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Research Article

Synchronization of Dissipative Dynamical Systems Driven by Non-Gaussian Lévy Noises

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Dynamical systems driven by Gaussian noises have been considered extensively in modeling, simulation, and theory. However, complex systems in engineering and science are often subject to non-Gaussian fluctuations or uncertainties. A coupled dynamical system under a class of Lévy noises is considered. After discussing cocycle property, stationary orbits, and random attractors, a synchronization phenomenon is shown to occur, when the drift terms of the coupled system satisfy certain dissipativity and integrability conditions. The synchronization result implies that coupled dynamical systems share a dynamical feature in some asymptotic sense.

1. Introduction

Synchronization of coupled dynamical systems is an ubiquitous phenomenon that has been observed in biology, physics, and other areas. It concerns coupled dynamical systems that share a dynamical feature in an asymptotic sense. A descriptive account of its diversity of occurrence can be found in the recent book [1]. Recently Caraballo and Kloeden [2, 3] proved that synchronization in coupled deterministic dissipative dynamical systems persists in the presence of various Gaussian noises (in terms of Brownian motion), provided that appropriate concepts of random attractors and stochastic stationary solutions are used instead of their deterministic counterparts.

In this paper we investigate a synchronization phenomenon for coupled dynamical systems driven by nonGaussian noises. We show that couple dissipative systems exhibit synchronization for a class of Lévy motions.

This paper is organized as follows. We first recall some basic facts about random dynamical systems (RDSs) as well as formulate the problem of synchronization of stochastic dynamical systems driven by Lévy noises in Section 2. The main result (Theorem 3.3) and an example are presented in Section 3.

Throughout this paper, the norm of a vector x in Euclidean space $\mathbb{R}^d$ is always denote by $|x|$.

2. Dynamical Systems Driven by Lévy Noises

Dynamical systems driven by nonGaussian Lévy motions have attracted much attention recently [4, 5]. Under certain conditions, the SDEs driven by Lévy motion generate stochastic flows [4, 6], and also generate random dynamical systems (or cocycles) in the sense of Arnold [7]. Recently, exit time estimates have been investigated by Imkeller and Pavlyukevich [8], and Imkeller et al. [9], and Yang and Duan [10] for SDEs driven by Lévy motion. This shows some qualitatively different dynamical behavior between SDEs driven by Gaussian and nonGaussian noises.

2.1. Lévy Processes

A Lévy process or motion on $\mathbb{R}^d$ is characterized by a drift parameter $\gamma \in \mathbb{R}^d$, a covariance $d \times d$ matrix A, and a nonnegative Borel measure ν, defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and concentrated on $\mathbb{R}^d \setminus \{0\}$, which satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} \left(|y|^2 \wedge 1\right) \nu(dy) < \infty, \tag{2.1}$$

or equivalently

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|y|^2}{1 + |y|^2} \nu(dy) < \infty. \tag{2.2}$$

This measure ν is the so-called the Lévy jump measure of the Lévy process L_t. Moreover Lévy process L_t has the following Lévy-Itô decomposition:

$$L_t = \gamma t + B_t + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx), \tag{2.3}$$

where $N(dt, dx)$ is Poisson random measure,

$$\widetilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt, \tag{2.4}$$

is the compensated Poisson random measure of L_t, and B_t is an independent Brownian motion on $\mathbb{R}^d$ with covariance matrix A (see [4, 10–12]). We call (A, ν, γ) the *generating triplet*.

General semimartingales, especially Lévy motions, are thought to be appropriate models for nonGaussian processes with jumps [11]. Let us recall that a Lévy motion L_t is a nonGaussian process with independent and stationary increments. Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L_t - L_{t_0}| \geq \delta) \to 0$ as $t \to t_0$ for any positive δ. With a suitable modification [4], these paths may be taken as càdlàg, that is, paths are continuous on the right and have limits on the left. This continuity is weaker than the usual continuity in time. In fact, a càdlàg function has finite or at most countable discontinuities on any time interval (see, e.g., [4, page 118]). This generalizes the Brownian motion B_t, since B_t satisfies all these three conditions, but *additionally*, (i) almost every sample path of the Brownian motion is continuous in time in the usual sense, and (ii) the increments of Brownian motion are Gaussian distributed.

The next useful lemma provides some important pathwise properties of L_t with two-sided time $t \in \mathbb{R}$.

