 *in* $\mathcal{R}$ *such that* $|e_0| + |e_1| + |e_2| \leq t$*,* $e_0 + e_2 h_0 = s_0$ *and* $e_1 + e_2 h_1 = s_1$*.*

As they are presented, those problems have the appearance of *sparse polynomials* problem, but in fact they are equivalent to generic quasi-cyclic decoding and codeword finding problems.

In the current state of the art, the best known techniques for solving those problems are variants of Prange's Information Set Decoding (ISD) [34]. We remark that, though the best attacks consist in solving one of the search problems, the security reduction of our scheme requires the decision version of Problem 2.

5.2 Information Set Decoding

The best asymptotic variant of ISD is due to May and Ozerov [30], but it has a polynomial overhead which is difficult to estimate precisely. In practice, the BJMM variant [5] is probably the best for relevant cryptographic parameters. The work factor for classical (*i.e.* non quantum) computing of any variant $\mathcal{A}$ of ISD for decoding t errors (or finding a word of weight t) in a binary code of length n and dimension k can be written

$$\mathrm{WF}_{\mathcal{A}}(n, k, t) = 2^{ct(1+o(1))}$$

where c depends on the algorithm, on the code rate $R = k/n$ and on the error rate t/N. It has been proven in [36] that, asymptotically, for sublinear weight $t = o(n)$ (which is the case here as $w \approx t \approx \sqrt{n}$), we have $c = \log_2 \frac{1}{1-R}$ for all variants of ISD.

In practice, when t is small, using 2^{ct} with $c = \log_2 \frac{1}{1-R}$ gives a remarkably good estimate for the complexity. For instance, non asymptotic estimates derived

from [23] gives $\mathrm{WF}_{\mathrm{BJMM}}(65542, 32771, 264) = 2^{263.3}$ "column operations" which is rather close to 2^{264}. This closeness is expected asymptotically, but is circumstantial for fixed parameters. It only holds because various factors compensate, but it holds for most MDPC parameters of interest.

5.2.1 Exploiting the Quasi-Cyclic Structure.

Both codeword finding and decoding are a bit easier (by a polynomial factor) when the target code is quasi-cyclic. If there is a word of weight w in a QC code then its r quasi-cyclic shifts are in the code. In practice, this gives a factor r speedup compared to a random code. Similarly, using Decoding One Out of Many (DOOM) [35] it is possible to produce r equivalent instances of the decoding problem. Solving those r instances together saves a factor $\sqrt{r}$ in the workload.

5.2.2 Exploiting Quantum Computations.

Recall first that the NIST proposes to evaluate the quantum security as follows:

1. A quantum computer can only perform quantum computations of limited depth. They introduce a parameter, MAXDEPTH, which can range from 2^{40} to 2^{96}. This accounts for the practical difficulty of building a full quantum computer.

2. The amount (or bits) of security is not measured in terms of absolute time but in the time required to perform a specific task.

Regarding the second point, the NIST presents 6 security categories which correspond to performing a specific task. For example Task 1, related to Category 1, consists of finding the 128 bit key of a block cipher that uses AES-128. The security is then (informally) defined as follows:

Definition 5. *A cryptographic scheme is secure with respect to Category k iff. any attack on the scheme requires computational resources comparable to or greater than those needed to solve Task k.*

In what follows we will estimate that our scheme reaches a certain security level according to the NIST metric and show that the attack takes more quantum resources than a quantum attack on AES. We will use for this the following proposition.

Proposition 1. *Let f be a Boolean function which is equal to 1 on a fraction α of inputs which can be implemented by a quantum circuit of depth D_f and whose gate complexity is C_f. Using Grover's algorithm for finding an input x of f for which*

$f(x) = 1$ can not take less quantum resources than a Grover's attack on AES-N as soon as

$$\frac{D_f \cdot C_f}{\alpha} \geq 2^N D_{AES-N} \cdot C_{AES-N}$$

where D_{AES-N} and C_{AES-N} are respectively the depth and the complexity of the quantum circuit implementing AES-N.

This proposition is proved in Section B of the appendix. The point is that (essentially) the best quantum attack on our scheme consists in using Grover's search on the information sets computed in Prange's algorithm (this is Bernstein's algorithm [7]). Theoretically there is a slightly better algorithm consisting in quantizing more sophisticated ISD algorithms [24], however the improvement is tiny and the overhead in terms of circuit complexity make Grover's algorithm used on top of the Prange algorithm preferable in our case.

5.3 Defeating the GJS Reaction Attack

BIKE uses an ephemeral KEM key pair, i.e. a KEM key generation is performed for each key exchange. As a result, the GJS reaction attack is inherently defeated: a GJS adversary would have (at most) a single opportunity to observe decryption, thus not being able to create statistics about different error patterns. We note that, for efficiency purposes, an initiator may want to precompute KEM key pairs before engaging in key exchange sessions. We remark that policies to securely store the pregenerated KEM key pair must be in place, in order to avoid that an adversary access a KEM key pair to be used in a future communication.

5.4 Choice of Parameters

We denote $\mathrm{WF}(n, k, t)$ the workfactor of the best ISD variant for decoding t errors in a binary code of length n and dimension k. In the following we will consider only codes of transmission rate 0.5, that is length $n = 2r$ and dimension r. In a classical setting, the best solver for Problem 3 has a cost $\mathrm{WF}(2r, r, t)/\sqrt{r}$, the best solver for Problem 4 has a cost $\mathrm{WF}(2r, r, w)/r$, and the best solver for Problem 5 has a cost $\mathrm{WF}(3r, r, 3t/2)/\sqrt{r}$. As remarked above, with $\mathrm{WF}(n, k, \ell) \approx 2^{\ell \log_2 \frac{n}{n-k}}$ we obtain a crude but surprisingly accurate, parameter selection rule. We target security levels corresponding to AES λ with $\lambda \in \{128, 192, 256\}$. To reach λ bits of classical security, we choose w, t and r such that

- for BIKE-1 and BIKE-2, Problem 3 with block size r and weight t and Problem 4 with block size r and weight w must be hard enough, that is

$$\lambda \approx t - \frac{1}{2}\log_2 r \approx w - \log_2 r. \tag{2}$$

- for BIKE-3, Problem 5 with block size r and weight $3t/2$ and Problem 3 with block size r and weight w must be hard enough, that is

$$\lambda \approx \frac{3t}{2}\log_2\frac{3}{2} - \frac{1}{2}\log_2 r \approx w - \frac{1}{2}\log_2 r. \tag{3}$$

Those equation have to be solved in addition with the constraint that r must be large enough to decode t errors in $(2, 1, r, w)$-QC-MDPC code with a negligible failure rate. Finally, we choose r such that 2 is primitive modulo r. First, this will force r to be prime, thwarting the so-called squaring attack [26]. Also, it implies that $(X^r - 1)$ only has two irreducible factors (one of them being $X - 1$). This is an insurance against an adversary trying to exploit the structure of $\mathbb{F}_2[X]/\langle X^r - 1\rangle$ when $(X^r - 1)$ has small factors, other than $(X - 1)$. This produces the parameters proposed in the document.

The quantum speedup is at best quadratic for the best solvers of the problems on which our system, from the arguments of §5.2.2, it follows our set of parameters correspond the security levels 1, 3, and 5 described in the NIST call for quantum safe primitives.

6 Formal Security (2.B.4)

6.1 IND-CPA Security

We start with the following definition, where we denote by $\mathcal{K}$ the domain of the exchanged symmetric keys and by λ the security level of the scheme.

Definition 6. *A key-encapsulation mechanism is IND-CPA (passively) secure if the outputs of the two following games are computationally indistinguishable.*

Game $\mathbf{G}_{real}$	**Game $\mathbf{G}_{fake}$**
$(sk, pk) \leftarrow \textsc{Gen}(\lambda)$	$(sk, pk) \leftarrow \textsc{Gen}(\lambda)$
$(c, K) \leftarrow \textsc{Encaps}(pk)$	$(c, K) \leftarrow \textsc{Encaps}(pk)$
	$K^* \xleftarrow{\$} \mathcal{K}$
Output (pp, pk, c, K)	Output (pp, pk, c, K^*)

Rather than analyzing all three variants of BIKE separately, we state a single theorem, and highlight the differences in the proof.

Theorem 2. *BIKE is IND-CPA secure in the Random Oracle Model under the* $(2,1)$*-QCCF,* $(2,1)$*-QCSD and* $(3,1)$*-QCSD assumptions.*

Proof. To begin, note that we model the hash function $\mathbf{K}$ as a random oracle. The goal of our proof is to prove that an adversary distinguishing one game from another can be exploited to break one or more of the problems above in polynomial time (see Section 5.1 for definitions). Let $\mathcal{A}$ be a probabilistic polynomial time adversary against the IND-CPA of our scheme and consider the following games where we consider that $\mathcal{A}$ receives the encapsulation at the end of each game.

Game G_1: This corresponds to an honest run of the protocol, and is the same as Game $\mathbf{G}_{real}$. In particular, the simulator has access to all keys and randomness.

Game G_2: In this game, the simulator picks uniformly at random the public key, specifically (f_0, f_1) for BIKE-1 and BIKE-3, and h for BIKE-2. The rest of the game then proceeds honestly.

An adversary distinguishing between these two games is therefore able to distinguish between a well-formed public key and a randomly-generated one. Note that the public key in $\boldsymbol{G}_1$ corresponds to a valid $(2,1)$-QCCF instance for BIKE-1 and BIKE-2, and to a $(2,1)$-QCSD instance for BIKE-3, while it is random in $\boldsymbol{G}_2$. Thus we have respectively

$$\mathrm{Adv}^{G_1-G_2}(\mathcal{A}) \leq \mathrm{Adv}^{(2,1)\text{-QCCF}}(\mathcal{A}')$$

and

$$\mathrm{Adv}^{G_1-G_2}(\mathcal{A}) \leq \mathrm{Adv}^{(2,1)\text{-QCSD}}(\mathcal{A}')$$

where $\mathcal{A}'$ is a polynomial time adversary for the $(2,1)$-QCCF/ $(2,1)$-QCSD problem.

Game G_3: Now, the simulator also picks uniformly at random the ciphertext: again, this is (c_0, c_1) for BIKE-1 and BIKE-3, and c for BIKE-2. The encapsulated key K is still generated honestly.

If an adversary is able to distinguish game $\boldsymbol{G}_2$ from game $\boldsymbol{G}_3$, then it can solve one of the QCSD problems.
In fact, for BIKE-2, the ciphertext is exactly a syndrome that follows the $(2,1)$-QCSD distribution in game $\boldsymbol{G}_2$ and the uniform distribution in $\boldsymbol{G}_3$. The same can be easily shown for BIKE-1. Thus for both variants we have

$$\mathrm{Adv}^{G_2-G_3}(\mathcal{A}) \leq \mathrm{Adv}^{(2,1)\text{-QCSD}}(\mathcal{A}'')$$

where $\mathcal{A}''$ is a polynomial time adversary for the $(2,1)$-QCSD problem.