Lemma 2.1 (pathwise boundedness and convergence). *Let L_t be a two-sided Lévy motion on $\mathbb{R}^d$ for which $\mathbb{E}|L_1| < \infty$ and $\mathbb{E}L_1 = 0$. Then we have the following.*

(i) $\lim_{t\to\pm\infty}(1/t)L_t = 0, a.s.$

(ii) *The integrals $\int_{-\infty}^{t} e^{-\lambda(t-s)}\, dL_s(\omega)$ are pathwisely uniformly bounded in $\lambda > 1$ on finite time intervals $[T_1, T_2]$ in $\mathbb{R}$.*

(iii) *The integrals $\int_{T_1}^{t} e^{-\lambda(t-s)}\, dL_s(\omega) \to 0$ as $\lambda \to \infty$, pathwise on finite time intervals $[T_1, T_2]$ in $\mathbb{R}$.*

Proof. (i) This convergence result comes from the law of large numbers, in [11, Theorem 36.5].

(ii) Since the function $h(t) = e^{-\lambda t}$ is continuous in t, integrating by parts we obtain

$$\int_{-\infty}^{t} e^{-\lambda(t-s)}\, dL_s(\omega) = L_t(\omega) - \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} L_s(\omega) ds. \tag{2.5}$$

Then by (i) and the fact that every càdlàg function is bounded on finite closed intervals, we conclude (ii).

(iii) Integrating again by parts, it follows that

$$\int_{T_1}^{t} e^{-\lambda(t-s)}\, dL_s(\omega) = (L_t - L_{T_1}) e^{-\lambda(t-T_1)} + \lambda \int_{T_1}^{t} e^{-\lambda(t-s)} (L_t(\omega) - L_s(\omega)) ds, \tag{2.6}$$

from which the result follows. □

Remark 2.2. The assumptions on L_t in the above lemma are satisfied by a wide class of Lévy processes, for instance, the symmetric α-stable Lévy motion on $\mathbb{R}^d$ with $1 < \alpha < 2$. Indeed, in this case, we have $\int_{|x|>1} |x| \nu(dx) < \infty$, and then $E|L_1| < \infty$, see [11, Theorem 25.3].

For the canonical sample space of Lévy processes, that is, $\Omega = D(\mathbb{R}, \mathbb{R}^d)$ of càdlàg functions which are defined on $\mathbb{R}$ and taking values in $\mathbb{R}^d$ is not separable, if we use the usual compact-open metric. However, it is complete and separable when endowed with the Skorohod metric (see, e.g., [13], [14, page 405]), in which case we call $D(\mathbb{R}, \mathbb{R}^d)$ a Skorohod space.

2.2. Random Dynamical Systems

Following Arnold [7], a random dynamical system (RDS) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consists of two ingredients: a driving flow θ_t on the probability space Ω, that is, θ_t is a deterministic dynamical system, and a cocycle mapping $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, namely, φ satisfies the conditions

$$\varphi(0, \omega) = id_{\mathbb{R}^d}, \quad \varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \tag{2.7}$$

for all $\omega \in \Omega$ and all $s, t \in \mathbb{R}$. This cocycle is required to be at least measurable from the σ-field $\mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(\mathbb{R}^d)$ to the σ-field $\mathcal{B}(\mathbb{R}^d)$.

For random dynamical systems driven by Lévy noise we take $\Omega = D(\mathbb{R}, \mathbb{R}^d)$ with the Skorohod metric as the canonical sample space and denote by $\mathcal{F} := \mathcal{B}(D(\mathbb{R}, \mathbb{R}^d))$ the associated Borel σ-field. Let μ_L be the (Lévy) probability measure on $\mathcal{F}$ which is given by the distribution of a two-sided Lévy process with paths in $D(\mathbb{R}, \mathbb{R}^d)$.

The driving system $\theta = (\theta_t, t \in \mathbb{R})$ on Ω is defined by the shift

$$(\theta_t \omega)(s) := \omega(t+s) - \omega(t). \tag{2.8}$$

The map $(t, \omega) \to \theta_t \omega$ is continuous, thus measurable ([7, page 545]), and the (Lévy) probability measure is θ-invariant, that is,

$$\mu_L\left(\theta_t^{-1}(A)\right) = \mu_L(A) \tag{2.9}$$

for all $A \in \mathcal{F}$, see [4, page 325].