As we will see, a similar situation occurs for BIKE-3. In fact, the adversary has access to:

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & f_0 \\ 0 & 1 & f_1 \end{pmatrix} (e, e_0, e_1)^\top$$

Here (c_0, c_1) follows the $(3,1)$-QCSD distribution in game $\boldsymbol{G}_2$ and the uniform distribution over $(\mathbb{F}_2^n)^2$ in $\boldsymbol{G}_3$. Hence

$$\mathrm{Adv}^{G_2-G_3}(\mathcal{A}) \leq \mathrm{Adv}^{(3,1)\text{-QCSD}}(\mathcal{A}'')$$

where $\mathcal{A}''$ is a polynomial time adversary for the $(3,1)$-QCSD problem.

Game $\boldsymbol{G}_4$: Finally, we replace the value of K with a uniformly random value K^*. Since $\mathbf{K}$ is modeled as a random oracle, its output is pseudorandom, and an adversary only has negligible advantage ϵ, so for all three variants

$$\mathrm{Adv}^{G_3-G_4}(\mathcal{A}) \leq \epsilon.$$

Thus in the end we have

$$\mathrm{Adv}^{\text{IND-CPA}}(\mathcal{A}) \leq \mathrm{Adv}^{(2,1)\text{-QCCF}}(\mathcal{A}') + \mathrm{Adv}^{(2,1)\text{-QCSD}}(\mathcal{A}'') + \epsilon. \tag{4}$$

or

$$\mathrm{Adv}^{\text{IND-CPA}}(\mathcal{A}) \leq \mathrm{Adv}^{(2,1)\text{-QCSD}}(\mathcal{A}') + \mathrm{Adv}^{(3,1)\text{-QCSD}}(\mathcal{A}'') + \epsilon. \tag{5}$$

respectively for BIKE-1/BIKE-2 and BIKE-3. □

6.2 Public Keys and Subcodes

In this section, we prove that one can efficiently sample an *invertible* element from $\mathbb{F}_2[X]/\langle X^r-1\rangle$ by taking any polynomial $h \xleftarrow{\$} \mathbb{F}_2[X]/\langle X^r-1\rangle$ such that $|h|$ is odd. If this element was not invertible, the public code produced in BIKE-1 and BIKE-3 would be a subcode of the private one.

Lemma 1. *Let $h \in \mathbb{F}_2[X]$ have even weight. Then h is not invertible modulo X^r-1.*

Proof. We show that $(X-1) \mid h$ by induction on $|h|$. For $|h|=0$ trivially $(X-1) \mid h$. Assume that $(X-1) \mid h$ whenever $|h|=2k$ for some $k \geqslant 0$. Now consider any $h \in \mathbb{F}_2[X]$ with weight $|h|=2(k+1)$, and take two distinct terms X^i, X^j of h such that $i<j$. Define $h'=h-X^i-X^j$, so that $|h'|=2k$. Then $(X-1) \mid h'$ by induction, i.e. $h'=(X-1)h''$ for some $h'' \in \mathbb{F}_2[X]$. Hence $h=h'+X^i+X^j = (X-1)h''+X^i(X^{j-i}+1) = (X-1)h''+X^i(X-1)(X^{j-i-1}+\cdots+1) = (X-1)(h''+X^i(X^{j-i-1}+\cdots+1))$, and therefore $(X-1) \mid h$. □

Theorem 3. *Let r a prime such that $(X^r-1)/(X-1) \in \mathbb{F}_2[X]$ is irreducible. Then any $h \in \mathbb{F}_2[X]$ with $\deg(h)<r$ is invertible modulo X^r-1 iff $h \neq X^{r-1}+\cdots+1$ and $|h|$ is odd.*

Proof. Take a term X^i of h. Then $\left|h+X^i\right| = |h|-1$ is even, and by Lemma 1 $(X-1) \mid (h+X^i)$. Hence $h \bmod (X-1) = X^i \bmod (X-1) = 1$, meaning that h is invertible modulo $X-1$.

Now, since $(X^r-1)/(X-1) = X^{r-1}+\cdots+1$ is irreducible, if $\deg(h)<r-1$ then $\gcd(h, X^{r-1}+\cdots+1)=1$, and if $\deg(h)=r-1$, then $\gcd(h, X^{r-1}+\cdots+1) = \gcd(h+X^{r-1}+\cdots+1, X^{r-1}+\cdots+1)=1$, since $\deg(h+X^{r-1}+\cdots+1)<r-1$. Hence h is invertible modulo $X^{r-1}+\cdots+1$.

Therefore, the combination of the inverses of h modulo $X-1$ and modulo $X^{r-1}+\cdots+1$ via the Chinese remainder theorem is well defined, and by construction it is the inverse of h modulo $(X-1)(X^{r-1}+\cdots+1)=X^r-1$. □

Corollary 1. *One can efficiently sample an invertible element from $\mathbb{F}_2[X]/\langle X^r-1\rangle$ by taking any polynomial $h \xleftarrow{\$} \mathbb{F}_2[X]/\langle X^r-1\rangle$ such that $|h|$ is odd.*

7 Advantages and Limitations (2.B.6)

This document presents BIKE, a suite of IND-CPA secure key encapsulation mechanisms (KEM) composed by BIKE-1, BIKE-2 and BIKE-3. Each variant has its own pros and cons.

In common, all BIKE variants are based on quasi-cyclic moderate density parity-check (QC-MDPC codes), which can be efficiently decoded through bit flipping decoding techniques. This kind of decoder is extremely simple: it estimates what are the positions most likely in error, flip them and observes whether the result is better (smaller syndrome weight) than before or not. This process converges very quickly; in particular, Section 2.5 presents a 1-iteration bit flipping decoder.

Another characteristic in common to all BIKE variants is the fact that they rely on ephemeral keys. This leads to two things: at first, it inherently defeats the GJS reaction attack mentioned in section 5, which is an attack that needs to observe a large number of decodings for a same private key (something impossible when ephemeral keys are used). The other aspect of this choice is that key generation must be efficient since it is executed at every key encapsulation. Previous works based on QC-MDPC codes compute a polynomial inversion operation in order to obtain a QC-MDPC public key in systematic form. The polynomial inversion is an expensive operation. BIKE-1 completely avoids the polynomial inversion by not relying on public keys in systematic form. Instead, it hides the private sparse structure by multiplying it by a dense polynomial of odd weight sampled uniformly at random. This leads to an increased public key size but results in a very efficient key generation process (it becomes the fastest process among key generation, encapsulation and decapsulation operations). BIKE-2 uses public keys in systematic form, but thanks to our batch key generation technique discussed in Section 3.4, the amortized cost can decrease up to 84%, becoming less expensive than the bit flipping decoder. Besides the bit flipping algorithm and the eventual polynomial inversion (restricted to BIKE-2), all other operations in the BIKE suite consist of simple products of binary vectors, an operation that can be easily optimized for all sorts of hardware and software applications.

Regarding communication bandwidth, in BIKE-1 and BIKE-3 all public keys, private keys and cryptograms are n bits long, corresponding to the bandwidth of the messages exchanged by the parties. BIKE-2 offers smaller public keys and ciphertexts, r bits only, corresponding to the bandwidth of the messages exchanged by the parties as well. Two messages are exchanged per key encapsulation of same size (either n or r bits). In practice, these numbers range from 1.24 KB per message in BIKE-2 security level 1, up to 8.82 KB per message in BIKE-3 security level 5. These numbers seem fairly reasonable when compared to the the average size of a page in the Internet (currently near 2MB [2]), just as an example.

Regarding security, all BIKE variants rely their security on very well-known coding-theory problems: quasi-cyclic syndrome decoding and quasi-cyclic codeword finding problems. The best strategies to solve these problems are based on

Information Set Decoding (ISD) techniques, a research field that has a very long history (Prange's seminal work dates back 1962) and which has seem very little improvement along the years. Moreover, we show that in the quantum setting, Grover's algorithm used on top of the seminal Prange ISD algorithm is still the most preferable choice in our case.

One point of attention in BIKE is the fact that, nowadays, the bit flipping decoding techniques do not attain a negligible decoding failure rate. This makes it challenge to achieve higher security notions such as IND-CCA. This may also limits the usage of BIKE in certain applications such as, for instance, Hybrid Encryption, where both KEM and DEM need to satisfy IND-CCA security to guarantee chosen-ciphertext security for the hybrid encryption scheme. We stress however that it seems possible (although not simple) to prove that certain decoding techniques can in fact attain negligible decoding failure rates for QC-MDPC codes.

Regarding intellectual property, to the best of our knowledge, BIKE-1 and BIKE-2 are not covered by any patent. BIKE-3 is covered by a patent whose owners are willing to grant a non-exclusive license for the purpose of implementing the standard *without compensation* and under reasonable terms and conditions that are demonstrably free of any unfair discrimination, as denoted in the accompanying signed statements. We emphasize that BIKE-1 and BIKE-2 are **not covered** by the aforementioned patent, and that the BIKE team is willing to drop BIKE-3 if this ever becomes a disadvantage when comparing our suite with other proposals.

Overall, taking all these considerations into account, we believe that BIKE is a promising candidate for post-quantum key exchange standardization.

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A Proof of Theorem 1

Let us recall the theorem we want to prove.

Theorem 1. *Under assumption 1, the probability P_{err} that the bit flipping algorithm fails to decode with fixed threshold $\tau = \frac{1}{2}$ is upper-bounded by*

$$P_{err} \leq \frac{1}{\sqrt{\alpha\pi t}} e^{\frac{\alpha t w}{8}\ln\left(1-\varepsilon^2\right)+\frac{\alpha t}{8}\ln(n)+O(t)},$$

where $\varepsilon \overset{def}{=} e^{-\frac{2wt}{n}}$.

We will denote in the whole section by $h(x)$ the entropy (in nats) of a Bernoulli random variable of parameter x, that is $h(x) \overset{\text{def}}{=} -x\ln x - (1-x)\ln(1-x)$.

A.1 Basic tools

A particular quantity will play a fundamental role here, the Kullback-Leibler divergence (see for instance [13])

Definition 7. Kullback-Leibler divergence
Consider two discrete probability distributions $\mathbf{p}$ and $\mathbf{q}$ defined over a same discrete space $\mathcal{X}$. The Kullback-Leibler divergence between $\mathbf{p}$ and $\mathbf{q}$ is defined by

$$D(\mathbf{p}\|\mathbf{q}) = \sum_{x\in\mathcal{X}} p(x)\ln\frac{p(x)}{q(x)}.$$

We overload this notation by defining for two Bernoulli distributions $\mathcal{B}(p)$ and $\mathcal{B}(q)$ of respective parameters p and q

$$D(p\|q) \overset{def}{=} D(\mathcal{B}(p)\|\mathcal{B}(q)) = p\ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right).$$

We use the convention (based on continuity arguments) that $0\ln\frac{0}{p} = 0$ and $p\ln\frac{p}{0} = \infty$.