We say that a family $\widehat{A} = \{A(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $A(\omega)$ of $\mathbb{R}^d$ is *invariant* for a RDS (θ, φ), if $\varphi(t, \omega, A(\omega)) = A(\theta_t \omega)$ for all $t > 0$ and that it is a random attractor if in addition it is pathwise pullback attracting in the sense that

$$H_d^*\big(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)\big) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \tag{2.10}$$

for all suitable families (called the attracting universe) of $\widehat{D} = \{D(\omega), \omega \in \Omega\}$ of nonempty measurable bounded subsets $D(\omega)$ of $\mathbb{R}^d$, where $H_d^*(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|$ is the Hausdorff semi-distance on $\mathbb{R}^d$.

The following result about the existence of a random attractor may be proved similarly as in [2, 15–18].

Lemma 2.3 (random attractor for càdlàg RDS). *Let (θ, φ) be an RDS on $\Omega \times \mathbb{R}^d$ and let φ be continuous in space, but càdlàg in time. If there exits a family $\widehat{B} = \{B(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $B(\omega)$ of $\mathbb{R}^d$ and a $T_{\widehat{D},\omega} \geq 0$ such that*

$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \quad \forall t \geq T_{\widehat{D},\omega}, \tag{2.11}$$

for all families $\widehat{D} = \{D(\omega), \omega \in \Omega\}$ in a given attracting universe, then the RDS (θ, φ) has a random attractor $\widehat{A} = \{A(\omega), \omega \in \Omega\}$ with the component subsets defined for each $\omega \in \Omega$ by

$$A(\omega) = \bigcap_{s>0} \overline{\bigcup_{t \geq s} \varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}. \tag{2.12}$$

Forevermore if the random attractor consists of singleton sets, that is, $A(\omega) = \{X^*(\omega)\}$ for some random variable X^*, then $X_t^*(\omega) = X^*(\theta_t\omega)$ is a stationary stochastic process.

We also need the following Gronwall's lemma from [19].

Lemma 2.4. *Let $x(t)$ satisfy the differential inequality*

$$\frac{d}{dt_+} x \leq g(t)x + h(t), \tag{2.13}$$

where $(d/dt_+)x := \lim_{h\downarrow 0^+}((x(t+h) - x(t))/h)$ is right-hand derivative of x. Then

$$x(t) \leq x(0) \exp\left[\int_0^t g(r)dr\right] + \int_0^t \exp\left[\int_s^t g(r)dr\right] h(s)ds. \tag{2.14}$$

2.3. Dissipative Synchronization

Suppose that we have two autonomous ordinary differential equations in $\mathbb{R}^d$,

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = g(y), \tag{2.15}$$

where the vector fields f and g are sufficiently regular (e.g., differentiable) to ensure the existence and uniqueness of *local* solutions, and additionally satisfy one-sided dissipative Lipschitz conditions

$$\max\{\langle x_1 - x_2, f(x_1) - f(x_2)\rangle, \langle x_1 - x_2, g(x_1) - g(x_2)\rangle\} \leq -l|x_1 - x_2|^2 \tag{2.16}$$

on $\mathbb{R}^d$ for some $l > 0$. These dissipative Lipschitz conditions ensure existence and uniqueness of *global* solutions. Each of the systems has a unique globally asymptotically stable equilibria, $\overline{x}$ and $\overline{y}$, respectively [18]. Then, the coupled *deterministic* dynamical system in $\mathbb{R}^{2d}$

$$\frac{dx}{dt} = f(x) + \lambda(y - x), \qquad \frac{dy}{dt} = g(x) + \lambda(x - y) \tag{2.17}$$

with parameter $\lambda > 0$ also satisfies a one-sided dissipative Lipschitz condition and, hence, also has a unique equilibrium $(\overline{x}^{\lambda}, \overline{y}^{\lambda})$, which is globally asymptotically stable [18]. Moreover, $(\overline{x}^{\lambda}, \overline{y}^{\lambda}) \to (\overline{z}, \overline{z})$ as $\lambda \to \infty$, where $\overline{z}$ is the unique globally asymptotically stable equilibrium of the "averaged" system in $\mathbb{R}^d$

$$\frac{dz}{dt} = \frac{1}{2}\,(f(z) + g(z)). \tag{2.18}$$

This phenomenon is known as synchronization for the coupled deterministic system (2.17). The parameter λ often appears naturally in the context of the problem under consideration. For example in control theory it is a control parameter which can be chosen by the engineer, whereas in chemical reactions in thin layers separated by a membrane it is the reciprocal of the thickness of the layers; see [20].