We will need the following approximations/results of the Kullback-Leibler divergence

Lemma 2. *For any $\delta \in (-1/2, 1/2)$ we have*

$$D\left(\frac{1}{2}\middle\|\frac{1}{2}+\delta\right) = -\frac{1}{2}\ln(1-4\delta^2). \tag{6}$$

For constant $\alpha \in (0,1)$ and δ going to 0 by staying positive, we have

$$D(\alpha\|\delta) = -h(\alpha) - \alpha \ln \delta + O(\delta). \tag{7}$$

For $0 < y < x$ and x going to 0 we have

$$D(x\|y) = x \ln \frac{x}{y} + x - y + O\left(x^2\right). \tag{8}$$

Proof. Let us first prove (6).

$$\begin{aligned} D\left(\frac{1}{2}\middle\|\frac{1}{2}+\delta\right) &= \frac{1}{2}\ln\frac{1/2}{1/2+\delta} + \frac{1}{2}\ln\frac{1/2}{1/2-\delta} \\ \mathbb{P} &= -\frac{1}{2}\ln(1+2\delta) - \frac{1}{2}\ln(1-2\delta) \\ &= -\frac{1}{2}\ln(1-4\delta^2). \end{aligned}$$

To prove (7) we observe that

$$\begin{aligned} D(\alpha\|\delta) &= \alpha\ln\left(\frac{\alpha}{\delta}\right) + (1-\alpha)\ln\left(\frac{1-\alpha}{1-\delta}\right) \\ &= -h(\alpha) - \alpha\ln\delta - (1-\alpha)\ln(1-\delta) \\ &= -h(\alpha) - \alpha\ln\delta + O(\delta). \end{aligned}$$

For the last estimate we proceed as follows

$$\begin{aligned} D(x\|y) &= x\ln\frac{x}{y} + (1-x)\ln\frac{1-x}{1-y} \\ &= x\ln\frac{x}{y} - (1-x)\left(-x+y+O\left(x^2\right)\right) \\ &= x\ln\frac{x}{y} + x - y + O\left(x^2\right). \end{aligned}$$

□

The Kullback-Leibler appears in the computation of large deviation exponents. In our case, we will use the following estimate which is well known and which can be found for instance in [8]

Lemma 3. *Let p be a real number in $(0,1)$ and $X_1, \ldots X_n$ be n independent Bernoulli random variables of parameter p. Then, as n tends to infinity:*

$$\mathbb{P}(X_1 + \ldots X_n \geq \tau n) = \frac{(1-p)\sqrt{\tau}}{(\tau-p)\sqrt{2\pi n(1-\tau)}} e^{-nD(\tau\|p)}(1+o(1)) \text{ for } p < \tau < 1, \tag{9}$$

$$\mathbb{P}(X_1 + \ldots X_n \leq \tau n) = \frac{p\sqrt{1-\tau}}{(p-\tau)\sqrt{2\pi n\tau}} e^{-nD(\tau\|p)}(1+o(1)) \text{ for } 0 < \tau < p. \tag{10}$$

A.2 Estimation of the probability that a parity-check equation of weight w gives an incorrect information

A.2.1 Main result

We start our computation by computing the probability that a parity-check equation gives an incorrect information about a bit. We say here that a parity-check equation $\mathbf{h}$ (viewed as a binary word) gives an incorrect information about an error bit e_i that is involved in $\mathbf{h}$ if $\langle \mathbf{h}, \mathbf{e} \rangle \neq e_i$, where $\mathbf{e}$ is the error. This is obtained through the following lemma.

Lemma 4. *Consider a word $\mathbf{h} \in \mathbb{F}_2^n$ of weight w and an error $\mathbf{e} \in \mathbb{F}_2^n$ of weight t chosen uniformly at random. Assume that both w and t are of order $\sqrt{n}$: $w = \Theta(\sqrt{n})$ and $t = \Theta(\sqrt{n})$. We have*

$$\mathbb{P}_{\mathbf{e}}(\langle \mathbf{h}, \mathbf{e} \rangle = 1) = \frac{1}{2} - \frac{1}{2} e^{-\frac{2wt}{n}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Remark 2. *Note that this probability is in this case of the same order as the probability taken over errors $\mathbf{e}$ whose coordinates are drawn independently from a Bernoulli distribution of parameter t/n. In such a case, from the piling-up lemma [28] we have*

$$\begin{aligned} \mathbb{P}_{\mathbf{e}}(\langle \mathbf{h}, \mathbf{e} \rangle = 1) &= \frac{1 - \left(1 - \frac{2t}{n}\right)^w}{2} \\ &= \frac{1}{2} - \frac{1}{2} e^{w \ln(1 - 2t/n)} \\ &= \frac{1}{2} - \frac{1}{2} e^{-\frac{2wt}{n}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right). \end{aligned}$$

Let us bring now the following fundamental quantities for $b \in \{0, 1\}$

$$p_b \stackrel{\text{def}}{=} \mathbb{P}(\langle \mathbf{h}, \mathbf{e} \rangle = 1 | e_1 = b) \tag{11}$$

where without loss of generality we assume that $h_1 = 1$ and $\mathbf{e}$ is an error of weight t and length n chosen uniformly at random.

The proof of this lemma will be done in the following subsection. From this lemma it follows directly that

Corollary 2. *Assume that $w = \Theta(\sqrt{n})$ and $t = \Theta(\sqrt{n})$. Then*

$$p_b = \frac{1}{2} - (-1)^b \varepsilon \left(\frac{1}{2} + O\left(\frac{1}{\sqrt{n}}\right)\right), \tag{12}$$

where $\varepsilon \stackrel{def}{=} e^{-\frac{2wt}{n}}$.

A.2.2 Proof of Lemma 4

The proof involves properties of the Krawtchouk polynomials. We recall that the (binary) Krawtchouk polynomial of degree i and order n (which is an integer), $P_i^n(X)$ is defined for $i \in \{0, \cdots, n\}$ by:

$$P_i^n(X) \stackrel{\text{def}}{=} \frac{(-1)^i}{2^i} \sum_{j=0}^{i} (-1)^j \binom{X}{j} \binom{n-X}{i-j} \quad \text{where } \binom{X}{j} \stackrel{\text{def}}{=} \frac{1}{j!} X(X-1)\cdots(X-j+1). \tag{13}$$

Notice that it follows on the spot from the definition of a Krawtchouk polynomial that

$$P_k^n(0) = \frac{(-1)^k \binom{n}{k}}{2^k}. \tag{14}$$

Let us define the bias δ by

$$\delta \stackrel{\text{def}}{=} 1 - 2\mathbb{P}_{\mathbf{e}}(\langle \mathbf{h}, \mathbf{e} \rangle = 1).$$