Caraballo and Kloeden [2], and Caraballo et al. [3] showed that this synchronization phenomenon persists under Gaussian Brownian noise, provided that asymptotically stable stochastic stationary solutions are considered rather than asymptotically stable steady state solutions. Recall that a stationary solution X^* of a SDE system may be characterized as a stationary orbit of the corresponding random dynamical system (θ, φ) (defined by the SDE system), namely, $\varphi(t, \omega, X^*(\omega)) = X^*(\theta_t \omega)$.

The aim of this paper is to investigate synchronization under nonGaussian Lévy noise. In particular, we consider a coupled SDE system in $\mathbb{R}^d$, driven by Lévy motion

$$\begin{aligned} dX_t &= \big(f(X_t) + \lambda(Y_t - X_t)\big)dt + a\,dL_t^1, \\ dY_t &= \big(g(Y_t) + \lambda(X_t - Y_t)\big)dt + b\,dL_t^2, \end{aligned} \tag{2.19}$$

where $a, b \in \mathbb{R}^d$ are constant vectors with no components equal to zero, L_t^1, L_t^2 are independent two-sided scalar Lévy motion as in Lemma 2.1, and f, g satisfy the one-sided dissipative Lipschitz conditions (2.16).

In addition to the one-sided Lipschitz dissipative condition (2.16) on the functions f and g, as in [2] we further assume the following integrability condition. There exists $m_0 > 0$ such that for any $m \in (0, m_0]$, and any càdlàg function $u : \mathbb{R} \to \mathbb{R}^d$ with subexponential growth it follows

$$\int_{-\infty}^{t} e^{ms} \left| f(u(s)) \right|^2 ds < \infty, \qquad \int_{-\infty}^{t} e^{ms} \left| g(u(s)) \right|^2 ds < \infty. \tag{2.20}$$

Without loss of generality, we assume that the one-sided dissipative Lipschitz constant $l \le m_0$.

In the next section we will show that the coupled system (2.19) has a unique stationary solution $(\widetilde{X}_t^{\lambda}, \widetilde{Y}_t^{\lambda})$ which is pathwise globally asymptotically stable with $(\widetilde{X}_t^{\lambda}, \widetilde{Y}_t^{\lambda}) \to (Z_t^{\infty}, Z_t^{\infty})$ as $\lambda \to \infty$, pathwise on finite time intervals $[T_1, T_2]$, where Z_t^{∞} is the unique pathwise globally asymptotically stable stationary solution of the "averaged" SDE in $\mathbb{R}^d$

$$dZ_t = \frac{1}{2}\left[f(Z_t) + g(Z_t)\right]dt + \frac{1}{2} a\,dL_t^1 + \frac{1}{2} b\,dL_t^2. \tag{2.21}$$

3. Systems Driven by Lévy Noise

For the coupled system (2.19), we have the following two lemmas about its stationary solutions.

Lemma 3.1 (existence of stationary solutions). *If the Assumption* (2.20) *holds, f and g are continuous and satisfy the one-sided Lipschitz dissipative conditions* (2.16) *with Lipschitz constant l, then the coupled stochastic system* (2.19) *has a unique stationary solution.*

Proof. First, the stationary solutions of the Langevin equations [4, 21]

$$dX_t = -\lambda X_t dt + a dL_t^1, \qquad dY_t = -\lambda Y_t dt + b dL_t^2 \tag{3.1}$$

are given by

$$\overline{X}_t^\lambda = a e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} dL_t^1, \qquad \overline{Y}_t^\lambda = b e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} dL_t^2. \tag{3.2}$$

The differences of the solutions of (2.19) and these stationary solutions satisfy a system of random ordinary differential equations, with right-hand derivative in time

$$\begin{aligned} \frac{d}{dt_+}\left(X_t - \overline{X}_t^\lambda\right) &= f(X_t) + \lambda(Y_t - X_t) + \lambda \overline{X}_t^\lambda, \\ \frac{d}{dt_+}\left(Y_t - \overline{Y}_t^\lambda\right) &= g(Y_t) + \lambda(X_t - Y_t) + \lambda \overline{Y}_t^\lambda. \end{aligned} \tag{3.3}$$

The equations (3.3) are equivalent to

$$\frac{d}{dt_+} U_t^\lambda = f(X_t) + \lambda\left(V_t^\lambda - U_t^\lambda\right) + \lambda \overline{Y}_t^\lambda, \qquad \frac{d}{dt_+} V_t^\lambda = g(Y_t) + \lambda\left(U_t^\lambda - V_t^\lambda\right) + \lambda \overline{X}_t^\lambda, \tag{3.4}$$

where $U_t^\lambda = X_t - \overline{X}_t^\lambda$ and $V_t^\lambda = Y_t - \overline{Y}_t^\lambda$. Thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt_+}\left(\left|U_t^\lambda\right|^2 + \left|V_t^\lambda\right|^2\right) &= \left(U_t^\lambda, f\left(U_t^\lambda + \overline{X}_t^\lambda\right) - f\left(\overline{X}_t^\lambda\right)\right) + \left(V_t^\lambda, g\left(V_t^\lambda + \overline{Y}_t^\lambda\right) - g\left(\overline{Y}_t^\lambda\right)\right) \\ &\quad + \left(U_t^\lambda, f\left(\overline{X}_t^\lambda\right) + \lambda \overline{Y}_t^\lambda\right) + \left(V_t^\lambda, g\left(\overline{Y}_t^\lambda\right) + \lambda \overline{X}_t^\lambda\right) - \lambda\left|U_t^\lambda - V_t^\lambda\right|^2 \\ &\le -\frac{l}{2}\left(\left|U_t^\lambda\right|^2 + \left|V_t^\lambda\right|^2\right) + \frac{2}{l}\left|f\left(\overline{X}_t^\lambda\right) + \lambda \overline{Y}_t^\lambda\right|^2 + \frac{2}{l}\left|g\left(\overline{Y}_t^\lambda\right) + \lambda \overline{X}_t^\lambda\right|^2. \end{aligned} \tag{3.5}$$

Hence, by Lemma 2.4,

$$\left|U_t^\lambda\right|^2+\left|V_t^\lambda\right|^2\le\left(\left|U_{t_0}^\lambda\right|^2+\left|V_{t_0}^\lambda\right|^2\right)e^{l(t-t_0)}+\frac{4e^{-lt}}{l}\int_{t_0}^t e^{ls}\left[\left|f\left(\overline{X}_t^\lambda\right)+\lambda\overline{Y}_t^\lambda\right|^2+\left|g\left(\overline{Y}_t^\lambda\right)+\lambda\overline{X}_t^\lambda\right|^2\right]ds. \tag{3.6}$$

Define

$$|R_\lambda(\omega)|^2=1+\frac{4}{l}\int_{-\infty}^0 e^{ls}\left[\left|f\left(\overline{X}^\lambda(\theta_s\omega)\right)+\lambda\overline{Y}^\lambda(\theta_s\omega)\right|^2+\left|g\left(\overline{Y}^\lambda(\theta_s\omega)\right)+\lambda\overline{X}^\lambda(\theta_s\omega)\right|^2\right]ds \tag{3.7}$$

and let $B_{2d}^\lambda(\omega)$ be a random closed ball in $\mathbb{R}^{2d}$ centered on the origin and of radius $R_\lambda(\omega)$.

Now we can use pathwise pullback convergence (i.e., with $t_0\to-\infty$) to show that $|U_t^\lambda|^2+|V_t^\lambda|^2$ is pathwise absorbed by the family $\widehat{B}_{2d}^\lambda=\{B_{2d}^\lambda(\omega),\omega\in\Omega\}$, that is, for appropriate families $\widehat{D}$, there exists $T_{\widehat{D},\omega}\ge 0$ such that

$$\varphi(t,\theta_{-t}\omega,D(\theta_{-t}\omega))\subset B_{2d}^\lambda(\omega),\quad \forall t\ge T_{\widehat{D},\omega}. \tag{3.8}$$

Hence, by Lemma 2.3, the coupled system has a random attractor $\widehat{A}^\lambda=\{A^\lambda(\omega),\ \omega\in\Omega\}$ with $A^\lambda(\omega)\subset B_{2d}^\lambda(\omega)$.