In other words $\mathbb{P}_{\mathbf{e}}(\langle \mathbf{h}, \mathbf{e} \rangle = 1) = \frac{1}{2}(1-\delta)$. These Krawtchouk polynomials are readily related to δ. We first observe that

$$\mathbb{P}_{\mathbf{e}}(\langle \mathbf{h}, \mathbf{e} \rangle = 1) = \frac{\sum_{\substack{j=1 \\ j \text{ odd}}}^{w} \binom{t}{j} \binom{n-t}{w-j}}{\binom{n}{w}}.$$

Moreover by observing that $\sum_{j=0}^{w} \binom{t}{j} \binom{n-t}{w-j} = \binom{n}{w}$ we can recast the following evaluation of a Krawtchouk polynomial as

$$\begin{aligned}
\frac{(-2)^w}{\binom{n}{w}} P_w^n(t) &= \frac{\sum_{j=0}^{w} (-1)^j \binom{t}{j} \binom{n-t}{w-j}}{\binom{n}{w}} \\
&= \frac{\sum_{\substack{j=0 \\ j \text{ even}}}^{w} \binom{t}{j} \binom{n-t}{w-j} - \sum_{\substack{j=1 \\ j \text{ odd}}}^{w} \binom{t}{j} \binom{n-t}{w-j}}{\binom{n}{w}} \\
&= \frac{\binom{n}{w} - 2\sum_{\substack{j=1 \\ j \text{ odd}}}^{w} \binom{t}{j} \binom{n-t}{w-j}}{\binom{n}{w}} \\
&= 1 - 2\mathbb{P}_{\mathbf{e}}(\langle \mathbf{h}, \mathbf{e} \rangle = 1) \\
&= \delta.
\end{aligned} \tag{15}$$

To simplify notation we will drop the superscript n in the Krawtchouk polynomial notation. It will be chosen as the length of the MDPC code when will use it in our case. An important lemma that we will need is the following one.

Lemma 5. *For all x in $\{1,\ldots,t\}$, we have*

$$\frac{P_w(x)}{P_w(x-1)} = \left(1+O\left(\frac{1}{n}\right)\right)\frac{n-2w+\sqrt{(n-2w)^2-4w(n-w)}}{2(n-w)}.$$

Proof. This follows essentially from arguments taken in the proof of [27][Lemma 36, §7, Ch. 17]. The result we use appears however more explicitly in [25][Sec. IV] where it is proved that if x is in an interval of the form $\left[0,(1-\alpha)\left(n/2-\sqrt{w(n-w)}\right)\right]$ for some constant $\alpha\in[0,1)$ independent of x, n and w, then

$$\frac{P_w(x+1)}{P_w(x)} = \left(1+O\left(\frac{1}{n}\right)\right)\frac{n-2w+\sqrt{(n-2w)^2-4w(n-w)}}{2(n-w)}.$$

For our choice of t this condition is met for x and the lemma follows immediately. □

We are ready now to prove Lemma 4.

Proof of Lemma 4. We start the proof by using (15) which says that

$$\delta = \frac{(-2)^w}{\binom{n}{w}}P_w^n(t).$$

We then observe that

$$\begin{aligned}
\frac{(-2)^w}{\binom{n}{w}}P_w^n(t) &= \frac{(-2)^w}{\binom{n}{w}}\frac{P_w^n(t)}{P_w^n(t-1)}\frac{P_w^n(t-1)}{P_w^n(t-2)}\cdots\frac{P_w^n(1)}{P_w^n(0)}P_w^n(0)\\
&= \frac{(-2)^w}{\binom{n}{w}}\left(\left(1+O\left(\frac{1}{n}\right)\right)\frac{n-2w+\sqrt{(n-2w)^2-4w(n-w)}}{2(n-w)}\right)^t P_w^n(0) \text{ (by Lemma 5)}\\
&= \left(1+O\left(\frac{1}{n}\right)\right)^t\left(\frac{n-2w+\sqrt{(n-2w)^2-4w(n-w)}}{2(n-w)}\right)^t \text{ (by (14))}\\
&= e^{t\ln\left(\frac{1-2\omega+\sqrt{(1-2\omega)^2-4\omega(1-\omega)}}{2(1-\omega)}\right)}\left(1+O\left(\frac{t}{n}\right)\right) \text{ where } \omega \stackrel{\text{def}}{=} \frac{w}{n}\\
&= e^{t\ln\left(\frac{1-2\omega+1-4\omega+O\left(\omega^2\right)}{2(1-\omega)}\right)}\left(1+O\left(\frac{t}{n}\right)\right)\\
&= e^{t\ln\left(\frac{1-3\omega+O\left(\omega^2\right)}{1-\omega}\right)}\left(1+O\left(\frac{t}{n}\right)\right)\\
&= e^{-2t\omega+O\left(\frac{tw^2}{n^2}\right)}\left(1+O\left(\frac{t}{n}\right)\right)\\
&= e^{-\frac{2wt}{n}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right),
\end{aligned}$$

where we used at the last equation that $t = \theta(\sqrt{n})$ and $w = \theta(\sqrt{n})$. □

A.3 Estimation of the probability that a bit is incorrectly estimated by the first step of the bit flipping algorithm