Note that, by Lemma 2.1, it can be shown that the random compact absorbing balls $B_{2d}^\lambda(\omega)$ are contained in the common compact ball for $\lambda\ge 1$.

However, the difference $(\Delta X_t,\Delta Y_t)=(X_t^1-X_t^2,Y_t^1-Y_t^2)$ of any pair of solutions satisfies the system of random ordinary differential equations

$$\begin{aligned}\frac{d}{dt_+}\Delta X_t&=f\left(X_t^1\right)-f\left(X_t^2\right)+\lambda(\Delta Y_t-\Delta X_t),\\ \frac{d}{dt_+}\Delta Y_t&=g\left(Y_t^1\right)-g\left(Y_t^2\right)-\lambda(\Delta Y_t-\Delta X_t),\end{aligned} \tag{3.9}$$

so

$$\begin{aligned}\frac{d}{dt_+}\left(|\Delta X_t|^2+|\Delta Y_t|^2\right)&=2\left(\Delta X_t,f\left(X_t^1\right)-f\left(X_t^2\right)\right)+2\left(\Delta Y_t,g\left(Y_t^1\right)-g\left(Y_t^2\right)\right)\\&\quad-2\lambda|\Delta X_t-\Delta Y_t|^2\\&\le-2l\left(|\Delta X_t|^2+|\Delta Y_t|^2\right)\end{aligned} \tag{3.10}$$

from which we obtain

$$|\Delta X_t|^2+|\Delta Y_t|^2\le\left(|\Delta X_0|^2+|\Delta Y_0|^2\right)e^{-2lt} \tag{3.11}$$

which means all solutions converge pathwise to each other as $t \to \infty$. Thus the random attractor consists of a singleton set formed by an ordered pair of stationary processes $(\tilde{X}_t^\lambda(\omega), \tilde{Y}_t^\lambda(\omega))$ or equivalently $(\tilde{X}^\lambda(\theta_t\omega), \tilde{Y}^\lambda(\theta_t\omega))$. □

Lemma 3.2 (a property of stationary solutions). *The stationary solutions of the coupled stochastic system* (2.19) *have the following asymptotic behavior:*

$$\tilde{X}_t^\lambda(\omega) - \tilde{Y}_t^\lambda(\omega) \longrightarrow 0 \quad \text{as } \lambda \longrightarrow \infty \tag{3.12}$$

pathwise on any bounded time interval $[T_1, T_2]$ *of* $\mathbb{R}$.

Proof. Since

$$d\left(\tilde{X}_t^\lambda - \tilde{Y}_t^\lambda\right) = \left(-2\lambda\left(\tilde{X}_t^\lambda - \tilde{Y}_t^\lambda\right) + f\left(\tilde{X}_t^\lambda\right) - g\left(\tilde{Y}_t^\lambda\right)\right)dt + a\,dL_t^1 - b\,dL_t^2, \tag{3.13}$$

we have

$$d\left(D_t^\lambda e^{2\lambda t}\right) = e^{2\lambda t}\left(f\left(\tilde{X}_t^\lambda\right) - g\left(\tilde{Y}_t^\lambda\right)\right) + ae^{2\lambda t}dL_t^1 - be^{2\lambda t}dL_t^2, \tag{3.14}$$

where $D_t^\lambda = \tilde{X}_t^\lambda - \tilde{Y}_t^\lambda$, so pathwise

$$\begin{aligned} \left|D_t^\lambda\right| \le{}& e^{-2\lambda(t-T_1)}\left|D_{T_1}^\lambda\right| + \int_{T_1}^t e^{-2\lambda(t-s)}\left(\left|f\left(\tilde{X}_s^\lambda\right)\right| + \left|g\left(\tilde{Y}_s^\lambda\right)\right|\right)ds \\ &+ |a|\left|\int_{T_1}^t e^{-2\lambda(t-s)}dL_t^1\right| + |b|\left|\int_{T_1}^t e^{-2\lambda(t-s)}dL_t^2\right|. \end{aligned} \tag{3.15}$$

By Lemma 2.1 we see that the radius $R_\lambda(\theta_t\omega)$ is pathwise uniformly bounded on each bounded time interval $[T_1, T_2]$, so we see that the right hand of above inequality converge to 0 as $\lambda \to \infty$ pathwise on the bounded time interval $[T_1, T_2]$. □

We now present the main result of this paper.

Theorem 3.3 (synchronization under Lévy noise). *Suppose that the coupled stochastic system in* $\mathbb{R}^{2d}$

$$\begin{aligned} dX_t &= \left(f(X_t) + \lambda(Y_t - X_t)\right)dt + a\,dL_t^1, \\ dY_t &= \left(g(Y_t) + \lambda(X_t - Y_t)\right)dt + b\,dL_t^2 \end{aligned} \tag{3.16}$$

defines a random dynamical system (θ, φ). *In addition, assume that the continuous functions* f, g *satisfy the integrability condition* (2.20) *as well as the one-sided Lipschitz dissipative condition* (2.16), *then the coupled stochastic system* (3.16) *is synchronized to a single averaged SDE in* $\mathbb{R}^d$

$$dZ_t = \frac{1}{2}\left[f(Z_t) + g(Z_t)\right]dt + \frac{a}{2}dL_t^1 + \frac{b}{2}dL_t^2, \tag{3.17}$$

in the sense that the stationary solutions of (3.16) *pathwise converge to that of* (3.17)*, that is,* $(\tilde{X}_t^\lambda, \tilde{Y}_t^\lambda) \to (Z_t^\infty, Z_t^\infty)$ *pathwise on any bounded time interval* $[T_1, T_2]$ *as parameter* $\lambda \to \infty$*.*

Proof. It is enough to demonstrate the result for any sequence $\lambda_n \to \infty$. Define

$$Z_t^\lambda := \frac{1}{2}\left[\tilde{X}_t^\lambda + \tilde{Y}_t^\lambda\right], \quad t \in \mathbb{R}. \tag{3.18}$$

Note that $Z_t^\lambda(\omega) = Z^\lambda(\theta_t\omega)$ satisfies the equation

$$dZ_t^\lambda = \frac{1}{2}\left[f\left(\tilde{X}_t^\lambda\right) + g\left(\tilde{Y}_t^\lambda\right)\right]dt + \frac{a}{2}dL_t^1 + \frac{b}{2}dL_t^2. \tag{3.19}$$

Also we define

$$\overline{Z}_t(\omega) = \overline{Z}(\theta_t\omega) := \frac{1}{2}\left[\overline{X}_t(\omega) + \overline{Y}_t(\omega)\right], \quad t \in \mathbb{R}, \tag{3.20}$$

where $\overline{X}_t$ and $\overline{Y}_t$ are the (stationary) solutions of the Langevin equations

$$dX_t = -X_t dt + a dL_t^1, \qquad dY_t = -Y_t dt + b dL_t^2, \tag{3.21}$$

that is,

$$\overline{X}_t = ae^{-t}\int_{-\infty}^t e^s dL_t^1, \qquad \overline{Y}_t = be^{-t}\int_{-\infty}^t e^s dL_t^2. \tag{3.22}$$

The difference $Z_t^\lambda - \overline{Z}_t$ satisfies

$$2\left(Z_t^\lambda - \overline{Z}_t\right) = 2\left(Z^\lambda - \overline{Z}\right) + \int_0^t \left(f\left(\tilde{X}_s^\lambda\right) + g\left(\tilde{Y}_s^\lambda\right) + \overline{X}_s + \overline{Y}_s\right)ds. \tag{3.23}$$

By Lemma 2.1, and the fact that these solutions belong to the common compact ball and every càdlàg function is bounded on finite closed intervals, we obtain

$$\left|f\left(\tilde{X}_t^\lambda(\omega)\right) + g\left(\tilde{Y}_t^\lambda(\omega)\right)\right| + \left|\overline{X}_t(\omega) + \overline{Y}_t(\omega)\right| \le M_{T_1,T_2}(\omega) < \infty, \tag{3.24}$$

which implies uniform boundedness as well as equicontinuity. Thus by the Ascoli-Arzela theorem [13], we conclude that for any sequence $\lambda_n \to \infty$, there is a random subsequence