We are here in the model where every bit is involved in $w/2$ parity-check equations and each parity-check equation is of weight w. We assume that the bit-flipping algorithm consists in computing for each bit i the syndrome bits corresponding to the parity-checks involving i and taking the majority vote of these syndrome bits. We model each vote of a parity-check by a Bernoulli variable equal to 1 if the information coming from this random variable says that the bit should be flipped. The parameter of this Bernoulli random variable depends on whether or not i is incorrect. When i is correct, then the Bernoulli random variable is of parameter p_0. When i is incorrect, then the Bernoulli random variable is of parameter p_1. We bring in the quantities

$$q_0 \stackrel{\text{def}}{=} \mathbb{P}(\text{flip the bit}|\text{bit was correct}) \tag{16}$$

$$q_1 \stackrel{\text{def}}{=} \mathbb{P}(\text{stay with the same value}|\text{bit was incorrect}) \tag{17}$$

Lemma 6. *For $b \in \{0,1\}$, we have*

$$q_b = O\left(\frac{(1-\varepsilon^2)^{w/4}}{\sqrt{\pi w}\varepsilon}\right).$$

Proof. For $b \in \{0,1\}$, we let X_1^b, $X_2^b, \ldots, X_{w/2}^b$ be independent random variables of parameter p_b. We obviously have

$$\begin{aligned} q_0 &\leq \mathbb{P}(\sum_{i=1}^{w/2} X_i^0 \geq w/4) \\ q_1 &\leq \mathbb{P}(\sum_{i=1}^{w/2} X_i^1 \leq w/4). \end{aligned}$$

By using Lemma 3 we obtain for q_0

$$\begin{aligned} q_0 &\leq \frac{(1-p_0)\sqrt{\frac{1}{2}}}{(\frac{1}{2}-p_0)\sqrt{2\pi\frac{w}{2}(1-\frac{1}{2})}} e^{-w/2D\left(\frac{1}{2}\|p_0\right)} \\ &\leq \frac{(1-p_0)}{\sqrt{\pi w}\varepsilon} e^{-w/2D\left(\frac{1}{2}\|\frac{1}{2}-\frac{1}{2}\varepsilon(1+O(1/w))\right)} \quad (18) \\ &\leq \frac{(1-p_0)}{\sqrt{\pi w}\varepsilon} e^{\frac{w\left(\ln(1-\varepsilon^2)+O\left(\frac{1}{w}\right)\right)}{4}} \quad (19) \\ &\leq O\left(\frac{(1-\varepsilon^2)^{w/4}}{\sqrt{\pi w}\varepsilon}\right) \quad (20) \end{aligned}$$

Whereas for q_1 we also obtain

$$\begin{aligned} q_1 &\leq \frac{p_1\sqrt{\frac{1}{2}}}{(p_1-\frac{1}{2})\sqrt{2\pi\frac{w}{2}\frac{1}{2}}} e^{-w/2D\left(\frac{1}{2}\|p_1\right)} \quad (21) \\ &\leq O\left(\frac{(1-\varepsilon^2)^{w/4}}{\sqrt{\pi w}\varepsilon}\right) \quad (22) \end{aligned}$$

□

A.4 Proof of Theorem 1

We are ready now to prove Theorem 1. We use here the notation of Assumption 1. Recall that e^0 denotes the true error vector. e^1 is the value of vector e after one round of iterative decoding in Algorithm 1. We let $\Delta e \stackrel{\text{def}}{=} e^0+e^1$. Call $X_1^0,\ldots,X_{n-t}^0$ the values after one round of iterative decoding of the $n-t$ bits which were without error initially (that is the bits i such that $e_i^0=0$) . Similarly let $X_1^1,\ldots,X_t^1$ be the values after one round of iterative decoding of the t bits which were initially in error (i.e. for which $e_i^0=1$). We let

$$\begin{aligned} S_0 &\stackrel{\text{def}}{=} X_1^0+\cdots+X_{n-t}^0 \\ S_1 &\stackrel{\text{def}}{=} X_1^1+\cdots+X_t^1 \end{aligned}$$

S_0 is the number of errors that were introduced after one round of iterative decoding coming from flipping the $n-t$ bits that were initially correct, that is the number of i's for which $e_i^0=0$ and $e_i^1=1$. Similarly S_1 is the number of errors that are left after one round of iterative decoding coming from not flipping the t bits that were initially incorrect, that is the number of i's for which $e_i^0=1$ and $e_i^1=0$.

Let S be the weight of Δe. By assumption 1 we have

$$P_{\text{err}} \leq \mathbb{P}(|\Delta e| \geq \alpha t) = \mathbb{P}(S \geq \alpha t),$$

for some α in $(0,1)$. We have

$$\begin{aligned}\mathbb{P}(S \geq \alpha t) &\leq \mathbb{P}(S_0 \geq \alpha t/2 \cup S_1 \geq \alpha t/2) \\ &\leq \mathbb{P}(S_0 \geq \alpha t/2) + \mathbb{P}(S_1 \geq \alpha t/2)\end{aligned}$$

By Assumption 1, S_0 is the sum of $n-t$ Bernoulli variables of parameter q_0. By applying Lemma 3 we obtain

$$\begin{aligned}\mathbb{P}(S_0 \geq \alpha t/2) &\leq \frac{(1-q_0)\sqrt{\frac{\alpha t}{2(n-t)}}}{(\frac{\alpha t}{2(n-t)} - q_0)\sqrt{2\pi(n-t)(1-\frac{\alpha t}{2(n-t)})}} e^{-(n-t)D\left(\frac{\alpha t}{2(n-t)} \middle\| q_0\right)} \\ &\leq \frac{1}{\sqrt{\alpha\pi t}} e^{-(n-t)D\left(\frac{\alpha t}{2(n-t)} \middle\| q_0\right)} \qquad (23)\end{aligned}$$

We observe now that

$$D\left(\frac{\alpha t}{2(n-t)} \middle\| q_0\right) \geq D\left(\frac{\alpha t}{2(n-t)} \middle\| O\left(\frac{(1-\varepsilon^2)^{w/4}}{\sqrt{\pi w}\varepsilon}\right)\right) \qquad (24)$$

where we used the upper-bound on q_0 coming from Lemma 6 and the fact that $D(x\|y) \geq D(x\|y')$ for $0 < y < y' < x < 1$. By using this and Lemma 2, we deduce

$$\begin{aligned}D\left(\frac{\alpha t}{2(n-t)} \middle\| q_0\right) &\geq \frac{\alpha t}{2(n-t)} \ln\left(\frac{\alpha t}{2(n-t)}\right) - \frac{\alpha t}{2(n-t)} \ln\left(O\left(\frac{(1-\varepsilon^2)^{w/4}}{\varepsilon\sqrt{w}}\right)\right) + O\left(\frac{\alpha t}{2(n-t)}\right) \\ &\geq \frac{\alpha t}{2(n-t)} \ln\left(\frac{t\sqrt{w}}{n}\right) - \frac{\alpha t w}{8(n-t)} \ln\left(1-\varepsilon^2\right) + O\left(\frac{t}{n}\right) \\ &\geq -\frac{\alpha t}{8(n-t)} \ln n - \frac{\alpha t w}{8(n-t)} \ln\left(1-\varepsilon^2\right) + O\left(\frac{t}{n}\right).\end{aligned}$$

By plugging in this expression in (23) we obtain

$$\mathbb{P}(S_0 \geq \alpha t/2) \leq \frac{1}{\sqrt{\alpha\pi t}} e^{\frac{\alpha t w}{8}\ln\left(1-\varepsilon^2\right) + \frac{\alpha t}{8}\ln(n) + O(t)}$$

On the other hand we have

$$\begin{aligned}\mathbb{P}(S_1 \geq \alpha t/2) &\leq \frac{(1-q_1)\sqrt{\frac{\alpha}{2}}}{(\frac{\alpha}{2} - q_1)\sqrt{2\pi t(1-\frac{\alpha}{2})}} e^{-tD\left(\frac{\alpha}{2} \| q_1\right)} \\ &\leq \frac{1}{\sqrt{\alpha\pi t}} e^{-tD\left(\frac{\alpha}{2} \| q_1\right)} \qquad (25)\end{aligned}$$

Similarly to what we did above, by using the upper-bound on q_1 of Lemma 6 and $D(x\|y) \geq D(x\|y')$ for $0 < y < y' < x < 1$, we deduce that

$$D\Big(\frac{\alpha}{2}\Big\|q_1\Big) \geq D\left(\frac{\alpha}{2}\middle\|O\left(\frac{(1-\varepsilon^2)^{w/4}}{\varepsilon\sqrt{w}}\right)\right)$$

By using together with Lemma 2 we obtain

$$\begin{aligned} D\Big(\frac{\alpha}{2}\Big\|q_1\Big) &\geq -h(\alpha/2) - \frac{\alpha}{2}\ln\left(O\left(\frac{(1-\varepsilon^2)^{w/4}}{\varepsilon\sqrt{w}}\right)\right) + O\left(\frac{(1-4\varepsilon^2)^{w/4}}{\varepsilon\sqrt{w}}\right) \\ &\geq -\frac{\alpha w}{8}\ln\left(1-\varepsilon^2\right) + \frac{\alpha}{8}\ln n + O\left(1\right). \end{aligned}$$

By using this lower-bound in (25), we deduce

$$\mathbb{P}(S_1 \geq \alpha t/2) \leq \frac{1}{\sqrt{\alpha\pi t}} e^{\frac{\alpha t w}{8}\ln\left(1-\varepsilon^2\right) + \frac{\alpha t}{8}\ln(n) + O(t)}.$$

B Proof of Proposition 1

Let us first recall the proposition we want to prove

Proposition 1. *Let f be a Boolean function which is equal to 1 on a fraction α of inputs which can be implemented by a quantum circuit of depth D_f and whose gate complexity is C_f. Using Grover's algorithm for finding an input x of f for which $f(x) = 1$ can not take less quantum resources than a Grover's attack on AES-N as soon as*

$$\frac{D_f \cdot C_f}{\alpha} \geq 2^N D_{AES-N} \cdot C_{AES-N}$$

where D_{AES-N} and C_{AES-N} are respectively the depth and the complexity of the quantum circuit implementing AES-N.

Proof. Following Zalka[37], the best way is to perform Grover's algorithm sequentially with the maximum allowed number of iterations in order not to go beyond MAXDEPTH. Grover's algorithm consists of iterations of the following procedure:

- Apply $U : |0\rangle|0\rangle \rightarrow \sum_{x\in\{0,1\}^n} \frac{1}{2^{n/2}}|x\rangle|f(x)\rangle$.
- Apply a phase flip on the second register to get $\sum_{x\in\{0,1\}^n} \frac{1}{2^{n/2}}(-1)^{f(x)}|x\rangle|f(x)\rangle$.
- Apply $U^\dagger$.

If we perform I iterations of the above for $I \leq \frac{1}{\sqrt{\alpha}}$ then the winning probability is upper bounded by αI^2. In our setting, we can perform $I = \frac{\mathsf{MAXDEPTH}}{D_f}$ sequentially before measuring, and each iteration costs time C_f. At each iteration, we succeed with probability αI^2 and we need to repeat this procedure $\frac{1}{\alpha I^2}$ times to get a result with constant probability. From there, we conclude that the total complexity Q is:

$$Q = \frac{1}{\alpha I^2} \cdot I \cdot C_f = \frac{D_f \cdot C_f}{\alpha \mathsf{MAXDEPTH}}. \tag{26}$$

A similar reasoning performed on using Grover's search on AES-N leads to a quantum complexity

$$Q_{AES-N} = \frac{2^N D_{AES-N} \cdot C_{AES-N}}{\mathsf{MAXDEPTH}}. \tag{27}$$

The proposition follows by comparing (26) with (27). □