$\lambda_{n_j}(\omega) \to \infty$, such that $Z_t^{\lambda_{n_j}}(\omega)-\overline{Z}_t(\omega) \to Z_t^{\infty}(\omega)-\overline{Z}_t(\omega)$ as $j \to \infty$. Thus $Z_t^{\lambda_{n_j}}(\omega) \to Z_t^{\infty}(\omega)$ as $j \to \infty$. Now, by Lemma 3.2

$$\begin{aligned} Z_t^{\lambda_{n_j}}(\omega) - \widetilde{Y}_t^{\lambda_{n_j}}(\omega) &= \frac{\widetilde{X}_t^{\lambda_{n_j}}(\omega) - \widetilde{Y}_t^{\lambda_{n_j}}(\omega)}{2} \longrightarrow 0, \\ Z_t^{\lambda_{n_j}}(\omega) - \widetilde{X}_t^{\lambda_{n_j}}(\omega) &= \frac{\widetilde{Y}_t^{\lambda_{n_j}}(\omega) - \widetilde{X}_t^{\lambda_{n_j}}(\omega)}{2} \longrightarrow 0, \end{aligned} \tag{3.25}$$

as $\lambda_{n_j} \to \infty$, so

$$\begin{aligned} \widetilde{X}_t^{\lambda_{n_j}}(\omega) &= 2Z_t^{\lambda_{n_j}}(\omega) - \widetilde{Y}_t^{\lambda_{n_j}}(\omega) \longrightarrow Z_t^{\infty}(\omega), \\ \widetilde{Y}_t^{\lambda_{n_j}}(\omega) &= 2Z_t^{\lambda_{n_j}}(\omega) - \widetilde{X}_t^{\lambda_{n_j}}(\omega) \longrightarrow Z_t^{\infty}(\omega), \end{aligned} \tag{3.26}$$

as $\lambda_{n_j} \to \infty$.

Using the integral representation of the equation, it can be verified that Z_t^{∞} is a solution of the averaged random differential equation (3.17) for all $t \in \mathbb{R}$. The drift of this SDE satisfies the dissipative one-sided condition (2.16). It has a random attractor consisting of a singleton set formed by a stationary orbit, which must be equal to Z_t^{∞}.

Finally, we note that all possible subsequences of $Z_t^{\lambda_n}$ have the same pathwise limit. Thus the full sequence $Z_t^{\lambda_n}$ converges to Z_t^{∞}, as $\lambda_n \to \infty$. This completes the proof. □

3.1. An Example

Consider two scalar SDEs:

$$dX_t = -(X_t+1)dt + dL_t^1, \qquad dY_t = -(Y_t+3)dt + 2dL_t^2, \tag{3.27}$$

which we rewrite as

$$dX_t = -X_t dt + dL_t^3, \qquad dY_t = -Y_t dt + 2dL_t^4, \tag{3.28}$$

where $L_t^3 = -t + L_t^1$ and $L_t^4 = -3t/2 + L_t^2$.

The corresponding coupled system (3.16) is

$$\begin{aligned} dX_t &= -X_t dt + \lambda(Y_t - X_t)dt + dL_t^3, \\ dY_t &= -Y_t dt + \lambda(X_t - Y_t)dt + 2dL_t^4 \end{aligned} \tag{3.29}$$

with the stationary solutions

$$\widetilde{X}_t^\lambda = \int_{-\infty}^t e^{-(\lambda+1)(t-s)} \cosh \lambda(t-s) dL_s^3 + 2\int_{-\infty}^t e^{-(\lambda+1)(t-s)} \sinh \lambda(t-s) dL_s^4,$$
$$\widetilde{Y}_t^\lambda = \int_{-\infty}^t e^{-(\lambda+1)(t-s)} \sinh \lambda(t-s) dL_s^3 + 2\int_{-\infty}^t e^{-(\lambda+1)(t-s)} \cosh \lambda(t-s) dL_s^4. \tag{3.30}$$

Let $\lambda \to \infty$, then

$$\left(\widetilde{X}_t^\lambda, \widetilde{Y}_t^\lambda\right) \longrightarrow (Z_t^\infty, Z_t^\infty), \tag{3.31}$$

where Z_t^∞, given by

$$Z_t^\infty = \int_{-\infty}^t \frac{1}{2} e^{-(t-s)} dL_s^3 + \int_{-\infty}^t e^{-(t-s)} dL_s^4, \tag{3.32}$$

is the stationary solution of the following averaged SDE:

$$dZ_t = -Z_t dt + \frac{1}{2} dL_t^3 + dL_t^4, \tag{3.33}$$

which is equivalent to the following SDE, in terms of the original Lévy motions L^1 and L^2,

$$dZ_t = -(Z_t + 2)dt + \frac{1}{2} dL_t^1 + dL_t^2. \tag{3.34}$$

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